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Abstract. In this paper, the concept of connected geodesic number, $gn_c(G)$, of a fuzzy graph G is introduced and its limiting bounds are identified. It is proved that all extreme nodes of G and all cut-nodes of the underlying crisp graph G^* belong to every connected geodesic cover of G. The connected geodesic number of complete fuzzy graphs, fuzzy cycles, fuzzy trees and of complete bipartite fuzzy graphs are obtained. It is proved that for any pair k, n of integers with $3 \leq k \leq n$, there exists a connected fuzzy graph $G : (V, \sigma, \mu)$ on n nodes such that $gn_c(G) = k$. Also, for any positive integers $2 \leq a < b \leq c$, it is proved that there exists a connected fuzzy graph $G : (V, \sigma, \mu)$ such that the geodesic number $gn(G) = a$ and the connected geodesic number $gn_c(G) = b$.

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1. Introduction

Zadeh in 1965 [35] developed a mathematical phenomenon for describing the uncertainties prevailing in day-to-day life situations by introducing the concept of fuzzy sets. The theory of fuzzy graphs was later on developed by Rosenfeld in the year 1975 [27] along with Yeh and Bang [34]. Rosenfeld also obtained the fuzzy analogue of several graph theoretic concepts like paths, cycles, trees and connectedness along with some of their properties [27] and the concept of fuzzy trees [23], automorphism of fuzzy graphs [10], fuzzy interval graphs [20], cycles and co-cycles of fuzzy graphs [21] etc has been established by several authors during the course of time. Akram *et.al.* in [4, 5, 6, 7, 8] introduced the concepts of bipolar fuzzy graphs and interval-valued fuzzy line graphs and established some of the properties satisfied by them. Fuzzy groups and the notion of a metric in fuzzy graphs were introduced by Bhattacharya [9]. Several other recent works on fuzzy graphs can be found in [2, 3, 14, 16, 24, 25, 30, 33, 36, 37].

The concept of strong arcs [13] was introduced by Bhutani and Rosenfeld in the year 2003. The definition of fuzzy end nodes and some of their properties were established by the same authors in [11]. The concept of geodesic distance was introduced by Bhutani and Rosenfeld in [12] and using this geodesic distance, Suvarna and Sunitha in [32] brought the concept of geodesic iteration number and geodesic number of a fuzzy graph into existence and studied some of the properties satisfied by them. The same concepts using μ-distance was introduced by Linda and Sunitha in [17].

Covering problems are among the fundamental problems in graph theory and some of them have also been introduced in fuzzy graphs such as fuzzy vertex covering problem, fuzzy edge covering problem, fuzzy minimum weight edge covering problem and so on. An important subclass of fuzzy covering problems is formed by path coverings, in particular, coverings with shortest paths or geodesics.The concept of geodesic numbers has many applications in location theory and convexity theory. In fact edge geodesic sets are more useful than geodesic sets in the case of transportation and routing problems. In this paper, the concept of connected geodesic number of a fuzzy graph is introduced and its limiting bounds are identified. It is proved that every connected geodesic cover of a connected fuzzy graph G contains all extreme nodes of G and all cut-nodes of G^*. The connected geodesic number of complete fuzzy graphs, fuzzy cycles, fuzzy trees and of complete bipartite fuzzy graphs are obtained.

2. Preliminaries

A graph is a pair (V, E), where V is a set and E is a relation on V. The elements of V are thought of as vertices of the graph and the elements of E are thought of as the edges. Sometimes, there can be vagueness in the description of vertices or in its relationships or in both. In such cases, designing a fuzzy graph model becomes useful as it is more realistic in natural situations. In this section, a brief summary of some basic definitions in fuzzy graph theory is given.

Definition 2.1 ([22])**.** A fuzzy graph $G : (V, \sigma, \mu)$ is a non-empty set V together with a pair of functions $\sigma : V \longrightarrow [0, 1]$ and $\mu : V \times V \longrightarrow [0, 1]$ such that for all $x, y \in V, \mu(x, y) \leq \sigma(x) \wedge \sigma(y)$. We call σ the fuzzy vertex set of G and μ the fuzzy edge set of G, respectively.

We assume that V is finite and non-empty, μ is reflexive $(i.e., \mu(x, x) = \sigma(x), \forall x)$ and symmetric $(i.e., \mu(x, y) = \mu(y, x), \forall(x, y))$. Also we denote the *underlying crisp graph* by $G^* : (\sigma^*, \mu^*)$ where $\sigma^* = \{x \in V/\sigma(x) > 0\}$ and $\mu^* = \{(x, y) \in V \times V/\mu(x, y) > 0\}$. Here we assume $\sigma^* = V$.

A fuzzy graph $G : (V, \sigma, \mu)$ is called trivial, if $|\sigma^*| = 1$. Otherwise it is called non-trivial.

Definition 2.2 ([22])**.** The fuzzy graph $H : (V, \tau, \nu)$ is called a partial fuzzy subgraph of $G : (V, \sigma, \mu)$, if $\tau \subseteq \sigma$ and $\nu \subseteq \mu$.

Similarly, the fuzzy graph $H : (P, \tau, \nu)$ is called a fuzzy subgraph of $G : (V, \sigma, \mu)$ induced by P, if $P \subseteq V$, $\tau(x) = \sigma(x)$, for all $x \in P$ and $\nu(x, y) = \mu(x, y)$, for all $x, y \in P$.

