**A**nnals of **F**uzzy **M**athematics and **I**nformatics
Volume 16, No. 1, (August 2018) pp. 71–86
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2018.16.1.71

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Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 16, No. 1, August 2018

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# New transitivity property of intuitionistic fuzzy multiplicative preference relation and its application in missing value estimation

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Received 24 April 2018; Revised 13 May 2018; Accepted 20 June 2018

Abstract. In the decision-making process, consistency is a crucial issue which causes wide public concern of exports. The lack of consistency in preference relations results in an inconsistent solution. In this paper, we propose a characterization of the consistency property using newly defined transitivity property for intuitionistic fuzzy multiplicative preference relations (IFMPR) together with complementing missing elements for incomplete IFMPR. In 2015 Jiang et al. worked on incomplete intuitionistic fuzzy multiplicative preference relations (incomplete IFMPRs) in which the IFMPRs split into two multiplicative preference relation (MPRs), and the missing elements were calculated by using the consistency of MPRs. Using new transitivity property of IFMPR, and we have developed two different methods to find the missing element of IFMPRs. The first method is two-step procedure method containing estimating step followed by adjusting step. In estimating step, the missing elements of incomplete IFMPRs are calculated by using a new transitive property instead of splitting IFMPRs into two multiplicative preference relations(MPRs). Sometimes the initial value may not satisfy the conditions of IFMPRs. An optimization model is developed in the second step to adjust the initial values that are solved by MATLAB optimization tool. The second proposed method is goal programming model based on new transitivity property to calculate the missing elements directly. Acceptably Consistent with complete IFMPRs is also checked. Two numerical examples are carried out to illustrate the above-said methods.

2010 AMS Classification: 03E72, 90C70

Keywords: Multiplicative preference relation(MPR), Intuitionistic fuzzy multiplicative preference relation (IFMPR), Consistent, Acceptably consistent, Transitivity, Goal programming.

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## 1. Introduction

Decision making is one of the most important tasks for individuals and organizations and is an interdisciplinary research area attracting researchers from almost all fields from psychologists, economists, to computer scientists. Among all the fields from psychologists, economists, to computer scientists, the fuzzy decision is an important branch of fuzzy theory. Liu and Liao [11] gave a bibliometric analysis of fuzzy decision-related research to find out some underlying patterns and dynamics in the direction of the fuzzy decision. Yu and Liao [28] make a scientometric review on IFS studies to reveal the most cited papers based on the 1318 references retrieved from SCIE and SSCI databases via Web of science. Liao, Zhang, and Luo [10] present some new distance measures between intuitionistic fuzzy multiplicative sets, which incorporate the projection-based distance measure and the psychological distance measures. In 1980 Saaty[16] introduced the Analytic hierarchy process, is the most important working Multi-criteria decision making (MCDM) method. Mathematically to describe the decision making problem, different type of preference relation have been proposed (see [13, 15, 16, 20, 21, 22, 23]). In different decision-making problems, there are different types of preference relations indicate different forms of preference information in matrices and exert their power. The preference relations are classified into two forms: fuzzy preference relations (FPR) and multiplicative preference relations (MPRs). In FPRs, [13] the decision maker express the information by using 0-1 ratio scale whereas the MPR [15] the intensity of the pairwise comparison of objects by using $1/9-9$ ratio scale (also called 1-9 ratio scale). Due to the time pressure and lack of information or knowledge, sometimes it is complicated for the decision makers to provide their preferences. To express their preference information, interval-valued preference relations [17, 23] allow the decision makers to use the interval numbers.

All the elements of both FPRs and MPRs are single values, which only involves the intensities of preferences relation. But it is possible that, in the decision-making process, the decision maker not sure about the preference information namely, uncertainty degrees (hesitation degrees). Intuitionistic fuzzy preference relations (IFPRs) [22] and intuitionistic fuzzy multiplicative preference relations (IFMPRs) [20] are defined to indicate the positive information $x_i$ is preferred to $x_j$, the degree of negative details $x_i$ is not preferred to $x_j$ and simultaneously the degree that cannot determine by the decision maker. Recently, Zhang et al.[33] created intuitionistic fuzzy multiplicative ORESTE strategy and further featured by a contextual analysis concerning the patients' prioritization.

Consistency plays a significant role in decision-making process. Consistency measures the level of agreement among the preference data given by the individual DMs [19]. A good amount of researchers have paid their attention to the use of consistency of preference relations in decision making under uncertain environments. For good understanding one may refer to [4, 8, 26, 27].

In 2015 Jiang et al. [9] discussed the consistency property of an IFMPR especially the acceptable consistency. Based on it, two approaches have developed to complement all missing elements of incomplete IFMPRs. Ren[14] verified that the intuitionistic fuzzy multiplicative weighted geometric aggregation (IFMWGA) operator is of excellent characteristics in remaining the consistency of the IFMPRs. To adjust the inconsistent IFMPRs into an acceptably consistent one, they proposed an iterative process and also, they Provide an adjustment process to restore and improves the consistency of inconsistent IFMPR. Zhang et al. [32] suggested several goal programming models to manage unity and consensus of IFMPRs and develop consistency and consensus-based approach for dealing with group decision-making (GDM) with IFMPRs. Zhang and Guo [29] has developed a linear programming-based algorithm to check and improve the consistency of an IFMPR. Also, Zhang and Guo [29] discuss the relationships between an IFMPR and a normalized intuitionistic fuzzy multiplicative weight vector and develop two approaches to group decision making based on complete and incomplete IFMPRs, respectively. In this paper, our work focuses on only IFMPRs.

Sometimes decision maker may not yet have a good understanding of a particular question, and this he/she is unable to make a direct comparison between every two objects; therefore it is sometimes necessary to allow the decision maker to skip some dubious comparisons flexibly. In this case, incomplete preference relations are obtained, and the whole process may slow down. In the decision-making process, to present a complete preference relation, a decision maker should make $\frac{n(n-1)}{2}$ judgments at each level, and when $n$ is large, it becomes an onerous task. Therefore sometimes, due to lack of time and busy schedule of the decision maker, incomplete preference relations are obtained. Our work focus on incomplete IFMPR.

In this paper, the new consistency property of IFMPR is defined. Based on which two novel method is provided for estimating the missing element of incomplete IFMPRs, where one traditional "two-step procedure methods" is divided into two sub-steps such as: (i) "Estimating step": Initial values are evaluated for the missing elements of the incomplete IFMPRs without splitting into two MPRs; (ii) "Adjusting step": An optimization model is developed to adjust the initial values derived from the estimating step which is solved by MATLAB optimization tool. We have also developed a goal programming model to estimate the missing element without splitting into two MPRs. Both novel methods give the equivalent result. Then the acceptably consistent of IFMPR has been checked. Techniques are illustrated with suitable examples.

Paper is organised as follows. In section 2, some basic concepts are defined briefly, and new transitivity property of IFMPR is defined. In section 3, we have developed an algorithm to find the missing elements by using newly defined transitivity property of IFMPR and proposed an optimization model for adjusting the initial values. In section 4 a goal programming model is intended to complete the incomplete IFMPRs. Two numerical examples illustrate the developed procedures. In the section 5, we have given a comparative analysis of our work with the work of [9] and [12]. Concluding remarks are provided in the last part.

