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Abstract. Many generalizations of ideals of a lattice to an arbitrary poset have been studied by different scholars. In this paper, we introduce several L-fuzzy ideals of a poset which generalize the notion of an L-fuzzy ideal of a lattice and give several characterizations of them.

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Keywords: Poset, Ideal, L-fuzzy closed Ideal, L-fuzzy Frink Ideal, L-fuzzy V-Ideal, L-fuzzy M-ideal, L-fuzzy Semi-ideal, L-Fuzzy ideal, u-L-fuzzy ideal.

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1. Introduction

We have found several generalizations of ideals of a lattice to arbitrarily partially ordered set (poset) in a literature which has been studied by different authors. Closed ideals or normal ideals of a poset were introduced by Birkhoff [2], who gives credit to Stone[15] for the case of Boolean algebras. Next, in 1954, the second type of ideal of a poset called Frink ideal has been introduced by Frink [6]. Following this Venkatanarasimhan developed the theory of semi-ideals and ideals for posets [17] and [18], in 1970. These ideals are called ideals in the sense of Venkataranasimhan or V-ideals for short. Next, the concept of ideals of a poset have been suggested by Erné [4] in 1979 which are called m-ideal. This ideal generalize almost all ideals of a poset suggested by different authors. Latter, Halaś [9] in 1994, introduced a new ideal of a poset which seems to be a suitable generalization of the usual concept of ideal in a lattice. We will simply call it ideal in the sense of Halaš.

On the other hand, the notion of fuzzy ideals of a lattice has been studied by different authors in series of papers [1, 14, 16, 19].

In this paper we introduce several generalizations of fuzzy ideals of a lattice to an arbitrary poset whose truth values are in a complete lattice satisfying the infinite meet distributive law and give several characterizations of them. We also prove that the set of all L-fuzzy ideals of a poset forms a complete lattice with respect to point-wise ordering. Throughout this work L stands for a non-trivial complete lattice satisfying the infinite meet distributive law: $a \wedge \sup S = \sup\{a \wedge s : s \in S\}$, for any $a \in L$ and for any subset S of L.

2. Preliminaries

We briefly recall certain necessary concepts, terminologies and notations from [2, 3, 8].

A binary relation " $\leq$ " on a set Q is called a partial order, if it is reflexive, anti-symmetric and transitive. A pair $(Q, \leq)$ is called a partially ordered set or simply a poset, if Q is a non-empty set and $\leq$ is a partial order on Q. When confusion is unlikely, we use simply the symbol Q to denote a poset $(Q, \leq)$.

Let Q be a poset and $A \subseteq Q$. Then the set $A^u = \{x \in Q : x \geq a \ \forall a \in A\}$ is called the upper cone of A and the set $A^l = \{x \in Q : x \leq a \ \forall a \in A\}$ of A is called the lower cone of A. A^{ul} shall mean $\{A^u\}^l$ and A^{lu} shall mean $\{A^l\}^u$. Let $a, b \in Q$. Then the upper cone $\{a\}^u$ is simply denoted by a^u and the upper cone $\{a, b\}^u$ is denoted by $(a, b)^u$. Similar notations are used for lower cones. We note that $A \subseteq A^{ul}$ and $A \subseteq A^{lu}$ and if $A \subseteq B$ in Q, then $A^l \supseteq B^l$ and $A^u \supseteq B^u$. Moreover, $A^{lul} = A^l$, $A^{ulu} = A^u$, $\{a^u\}^l = a^l$ and $\{a^l\}^u = a^u$.

An element x_0 in Q is called the least upper bound of A or supremum of A, denoted by $supA$ (receptively, the greatest lower bound of A or infimum of A, denoted by $infA$), if $x_0 \in A^u$ and $x_0 \leq x$, for each $x \in A^u$ (respectively, if $x_0 \in A^l$ and $x \leq x_0$, for each $x \in A^l$).

An element x_0 in Q is called the largest (respectively, the smallest) element, if $x \leq x_0$ (respectively, $x_0 \leq x$), for all $x \in Q$. The largest (respectively, the smallest) element, if it exists in Q, is denoted by 1 (respectively, by 0).

A poset $(Q \leq)$ is called bounded, if it has 0 and 1. Note that if $A = \varnothing$, we have $A^{ul} = (\varnothing^u)^l = Q^l$ which is either empty or consists of the least element 0 of Q alone, if it exists.

Now we recall definitions of ideals of a poset that are introduced by different scholars.

Definition 2.1. (i) [2] A subset I of a poset Q is called a closed or normal ideal of Q, if $I^{ul} \subseteq I$ (or equivalently, $I^{ul} = I$, since $I \subseteq I^{ul}$).

(ii) [6] A subset I of a poset Q is called a Frink ideal in Q if $F^{ul} \subseteq I$, whenever F is a finite subset of I. ,

(iii) [17] A non-empty subset I of a poset Q is called a semi-ideal or an order ideal of Q, if $a \leq b$ and $b \in I$ implies $a \in I$.

(iv) [18] A subset I of a poset Q is called a V-ideal or an ideal in the sense of Venkatannarasimhan, if I is a semi-ideal and for any non-empty subset $A \subseteq I$, if $\sup A$ exists, then $\sup A \in I$.

(v) [9] A subset I of a poset Q is called an ideal in Q in the sense of Halaš, if $(a, b)^{ul} \subseteq I$, whenever $a, b \in I$

Note that every ideal of a poset Q contains Q^l. The following definition generalize all the definitions of ideal given above.

Definition 2.2 ([4]). Let Q be a poset and m denote any cardinal number. Then a subset I of a poset Q is called an m-ideal in Q, if for any subset A of I of cardinality strictly less than m, written as $A \subset_m I$, we have $A^{ul} \subseteq I$.

Remark 2.3 ([5]). The following special cases are included in this general definition:

(1) 2-ideals are semi-ideals containing Q^l.
(2) 3-ideals are ideals in the sense of Halaś containing Q^l.
(3) ω-ideals are Frinkideals containing Q^l where ω the least infinite cardinal number.
(4) Ω-ideals are closed ideals, where the symbol Ω mean if I has cardinality κ then Ω is a cardinal greater than κ.
(5) V-ideals are 2-ideals which are closed under finite supremum and containing Q^l.

Remark 2.4. The following remarks are due to Halaš and Rachunek [11].

(1) if Q is a lattice then a non-empty subset I of Q is an ideal as a poset if and only if it is an ideal as a lattice.
(2) if a poset Q does not have the least element then the empty subset $\varnothing$ is an ideal in Q (since $\varnothing^{ul} = (\varnothing^u)^l = Q^l = \varnothing$).