A fuzzy subgraph $H : (P, \tau, \nu)$ of a fuzzy graph $G : (V, \sigma, \mu)$ is in fact a special case of a partial fuzzy subgraph obtained as follows:

$$\tau(x) = \begin{cases} \sigma(x) & if\ x \in P \\ 0 & if\ x \in V - P, \end{cases}$$

$$\nu(x, y) = \begin{cases} \mu(x, y) & if\ (x, y) \in P \times P \\ 0 & if\ (x, y) \in V \times V - P \times P. \end{cases}$$

A fuzzy subgraph $H : (V, \tau, \nu)$ of $G : (V, \sigma, \mu)$ is said to span G, if $\sigma = \tau$. In this case, we call $H : (V, \tau, \nu)$ a spanning fuzzy subgraph of $G : (V, \sigma, \mu)$.

Definition 2.3 ([22]). A fuzzy graph $G : (V, \sigma, \mu)$ is a complete fuzzy graph, if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$, $\forall u, v \in \sigma^*$.

Definition 2.4 ([22]). A sequence of distinct nodes $u_0, u_1, ..., u_n$ such that $\mu(u_{i-1}, u_i) > 0, i = 1, 2, 3, ..., n$ is called a path P_n of length n.

Definition 2.5 ([23]). An arc (u, v) of $G : (V, \sigma, \mu)$ with least non-zero membership value is the weakest arc of G.

The degree of membership of a weakest arc in the path is defined as the strength of the path.

The path becomes a cycle, if $u_0 = u_n, n \geq 3$ and a cycle is called a fuzzy cycle, if it contains more than one weakest arc.

Definition 2.6 ([22]). The strength of connectedness between two nodes u and v is the maximum of the strengths of all paths between u and v and is denoted by $CONN_G(u, v)$.

The fuzzy graph $G : (V, \sigma, \mu)$ is said to be connected, if $CONN_G(u, v) > 0$, for every u, v in σ^*.

Definition 2.7 ([13]). An arc (u, v) of a fuzzy graph is called strong, if its weight is at least as great as the strength of connectedness of its end nodes u, v, when the arc (u, v) is deleted.

A $u - v$ path P is called a strong path, if P contains only strong arcs.

Definition 2.8 ([18]). Depending on the $CONN_G(u, v)$ of an arc (u, v) in a fuzzy graph G, three different types of arcs denoted by α, β and δ are defined. Note that $CONN_{G-(u,v)}(u, v)$ denote the strength of connectedness between u and v in the fuzzy graph G obtained by deleting the arc (u, v). Then

(i) an arc (u, v) in G is α-strong, if $CONN_{G-(u,v)}(u, v) < \mu(u, v)$,

(ii) an arc (u, v) in G is β-strong, if $CONN_{G-(u,v)}(u, v) = \mu(u, v)$,

(iii) an arc (u, v) in G is δ-arc, if $CONN_{G-(u,v)}(u, v) > \mu(u, v)$,

(iv) a δ-arc (u, v) is called a δ^*-arc, if $\mu(u, v) > \mu(x, y)$, where (x, y) is a weakest arc of G,

(v) an arc (u, v) in G is said to be strong, if it is either α-strong or β-strong,

(vi) a path P is called a strong path, if all arcs of P are either α-strong or β-strong.

Definition 2.9 ([27]). A connected fuzzy graph $G : (V, \sigma, \mu)$ is called a fuzzy tree, if it has a spanning fuzzy subgraph $F : (V, \sigma, \nu)$, which is a tree such that for all arcs (u, v) not in F, $CONN_F(u, v) > \mu(u, v)$.

Definition 2.10 ([27]). A node is a fuzzy cut node of $G : (V, \sigma, \mu)$, if removal of it reduces the strength of connectedness between some other pair of nodes.

Definition 2.11 ([11]). Two nodes u and v in a fuzzy graph $G : (V, \sigma, \mu)$ are neighbors (adjacent), if $\mu(u, v) > 0$ and v is called a strong neighbor of u, if the arc (u, v) is strong. Also $N(u)$ denotes the set of neighbors of u other than u, where as $N[u]$ denotes the set of neighbors of u including u. The degree of u is $deg(u) = |N(u)|$.

A node v is called a fuzzy end node of G, if it has at most one strong neighbor in G.

Definition 2.12 ([31]). A fuzzy graph G is said to be bipartite, if the vertex set V can be partitioned into two non-empty sets V_1 and V_2 such that $\mu(v_1, v_2) = 0$, if $v_1, v_2 \in V_1$ or $v_1, v_2 \in V_2$.

Further if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ $\forall u \in V_1$ and $\forall v \in V_2$, then G is called a complete bipartite fuzzy graph and is denoted by K_{σ_1,σ_2}, where σ_1 and σ_2 are respectively the restrictions of σ to V_1 and V_2.

Definition 2.13 ([12]). A strong path P from x to y is called geodesic, if there is no shorter strong path from x to y and the length of a $u - v$ geodesic is the geodesic distance from u to v, denoted by $d_g(u, v)$.

Definition 2.14 ([12]). The geodesic eccentricity (g-eccentricity), $e_g(u)$, of a node u in a connected fuzzy graph $G : (V, \sigma, \mu)$ is given by $e_g(u) = \max_{v \in V} d_g(u, v)$. A node u with maximum $e_g(u)$ is called a g-peripheral node or diametral node.

The g-diameter of G, $d_g(G) = \max\{e_g(v) : v \in V\}$ and the g-radius of G, $r_g(G) = \min\{e_g(v) : v \in V\}$.

Definition 2.15 ([15]). A crisp graph G is said to be connected, if any two distinct nodes of G are joined by a path.

A maximal connected subgraph of G is called a component of G and a cut-node of G is a node whose removal increases the number of components.

The following definitions and results have been taken from [12, 32].

Definition 2.16 ([12]). Let S be a set of nodes of a connected fuzzy graph G.