## 2. Preliminaries

Let $X = \{x_1, \cdots, x_n\}$ be a discrete set of alternatives/criteria in a decision making problem and set $N = \{1, \cdots, n\}$ be the set of indices. A decision maker needs to provide his/her preferences over the alternatives/criteria using the pairwise comparison method. The preferences values are provided by the decision maker from the ratio scale $[1/9, 9]$ introduced by Satty [16] to estimate and differentiate the intensity of preferences. Based on the above ratio scale, some basic concepts of IFMPRs are defined.

## 2.1. Intuitionistic fuzzy multiplicative preference relation.

**Definition 2.1** ([20]). An intuitionistic fuzzy multiplicative preference relation (IFMPR) is $\tilde{R} = [\tilde{r}_{ij}(x_i, x_j)]_{n\times n}$, where $\tilde{r}_{ij}(x_i, x_j) = (\mu(x_i, x_j), \nu(x_i, x_j)),\ i, j \in N$, is an intuitionistic fuzzy multiplicative number (IFMN), and $\mu(x_i, x_j)$ indicates certainty degree to which $x_i$ is preferred to $x_j$ and $\nu(x_i, x_j)$ is the certainty degree to which $x_i$ is not preferred to $x_j$, and they satisfy the following characteristics:

$$1/9 \leq \mu(x_i, x_j)\,, \quad \nu(x_i, x_j) \leq 9,\ \mu(x_i, x_j) = \nu(x_j, x_i)\,, \nu(x_i, x_j) = \mu(x_j, x_i)$$
$$\mu(x_i, x_i) = \nu(x_i, x_i) = 1\,, 0 < \mu(x_i, x_j)\,\nu(x_i, x_j) \leq 1\,, \quad \forall\, i, j \in N.$$

For the sake of convenience $\mu(x_i, x_j)$ and $\nu(x_i, x_j)$ are denoted by $\mu_{ij}$ and $\nu_{ij}$ respectively.

In 2013 Xu [24] gave the concept of the consistent property of IFMPR $\tilde{R} = (\tilde{r}_{ij})_{n\times n} = (\mu_{ij}, \nu_{ij})_{n\times n}$ based on the transitivity

$$(\mu_{ij}, \nu_{ij}) = (\mu_{ik}\mu_{kj}, \nu_{ik}\nu_{kj}),\ \text{for all } i, j, k \in N \text{ and } \ i \leq k \leq j. \tag{2.1}$$

It is to note that the equation (2.1) is restricted for the condition $i \leq k \leq j$, while the transitive property of a MPR is unconstrained which satisfies for all $i, j, k \in N$. If the equation (2.1) is utilized to check the consistency of an IFMPR for all $i, j, k \in N$, the transitivity and consistency properties some times do not hold. This is because that when '$k$' comes from the row of lower triangular matrix, the equation does not hold. For example:

$$\begin{pmatrix} (1,1) & (1/2,1) & (1,1/2) \\ (1,1/2) & (1,1) & (2,1/2) \\ (1/2,1) & (1/2,2) & (1,1) \end{pmatrix}_{3\times 3}$$

is a consistent IFMPR given in [24]. Jiang et al. [9] relax the condition $i \leq k \leq j$, it is follow that $a_{23} = (\mu_{21}\mu_{13}, \nu_{21}\nu_{13}) = (1, 1/4)$. But $a_{23} = (2, 1/2) \neq (1, 1/4)$. To over come this type of transitivity limitation, Jiang et al. [9] proposed a more general consistency property of an IFMPR split into two MPRs by using the formula

$$a_{ij} = \begin{cases} \mu_{ij} & i < j \\ 1 & i = j \\ 1/\nu_{ij} & i > j \end{cases} \ \text{and}\ b_{ij} = \begin{cases} \nu_{ij} & i < j \\ 1 & i = j \\ 1/\mu_{ij} & i > j\,. \end{cases} \tag{2.2}$$

where the MPR $A = (a_{ij})_{n\times n}$ is preferred information matrix given by the decision maker with respect to the alternative $x_i$ over $x_j$ and $B = (b_{ij})_{n\times n}$ is the non-preferred information matrix given by the decision maker with respect to the alternative $x_i$ over $x_j$. Based on the above concept, Jiang et al. [9] defined the consistent IFMPR.

**Definition 2.2** ([9]). A IFMPR $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ is said to be consistent, if both MPRs $A$ and $B$ given by the equation (2.2) are consistent such that

$$a_{ij} = a_{ik}a_{kj},\ b_{ij} = b_{ik}b_{kj} \ \ \forall i, j, k \in N$$

In this work, instead of splitting IFMPRs into MPRs we have defined an new consistency property of IFMPR.

**Definition 2.3.** An IFMPRs $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ is called consistent, if it satisfy the transitivity property, where $\tilde{r}_{ij}$ is

$$(2.3) \qquad (\mu_{ij}, \nu_{ij}) = \begin{cases} (\mu_{ik}, \nu_{ik}) \otimes (\mu_{kj}, \nu_{kj}) & \text{if } i \leq k, k \leq j \\ (\frac{1}{\mu_{ki}}, \frac{1}{\nu_{ki}}) \otimes (\mu_{kj}, \nu_{kj}) & \text{if } i \geq k, k \leq j \\ (\mu_{ik}, \nu_{ik}) \otimes (\frac{1}{\mu_{jk}}, \frac{1}{\nu_{jk}}) & \text{if } i < k, k > j. \end{cases}$$

In this work for convenience, we have used the multiplication of two IFMNs as the multiplication of two order pairs. Let $a = (\mu, \nu)$, $a_1 = (\mu_1, \nu_1)$ and $a_2 = (\mu_2, \nu_2)$ be intuitionistic fuzzy multiplicative numbers (IFMNs) and $\lambda > 0$. Then

$$a_1 \otimes a_2 = (\mu_1, \nu_1) \otimes (\mu_2, \nu_2) = (\mu_1\mu_2, \nu_1\nu_2),$$
$$a^\lambda = \left(\frac{2\mu^\lambda}{(2+\mu)^\lambda - \mu^\lambda}, \frac{(1+2\nu)^\lambda - 1}{2}\right)$$

is given by Xia[20], for $\lambda > 0$.

In decision-making problem, it may be the case that decision maker may not have a good understanding on a particular question, and therefore he/she is unable to make a direct comparison between two alternatives or criteria. Consequently, it is more appropriate and flexible to skip some similarities, and in that cases, the decision maker may prefer to express their judgments with incomplete preference relation. According to previously discussed, the decision makers may provide less than $\frac{n(n-1)}{2}$ judgments in practical decision making and the incomplete IFMPR will be presented. So, it is essential to investigate the incomplete preference relations as a useful tool in decision-making problem, and many research results have been developed. Herrera-Viedma et al. [7] proposed the definition of incomplete preference relation. The concept of IFMPRs is extended to the situations where the preference information given by decision maker is incomplete. Jiang et al. [9] propose to extend the above situation to incomplete IFMPR where some elements are missing in the preference relation matrix.

**Definition 2.4** ([9])**.** An IFMPR $\tilde{r}_{ij} = (\mu_{ij}, \nu_{ij})_{n\times n}$ is called an incomplete IFMPR, if some elements in it are missing and all available elements satisfy the characteristics of IFMPR stated in Definition 2.1.

## 3. Complement of an incomplete IFMPRs

In this section, we define the consistent property of incomplete IFMPR.