Definition 2.5. Let A be any subset of a poset Q. Then the smallest ideal containing A is called an ideal generated by A and is denoted by $(A]$. The ideal generated by a singleton set $A = \{a\}$, is called principal ideal and is denoted by $(a]$.

Note that for any subset A of Q if $\sup A$ exists then $A^{ul} = (\sup A]$.

The followings are some characterizations of ideals generated by a subset A of a poset Q. We write $F \subset\subset A$ to mean F is a finite subset of A.

(1) $(A]_C = \bigcup\{B^{ul} : B \subseteq A\}$ is the closed ideal or normal ideal generated by A where the union is taken overall subsets B of A.
(2) $(A]_F = \bigcup\{F^{ul} : F \subset\subset A\}$ is the Frink ideal generated by A, where the union is taken overall finite subsets F of A
(3) Define $C_1 = \bigcup\{(a,b)^{ul} : a,b \in A\}$ and $C_n = \bigcup\{(a,b)^{ul} : a,b \in C_{n-1}\}$ for each positive integer $n \geq 2$, inductively. Then $(A]_H = \bigcup\{C_n : n \in \mathcal{N}\}$ is the ideal generated by A in the sense of Halaś, where $\mathcal{N}$ denotes the set of positive integers.
(4) If $a \in Q$, then $(a] = \{x \in Q : x \leq a\} = a^l$ is the principal ideal generated by a.

Lemma 2.6 ([10]). *Let $\mathcal{I}(Q)$ be the set of all ideals of a poset Q in the sense of Halaś and $I, J \in \mathcal{I}(Q)$. Then the supremum $I \vee J$ of I and J in $\mathcal{I}(Q)$ is:*

$$I \vee J = \bigcup\{C_n : n \in \mathcal{N}\},$$

where $C_1 = \bigcup\{(a,b)^{ul} : a,b \in I \cup J\}$ and $C_n = \bigcup\{(a,b)^{ul} : a,b \in C_{n-1}\}$, for each positive integer $n \geq 2$.

Definition 2.7 ([9]). An ideal I of a poset Q is called a u-ideal, if $(x,y)^u \cap I \neq \varnothing$, for all $x, y \in I$.

Note that an easy induction shows I is a u-ideal, if $F^u \cap I \neq \varnothing$, for any finite subset F of I.

Theorem 2.8 ([9]). *Let $\mathcal{I}(Q)$ be the set of all ideals of Q in the sense of Halaś and I, J be u-ideals of a poset Q. Then the supremum $I \vee J$ of I and J in $\mathcal{I}(Q)$ is:*

$$I \vee J = \bigcup \{(a, b)^{ul} : a \in I, b \in J\}.$$

Definition 2.9 ([7]). Let X be a non-empty set. An L-fuzzy subset μ of X is a mapping from X into L, where L is a complete lattice satisfying the infinite meet distributive law.

Note that if L is a unit interval of real numbers, then μ is the usual fuzzy subset of X originally introduced by Zadeh [20].

Definition 2.10 ([16]). Let μ be an L-fuzzy subset of X. Then for each $\alpha \in L$, the set $\mu_\alpha = \{x : \mu(x) \geq \alpha\}$ is called the level subset of μ at α.

Lemma 2.11 ([12]). *Let μ be an L- fuzzy subset of a poset Q. Then $\mu(x) = \sup\{\alpha \in L : x \in \mu_\alpha\}$, for all $x \in Q$.*

Definition 2.12 ([7]). Let L be a complete lattice satisfying the infinite meet distributivity and X be a non-empty set. For any L-fuzzy subsets μ and σ, define $\mu \subseteq \sigma$ if and only $\mu(x) \leq \sigma(x)$, for all $x \in X$.

It can be easily verified that $\subseteq$ is a partial order on the set L^X of L- fuzzy subsets of X and is called the point wise ordering.

Definition 2.13 ([13]). Let μ and σ be an L-fuzzy subsets a non-empty set X. The union of fuzzy subsets μ and σ of X, denoted by $\mu \cup \sigma$, is a fuzzy subset of X defined by: for all $x \in X$,

$$(\mu \cup \sigma)(x) = \mu(x) \vee \sigma(x)$$

and the intersection of fuzzy subsets μ and σ of X, denoted by $\mu \cap \sigma$, is a fuzzy subset of X defined by: for all $x \in X$,

$$(\mu \cap \sigma)(x) = \mu(x) \wedge \sigma(x).$$

More generally, the union and intersection of any family $\{\mu_i\}_{i \in \Delta}$ of L-fuzzy subsets of X, denoted by $\bigcup_{i \in \Delta} \mu_i$ and $\bigcap_{i \in \Delta} \mu_i$ respectively, are defined by:

$$(\bigcup_{i \in \Delta} \mu_i)(x) = \sup_{i \in \Delta} \mu_i(x) \text{ and } (\bigcap_{i \in \Delta} \mu_i)(x) = \inf_{i \in \Delta} \mu_i(x),$$

for all $x \in X$, respectively.

Definition 2.14 ([16]). An L-fuzzy subset μ of a lattice X with 0 is said to be an L -fuzzy ideal of X, if $\mu(0) = 1$ and $\mu(a \vee b) = \mu(a) \wedge \mu(b)$, for all $a, b \in X$.

Definition 2.15. Let μ be an L- fuzzy subset of a lattice X. The smallest fuzzy ideal of X containing μ is called a fuzzy ideal generated by μ and is denoted by $(\mu]$.

Lemma 2.16. *Let $\mathcal{FI}(Q)$ be the set of all L-fuzzy ideals of a lattice X and μ be an L fuzzy subset of X. Then $(\mu] = \bigcap\{\theta \in \mathcal{FI}(Q) : \mu \subseteq \theta\}$.*

3. L-fuzzy ideals of a poset

In this section, we introduce several notions of L-fuzzy ideals of a poset and give several characterizations of them. Throughout this paper Q stands for a poset $(Q \leq)$ with 0 unless otherwise stated.

We shall begin with the following definition.

Definition 3.1. An L- fuzzy subset μ of Q is called an L- fuzzy closed ideal, if it satisfies the following conditions:
(i) $\mu(0) = 1$,
(ii) for any subset A of Q, $\mu(x) \geq \inf\{\mu(a) : a \in A\}$ $\forall x \in A^{ul}$.

Lemma 3.2. *A subset I of Q is a closed ideal of Q if and only if its characteristic map χ_I is a closed L-fuzzy ideal of Q.*

Proof. Suppose I is a closed ideal of Q. Since $0 \in I^{ul} \subseteq I$, we have $\chi_I(0) = 1$. Let A be any subset of Q and $x \in A^{ul}$.

If $A \subseteq I$, then we have $x \in A^{ul} \subseteq I^{ul} \subseteq I$. Thus $\chi_I(x) = 1 = \inf\{\chi_I(a) : a \in A\}$.