(i) The *geodesic closure* (S) of S is the set of all nodes in S together with the nodes that lie on geodesics between nodes of S.

(ii) S is said to be convex, if S contains all nodes of every $u - v$ geodesic for all u, v in S. i.e, if $(S) = S$.

(iii) S is said to be geodesic cover (geodesic set) of G, if $(S) = V(G)$.

(iv) Any cover of G with minimum number of nodes is called a geodesic basis for G. Order of a geodesic basis is the number of nodes in it.

Definition 2.17 ([32]). The geodesic number of a fuzzy graph $G : (V, \sigma, \mu)$ is the order of a geodesic basis of G and is denoted by $gn(G)$.

Proposition 2.18 ([26]). *For any non-trivial fuzzy graph G on n nodes, $2 \leq gn(G) \leq n$.*

Proposition 2.19 ([32]). *For a fuzzy cycle G on n nodes, $gn(G) = 2$, if n is even and $gn(G) = 3$, if n is odd.*

Proposition 2.20 ([32]). *For a complete fuzzy graph G on n nodes, $gn(G) = n$.*

Proposition 2.21 ([12]). *A fuzzy tree has a unique geodesic basis consisting of its fuzzy end nodes.*

3. Connected geodesic number of a fuzzy graph

The concept of connected geodetic number in graph theory was introduced and studied by Mojdeh and Rad in [19], by Santhakumaran *et.al.* in [28, 29] and by Hossein Abdollahzadeh Ahangar *et.al.* in [1]. This concept is extended to fuzzy graphs using geodesic distance and is named as connected geodesic number.

Definition 3.1. A connected geodesic cover (connected geodesic set) of a fuzzy graph $G : (V, \sigma, \mu)$ is a geodesic cover S of G such that the fuzzy subgraph induced by S, $< S >$, is connected.

Definition 3.2. The minimum cardinality of a connected geodesic cover of G is the connected geodesic number of G and is denoted by $gn_c(G)$.

A connected geodesic cover of G of cardinality $gn_c(G)$ is called a connected geodesic basis of G.

Example 3.3. Consider the fuzzy graph $G : (V, \sigma, \mu)$ given in Fig.1.

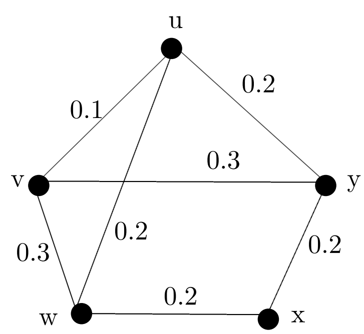

Fig.1.

Here, $S = \{w, y\}$ is a geodesic cover of G, since $(\{w, y\}) = V(G)$. However $< S >$ is not connected and then S is not a connected geodesic cover of G. But $S_1 = \{u, w, y\}$, $S_2 = \{v, w, y\}$ and $S_3 = \{x, w, y\}$ are all connected geodesic covers of G of minimum cardinality. Thus S_1, S_2 and S_3 are connected geodesic bases of G. So $gn_c(G) = 3$. Hence it can be concluded that there can be more than one connected geodesic basis for a fuzzy graph. Also note that the geodesic number and the connected geodesic number of a fuzzy graph need not be the same. Here, $gn(G) = 2 \neq gn_c(G) = 3$.

Proposition 3.4. *For any connected fuzzy graph $G : (V, \sigma, \mu)$ on n nodes, $2 \leq gn(G) \leq gn_c(G) \leq n$.*

Proof. By Proposition 2.18, it is clear that $gn(G) \geq 2$. Now by Definition 3.1, every connected geodesic cover is also a geodesic cover of G and so $gn(G) \leq gn_c(G)$. Also note that $V(G)$ induces a connected geodesic cover of G and then it is obvious that $gn_c(G) \leq n$. Thus, $2 \leq gn(G) \leq gn_c(G) \leq n$. $\square$

Corollary 3.5. *Let $G : (V, \sigma, \mu)$ be any connected fuzzy graph. If $gn_c(G) = 2$, then $gn(G) = 2$.*

Remark 3.6. The converse of Corollary 3.5 is not true. For example, the geodesic number of a path P on 3 nodes is 2, where the geodesic basis S is the set of end-nodes of P. However S is not a connected geodesic cover, since the fuzzy subgraph induced by S, $< S >$, is not connected. Then $gn_c(G) = 3 \neq 2 = gn(G)$.

Corollary 3.7. *Let $G : (V, \sigma, \mu)$ be any connected fuzzy graph on n nodes. If $gn(G) = n$, then $gn_c(G) = n$.*

Definition 3.8. A node v in a fuzzy graph G is called an extreme node, if the fuzzy subgraph induced by its neighbors is a complete fuzzy graph.

Proposition 3.9. *Each extreme node of a fuzzy graph $G : (V, \sigma, \mu)$ belongs to every geodesic cover of G.*

Proof. Let S be a geodesic cover of G and v be an extreme node of G. Let $\{v_1, v_2, ..., v_n\}$ be the neighbors of v and (v, v_i) $(1 \leq i \leq n)$ be the edges incident on v. Since v is an extreme node, v_i and v_j are adjacent for $i \neq j$ $(1 \leq i, j \leq n)$. Then any geodesic which contains v, is either (v_i, v) $(1 \leq i \leq n)$ or $u_1, u_2, ..., u_m, v_i, v$ where each u_i $(1 \leq i \leq m)$ is different from v_i. Thus it follows that $v \in S$. □

Proposition 3.10. *Every extreme node of a connected fuzzy graph $G : (V, \sigma, \mu)$ belongs to every connected geodesic cover of G.*

Proof. Since every connected geodesic cover is also a geodesic cover, the result follows from Proposition 3.9. □