**Definition 3.1.** An incomplete IFMPR is said to be consistent, if all the known element satisfy the equation (2.3).

In an incomplete MPR $P = (p_{ij})_{n\times n}$, the element $p_{ij}$ and $p_{kl}$ are called adjoining if $(i, j) \cap (k, l)$ is non empty set, e.g if $p_{i_0j_0} = p_{i_0k} \times p_{kj_0}$, where $p_{i_0k}$ and $p_{kj_0}$ are adjoining known elements and $p_{i_0j_0}$ be an unknown elements, then $p_{i_0j_0}$ can be found directly and the corresponding incomplete MPR is called acceptable. MPR $P$ are called an unacceptable incomplete multiplicative preference relation, if there does not exist adjoining known element such that unknown factor can be calculated

[21]. In that cases, therefore, it is necessary to return the unacceptable incomplete multiplicative preference relation to the decision maker for revaluation until an acceptable incomplete multiplicative preference relation can be obtained. Wang and Xu [18] showed that if an incomplete MPRs are acceptable, then there exists at least one known element(except diagonal elements) in each line/column of MPR matrix $P$, i.e., their exit at least $n-1$ judgment provided by the decision maker. Later according to Cai and Deng[2], Xu[21] and Alonso et al.[1] prove that an incomplete multiplicative preference relation which is acceptable. Then there exists at least a set of an $n-1$ number of non-leading diagonal known elements, where each of the criteria is compared at least once, which includes the case when a complete row or column of preference values is known. In this work, we have applied the same above-said applications in the incomplete IFMPRs scenario.

Taking inspiration from the work of Jiang et al. [9], we propose a two-step procedure method to estimate the missing values in an incomplete IFMPR without splitting IFMPRs into two MPRs. The idea is first to evaluate their value using the simple connecting path approach and subsequently improve upon them using an optimization problem.

3.1. **Estimating step.** To complement the missing elements in an incomplete MPR, Harker [5, 6] designed a geometric mean method based on the connective paths. The general structure of a connecting path of length $\theta+1$, denoted by $cp_{(\theta+1)}$, has the following form: $cp_{(\theta+1)}$: $*_{ij} = p_{i\,k_1}p_{k_1\,k_2}\dots p_{k_\theta\,j}$ , where $*_{ij}$ denotes the missing element to be estimated and the elements on the right hand side are known entities in the path connecting $i$ with $j$, where $i,\,j,\,k_1,\dots,k_\theta \in N,\ 0\le \ell \le n-2$. The connecting path of length two is an elementary connecting path $cp_{(2)} : *_{ij} = p_{i\,k_1}p_{k_1\,j}$ for $k_1 \in N$, and $k_1 \neq i,\,j$. Harker [5, 6] argued that the value of the missing element $*_{ij}$ is the geometric mean of all elementary connecting paths related to it with no vertex repeats more than once in the path. Consequently, $*_{ij} = \left(\prod_{r=1}^{n_\theta} cp_{(r)}\right)^{1/n_\theta}$, where $n_\theta$ is the number of all possible connecting fully known paths (that is, no missing entries along the path) from $i$ to $j$. A major limitation of this method is that the number of connecting paths of different lengths between $i$ and $j$ may be extremely large and computationally intractable for many real problems. For instance, Deschrijver and Kerre [3] presented an example of a matrix of size 10 only with the number of connecting paths exceeding 109,000. Jiang et al. [9] improved this method for incomplete IFMPR by taking elementary connecting path instead of all connecting paths of all sizes. In that case the matrix of size 10, the number of all elementary connecting path would not surpass 8, which is much less than 109,000. Based on the new consistency property of IFMPRs, the initial value of the missing element of incomplete IFMPRs can be calculated by using a geometric mean method which is denoted by $\tilde{r}^{'}_{ij}$, where $\tilde{r}^{'}_{ij} =$

$$(3.1)\quad (\mu^*_{ij},\nu^*_{ij}) = \begin{cases} \left(\prod_{k\in T_{ij}}\{(\mu_{ik},\nu_{ik})\otimes(\mu_{kj},\nu_{kj})\}\right)^{1/t_{ij}} & \text{if } i\le k, k\le j;\\ \left(\prod_{k\in T_{ij}}\{(\frac{1}{\mu_{ki}},\frac{1}{\nu_{ki}})\otimes(\mu_{kj},\nu_{kj})\}\right)^{1/t_{ij}} & \text{if } i\ge k, k\le j;\\ \left(\prod_{k\in T_{ij}}\{(\mu_{ik},\nu_{ik})\otimes(\frac{1}{\mu_{jk}},\frac{1}{\nu_{jk}})\}\right)^{1/t_{ij}} & \text{if } i< k, k> j, \end{cases}$$

where $T_{ij} = \{k|(\mu_{ik},\nu_{ik}),(\mu_{kj},\nu_{kj}) \in \Omega\}$, $\Omega$ is the set of known element and $t_{ij}$ is the number of element present in the set $T_{ij}$ which indicates that there may exist different pairs of adjoining known elements to find out the unknown elements. The initial values are denoted by $\left(\mu_{ij}^{*(0)}, \nu_{ij}^{*(0)}\right)$.

**Remark 3.2.** It is to note that in equation 3.1, $\mu_{kj} \times \frac{1}{\mu_{ki}} \neq \mu_{kj} \times \mu_{ik}$, and it should follow in other expression also.

3.2. **Adjusting step.** The IFMPR is consistent if equation 2.3 is satisfied. Sometimes initial values of the missing element may not satisfy the conditions of IFMPRs. To overcome this difficulty we have developed a local optimization model(Model(M)) by minimizing the error.

$$
\begin{aligned}
&\text{Model(M)}\\
&\text{Min} \sum_{i,k=1}^{n} \sum_{j=i+1}^{n} \left(\varepsilon_{ij_{i\le k,k\le j}}^{k} + \varepsilon_{ij_{i\ge k,k\le j}}^{k} + \varepsilon_{ij_{i<k,k>j}}^{k}\right)\\
&s.t.\\
&\varepsilon_{ij_{i\le k,k\le j}}^{k} = \mid (\mu_{ij}, \nu_{ij}) - (\mu_{ik}\mu_{kj}, \nu_{ik}\nu_{kj}) \mid;\\
&\varepsilon_{ij_{i\ge k,k\le j}}^{k} = \left|(\mu_{ij}, \nu_{ij}) - \left(\frac{\mu_{kj}}{\mu_{ki}}, \frac{\nu_{kj}}{\nu_{ki}}\right)\right|;\\
&\varepsilon_{ij_{i<k,k>j}}^{k} = \left|(\mu_{ij}, \nu_{ij}) - \left(\frac{\mu_{ik}}{\mu_{jk}}, \frac{\nu_{ik}}{\nu_{jk}}\right)\right|;\\
&\mu_{ij}\nu_{ij} \le 1;\ 1/9 \le \mu_{ij};\ \nu_{ij} \le 9;\\
&\mu_{ij}^{(0)} = \mu_{ij}^{*(0)};\ \nu_{ij}^{(0)} = \nu_{ij}^{*(0)};\ i \neq j \neq k;\ i,j,k \in N,
\end{aligned}
$$

where $\mu_{ij}^{*(0)}$ and $\nu_{ij}^{*(0)}$ are the initial value obtain from estimating step. For proper understanding, we have represented an algorithm that illustrates the above methods.

**Algorithm 1**

Step 1: Consider an incomplete IFMPRs of $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ in which some elements are missing and the known elements satisfy the transitivity property of the equation (2.3).

Step 2: The initial value $r_{ij}^{'}$ of the missing element of incomplete IFMPRs is calculated using the equation (3.1).

Step 3: Initial values obtained in step 2 are adjusted by the Model(M), and the adjusting values are denoted by $\tilde{r}_{ij}^{''}$.