If $A \nsubseteq I$, then there is $b \in A$ such that $b \notin I$. Thus $\chi_I(b) = 0$. This implies $\inf\{\chi_I(a) : a \in A\} = 0$. So $\chi_I(x) \geq 0 = \inf\{\chi_I(a) : a \in A\}$, for all x in A^{ul}. Hence for any $A \subseteq Q$, we have $\chi_I(x) \geq \inf\{\chi_I(a) : a \in A\}$, for all $x \in A^{ul}$. Therefore χ_I is a fuzzy closed ideal of Q.

Conversely, suppose χ_I is a fuzzy closed ideal. Since $\chi_I(0) = 1$, we have $0 \in I$, i.e., $\{0\} = Q^l \subseteq I$. Let $x \in I^{ul}$. Then by hypotheses, $\chi_I(x) \geq \inf\{\chi_I(a) : a \in I\} = 1$. This implies $\chi_I(x) = 1$. Thus $x \in I$. So $I^{ul} \subseteq I$. Hence I is a closed ideal. This proves the result. □

The following result Characterize the L- fuzzy closed ideal of Q in terms of its level subsets.

Lemma 3.3. *An L- fuzzy subset μ of Q is an L- fuzzy closed ideal of Q if and only if μ_α is a closed ideal of Q, for all $\alpha \in L$.*

Proof. Let μ be an L- fuzzy closed ideal of Q and $\alpha \in L$. Then $\mu(0) = 1 \geq \alpha$. Thus $0 \in \mu_\alpha$, i.e., $\{0\} = Q^l \subseteq \mu_\alpha$. Again let $x \in (\mu_\alpha)^{ul}$. Then $\mu(x) \geq \inf\{\mu(a) : a \in \mu_\alpha\} \geq \alpha$. Then $x \in \mu_\alpha$ Thus $(\mu_\alpha)^{ul} \subseteq \mu_\alpha$. So μ_α is a closed ideal.

Conversely, suppose that μ_α is a closed ideal of Q, for all $\alpha \in L$. In particular, μ_1 is a closed ideal. Since $\{0\} = Q^l \subseteq (\mu_1)^{ul} \subseteq \mu_1$, we have $0 \in \mu_1$. Then $\mu(0) = 1$. Again let A be any subset of Q. Put $\alpha = \inf\{\mu(a) : a \in A\}$. Then $\mu(a) \geq \alpha$, $\forall a \in A$. Thus $A \subseteq \mu_\alpha$. This implies $A^{ul} \subseteq \mu_\alpha^{ul} \subseteq \mu_\alpha$. Since $x \in A^{ul}$, $x \in \mu_\alpha$. So $\mu(x) \geq \alpha = \inf\{\mu(a) : a \in A\}$. Hence μ is an L-fuzzy closed ideal of Q. This proves the result. □

Corollary 3.4. *Let μ be a fuzzy closed ideal of a poset Q. Then μ is anti-tone in the sense that $\mu(x) \geq \mu(y)$, whenever $x \leq y$.*

Proof. Let $x, y \in Q$ such that $x \leq y$. Put $\mu(y) = \alpha$. Since μ a fuzzy closed ideal, we have μ_α is a closed ideal of Q, i.e., $(\mu_\alpha)^{ul} \subseteq \mu_\alpha$. Since $\mu(y) = \alpha$, $y \in \mu_\alpha$. Then $y^l = \{y\}^{ul} \subseteq (\mu_\alpha)^{ul} \subseteq \mu_\alpha$. Thus $x \leq y \Rightarrow x \in y^l \Rightarrow x \in \mu_\alpha$. So $\mu(x) \geq \alpha = \mu(y)$. This proves the result. □

Lemma 3.5. *The intersection of any family of fuzzy closed ideals is a fuzzy closed ideal.*

Theorem 3.6. *Let $(A]_C$ be a closed ideal generated subset A of Q and χ_A be its characteristics functions. Then $(\chi_A] = \chi_{(A]_C}$.*

Proof. Since $(A]_C$ is a closed ideal of Q containing A, by Lemma 3.2, we have $\chi_{(A]_C}$ is a fuzzy closed ideal. Since $A \subseteq (A]$, we have $\chi_A \subseteq \chi_{(A]_C}$. We remain to show that it is the smallest fuzzy closed ideal containing χ_A. Let μ be any L-fuzzy closed ideal such that $\chi_A \subseteq \mu$. Then $\mu(a) = 1$, for all $a \in A$. Now we claim $\chi_{(A]} \subseteq \mu$. Let $x \in Q$. If $x \notin (A]$, then $\chi_{(A]}(x) = 0 \leq \mu(x)$. If $x \in (A]_C$, then $x \in B^{ul}$, for some subset B of A. Thus $\mu(x) \geq \inf\{\mu(b) : b \in B\} = 1 = \chi_{(A]_C}(x)$. So $\chi_{(A]_C}(x) \leq \mu(x)$, for all $x \in Q$. Hence the claim holds. This completes the proof. □

In the following theorem we characterize the fuzzy closed ideal generated by a fuzzy subset of Q in terms of its level ideals.

Theorem 3.7. *Let μ be an L-fuzzy subset of Q. Then the L-fuzzy subset $\hat{\mu}$ of Q defined by $\hat{\mu}(x) = \sup\{\alpha \in L : x \in (\mu_\alpha]_C\}$, for all $x \in Q$ is a fuzzy closed ideal of Q generated by μ.*

Proof. We show $\hat{\mu}$ is the smallest fuzzy closed ideal containing μ. Let $x \in Q$ and put $\mu(x) = \beta$. Then $x \in \mu_\beta \subseteq (\mu_\beta]_C$. Thus $\beta \in \{\alpha \in L : x \in (\mu_\alpha]_C\}$. So

$$\mu(x) = \beta \leq \sup\{\alpha \in L : x \in (\mu_\alpha]_C\} = \hat{\mu}(x).$$

Hence $\mu \subseteq \hat{\mu}$.

Again since $0 \in Q^l \subseteq (\mu_\alpha]_C$, for all $\alpha \in L$, we have $\hat{\mu}(0) = 1$. Let A be any subset of Q and $x \in A^{ul}$. On the other hand,

$$\begin{aligned}\inf\{\hat{\mu}(a) : a \in A\} &= \inf\{\sup\{\alpha_a : a \in (\mu_{\alpha_a}]_C\} : a \in A\} \\ &= \sup\{\inf\{\alpha_a : a \in A\} : a \in (\mu_{\alpha_a}]_C\}.\end{aligned}$$

Put $\lambda = \inf\{\alpha_a : a \in A\}$. Then $\lambda \leq \alpha_a$, for all $a \in A$. Thus $(\mu_{\alpha_a}]_C \subseteq (\mu_\lambda]_C$, $\forall a \in A$. So $A \subseteq (\mu_\lambda]_C$ and thus $x \in A^{ul} \subseteq ((\mu_\lambda]_C)^{ul} \subseteq (\mu_\lambda]_C$. Hence

$$\begin{aligned}\inf\{\hat{\mu}(a) : a \in A\} &= \sup\{\inf\{\alpha_a : a \in A\} : a \in (\mu_{\alpha_a}]_C\} \\ &\leq \sup\{\lambda \in L : x \in (\mu_\lambda]_C\} \\ &= \hat{\mu}(x).\end{aligned}$$

Therefore $\hat{\mu}$ is a Fuzzy closed ideal.