Corollary 3.11. *For any connected fuzzy graph $G : (V, \sigma, \mu)$ with k extreme nodes, $gn_c(G) \geq max\{2, k\}$.*

Proof. The result follows from Propositions 3.4 and 3.10. □

Corollary 3.12. *The connected geodesic number of a complete fuzzy graph $G : (V, \sigma, \mu)$ on n nodes is n.*

Proof. Since each node in a complete fuzzy graph is an extreme node, the result follows from Proposition 3.10. Then the proof can also be obtained from Proposition 2.20 and Corollary 3.7. □

Proposition 3.13. *The connected geodesic number of a fuzzy cycle $G : (V, \sigma, \mu)$ on n nodes, $(n \geq 3)$, is given by:*

$$gn_c(G) = \begin{cases} \frac{n}{2} + 1 & \text{when } n \text{ is even} \\ \frac{(n-1)}{2} + 2 & \text{when } n \text{ is odd.} \end{cases}$$

Proof. Case(1): Suppose n is even. Let $n = 2k$ and let $v_1, v_2, v_3, ..., v_{2k}, v_1$ be the fuzzy cycle G on n nodes. Since all arcs in G are strong [11], v_{k+1} is the g-peripheral node of v_1 in G. Take $S = \{v_1, v_{k+1}\}$. Then clearly, S is a geodesic cover of G. However it is clear that $< S >$ is not connected. But $S_1 = \{v_1, v_2, ..., v_{k+1}\}$ is a connected geodesic cover of G. Thus $gn_c(G) \leq k + 1$.

Claim: S_1 is a connected geodesic cover of minimum cardinality.

If possible suppose S' is any connected geodesic cover of G with $|S'| < |S_1|$. Then S' contains at most k nodes. Thus no two nodes of S' are g-peripheral to each other. So S' is not a geodesic cover of G, which is a contradiction. Hence, S_1 is a connected geodesic cover of G of minimum cardinality. Therefore $gn_c(G) = k + 1$, i.e., $gn_c(G) = \frac{n}{2} + 1$.

Case(2): suppose n is odd. Let $n = 2k + 1$ and let $v_1, v_2, v_3, ..., v_{2k+1}, v_1$ be the fuzzy cycle G on n nodes. Again, since all arcs in G are strong [11], v_{k+1} and v_{k+2} are the g-peripheral nodes of v_1 in G. Take $S = \{v_1, v_{k+1}, v_{k+2}\}$. Then clearly, S is a geodesic cover of G but $< S >$ is not connected. Thus S is not a connected geodesic cover of G. However, $S_2 = \{v_1, v_2, ..., v_{k+1}, v_{k+2}\}$ is a connected geodesic cover of G. So $gn_c(G) \leq k + 2$.

Claim: S_2 is a connected geodesic cover of minimum cardinality.

If possible suppose S" is any connected geodesic cover of G with $|S"| < |S_2|$. Then S" contains at most $k+1$ nodes of G. Also S" contains at most 2 nodes say u and v that are g-peripheral to each other. Let $w \neq v$ be a g-peripheral node of u in G. Then the node w does not lie on any geodesic joining a pair of nodes of S". Thus S" is not a geodesic cover of G, which is a contradiction. So, $gn_c(G) = k + 2 = \frac{(n-1)}{2} + 2$. □

Proposition 3.14. *Let $G : (V, \sigma, \mu)$ be a connected fuzzy graph such that the underlying crisp graph G^* contains at least one cut-node and let S be a connected geodesic cover of G. If v is a cut-node of G^*, then every component of $G^* - \{v\}$ contains an element of S.*

Proof. Let v be a cut-node of G^* and let S be a connected geodesic cover of G. Suppose that there exists a component, say G_1^*, of $G^* - \{v\}$ such that G_1^* contains no node of S. Let $u \in V(G_1^*)$. Since S is a connected geodesic cover of G, there exists a pair of nodes x and y in S such that u lies on some $x - y$ geodesic $P : x = u_0, u_1, ..., u, ..., u_n = y$ in G. Since v is a cut-node of G^*, the $x - u$ geodesic sub path of P and the $u - y$ geodesic sub path of P both contain v. Then it follows that P is not a path, contrary to assumption. □

Proposition 3.15. *Let $G : (V, \sigma, \mu)$ be a connected fuzzy graph such that G^* contains at least one cut-node. Then every cut-node of G^* belongs to every connected geodesic cover of G.*

Proof. Let v be a cut-node of G^* and let $G_1^*, G_2^*, ..., G_r^*$ ($r \geq 2$) be the components of $G^* - \{v\}$. Let S be any connected geodesic cover of G. Then by Proposition 3.14, S contains at least one element from each G_i^* ($l \leq i \leq r$). Since $< S >$ is connected, it follows that $v \in S$. □

Corollary 3.16. *Let $G : (V, \sigma, \mu)$ be a connected fuzzy graph with k extreme nodes such that the underlying crisp graph G^* contains l cut-nodes. Then $gn_c(G) \geq max\{2, k + l\}$.*

Proof. This follows from Propositions 3.4, 3.10 and 3.15. □

Example 3.17. Consider the fuzzy graph $G : (V, \sigma, \mu)$ given in Fig.2.

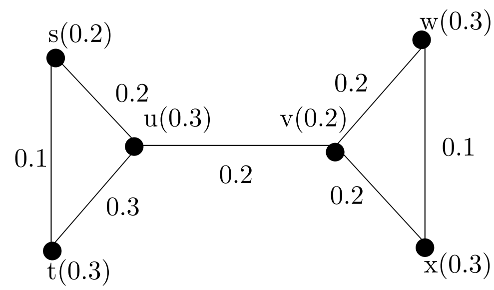

Fig.2.