Step 4: The complete IFMPRs $\tilde{R_c} = (\tilde{r}_{c,_{ij}})_{n\times n}$, is obtained where

$$
\tilde{r}_{c,_{ij}} = \begin{cases} \tilde{r}_{ij}^{''} & \tilde{r}_{ij} \notin \Omega \\ \tilde{r}_{ij} & \tilde{r}_{ij} \in \Omega. \end{cases} \tag{3.2}
$$

In the next section, we have developed a goal programming model to estimate the missing values.

## 4. Goal programming model to estimate the missing values

In 2015 Meng and Chen [12] construct a linear programming model to evaluate the missing value with incomplete MPRs, which is based on consistency index. To cope with incomplete intuitionistic fuzzy multiplicative preference relation (incomplete IFMPR), this section developed a deviation model to evaluate the missing value which is based on new transitivity property which was discussed in Section 2. Let $\tilde{R} = (\mu_{ij}, \nu_{ij})$ be an incomplete IFMPR. We know that $\tilde{R}$ is consistent if and only if the equation (2.3) holds for the strictly upper triangular elements. To minimize the errors, Approximate the equation (2.3). Define

(4.1)
$$\begin{aligned}
(\varepsilon_{ij})_{i\leq k,k\leq j} &= \delta_{ij}\left|(\mu_{ij},\nu_{ij}) - (\mu_{ik}\mu_{kj},\nu_{ik}\nu_{kj})\right| \\
(\varepsilon_{ij})_{i\geq k,k\leq j} &= \delta_{ij}\left|(\mu_{ij},\nu_{ij}) - \left(\frac{\mu_{kj}}{\mu_{ki}},\frac{\nu_{kj}}{\nu_{ki}}\right)\right| \\
(\varepsilon_{ij})_{i< k,k> j} &= \delta_{ij}\left|(\mu_{ij},\nu_{ij}) - \left(\frac{\mu_{ik}}{\mu_{jk}},\frac{\nu_{ik}}{\nu_{jk}}\right)\right| \\
&\text{for } i,j = 1,2,\cdots n,\ i<j
\end{aligned}$$

$$\text{where, } \delta_{ij} = \begin{cases} 1 & k \in T_{ij} \\ 0 & \text{otherwise,} \end{cases}$$

where $T_{ij} = \{k|(\mu_{ik},\nu_{ik}),(\mu_{kj},\nu_{kj}) \in \Omega\}$, $T_{ij}$ is the set of known element. Using the above equation (4.1), we construct the following goal programming model to estimate the missing value

$$\begin{aligned}
&\text{Min}(\varepsilon_{ij})_{i\leq k,k\leq j} = \delta_{ij}|(\mu_{ij},\nu_{ij}) - (\mu_{ik}\mu_{kj},\nu_{ik}\nu_{kj})| \\
&\text{Min}(\varepsilon_{ij})_{i\geq k,k\leq j} = \delta_{ij}\left|(\mu_{ij},\nu_{ij}) - \left(\frac{\mu_{kj}}{\mu_{ki}},\frac{\nu_{kj}}{\nu_{ki}}\right)\right| \\
&\text{Min}(\varepsilon_{ij})_{i< k,k> j} = \delta_{ij}\left|(\mu_{ij},\nu_{ij}) - \left(\frac{\mu_{ik}}{\mu_{jk}},\frac{\nu_{ik}}{\nu_{jk}}\right)\right| \\
&\text{subject to,} \\
&\frac{1}{9} \leq \mu_{ij};\ \nu_{ij} \leq 9;\ (\mu_{ij},\nu_{ij}) \in U; \\
&\text{where } U = \{(\mu_{ij},\nu_{ij})|(\mu_{ij},\nu_{ij}) \text{ is a missing value} \\
&\text{for } i,j = 1,2,\cdots n,\ i<j\}.
\end{aligned}$$

The solution of the minimization problem can be obtained by solving the goal programming Model(P).

**Model(P)**

$$\begin{aligned}
\text{Min } D = \sum_{i,k=1}^{n}\sum_{j=i+1}^{n} &\left(d_{ij,k}^{(+)}\right)_{i\leq k,k\leq j} + \left(d_{ij,k}^{(-)}\right)_{i\leq k,k\leq j} + \left(d_{ij,k}^{(+)}\right)_{i\geq k,k\leq j} \\
&+ \left(d_{ij,k}^{(-)}\right)_{i\geq k,k\leq j} + \left(d_{ij,k}^{(+)}\right)_{i<k,k>j} + \left(d_{ij,k}^{(-)}\right)_{i<k,k>j}
\end{aligned}$$

subject to,

$$\delta_{ij}\{(\mu_{ij},\nu_{ij}) - (\mu_{ik}\mu_{kj},\nu_{ik}\nu_{kj})\} - \left(d_{ij,k}^{(+)}\right)_{i\leq k,k\leq j} + \left(d_{ij,k}^{(-)}\right)_{i\leq k,k\leq j} = 0;$$

$$\delta_{ij}\left\{(\mu_{ij},\nu_{ij})-\left(\frac{\mu_{kj}}{\mu_{ki}},\frac{\nu_{kj}}{\nu_{ki}}\right)\right\}-\left(d^{(+)}_{ij,k}\right)_{i\geq k,k\leq j}+\left(d^{(-)}_{ij,k}\right)_{i\geq k,k\leq j}=0;$$
$$\delta_{ij}\left\{(\mu_{ij},\nu_{ij})-\left(\frac{\mu_{ik}}{\mu_{jk}},\frac{\nu_{ik}}{\nu_{jk}}\right)\right\}-\left(d^{(+)}_{ij,k}\right)_{i<k,k>j}+\left(d^{(-)}_{ij,k}\right)_{i<k,k>j}=0;$$
$\mu_{ij}\nu_{ij}\leq 1;\ 1/9\leq\mu_{ij};\ \nu_{ij}\leq 9;$
$$\left(d^{(+)}_{ij,k}\right)_{i\leq k,k\leq j},\ \left(d^{(-)}_{ij,k}\right)_{i\leq k,k\leq j},\ \left(d^{(+)}_{ij,k}\right)_{i\geq k,k\leq j},\ \left(d^{(-)}_{ij,k}\right)_{i\geq k,k\leq j},\ \left(d^{(+)}_{ij,k}\right)_{i<k,k>j},\ \left(d^{(-)}_{ij,k}\right)_{i<k,k>j}\geq 0;$$
where,