Again let θ be any fuzzy closed ideal of Q such that $\mu \subseteq \theta$. Then $\mu_\alpha \subseteq \theta_\alpha$. Thus $(\mu_\alpha]_C \subseteq (\theta_\alpha]_C = \theta_\alpha$. So for any $x \in Q$, $\hat{\mu}(x) = \sup\{\alpha \in L : x \in (\mu_\alpha]_C\} \leq \sup\{\alpha \in L : x \in \theta_\alpha\} = \theta(x)$. Hence $\hat{\mu} \subseteq \theta$. This proves that $\hat{\mu}$ is the smallest fuzzy closed ideal containing μ. Therefore $\hat{\mu} = (\mu]$. □

In the following we give an algebraic characterization of L-fuzzy Closed ideal generated by a fuzzy subset of Q.

Theorem 3.8. *Let μ be a fuzzy subset of Q. Then the fuzzy subset $\overline{\mu}$ defined by*

$$\overline{\mu}(x) = \begin{cases} 1 & if x = 0 \\ \sup\{\inf_{a \in A} \mu(a) : A \subseteq Q \text{ and } x \in A^{ul}\} & if x \neq 0 \end{cases}$$

is a fuzzy closed ideal of Q generated by μ.

Proof. It is enough to show that $\overline{\mu} = \hat{\mu}$, where $\hat{\mu}$ is a fuzzy subset defined in the above theorem. Let $x \in Q$. If $x = 0$, then $\bar{\mu}(x) = 1 = \hat{\mu}(x)$. Let $x \neq 0$. Put

$$A_x = \{\inf_{a\in A} \mu(a) : A \subseteq Q \ and \ \ x \in A^{ul}\} \text{ and } B_x = \{\alpha : x \in (\mu_\alpha]_C\}.$$

Now we show $\sup A_x = \sup B_x$. Let $\alpha \in A_x$. Then $\alpha = \inf_{a\in A} \mu(a)$, for some subset A of Q such that $x \in A^{ul}$. This implies that $\alpha \leq \mu(a)$, for all $a \in A$. Thus $A \subseteq \mu_\alpha \subseteq (\mu_\alpha]_C$. Since $(\mu_\alpha]_C$ is a closed ideal, we have $A^{ul} \subseteq ((\mu_\alpha]_C)^{ul} \subseteq (\mu_\alpha]_C$. So $x \in (\mu_\alpha]$, i.e., $\alpha \in B_x$. Hence $A_x \subseteq B_x$. Therefore $\sup A_x \leq \sup B_x$.

Again let $\alpha \in B_x$. Then $x \in (\mu_\alpha]_C$. Since $(\mu_\alpha]_C = \bigcup\{A^{ul} : A \subseteq \ \ \mu_\alpha\}$, we have $x \in A^{ul}$, for some subset A of μ_α. This implies $\mu(a) \geq \alpha$, for all $a \in A$. Thus $\inf\{\mu(a) : a \in A\} \geq \alpha$. Put $\beta = \inf\{\mu(a) : a \in A\}$. Then $\beta \in A_x$. Thus for each $\alpha \in B_x$, we get $\beta \in A_x$ such that $\alpha \leq \beta$. So $\sup A_x \geq \sup B_x$. Hence $\sup A_x = \sup B_x$ and thus $\overline{\mu} = \hat{\mu}$. Therefore $\overline{\mu} = (\mu]$. $\square$

The above result yields the following.

Theorem 3.9. *The set $\mathcal{FCI}(Q)$ of all L-fuzzy closed ideals of Q forms a complete lattice, in which the supremum $sup_{i\in\Delta}\mu_i$ and the inifimum $\inf_{i\in\Delta}\mu_i$ of any family $\{\mu_i : i \in \Delta\}$ of L-fuzzy closed ideals of Q respectively are given by:*

$$(sup_{i\in\Delta}\mu_i)(x)$$

$$= \overline{(\bigcup_{i\in\Delta} \mu_i)}(x) = \begin{cases} 1 & if x = 0 \\ \sup\{\inf_{a\in A}(\bigcup_{i\in\Delta} \mu_i)(a) : A \subseteq Q \ and \ \ x \in A^{ul}\} & if x \neq 0 \end{cases}$$

and $(\inf_{i\in\Delta} \mu_i)(x) = (\bigcap_{i\in\Delta} \mu_i)(x)$, *for all* $x \in Q$.

Corollary 3.10. *For any μ and θ in $\mathcal{FCI}(Q)$, the supremum $\mu \vee \theta$ and the infimum $\mu \wedge \theta$ of μ and θ ,respectively are:*

$$(\mu \vee \theta)(x)$$

$$= (\overline{\mu \cup \theta})(x) = \begin{cases} 1 & if x = 0 \\ \sup\{\inf_{a\in A}(\mu \cup \theta)(a) : A \subseteq Q \ and \ \ x \in A^{ul}\} & if x \neq 0 \end{cases}$$

and $(\mu \wedge \theta)(x) = (\mu \cap \theta)(x)$, *for all* $x \in Q$.

Now we introduce the fuzzy version of the ideals of a poset introduced by Frink [6].

Definition 3.11. An L- fuzzy subset μ of Q is called an L- fuzzy Firink ideal, if it satisfies the following conditions:

(i) $\mu(0) = 1$,

(ii) for any finite subset F of Q, $\mu(x) \geq \inf\{\mu(a) : a \in F\}\ \forall x \in F^{ul}$.

Lemma 3.12. *An L- fuzzy subset μ of Q is an L- fuzzy Frink ideal of Q if and only if μ_α is a Frink ideal of Q, for all $\alpha \in L$.*

Corollary 3.13. *A subset I of Q is a Frink ideal of Q if and only if its characteristic map χ_I is an L-fuzzy Frink ideal of Q.*

Lemma 3.14. *The intersection of any family of fuzzy Frink-ideals is a Fuzzy frink-ideal.*

Theorem 3.15. *Let $(A]_F$ be a Frink-ideal generated subset A of Q and χ_A be its characteristics functions. Then $(\chi_A] = \chi_{(A]_F}$.*

In the following theorems we give characterizations of fuzzy Frink ideals generated by a fuzzy subset of Q.