Here, the extreme nodes of G are s, t, w and x. Then the number of extreme nodes of G, $k = 4$. Thus the cut nodes of the underlying crisp graph G^* are u and v. So the number of cut nodes in G^*, $l = 2$.

Now, $\{s, t, w, x\}$ is a geodesic basis of G. However, it is not connected. Then u and v should be included in the geodesic basis to make it connected. Thus, $\{s, t, u, v, w, x\}$ is the unique connected geodesic basis of G. So $gn_c(G) = k + l = 4 + 2 = 6$.

Proposition 3.18. *The connected geodesic number of a fuzzy tree $G : (V, \sigma, \mu)$ on n nodes containing no δ-arcs is n.*

Proof. By Proposition 2.21, it follows that the set S of all fuzzy end nodes of $G : (V, \sigma, \mu)$ form the unique geodesic basis of G. Since G is a fuzzy tree containing no δ-arcs, the fuzzy end nodes of G are indeed the end nodes of $G*$. However S is not a connected geodesic cover since the fuzzy subgraph induced by S, $< S >$, is not connected. Also, by Proposition 3.15, every cut node of $G*$ belong to every connected geodesic cover of G. Hence, the entire node set $V(G)$ is the unique connected geodesic basis of G. Therefore $gn_c(G) = n$. □

Proposition 3.19. *Let $G : (V, \sigma, \mu)$ be a connected fuzzy graph on n nodes such that every node of G is either a cut-node of G^* or an extreme node of G. Then the connected geodesic number of G, $gn_c(G) = n$.*

Proof. The proof follows from Propositions 3.10 and 3.15. □

Remark 3.20. The converse of Proposition 3.19 need not be true. For example, the fuzzy graph G given in Fig.3 on 8 nodes is a fuzzy tree and so by Proposition 3.18, the connected geodesic number $gn_c(G) = 8$. Note that the nodes u, v, w and x are cut-nodes of G^*. However, the nodes s, t, y and z are not extreme nodes as the subgraph induced by their neighbors are not complete fuzzy graphs.

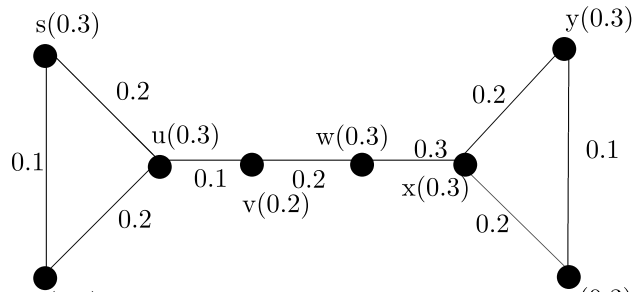

Fig.3.

Proposition 3.21. *Let $K_{\sigma_1,\sigma_2} = (V_1 \cup V_2, \sigma, \mu)$ be a complete bipartite fuzzy graph. Then*

(1) $gn_c(K_{\sigma_1,\sigma_2}) = 2$, *if* $|V_1| = |V_2| = 1$,
(2) $gn_c(K_{\sigma_1,\sigma_2}) = |V_2| + 1$, *if* $|V_1| = 1$ *and* $|V_2| \geq 2$,
(3) $gn_c(K_{\sigma_1,\sigma_2}) = min\{r, s\} + 1$, *if* $|V_1| = r$ *and* $|V_2| = s$, *where* $r, s \geq 2$.

Proof. (1) The proof follows from Corollary 3.12.

(2) The proof follows from Proposition 3.18.

(3) Let $r, s > 2$. First assume that $r < s$. Let $V_1 = \{u_1, u_2, ..., u_r\}$ and $V_2 = \{w_1, w_2, ..., w_s\}$ be a bipartition of K_{σ_1,σ_2}. Take $S = V_1 \cup \{w_1\}$.

Claim: S is a connected geodesic basis of K_{σ_1,σ_2}.

It is clear that any node $w_j (1 \leq j \leq s)$ lies on the geodesic $u_i w_j u_k$ for any $k \neq i$ so that S is a geodesic cover of K_{σ_1,σ_2}. Since $< S >$ is connected, S is a connected geodesic cover of K_{σ_1,σ_2}. To show that S is a connected geodesic cover of K_{σ_1,σ_2} having minimum cardinality, let T be any set of nodes such that $|T| < |S|$.

If $T \subset V_1$, then $< T >$ is not connected and so T is not a connected geodesic cover of K_{σ_1,σ_2}.

If $T \subset V_2$, then again by a similar argument, T is not a connected geodesic cover of K_{σ_1,σ_2}.

If $T \supseteq V_1$, then since $|T| < |S|$, we have $T = V_1$, which is not a connected geodesic cover of K_{σ_1,σ_2}.

If $T \supseteq V_2$, then $|T| \geq |V_2| = s > r$, i.e., $|T| > |S|$ which is a contradiction. Thus $T \subset V_1 \cup V_2$ such that T contains at least one node from each of V_1 and V_2. Since $|T| < |S|$, there exist nodes $u_i \in V_1$ and $w_j \in V_2$ such that $u_i \notin T$ and $w_j \notin T$. So clearly, at least one of the end nodes of the edge (u_i, w_j) does not lie on a geodesic connecting two nodes of T so that T is not a connected geodesic cover. Hence in any case, T is not a connected geodesic cover of K_{σ_1,σ_2}. Therefore S is a connected geodesic basis so that $gn_c(K_{\sigma_1,\sigma_2}) = |S| = r + 1$.