$$\left(d^{(+)}_{ij,k}\right)_{i\leq k,k\leq j}=[(\mu_{ij},\nu_{ij})-(\mu_{ik}\mu_{kj},\nu_{ik}\nu_{kj})]\vee 0;\ \left(d^{(-)}_{ij,k}\right)_{i\leq k,k\leq j}=[(\mu_{ik}\mu_{kj},\nu_{ik}\nu_{kj})-(\mu_{ij},\nu_{ij})]\vee 0;$$
$$\left(d^{(+)}_{ij,k}\right)_{i\geq k,k\leq j}=\left[(\mu_{ij},\nu_{ij})-(\frac{\mu_{kj}}{\mu_{ki}},\frac{\nu_{kj}}{\nu_{ki}})\right]\vee 0;\ \left(d^{(-)}_{ij,k}\right)_{i\geq k,k\leq j}=\left[(\frac{\mu_{kj}}{\mu_{ki}},\frac{\nu_{kj}}{\nu_{ki}})-(\mu_{ij},\nu_{ij})\right]\vee 0;$$
$$\left(d^{(+)}_{ij,k}\right)_{i<k,k>j}=\left[(\mu_{ij},\nu_{ij})-(\frac{\mu_{ik}}{\mu_{jk}},\frac{\nu_{ik}}{\nu_{jk}})\right]\vee 0;\ \left(d^{(-)}_{ij,k}\right)_{i<k,k>j}=\left[(\frac{\mu_{ik}}{\mu_{jk}},\frac{\nu_{ik}}{\nu_{jk}})-(\mu_{ij},\nu_{ij})\right]\vee 0;$$
For the sake of convenience, here we use,
$$\left(d^{(+)}_{ij,k}\right)_{i\leq k,k\leq j}=\left(d^{(+)}_{\mu_{ij,k}},d^{(+)}_{\nu_{ij,k}}\right)_{i\leq k,k\leq j},\ \left(d^{(-)}_{ij,k}\right)_{i\leq k,k\leq j}=\left(d^{(-)}_{\mu_{ij,k}},d^{(-)}_{\nu_{ij,k}}\right)_{i\leq k,k\leq j},$$
$$\left(d^{(+)}_{ij,k}\right)_{i\geq k,k\leq j}=\left(d^{(+)}_{\mu_{ij,k}},d^{(+)}_{\nu_{ij,k}}\right)_{i\geq k,k\leq j},\ \left(d^{(-)}_{ij,k}\right)_{i\geq k,k\leq j}=\left(d^{(-)}_{\mu_{ij,k}},d^{(-)}_{\nu_{ij,k}}\right)_{i\geq k,k\leq j},$$
$$\left(d^{(+)}_{ij,k}\right)_{i<k,k>j}=\left(d^{(+)}_{\mu_{ij,k}},d^{(+)}_{\nu_{ij,k}}\right)_{i<k,k>j},\ \left(d^{(-)}_{ij,k}\right)_{i<k,k>j}=\left(d^{(-)}_{\mu_{ij,k}},d^{(-)}_{\nu_{ij,k}}\right)_{i<k,k>j},$$
where
$$\left(d^{(+)}_{\mu_{ij,k}}\right)_{i\leq k,k\leq j}=[\log\mu_{ij}-(\log\mu_{ik}+\log\mu_{kj})]\vee 0;$$
$$\left(d^{(-)}_{\mu_{ij,k}}\right)_{i\leq k,k\leq j}=[(\log\mu_{ik}+\log\mu_{kj})-\log\mu_{ij}]\vee 0;$$
$$\left(d^{(+)}_{\nu_{ij,k}}\right)_{i\leq k,k\leq j}=[\log\nu_{ij}-(\log\nu_{ik}+\log\nu_{kj})]\vee 0;$$
$$\left(d^{(-)}_{\nu_{ij,k}}\right)_{i\leq k,k\leq j}=[(\log\nu_{ik}+\log\nu_{kj})-\log\nu_{ij}]\vee 0;$$
$$\left(d^{(+)}_{\mu_{ij,k}}\right)_{i\geq k,k\leq j}=[\log\mu_{ij}-(\log\mu_{kj}-\log\mu_{ki})]\vee 0;$$
$$\left(d^{(-)}_{\mu_{ij,k}}\right)_{i\geq k,k\leq j}=[(\log\mu_{kj}-\log\mu_{ki})-\log\mu_{ij}]\vee 0;$$
$$\left(d^{(+)}_{\nu_{ij,k}}\right)_{i\geq k,k\leq j}=[\log\nu_{ij}-(\log\nu_{kj}-\log\nu_{ki})]\vee 0;$$
$$\left(d^{(-)}_{\nu_{ij,k}}\right)_{i\geq k,k\leq j}=[(\log\nu_{kj}-\log\nu_{ki})-\log\nu_{ij}]\vee 0;$$
$$\left(d^{(+)}_{\mu_{ij,k}}\right)_{i<k,k>j}=[\log\mu_{ij}-(\log\mu_{ik}-\log\mu_{jk})]\vee 0;$$
$$\left(d^{(-)}_{\mu_{ij,k}}\right)_{i<k,k>j}=[(\log\mu_{ik}-\log\mu_{jk})-\log\mu_{ij}]\vee 0;$$
$$\left(d^{(+)}_{\nu_{ij,k}}\right)_{i<k,k>j}=[\log\nu_{ij}-(\log\nu_{ik}-\log\nu_{jk})]\vee 0;$$
$$\left(d^{(-)}_{\nu_{ij,k}}\right)_{i<k,k>j}=[(\log\nu_{ik}-\log\nu_{jk})-\log\nu_{ij}]\vee 0.$$

To illustrate the above procedure we have presented two examples.

**Example 4.1.** Let us consider a decision making problem with five sets of alternatives $x_i(i = 1, 2, \cdots, 5)$. The decision maker judges these five alternatives by pairwise comparison and provides his/her judgement as $\tilde{r}_{12} = (\mu_{12}, \nu_{12}) = (5, 1/7)$, $\tilde{r}_{14} = (\mu_{14}, \nu_{14}) = (3, 1/7)$, $\tilde{r}_{23} = (\mu_{23}, \nu_{23}) = (9/5, 3/7)$, $\tilde{r}_{25} = (\mu_{25}, \nu_{25}) = (1/5, 3)$, $\tilde{r}_{35} = (\mu_{35}, \nu_{35}) = (1/9, 7)$, $\tilde{r}_{45} = (\mu_{45}, \nu_{45}) = (1/7, 3)$. The matrix representation of the above information is given by

$$\tilde{R} = \begin{bmatrix} (1,1) & (5,1/7) & (*,*) & (3,1/7) & (*,*) \\ (1/7,5) & (1,1) & (9/5,3/7) & (*,*) & (1/5,3) \\ (*,*) & (3/7,9/5) & (1,1) & (*,*) & (1/9,7) \\ (1/7,3) & (*,*) & (*,*) & (1,1) & (1/7,3) \\ (*,*) & (3,1/5) & (7,1/9) & (3,1/7) & (1,1) \end{bmatrix}_{5\times 5}$$

The initial value of missing element are calculated using the equation (3.1), are given in Table 1.

Table 1. Calculation of Missing element(initial value)

| Missing element | Adjoining element | Calculated value |
|---|---|---|
| $\left(\mu_{13}^{*(0)}, \nu_{13}^{*(0)}\right)$ | $(\mu_{12}, \nu_{12})$, $(\mu_{23}, \nu_{23})$ | $(9, 3/49)$ |
| $\left(\mu_{15}^{*(0)}, \nu_{15}^{*(0)}\right)$ | $(\mu_{12}, \nu_{12})$, $(\mu_{25}, \nu_{25})$<br>$(\mu_{14}, \nu_{14})$, $(\mu_{45}, \nu_{45})$ | $(1.44877, 0.0297)$ |
| $\left(\mu_{24}^{*(0)}, \nu_{24}^{*(0)}\right)$ | $(\mu_{21}, \nu_{21})$, $(\mu_{14}, \nu_{14})$<br>$(\mu_{25}, \nu_{25})$, $(\mu_{54}, \nu_{54})$ | $(2.38454, 0.366025)$ |
| $\left(\mu_{34}^{*(0)}, \nu_{34}^{*(0)}\right)$ | $(\mu_{35}, \nu_{35})$, $(\mu_{54}, \nu_{54})$ | $(0.777, 2.333)$ |

Some initial values does not satisfies the property of IFMPRs e.g $\boldsymbol{\mu_{34} \times \nu_{34} \nleq 1}$. To adjust these values solve this problem using an optimization Model(M) that minimize the error.