Theorem 3.16. *For any fuzzy subset μ of Q, define a fuzzy subset $\hat{\mu}$ of Q by $\hat{\mu}(x) = \sup\{\alpha \in L : x \in (\mu_\alpha]_F\}$, for all $x \in Q$. Then $\hat{\mu}$ is a Frink fuzzy ideal of Q generated by μ.*

In the following we give an algebraic characterization of fuzzy ideals generated by fuzzy sets. We write $F \subset\subset Q$ to mean that F a finite subset of Q.

Theorem 3.17. *Let μ be a fuzzy subset of Q. Then the fuzzy subset $\overline{\mu}$ defined by:*

$$\overline{\mu}(x) = \begin{cases} 1 & if x = 0 \\ \sup\{\inf_{a \in F} \mu(a) : F \subset\subset Q \text{ and } x \in F^{ul}\} & if x \neq 0 \end{cases}$$

is a Frink fuzzy ideal of Q generated by μ.

The above result yields the following.

Theorem 3.18. *The set $\mathcal{FFI}(\mathcal{Q})$ of all L-fuzzy Frink ideal of Q forms a complete lattice, in which the supremum $sup_{i\in\Delta}\mu_i$ and the inifimum $\inf_{i\in\Delta} \mu_i$ of any family $\{\mu_i : i \in \Delta\}$ of L-fuzzy Frink ideals of Q are given by:*

$$\sup_{i\in\Delta} \mu_i = \overline{\bigcup_{i\in\Delta} \mu_i} \text{ and } \inf_{i\in\Delta} \mu_i = \bigcap_{i\in\Delta} \mu_i.$$

Corollary 3.19. *For any μ and θ in $\mathcal{FFI}(Q)$, the supremum $\mu \vee \theta$ and the infimum $\mu \wedge \theta$ of μ and θ, respectively are:*

$$\mu \vee \theta = \overline{\mu \cup \theta} \text{ and } \mu \wedge \theta = \mu \cap \theta.$$

Now we introduce the fuzzy version of semi-ideals and V-ideals of a poset introduced by Venkatanarasimhan [17, 18].

Definition 3.20. An L- fuzzy subset μ of Q is called an L- fuzzy semi-ideal or L-fuzzy order ideal, if $\mu(x) \geq \mu(y)$, whenever $x \leq y$ in Q.

Definition 3.21. An L- fuzzy subset μ of Q is called an L- fuzzy V-ideal, if it satisfies the following conditions:
(i) $\mu(0) = 1$,
(ii) for any $x, y \in Q$, $\mu(x) \geq \mu(y)$, whenever $x \leq y$,
(iii) for any non-empty finite subset F of Q, if $\sup F$ exists, then

$$\mu(\sup F) \geq \inf\{\mu(a) : a \in F\}.$$

Theorem 3.22. *Every L-fuzzy Frink ideal is an L-fuzzy V-ideal.*

Proof. Let μ is an L-fuzzy Frink ideal and let $x, y \in Q$ such that $x \leq y$. Put $\mu(y) = \alpha$. Since μ an L-fuzzy Frink ideal, we have μ_α is a Frink ideal of Q. Since $\mu(y) = \alpha$, $y \in \mu_\alpha$. Then $\{y\} \subseteq \mu_\alpha$. Thus $\{y\}^{ul} \subseteq \mu_\alpha$. Since $x \leq y$, $x \in y^l = y^{ul} \subseteq \mu_\alpha$. So $\mu(x) \geq \alpha = \mu(y)$.

Again let F be any nonempty subset of Q such that $\sup F$ exists in Q. Then $F^{ul} = (\sup A]$. Thus $\sup F \in F^{ul}$ and $\mu(\sup F) \geq \inf\{\mu(a) : a \in F\}$. So μ is an L-fuzzy V-ideal. □

Now we introduce the fuzzy version ideals of a poset introduced by Halaš [9] which seems to be a suitable generalization of the usual concept of L-fuzzy ideal of a lattice.

Definition 3.23. An L- fuzzy subset μ of Q is called an L- fuzzy ideal in the sense of Halaś, if it satisfies the following conditions:
(i) $\mu(0) = 1$,
(ii) for any $a, b \in Q$, $\mu(x) \geq \mu(a) \wedge \mu(b)$, for all $x \in (a, b)^{ul}$.

In the rest of this paper, an L- fuzzy ideal of a poset will mean an L-fuzzy ideal in the sense of Halaś given in the above definition.

Lemma 3.24. *An L- fuzzy subset μ of Q is an L- fuzzy ideal of Q if and only if μ_α is an ideal of Q in the sense of Halaś, for all $\alpha \in L$.*

Corollary 3.25. *A subset I of Q is an ideal of Q in the sense of Halaś if and only if its characteristic map χ_I is an L-fuzzy ideal of Q.*

Lemma 3.26. *If μ is an L- fuzzy ideal of Q, then the following assertions hold:*
(1) *for any $x, y \in Q$, $\mu(x) \geq \mu(y)$, whenever $x \leq y$,*
(2) *for any $x, y \in Q$, $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$, whenever $x \vee y$ exists.*

Theorem 3.27. *Let $(Q, \leq)$ be a lattice. Then an L-fuzzy subset μ of Q is an L-fuzzy ideal in the poset Q if and only it an L-fuzzy ideal in the lattice Q.*

Proof. Let μ be an L-fuzzy ideal in the poset Q and $a, b \in Q$. Then $\mu(0) = 1$. Since $a \vee b \in (a \vee b] = (a, b)^{ul}$, we have $\mu(a \vee b) \geq \mu(a) \wedge \mu(b)$. Since μ is anti-tone, we have $\mu(a) \geq \mu(a \vee b)$ and $\mu(b) \geq \mu(a \vee b)$. Thus $\mu(a) \wedge \mu(b) \geq \mu(a \vee b)$. So $\mu(a \vee b) = \mu(a) \wedge \mu(b)$. Hence μ is an L-fuzzy ideal in the lattice Q.

Conversely, suppose μ is an L-fuzzy ideal in the lattice Q. Let $a, b \in Q$ and $x \in (a, b)^{ul}$. Then $x \leq y$, for all $y \in (a, b)^u$. Since $a \vee b \in (a, b)^u$, we have $x \leq a \vee b$. Thus $\mu(x) \geq \mu(a \vee b) = \mu(a) \wedge \mu(b)$. So μ is an L-fuzzy ideal in the poset Q. This completes the proof. □

Lemma 3.28. *The intersection of any family of L-fuzzy ideals is an L- fuzzy deal.*

Theorem 3.29. *Let $(A]_H$ be an ideal generated subset A of Q in the sense of Halaś and χ_A be its characteristics functions. Then $(\chi_A] = \chi_{(A]_H}$.*

Definition 3.30. Let μ be a fuzzy subset of Q and $\mathcal{N}$ be a set of positive integers. Define a fuzzy subset C_1^μ of Q by $C_1^\mu(x) = \sup\{\mu(a) \wedge \mu(b) : x \in (a, b)^{ul}\}$, $\forall\, x \in Q$. Inductively, let $C_{n+1}^\mu(x) = \sup\{C_n^\mu(a) \wedge C_n^\mu(b) : x \in (a, b)^{ul}\}$, for each $n \in \mathcal{N}$.