Now, if $r = s$, we can prove similarly that $S = \{x\} \cup V_2$, where $x \in V_1$ or $S = V_1 \cup \{y\}$, where $y \in V_2$ is a connected geodesic basis of G. □

Proposition 3.22. *If $G : (V, \sigma, \mu)$ is a connected fuzzy graph on $n \geq 3$ nodes containing no δ-arcs such that v is a cut-node of G^* of degree $n-1$, then $gn_c(G) = n$.*

Proof. Let S be any connected geodesic cover of G and v be a cut node of G^* of degree $n-1$. Then, by Proposition 3.15, $v \in S$.

Claim: $S = V(G)$ is a connected geodesic basis of G .

Otherwise, there exists a set $T \subset V(G)$ such that T is a connected geodesic cover of G. By Proposition 3.15, $v \in T$. Since $T \subset V(G)$, there exists a node $u \in V$ such that $u \notin T$. Since T is a connected geodesic cover of G, the node u lies on a geodesic joining a pair of nodes x and y of T.

Let the geodesic be $P : x, ..., v, u, ..., y$. Then we have $u \neq x, y$.

If $x = v$, then, since v is adjacent to every node of G, the arc (v, y) is the only geodesic joining v and y.

If $x \neq v$, then $x - v - y$ is the only geodesic joining x and y. Thus in any case P is not an $x - y$ geodesic, which is a contradiction. So $S = V(G)$ is the only connected geodesic basis of G. Hence $gn_c(G) = n$. □

Remark 3.23. The converse of Proposition 3.22 is false. For the fuzzy graph G given in Fig.4, $S = \{u, v, w, x, y\}$ is a connected geodesic basis of G and then, $gn_c(G) = 5$. But no node of G has degree $5 - 1$.

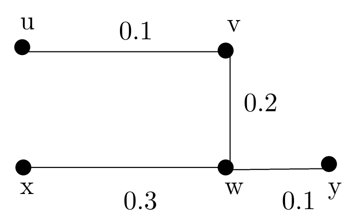

Fig.4.

Proposition 3.24. *For any pair k, n of integers with $3 \leq k \leq n$, there exists a connected fuzzy graph $G : (V, \sigma, \mu)$ on n nodes such that $gn_c(G) = k$.*

Proof. Construct a connected fuzzy graph $G : (V, \sigma, \mu)$ on n nodes having connected geodesic number k as follows:
Let $P_k : (V_1, \sigma_1, \mu_1)$ be a path on k nodes say $u_1, u_2, u_3, ..., u_k$ with $\sigma_1(u_i) = 0.i$, $(1 \leq i \leq k)$ and $\mu_1(u_i, u_{i+1}) = \sigma_1(u_i) \wedge \sigma_1(u_{i+1})$, $(1 \leq i \leq k-1)$.
Now, add $n - k$ new nodes $v_1, v_2, ..., v_{n-k}$ and join each v_j, $(1 \leq j \leq n-k)$ with u_1 and u_3, thereby obtaining a fuzzy graph $G : (V, \sigma, \mu)$ with each node having membership value $\sigma(v_j) = \wedge\{\sigma_1(u_i)\}$, $(1 \leq i \leq k)$. Put $\sigma = \sigma_1$ and $\mu = \mu_1$ on P_k. Set the membership values of remaining edges as $\mu(u_i, v_j) = \sigma_1(u_i) \wedge \sigma(v_j)$, $i = 1, 3$ and $1 \leq j \leq n-k$. The fuzzy graph $G : (V, \sigma, \mu)$ thus obtained is as shown in Fig.5.

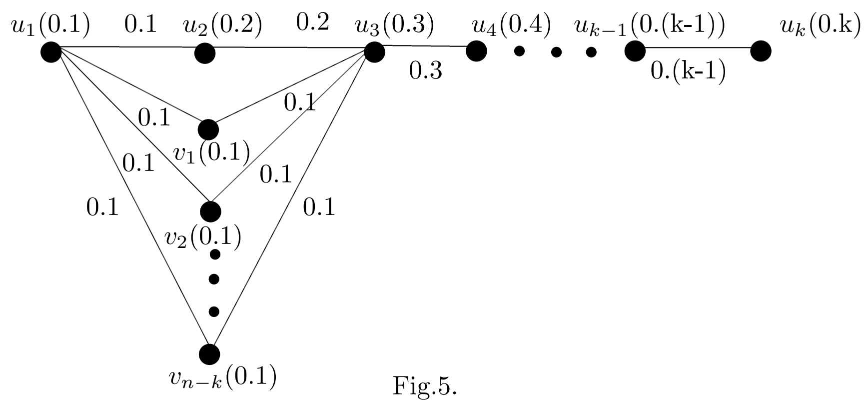

Fig.5.

Then $G : (V, \sigma, \mu)$ is a connected fuzzy graph on n nodes. Here, $S_1 = \{u_3, u_4, ..., u_{k-1}\}$ is the set of all cut nodes of the underlying crisp graph G^* and $S_2 = \{u_k\}$ is the only extreme node of G. Take $S = S_1 \cup S_2 = \{u_3, u_4, ..., u_k\}$. Now, it follows from Proposition 3.15 that every cut node of G^* belongs to every connected geodesic cover of G and from Proposition 3.10, it follows that every extreme node of G belongs to every connected geodesic cover of G. Since S consists of cut nodes of G^* and extreme nodes of G, any connected geodesic cover of G should contain S and thus $gn_c(G) \geq |S| = k - 2$.
But S is not a geodesic cover of G, since $(S) = S \neq V(G)$ and hence S is not a connected geodesic cover of G. Hence $gn_c(G) > k - 2$.
Note that neither $S \cup \{v_j\}$, $(1 \leq j \leq n - k)$ nor $S \cup \{u_2\}$ is a geodesic cover of G. Now, $T = S \cup \{u_1\}$ is a geodesic cover of G but $< T >$ is disconnected. However, $T \cup \{u_2\}$ is a connected geodesic cover of G of minimum cardinality. Therefore, the connected geodesic number, $gn_c(G) = k$. □