**Model(M)**

$$\begin{aligned} &\text{Min } \{|\{(9-\mu_{13})^2 + (3/49 - \nu_{13})^2\}^{0.5}| + |\{(1-\mu_{15})^2 + (3/7 - \nu_{15})^2\}^{0.5}| \\ &+|\{(3/7-\mu_{15})^2 + (3/7-\nu_{15})^2\}^{0.5}| + |\{(3/5-\mu_{24})^2 + (1-\nu_{24})^2\}^{0.5}| \\ &+|\{(7/5-\mu_{24})^2 + (1-\nu_{24})^2\}^{0.5}| + |\{(0.777-\mu_{34})^2 + (2.333-\nu_{34})^2\}^{0.5}|\}; \\ &\text{s.t.} \\ &\mu_{13} \times \nu_{13} \leq 1; \mu_{15} \times \nu_{15} \leq 1; \mu_{24} \times \nu_{24} \leq 1; \\ &\mu_{34} \times \nu_{34} \leq 1; 1/9 \leq \mu_{13}; \nu_{13} \leq 9; \\ &1/9 \leq \mu_{15}; \nu_{15} \leq 9; 1/9 \leq \mu_{24}; \nu_{24} \leq 9; \\ &1/9 \leq \mu_{34}; \nu_{34} \leq 9; {\mu_{13}}^{*(0)} = 9, {\nu_{13}}^{*(0)} = 3/49; \\ &{\mu_{15}}^{*(0)} = 1.44877, {\nu_{15}}^{*(0)} = 0.0297; \\ &{\mu_{24}}^{*(0)} = 2.3845, {\nu_{24}}^{*(0)} = 0.366; \\ &{\mu_{34}}^{*(0)} = 0.777, {\nu_{13}}^{*(0)} = 2.333. \end{aligned}$$

After solving the above optimization model, the adjusting values are given $\mu_{13} = 9, \nu_{13} = 0.111, \mu_{15} = 0.74, \nu_{15} = 0.429, \mu_{24} = 0.912, \nu_{24} = 1, \mu_{34} = 0.441, \nu_{34} = 2.268$. This model is solved by MATLAB optimization tool box. The complete IFMPR $\tilde{R}_c$ is given below

$$\begin{bmatrix} (1,1) & (5,1/7) & (9,0.111) & (3,1/7) & (0.74,0.429) \\ (1/7,5) & (1,1) & (9/5,3/7) & (0.912,1) & (1/5,3) \\ (0.111,9) & (3/7,9/5) & (1,1) & (0.441,2.268) & (1/9,7) \\ (1/7,3) & (1,0.912) & (2.268,0.441) & (1,1) & (1/7,3) \\ (0.429,0.74) & (3,1/5) & (7,1/9) & (3,1/7) & (1,1) \end{bmatrix}_{5\times 5}$$

To check the consistency degree, the complete IFMPR is split into two MPRs and their corresponding $CR$ values are given in row-1 of table 2.

$$C = \begin{pmatrix} 1 & 5 & 9 & 3 & 0.74 \\ \frac{1}{5} & 1 & \frac{9}{5} & 0.912 & \frac{1}{5} \\ \frac{1}{9} & \frac{5}{9} & 1 & 0.441 & \frac{1}{9} \\ \frac{1}{3} & \frac{1}{0.912} & \frac{1}{0.441} & 1 & \frac{1}{7} \\ \frac{1}{0.74} & 5 & 9 & 7 & 1 \end{pmatrix}_{5\times 5} \quad D = \begin{pmatrix} 1 & \frac{1}{7} & 0.111 & \frac{1}{7} & 0.429 \\ 7 & 1 & \frac{3}{7} & 1 & 3 \\ \frac{1}{0.111} & \frac{7}{3} & 1 & 2.268 & 7 \\ 7 & 1 & \frac{1}{2.268} & 1 & 3 \\ \frac{1}{0.429} & \frac{1}{3} & \frac{1}{7} & \frac{1}{3} & 1 \end{pmatrix}_{5\times 5}$$

The Example 4.1 is also solved by the goal programming problem and the missing elements are $\mu_{13} = 9, \nu_{13} = 0.111, \mu_{15} = 1, \nu_{15} = 0.429, \mu_{24} = 1, \nu_{24} = 1, \mu_{34} = 0.428, \nu_{34} = 2.333$. This model(P) is solved using Lingo software. The consistency ratio of both MPRs obtain from two different methods such as two-step procedure method, and goal programming model method are given in Table 2. Therefore, the complete IFMPR $\tilde{R}_c$ is acceptably consistent.

TABLE 2. Consistency ratio

| | | |
|---|---|---|
| Two-step procedure Method | CR(C) | 0.0114 |
| | CR(D) | 0.0094 |
| Goal programming model (Model(P)) | CR($C'$) | 0.0144 |
| | CR($D'$) | 0.0094 |

In Table 2 $C'$,$D'$ are two multiplicative preference relations are obtained by splitting the complete IFMPRs where missing element are found from the model(P).

**Example 4.2.** Let us consider a decision making problem with seven sets of alternatives $x_i (i = 1, 2, \cdots, 7)$. The decision maker judge these seven alternatives by pairwise comparison and provides his/her judgement as follows: $\tilde{r}_{12} = (\mu_{12}, \nu_{12}) = (3/5, 1/4)$, $\tilde{r}_{16} = (\mu_{16}, \nu_{16}) = (1/5, 1/2)$, $\tilde{r}_{23} = (\mu_{23}, \nu_{23}) = (1/2, 8/5)$, $\tilde{r}_{26} = (\mu_{26}, \nu_{26}) = (1/3, 2)$, $\tilde{r}_{34} = (\mu_{34}, \nu_{34}) = (2/9, 15/4)$, $\tilde{r}_{36} = (\mu_{36}, \nu_{36}) = (2/3, 5/4)$, $\tilde{r}_{45} = (\mu_{45}, \nu_{45}) = (7, 1/7)$, $\tilde{r}_{46} = (\mu_{46}, \nu_{46}) = (3, 1/3)$, $\tilde{r}_{56} = (\mu_{56}, \nu_{56}) = (3/7, 7/3)$, $\tilde{r}_{67} = (\mu_{67}, \nu_{67}) = (1/7, 3)$. The matrix representation of the above information is given by $\tilde{R}_1$.

$$\tilde{R}_1 = \begin{bmatrix} (1,1) & (3/5,1/4) & (*,*) & (*,*) & (*,*) & (1/5,1/2) & (*,*) \\ (1/4,3/5) & (1,1) & (1/2,8/5) & (*,*) & (*,*) & (1/3,2) & (*,*) \\ (*,*) & (8/5,1/2) & (1,1) & (2/9,15/4) & (*,*) & (2/3,5/4) & (*,*) \\ (*,*) & (*,*) & (15/4,2/9) & (1,1) & (7,1/7) & (3,1/3) & (*,*) \\ (*,*) & (*,*) & (*,*) & (1/7,7) & (1,1) & (3/7,7/3) & (*,*) \\ (1/2,1/5) & (2,1/3) & (5/4,2/3) & (1/3,3) & (7/3,3/7) & (1,1) & (1/7,3) \\ (*,*) & (*,*) & (*,*) & (*,*) & (*,*) & (3,1/7) & (1,1) \end{bmatrix}_{7\times 7}$$

The initial value of missing element is calculated by using the equation (3.1) which is given in Table 3.