Now we give a characterization of an L-fuzzy ideal generated by a fuzzy subset of a poset Q.

Theorem 3.31. *The set $\{C_n^\mu : n \in \mathcal{N}\}$ form a chain and the fuzzy subset $\hat{\mu}$ defined by: for all $x \in Q$,*

$$\hat{\mu}(x) = \sup\{C_n^\mu(x) : n \in \mathcal{N}\}$$

is a fuzzy ideal generated by μ.

Proof. Let $x \in Q$ and $n \in \mathcal{N}$. Then

$$\begin{aligned} C^{\mu}_{n+1}(x) &= \sup\{C^{\mu}_n(a) \wedge C^{\mu}_n(b) : x \in (a,b)^{ul}\} \\ &\geq C^{\mu}_n(x) \wedge C^{\mu}_n(x) \ (since \ \ x \in x^l = (x,x)^{ul}) \\ &= C^{\mu}_n(x), \ \forall \, x \in Q. \end{aligned}$$

Thus $C^{\mu}_n \subseteq C^{\mu}_{n+1}$, for each $n \in \mathcal{N}$. So $\{C^{\mu}_n : n \in \mathcal{N}\}$ is a chain.

Now we show $\hat{\mu}$ is the smallest fuzzy ideal containing μ. Since

$$\begin{aligned} \hat{\mu}(x) &= \sup\{C^{\mu}_n(x) : n \in \mathcal{N}\} \\ &\geq C^{\mu}_1(x) \\ &= \sup\{\mu(a) \wedge \mu(b) : x \in (a,b)^{ul}\} \\ &\geq \mu(x) \wedge \mu(x) \ \ (since \ x \in (x,x)^{ul}) \\ &= \mu(x), \ \ \forall \, x \in Q, \end{aligned}$$

we have $\mu \subseteq \hat{\mu}$. Let $a, b \in L$ and $x \in (a,b)^{ul}$. Then

$$\begin{aligned} \hat{\mu}(x) &= \sup\{C^{\mu}_n(x) : n \in \mathcal{N}\} \\ &\geq C^{\mu}_n(x) \ \ for \ all \ n \in \mathcal{N} \\ &= \sup\{C^{\mu}_{n-1}(y) \wedge C^{\mu}_{n-1}(z) : x \in (y,z)^{ul}\} \ for \ all \ n \geq 2. \\ &\geq C^{\mu}_{n-1}(a) \wedge C^{\mu}_{n-1}(b) \ \ \forall n \geq 2 \ (since \ x \in (a,b)^{ul}) \\ &= C^{\mu}_m(a) \wedge C^{\mu}_m(b), \ \ \forall \, m \in \mathcal{N}. \end{aligned}$$

Thus

$$\begin{aligned} \hat{\mu}(x) &\geq \sup\{C^{\mu}_m(a) \wedge C^{\mu}_m(b) : m \in \mathcal{N}\} \\ &= \sup\{C^{\mu}_m(a) : m \in \mathcal{N}\} \wedge \sup\{C^{\mu}_m(b) : m \in \mathcal{N}\} \\ &= \hat{\mu}(a) \wedge \hat{\mu}(b). \end{aligned}$$

So $\hat{\mu}$ is a fuzzy ideal.

Again let θ be any fuzzy ideal of Q such that $\mu \subseteq \theta$. Now let $a, b \in Q$ and $x \in (a,b)^{ul}$. Then $\theta(x) \geq \theta(a) \wedge \theta(b) \geq \mu(a) \wedge \mu(b)$. This implies

$$\theta(x) \geq \sup\{\mu(a) \wedge \mu(b) : x \in (a,b)^{ul}\} = C^{\mu}_1(x), \ \forall x \in (a,b)^{ul}.$$

Again for any $x \in (a,b)^{ul}$, we have $\theta(x) \geq \theta(a) \wedge \theta(b) \geq C^{\mu}_1(a) \wedge C^{\mu}_1(b)$. This implies

$$\theta(x) \geq \sup\{C^{\mu}_1(a) \wedge C^{\mu}_1(b) : x \in (a,b)^{ul}\} = C^{\mu}_2(x).$$

Thus by induction, we have $\theta(x) \geq C^{\mu}_n(x) \ \forall n \in \mathcal{N}$ and $\forall \ x \in (a,b)^{ul}$. So for any $x \in Q$,

$$\begin{aligned} \hat{\mu}(x) &= sup\{C^{\mu}_n(x) : n \in \mathcal{N}\} \\ &= \sup\{C^{\mu}_n(a) \wedge C^{\mu}_n(b) : x \in (a,b)^{ul}\} \\ &\leq \sup\{\theta(a) \wedge \theta(b) : x \in (a,b)^{ul}\} \ (since, \ a, b \in (a,b)^{ul}.) \\ &\leq \theta(x). \end{aligned}$$

Hence $\hat{\mu} \subseteq \theta$. This completes the proof. □

The above result yields the following.

Theorem 3.32. *The set $\mathcal{FI}(Q)$ of all L-fuzzy ideal of Q forms a complete lattice, in which the supremum $sup_{i\in\Delta}\mu_i$ and the inifimum $\inf_{i\in\Delta}\mu_i$ of any family $\{\mu_i : i \in \Delta\}$ in $\mathcal{FI}(Q)$ respectively are: for all $x \in Q$,*

$$(sup_{i\in\Delta}\mu_i)(x) = \sup\{C_n^{\bigcup_{i\in\Delta}\mu_i}(x) : n \in \mathcal{N}\} \text{ and } (\inf_{i\in\Delta}\mu_i)(x) = (\bigcap_{i\in\Delta}\mu_i)(x).$$

Corollary 3.33. *For any μ and $\theta \in \mathcal{FI}(Q)$ the supremum $\mu \vee \theta$ and the infimum $\mu \wedge \theta$ of μ and θ respectively are: for all $x \in Q$,*

$$(\mu \vee \theta)(x) = \sup\{C_n^{\mu\cup\theta}(x) : n \in \mathcal{N}\} \text{ and } (\mu \wedge \theta)(x) = (\mu \cap \theta)(x).$$

Theorem 3.34. *The following implications hold, where none of them is an equivalence:*

(1) *L- fuzzy closed ideal $\Longrightarrow$ L-fuzzy Frink ideal $\Longrightarrow$ L-fuzzy V-ideal $\Longrightarrow$ L-fuzzy semi-ideal,*
(2) *L- fuzzy closed ideal $\Longrightarrow$ L-fuzzy Frink ideal $\Longrightarrow$ L-fuzzy ideal $\Longrightarrow$ L- fuzzy semi-ideal.*

The following examples show that the converse of the above implications do not hold in general.