Proposition 3.25. *For any positive integers a, b and c, $2 \leq a < b \leq c$, there exists a connected fuzzy graph $G : (V, \sigma, \mu)$ such that $gn(G) = a$ and $gn_c(G) = b$.*

Proof. If $2 \leq a < b = c$, then consider $G : (V, \sigma, \mu)$ to be any fuzzy tree on $'\, b\, '$ nodes with $'\, a\, '$ fuzzy end nodes such that G contains no δ-arcs. Then by Proposition 2.21, $gn(G) = a$ and by Proposition 3.18, $gn_c(G) = b$.
If $2 \leq a < b < c$, then consider the following cases.

Case(1): Let $a > 2$ and $b - a \geq 2$. Then $b - a + 2 \geq 4$.

Construct a connected fuzzy graph $G : (V, \sigma, \mu)$ such that its geodesic number $gn(G) = a$ and its connected geodesic number $gn_c(G) = b$ as follows:
Let $P_{b-a+2} : (V_1, \sigma_1, \mu_1)$ be a path on $b - a + 2$ nodes say $v_1, v_2, ..., v_{b-a+2}$ with $\sigma_1(v_i) = 0.i$, $(1 \leq i \leq b-a+2)$ and $\mu_1(v_i, v_{i+1}) = \sigma_1(v_i) \wedge \sigma_1(v_{i+1})$, $(1 \leq i \leq b-a+1)$.
Now, construct the fuzzy graph $G : (V, \sigma, \mu)$ with
$V = V_1 \cup \{w_1, w_2, ..., w_{c-b+1}, u_1, u_2, ..., u_{a-2}\}$ where $\sigma(w_j) = \sigma(u_k) = \wedge\{\sigma_1(v_i)\}$, $(1 \leq i \leq b - a + 2, 1 \leq j \leq c - b + 1, 1 \leq k \leq a - 2)$, $\sigma = \sigma_1$ on P_{b-a+2} and $\mu(w_j, v_1) = \sigma(w_j) \wedge \sigma_1(v_1)$, $\mu(w_j, v_3) = \sigma(w_j) \wedge \sigma_1(v_3), (1 \leq j \leq c - b + 1)$,

$\mu(u_k, v_2) = \sigma(u_k) \wedge \sigma_1(v_2), (1 \leq k \leq a-2)$, $\mu = \mu_1$ on P_{b-a+2}. The fuzzy graph $G : (V, \sigma, \mu)$ is as shown in Fig.6.

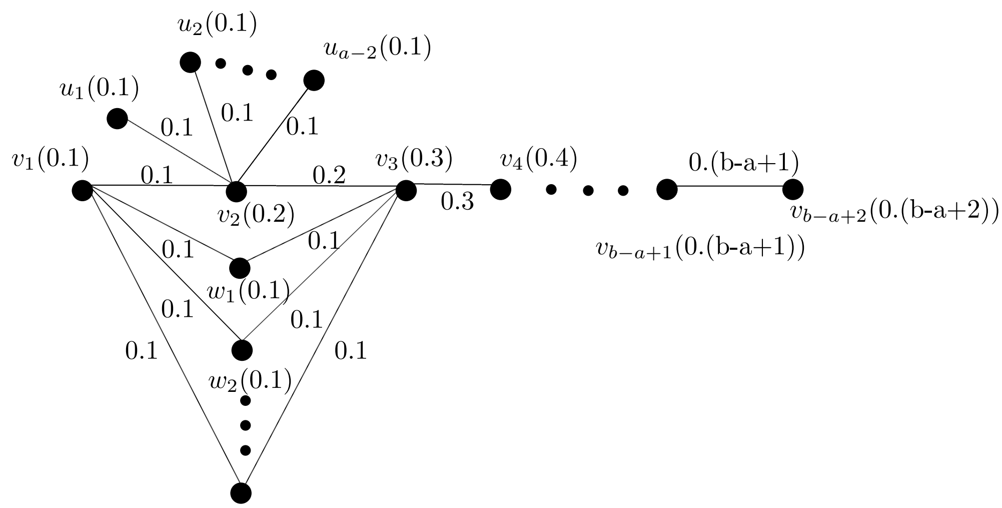

Fig.6.

Note that $S = \{u_1, u_2, ..., u_{a-2}, v_{b-a+2}\}$ is the set of all extreme nodes of G. By Proposition 3.10, every connected geodesic cover of G contains S.
It is clear that S is not a geodesic cover of G since
$(S) = \{u_1, u_2, ..., u_{a-2}, v_2, v_3, ..., v_{b-a+2}\} \neq V(G)$, and hence not a connected geodesic cover of G. But note that $S \cup \{v_1\}$ is a geodesic cover of G of minimum cardinality and hence the geodesic number, $gn(G) = a$.
Here, $S_1 = \{v_2, v_3, ..., v_{b-a+1}\}$ is the set of all cut nodes of the underlying crisp graph G^*. Now let $S_2 = S \cup S_1$. Thus S_2 contains all cut nodes of G^* and also all extreme nodes of G. Then it follows from Proposition 3.10 and Proposition 3.15 that every connected geodesic cover of G contains S_2. It is clear that S_2 is not a geodesic cover of G, since $(S_2) = S_2 \neq V(G)$. But $S_2 \cup \{v_1\}$ is a geodesic cover of G of minimum cardinality and also note that $S_2 \cup \{v_1\}$ is connected. So, $S_2 \cup \{v_1\}$ is a connected geodesic basis of G so that $gn_c(G) = |S_2 \cup \{v_1\}| = a-2+1+b-a+1 = b$.