Table 3. Calculation of Missing element(initial value)

| Missing element | Adjoining element | Calculated value |
|---|---|---|
| $\left(\mu_{13}^{*(0)}, \nu_{13}^{*(0)}\right)$ | $(\mu_{12}, \nu_{12}),\ (\mu_{23}, \nu_{23})$ | $(3/10, 2/5)$ |
| $\left(\mu_{14}^{*(0)}, \nu_{14}^{*(0)}\right)$ | $(\mu_{16}, \nu_{16}),\ (\mu_{64}, \nu_{64})$ | $(1/15, 3/2)$ |
| $\left(\mu_{15}^{*(0)}, \nu_{15}^{*(0)}\right)$ | $(\mu_{16}, \nu_{16}),\ (\mu_{65}, \nu_{65})$ | $(7/15, 3/14)$ |
| $\left(\mu_{17}^{*(0)}, \nu_{17}^{*(0)}\right)$ | $(\mu_{16}, \nu_{16}),\ (\mu_{67}, \nu_{67})$ | $(1/35, 3/2)$ |
| $\left(\mu_{24}^{*(0)}, \nu_{24}^{*(0)}\right)$ | $(\mu_{23}, \nu_{23}),\ (\mu_{34}, \nu_{34})$ $(\mu_{26}, \nu_{26}),\ (\mu_{46}, \nu_{46})$ | $(0.17, 3.77)$ |
| $\left(\mu_{25}^{*(0)}, \nu_{25}^{*(0)}\right)$ | $(\mu_{26}, \nu_{26}),\ (\mu_{56}, \nu_{56})$ | $(7/9, 6/7)$ |
| $\left(\mu_{27}^{*(0)}, \nu_{27}^{*(0)}\right)$ | $(\mu_{26}, \nu_{26}),\ (\mu_{67}, \nu_{67})$ | $(1/21, 6)$ |
| $\left(\mu_{35}^{*(0)}, \nu_{35}^{*(0)}\right)$ | $(\mu_{34}, \nu_{34}),\ (\mu_{45}, \nu_{45})$ $(\mu_{36}, \nu_{36}),\ (\mu_{56}, \nu_{56})$ | $(5.69, 0.13)$ |
| $\left(\mu_{37}^{*(0)}, \nu_{37}^{*(0)}\right)$ | $(\mu_{36}, \nu_{36}),\ (\mu_{67}, \nu_{67})$ | $(2/21, 15/4)$ |
| $\left(\mu_{47}^{*(0)}, \nu_{47}^{*(0)}\right)$ | $(\mu_{46}, \nu_{46}),\ (\mu_{67}, \nu_{67})$ | $(3/7, 1)$ |
| $\left(\mu_{57}^{*(0)}, \nu_{57}^{*(0)}\right)$ | $(\mu_{56}, \nu_{56}),\ (\mu_{67}, \nu_{67})$ | $(3/49, 7)$ |

Some initial values does not satisfies the property of IFMPRs e.g $\boldsymbol{\mu_{14}, \mu_{17}, \mu_{27}, \mu_{37}, \mu_{57} \ngeq \frac{1}{9}}$. To adjust these value we have solved optimization Model(M), that minimize the error.

$$\begin{aligned}
\text{Min } \{ & 2|\{(3/10-\mu_{13})^2+(2/5-\nu_{13})^2\}^{0.5}| + |\{(1/15-\mu_{14})^2+(3/2-\nu_{14})^2\}^{0.5}| \\
& +|\{(7/15-\mu_{15})^2+(3/14-\nu_{15})^2\}^{0.5}| + |\{(1/35-\mu_{17})^2+(3/2-\nu_{17})^2\}^{0.5}| \\
& +2|\{(1/9-\mu_{24})^2+(6-\nu_{24})^2\}^{0.5}| + |\{(7/9-\mu_{25})^2+(6/7-\nu_{25})^2\}^{0.5}| \\
& +|\{(1/21-\mu_{27})^2+(6-\nu_{27})^2\}^{0.5}| + 2|\{(14/9-\mu_{35})^2+(15/28-\nu_{35})^2\}^{0.5}| \\
& +|\{(2/21-\mu_{37})^2+(15/4-\nu_{37})^2\}^{0.5}| + |\{(3/7-\mu_{47})^2+(1-\nu_{47})^2\}^{0.5}| \\
& +|\{(3/49-\mu_{57})^2+(7-\nu_{57})^2\}^{0.5}|\},
\end{aligned}$$

subject to

$$\begin{aligned}
& \mu_{13}\times\nu_{13}\le 1, \mu_{14}\times\nu_{14}\le 1, \mu_{15}\times\nu_{15}\le 1, \mu_{17}\times\nu_{17}\le 1, \mu_{24}\times\nu_{24}\le 1, \mu_{25}\times\nu_{25}\le 1, \\
& \mu_{27}\times\nu_{27}\le 1, \mu_{35}\times\nu_{35}\le 1, \mu_{37}\times\nu_{37}\le 1, \mu_{47}\times\nu_{47}\le 1, \mu_{57}\times\nu_{57}\le 1, \\
& 1/9\le\mu_{13},\nu_{13}\le 9, 1/9\le\mu_{14},\nu_{14}\le 9, 1/9\le\mu_{15},\nu_{15}\le 9, 1/9\le\mu_{17},\nu_{17}\le 9, \\
& 1/9\le\mu_{24},\nu_{24}\le 9, 1/9\le\mu_{25},\nu_{25}\le 9, 1/9\le\mu_{27},\nu_{27}\le 9, 1/9\le\mu_{35},\nu_{35}\le 9, \\
& 1/9\le\mu_{37},\nu_{37}\le 9, 1/9\le\mu_{47},\nu_{47}\le 9, 1/9\le\mu_{57},\nu_{57}\le 9, \\
& {\mu_{13}}^{*(0)} = 0.5237, {\nu_{13}}^{*(0)} = 0.0745; {\mu_{14}}^{*(0)} = 1/15, {\nu_{14}}^{*(0)} = 3/2, \\
& {\mu_{15}}^{*(0)} = 7/15, {\nu_{15}}^{*(0)} = 3/14; {\mu_{17}}^{*(0)} = 1/35, {\nu_{17}}^{*(0)} = 3/2; \\
& {\mu_{24}}^{*(0)} = 0.17, {\nu_{24}}^{*(0)} = 3.77; {\mu_{25}}^{*(0)} = 7/9, {\nu_{25}}^{*(0)} = 6/7;
\end{aligned}$$

$\mu_{27}{}^{*(0)} = 1/21, \nu_{27}{}^{*(0)} = 6; \mu_{35}{}^{*(0)} = 5.69, \nu_{35}{}^{*(0)} = 0.127;$
$\mu_{37}{}^{*(0)} = 2/21, \nu_{37}{}^{*(0)} = 15/4; \mu_{47}{}^{*(0)} = 3/7, \nu_{47}{}^{*(0)} = 1; \mu_{57}{}^{*(0)} = 3/49, \nu_{57}{}^{*(0)} = 7.$

The adjusting value are given by:
$(\mu_{13}, \nu_{13}) = (0.3, 0.4)$, $(\mu_{14}, \nu_{14}) = (0.111, 1.5)$, $(\mu_{15}, \nu_{15}) = (0.467, 0.214)$,
$(\mu_{17}, \nu_{17}) = (0.111, 1.5)$, $(\mu_{24}, \nu_{24}) = (0.111, 6)$, $(\mu_{25}, \nu_{25}) = (0.778, 0.857)$,
$(\mu_{27}, \nu_{27}) = (0.111, 6)$, $(\mu_{35}, \nu_{35}) = (1.556, 0.536)$, $(\mu_{37}, \nu_{37}) = (0.111, 3.748)$,
$(\mu_{47}, \nu_{47}) = (0.429, 1), (\mu_{57}, \nu_{57}) = (0.111, 7)$.