Example 3.35. Consider the Poset $([0, 1], \leq)$ with the usual ordering. Define a fuzzy subset $\mu : [0, 1] \longrightarrow [0, 1]$ by:

$$\mu(x) = \begin{cases} 1 & if x \in [0, \frac{1}{2}) \\ 0 & if x \in [\frac{1}{2}, 1]. \end{cases}$$

Then μ is L-fuzzy Frink ideal but not L- fuzzy closed ideal.

Example 3.36. Consider the poset $(Q, \leq)$ depicted in the figure below. Define a fuzzy subset $\mu : Q \longrightarrow [0, 1]$ by: $\mu(0) = \mu(a) = 1$, $\mu(a') = \mu(b') = \mu(c') = \mu(d') = \mu(1) = 0.2$, $\mu(b) = 0.6$, $\mu(c) = 0.5$ and $\mu(d) = 0.7$.

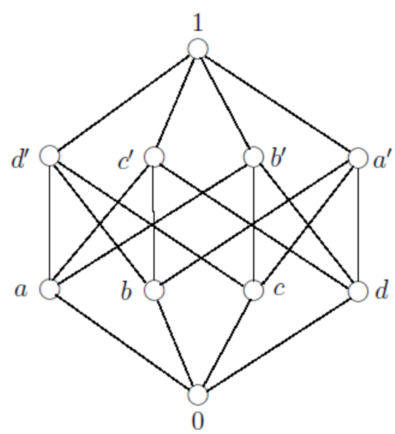

Figure 1

Then μ is L-fuzzy ideal but not L-fuzzy Frink-ideal.

Example 3.37. Consider the poset $(Q, \leq)$ depicted in the figure below. Define a fuzzy subset $\mu : Q \longrightarrow [0, 1]$ by: $\mu(0) = 1$, $\mu(a) = \mu(b) = 0.8$ and $\mu(c) = 0.6$.

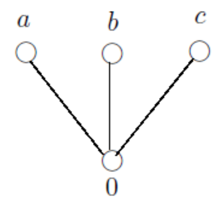

Figure 2

Then μ is L-fuzzy V-ideal but not L-fuzzy Frink-ideal.

Example 3.38. Consider the poset $(Q, \leq)$ depicted in the figure below. Define a fuzzy subset $\mu : Q \longrightarrow [0,1]$ by: $\mu(0) = \mu(a) = 1$, $\mu(b) = 0.8$, $\mu(c) = 0.9$, $\mu(d) = \mu(e) = 0.2$ and $\mu(1) = 0$.

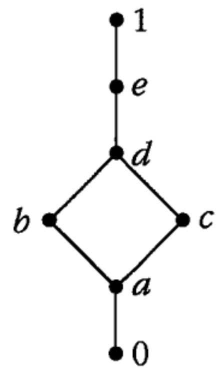

Figure 3

Then μ is L-fuzzy semi-ideal but not L-fuzzy ideal.

Theorem 3.39. *Let $x \in Q$ and $\alpha \in L$. Define an L- fuzzy subset α_x of Q by*

$$\alpha_x(y) = \begin{cases} 1 & if\ y \in (x] \\ \alpha & if\ y \notin (x], \end{cases}$$

for all $y \in Q$. Then α_x is an L-fuzzy ideal of Q.

Proof. By the definition of α_x, we clearly have $\alpha_x(0) = 1$. Let $a, b \in Q$ and $y \in (a,b)^{ul}$.

If $a, b \in (x]$, then $(a,b)^{ul} \subseteq (x]$ and $\alpha_x(a) = \alpha_x(b) = 1$. Thus $\alpha_x(y) = 1 = 1 \wedge 1 = \alpha_x(a) \wedge \alpha_x(b)$.

If $a \notin (x]$ or $b \notin (x]$, then $\alpha_x(a) = \alpha$ or $\alpha_x(b) = \alpha$. Thus

$$\alpha_x(y) \geq \alpha = \alpha_x(a) \wedge \alpha_x(b).$$

So in either cases, we have $\alpha_x(y) \geq \alpha_x(a) \wedge \alpha_x(b)$, for all $y \in (a,b)^{ul}$. Hence α_x is an L-fuzzy ideal. □

Definition 3.40. The L-fuzzy ideal α_x defined above is called the α-level principal fuzzy ideal corresponding to x.

Definition 3.41. An L-fuzzy ideal μ of a poset Q is called a u-L-fuzzy ideal, if for any $a, b \in Q$, there exists $x \in (a,b)^u$ such that $\mu(x) = \mu(a) \wedge \mu(b)$.

Note that this property is immediately extends from $\{a, b\}$ to any finite subset of Q. That is, if μ is a u-L-fuzzy ideal then there exists $x \in F^u$ such that $\mu(x) = \mu(a) \wedge \mu(b)$.

Lemma 3.42. *An L- fuzzy ideal μ of Q is a u-L-fuzzy ideal of Q if and only if μ_α is a u-ideal of Q, for all $\alpha \in L$.*

Proof. Suppose μ is a u-L-fuzzy ideal and $\alpha \in L$. Since μ is an L- fuzzy ideal, μ_α is an ideal of Q. Let $a, b \in \mu_\alpha$. Then $\mu(a) \geq \alpha$ and $\mu(b) \geq \alpha$. Thus $\mu(a) \wedge \mu(b) \geq \alpha$. Since μ is a u- L- fuzzy ideal, there exists $x \in (a, b)^u$ such that $\mu(x) = \mu(a) \wedge \mu(b)$. So $\mu(x) \geq \alpha$. Hence $x \in \mu_\alpha \cap (a, b)^u$ and thus $\mu_\alpha \cap (a, b)^u \neq \varnothing$. Therefore μ_α is a u-L- fuzzy ideal of a poset Q.