Case(2): Let $a > 2$ and $b - a = 1$. Since $c > b$, we have $c - b + 1 \geq 2$.

Consider the fuzzy graph $H = G - \{v_4, v_5, ..., v_{b-a+2}\}$ which is as shown in Fig.7.

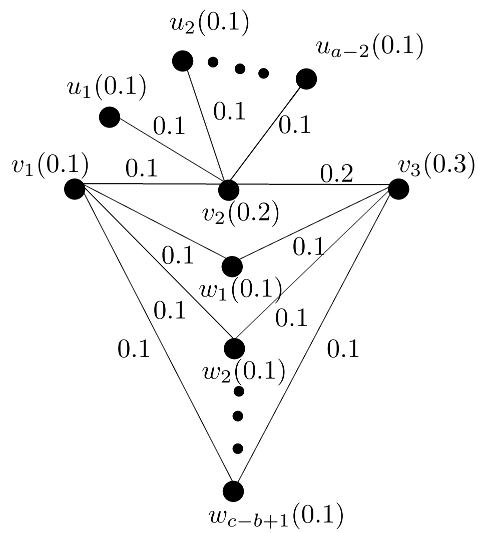

Fig.7.

Note that here, $S_3 = \{u_1, u_2, ..., u_{a-2}, v_1, v_3\}$ is a geodesic basis of G and $S_4 = S_3 \cup \{v_2\}$ is a connected geodesic basis of G. Hence $gn(G) = |S_3| = a$ and $gn_c(G) = |S_4| = a + 1 = b$.

Case(3): Let $a = 2$ and $b - a = 1$. Then $b = 3$.

Consider the fuzzy graph $H^{'} = H - \{u_1, u_2, ..., u_{a-2}\}$ which is as shown in Fig.8.

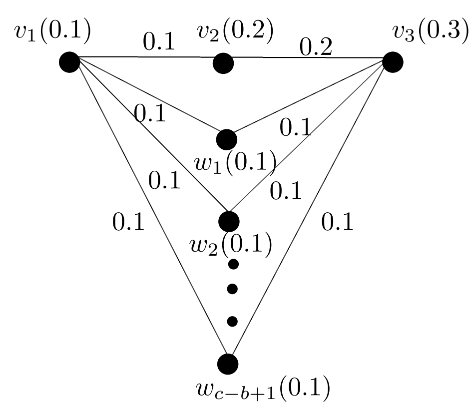

Fig.8.

Note that here $S_5 = \{v_1, v_3\}$ is a geodesic basis of G and $S_6 = \{v_1, v_2, v_3\}$ is a connected geodesic basis of G. Hence $gn(G) = |S_5| = 2 = a$ and $gn_c(G) = |S_6| = 3 = b$.

Case(4): Let $a = 2$ and $b - a \geq 2$. Then $b \geq 4$.

Consider the fuzzy graph $G^{'} = G - \{u_1, u_2, ..., u_{a-2}\}$ which is as shown in Fig.9.

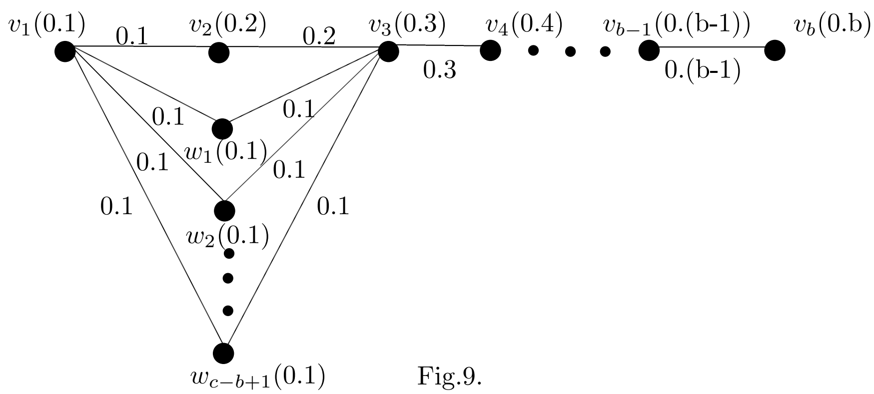

Fig.9.

Here, $S_7 = \{v_1, v_b\}$ is a geodesic basis of G and $S_8 = \{v_1, v_2, ..., v_b\}$ is a connected geodesic basis of G. Hence $gn(G) = |S_7| = 2 = a$ and $gn_c(G) = |S_8| = b$. □

4. Conclusions

In this paper, the concept of connected geodesic number of a fuzzy graph is introduced and its limiting bounds are identified. It is proved that all extreme nodes of a connected fuzzy graph G and all cut-nodes of its underlying crisp graph G^* belong to its connected geodesic cover. The connected geodesic number of complete fuzzy graphs, fuzzy cycles, fuzzy trees and of complete bipartite fuzzy graphs are obtained. For any pair k, n of integers with $3 \leq k \leq n$, it is proved that there exists a connected fuzzy graph $G : (V, \sigma, \mu)$ on n nodes such that $gn_c(G) = k$. Also, it is proved that for any positive integers $2 \leq a < b \leq c$, there exists a connected fuzzy graph $G : (V, \sigma, \mu)$ such that $gn(G) = a$ and $gn_c(G) = b$.

By introducing the concept of connected geodesic number in fuzzy graphs, one can obtain the fuzzy analogue of the concepts of forcing connected geodetic number and upper connected geodetic number that have already been established in graph theory.

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