Example 4.2 solved by Model(P) also and we obtained the same result. The complete IFMPR is given in the matrix $\tilde{R}_{c_1}$.

$$\tilde{R}_{c_1} = \begin{bmatrix} (1,1) & (3/5,1/4) & (0.3,0.4) & (0.111,1.5) & (0.467,0.214) & (1/5,1/2) & (0.111,1.5) \\ (1/4,3/5) & (1,1) & (1/2,8/5) & (0.111,6) & (0.778,0.857) & (1/3,2) & (0.111,6) \\ (0.4,0.3) & (8/5,1/2) & (1,1) & (2/9,15/4) & (1.556,0.536) & (2/3,5/4) & (0.111,3.748) \\ (1.5,0.111) & (6,0.111) & (15/4,2/9) & (1,1) & (7,1/7) & (3,1/3) & (0.429,1) \\ (0.214,0.467) & (0.857,0.778) & (0.536,1.556) & (1/7,7) & (1,1) & (3/7,7/3) & (0.111,7) \\ (1/2,1/5) & (2,1/3) & (5/4,2/3) & (1/3,3) & (7/3,3/7) & (1,1) & (1/7,3) \\ (1.5,0.111) & (6,0.111) & (3.748,0.111) & (1,0.429) & (7,0.111) & (3,1/7) & (1,1) \end{bmatrix}_{7\times 7}$$

To check the consistency degree of IFMPR, $\tilde{R}_{c_1}$ split into two MPRs $C$ and $D$.

$$C = \begin{pmatrix} 1 & \frac{3}{5} & 0.3 & 0.111 & 0.47 & \frac{1}{5} & 0.111 \\ \frac{5}{3} & 1 & \frac{1}{2} & 0.111 & 0.778 & \frac{1}{3} & 0.111 \\ \frac{1}{0.3} & 2 & 1 & \frac{2}{9} & 1.556 & \frac{2}{3} & 0.111 \\ \frac{1}{0.111} & \frac{1}{0.111} & \frac{9}{2} & 1 & 7 & 3 & 0.429 \\ \frac{1}{0.467} & \frac{1}{0.778} & \frac{1}{1.556} & \frac{1}{7} & 1 & \frac{3}{7} & 0.111 \\ 5 & 3 & \frac{3}{2} & \frac{1}{3} & \frac{7}{3} & 1 & \frac{1}{7} \\ \frac{1}{0.111} & \frac{1}{0.111} & \frac{1}{0.111} & \frac{1}{0.429} & \frac{1}{0.111} & 7 & 1 \end{pmatrix}_{7\times 7}$$

$$D = \begin{pmatrix} 1 & \frac{1}{4} & 0.4 & 1.5 & 0.214 & \frac{1}{2} & 1.5 \\ 4 & 1 & \frac{8}{5} & 6 & 0.857 & 2 & 6 \\ \frac{1}{0.4} & \frac{5}{8} & 1 & \frac{15}{4} & 0.536 & \frac{5}{4} & 3.748 \\ \frac{1}{1.5} & \frac{1}{6} & \frac{4}{15} & 1 & \frac{1}{7} & \frac{1}{3} & 1 \\ \frac{1}{0.214} & \frac{1}{0.857} & \frac{1}{0.536} & 7 & 1 & \frac{7}{3} & 7 \\ 2 & 1/2 & \frac{4}{5} & 3 & \frac{3}{7} & 1 & 3 \\ \frac{1}{1.5} & \frac{1}{6} & \frac{1}{3.748} & 1 & \frac{1}{7} & \frac{1}{3} & 1 \end{pmatrix}_{7\times 7}$$

$CR(C) = 0.0230$ and $CR(D) = 0$ both are acceptable threshold value. According to Satty[16] both $C$ and $D$ are acceptably consistent. Therefore $\tilde{R}_{c_1}$ is also acceptably consistent.

## 5. A Comparative analysis with existing methods

In this section, we compare our proposed method with Jiang et al. [9] for IFMPR and Meng and Chen [12] for MPRs.

(1) In 2015, Jiang et al. [9] discussed the consistency property, especially the acceptable consistency of an IFMPR by splitting into two MPRs. Based on it, Jiang et al. developed two approaches to complement all missing elements of incomplete IFMPRs. according to Jiang et al. [9], the incomplete IFMPR split into two MPRs, and the calculation of missing factor involves two steps, i.e. "estimating step" and " adjusting step." A geometric mean

method is used in the estimating step to calculate the initial values of missing element. Two different approaches are developed for improving the initial values: one is local optimization models, which is time-saving and other is an iterative method that can operate the whole optimization process suitably.

Using Jiang et al. [9] methods, Example 4.1 is solved where incomplete IFMPRs is split into two MPRs as $C$ and $D$. Using geometric mean method missing element is calculated and adjusting values are computed by using local optimization model (LOP1)[9]. Since the consistency ratio (CR) of $C$ and $D$ are 0.0197 and 0.0112. Hence the complete IFMPR is acceptably consistent. Consistency ratio of $C$ and $D$ obtained from our methods (both two-step procedure method and Goal programming model) are less than(see Table 2) from Jiang et al. [9] methods. Similarly, in Example 4.2, the complete IFMPR is also acceptably consistent. Both the model gives the equivalent result.

(2) To measure the multiplicative consistency of an MPR, Meng and Chen [12] proposed the notion of multiplicative geometric consistent index (MGCI). The consistency of an MPR is considered to be unacceptable if the MGCI of an MPR is less than the average value tabulated in Table 1 in their paper. The authors continued their study to include the case of incomplete MPR. They formulated multi-objective programming model to estimate the missing values. Using the goal programming approach and a suitable transformation, the proposed model was converted into an equivalent linear program. The missing values in an MPR were then obtained using the inverse transformation at the optimal solution of the linear program.

In Example 4.1, the incomplete IFMPR is split into two incomplete MPRs using the equation (2.2). The missing elements of two incomplete MPRs are obtained using Meng and Chen's linear programming model (LP)(see [12]). Consistency ratio of two MPRs obtained from Meng and Chen's model are 0.0199 and 0.0097 respectively which are less than from both two-step procedure method and Goal programming model (Model(P)).

## 6. Conclusion

In this paper, we have introduced a new transitivity property of IFMPR. Based on this, we have presented two approaches for completing incomplete IFMPRs. In the first approach, missing element can be calculated by using the new transitivity property, and an optimization model has been developed to adjust the initials values. In the second approach, the missing elements are evaluated by goal programming model based on new transitivity property. Two numerical examples are presented that illustrate the above method. Acceptably consistent with complete IFMPRs has been checked. Also, we have compared our method with Jiang et al. [9] and Meng and Chen [12] method.

The hesitant fuzzy preference relation (HFPR) is a useful tool for decision-makers to elicit their preference information over a set of alternatives. Recently lots of researchers have done in hesitant fuzzy preference relation (HFPR), that allow the decision makers (DMs) to provide several possible preference values over two alternatives. One may refer to [25, 30, 31]. Our work can extend to deal with HFPR and incomplete HFPRs. Also, the missing element of incomplete HFPR can find by using goal programming.

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