Conversely, suppose μ_α is a u- ideal of a poset Q, for all $\alpha \in L$. Then μ is an L- fuzzy ideal. Let $a, b \in Q$ and put $\alpha = \mu(a) \wedge \mu(b)$. Then $\mu_\alpha \cap (a, b)^u \neq \varnothing$. Let $x \in \mu_\alpha \cap (a, b)^u$. Then $x \in \mu_\alpha$ and $x \in (a, b)^u$. This implies $\mu(x) \geq \alpha = \mu(a) \wedge \mu(b)$ and $a \leq x, b \leq x$. Since μ is anti-tone, we have $\mu(a) \geq \mu(x)$ and $\mu(b) \geq \mu(x)$. Thus $\mu(a) \wedge \mu(b) \geq \mu(x)$. So there exists $x \in (a, b)^u$ such that $\mu(x) = \mu(a) \wedge \mu(b)$. Hence μ is a u-L-fuzzy ideal. □

Corollary 3.43. *Let $(Q, \leq)$ be a poset with 1 and let $x \in Q$ and $\alpha \in L$. Then the α-level principal fuzzy ideal corresponding to x is a u-L-fuzzy ideal.*

Remark 3.44. Every L-fuzzy ideal is not a u-L-fuzzy ideal. For example consider the poset $(Q \leq)$ depicted in the figure below and define a fuzzy subset $\mu : Q \longrightarrow [0, 1]$ and of Q by $\mu(0) = 1$, $\mu(a) = \mu(b) = 0.9$, $\mu(c) = \mu(d) = \mu(1) = 0.7$. Then μ is an L-fuzzy ideal but not a u- L-fuzzy ideal.

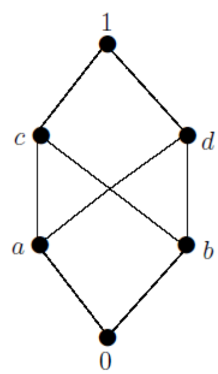

Figure 4

Theorem 3.45. *Every u- L-fuzzy ideal is an L- fuzzy Frink ideal.*

Proof. suppose μ is a u- L-fuzzy ideal. Let F be a finite subset of Q . Then there is $y \in F^u$ such that $\mu(y) = inf\{\mu(a) : a \in F\}$. Let $x \in F^{ul}$. Then $x \leq s$, $\forall s \in F^u$. Since $y \in F^u$, $x \leq y$. Thus $\mu(x) \geq \mu(y) = inf\{\mu(a) : a \in F\}$. So

$$\mu(x) \geq inf\{\mu(a) : a \in F\}.$$

Hence μ is an L-fuzzy Frink ideal. □

Theorem 3.46. *Let μ and θ be u- L-fuzzy ideals of Q. Then the supremum $\mu \vee \theta$ of μ and θ in $\mathcal{FI}(Q)$ is given by: for all $x \in Q$,*

$$(\mu \vee \theta)(x) = \sup\{\mu(a) \wedge \theta(b) : x \in (a,b)^{ul}\}.$$

Proof. Let σ be an L-fuzzy subset of Q defined by: for each $x \in Q$,

$$\sigma(x) = \sup\{\mu(a) \wedge \theta(b) : x \in (a,b)^{ul}\}.$$

We claim σ is the smallest L-fuzzy ideal of Q containing $\mu \cup \theta$. Let $x \in Q$. Then

$$\begin{aligned} \sigma(x) &= \sup\{\mu(a) \wedge \theta(b) : x \in (a,b)^{ul}\} \\ &\geq \mu(x) \wedge \theta(0), \ (since\ x \in (x,0)^{ul}) \\ &= \mu(x) \wedge 1 = \mu(x). \end{aligned}$$

Thus $\sigma \supseteq \mu$. Similarly, we can show $\sigma \supseteq \theta$. So $\sigma \supseteq \mu \cup \theta$.

Let $a, b \in Q$ and $x \in (a,b)^{ul}$. Then

$$\begin{aligned} \sigma(a) \wedge \sigma(b) &= \sup\{\mu(c) \wedge \theta(d) : a \in (c,d)^{ul}\} \wedge \sup\{\mu(e) \wedge \theta(f) : b \in (e,f)^{ul}\} \\ &= \sup\{\mu(c) \wedge \theta(d) \wedge \mu(e) \wedge \theta(f) : a \in (c,d)^{ul}, b \in (e,f)^{ul}\} \\ &\leq \sup\{\mu(c) \wedge \theta(d) \wedge \mu(e) \wedge \theta(f) : a, b \in (c,d,e,f)^{ul}\} \\ &= \sup\{\mu(c) \wedge \mu(e) \wedge \theta(d) \wedge \theta(f) : a, b \in (c,d,e,f)^{ul}\}. \end{aligned}$$

Since μ and θ are u-L-fuzzy ideals, for each c, e and d, f, there are $r \in (c,e)^u$ and $s \in (d,f)^u$ such that $\mu(r) = \mu(c) \wedge \mu(e)$ and $\theta(s) = \theta(d) \wedge \theta(f)$. Since $r \in (c,e)^u$ and $s \in (d,f)^u$, $\{c,d,e,f\}^{ul} \subseteq \{s,r\}^{ul}$. Thus $a, b \in \{s,r\}^{ul}$. So $(a,b)^{ul} \subseteq \{s,r\}^{ul}$ and thus $x \in \{s,r\}^{ul}$. Hence for all $x \in (a,b)^{ul}$,

$$\sigma(a) \wedge \sigma(b) \leq \sup\{\mu(r) \wedge \theta(s) : x \in (r,s)^{ul}\} \leq \sigma(x).$$

Therefore σ is an L-fuzzy ideal.

Let ϕ be any L-fuzzy ideal of Q such that $\mu \cup \theta \subseteq \phi$. Then for any $x \in Q$, we have

$$\begin{aligned} \sigma(x) &= \sup\{\mu(a) \wedge \theta(b) : x \in (a,b)^{ul}\} \\ &\leq \sup\{\phi(a) \wedge \phi(b) : x \in (a,b)^{ul}\} \\ &\leq \phi(x). \end{aligned}$$

Thus $\sigma \subseteq \phi$. So $\sigma = (\mu \cup \theta] = \mu \vee \theta$. Hence σ is the supremum of μ and θ in $\mathcal{FI}(Q)$. □

Now we complete this paper by introducing the following definition which generalize all the L-fuzzy ideals of a poset introduced above.

Definition 3.47. An L- fuzzy subset μ of Q is an L- fuzzy m-ideal, if it satisfies the following conditions:

(i) $\mu(0) = 1$,

(ii) for any subset A of Q of cardinality strictly less than m, we have $\mu(x) \geq \inf\{\mu(a) : a \in A\}$, $\forall x \in A^{ul}$, where m is any cardinal.

Remark 3.48. Note that the L- fuzzy Ω-ideals are nothing but the L-fuzzy closed ideal, the L- fuzzy ω-ideals are nothing but the L-Fuzzy Frink-ideals, the L- fuzzy 3-ideals are nothing but the L- fuzzy ideals and the L-fuzzy 2-ideals are nothing but the L-fuzzy semi-ideals.

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