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On the two most extended modal types of operators defined over interval-valued intuitionistic fuzzy sets

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Abstract. The definitions of the different modal type of intuitionistic fuzzy operators are given. The two most extended modal operators defined over interval valued intuitionistic fuzzy sets are given and a Theorem for equivalence of these two most extended modal operators is proved.

2010 AMS Classification: 03E72

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1. Introduction

Interval-Valued Intuitionistic Fuzzy Sets (IVIFSs; see [3, 4, 10]) are the most detailed described extension of the Intuitionistic Fuzzy Sets (IFSs; see [1, 2, 4, 5]) . They appeared in 1988, when Georgi Gargov (7.4.1947 – 9.11.1996) and the author read M. Gorzalczany's paper [12] on Interval-Valued Fuzzy Set (IVFS).

Let us have a fixed universe E and its subset A. Formally, the set

$$A = \{\langle x, M_A(x), N_A(x)\rangle \mid x \in E\},$$

where $M_A(x) \subset [0,1]$ and $N_A(x) \subset [0,1]$ are intervals and for all $x \in E$:

$$\sup M_A(x) + \sup N_A(x) \leq 1 \tag{1.1}$$

is called IVIFS and functions $M_A : E \to \mathcal{P}([0,1])$ and $N_A : E \to \mathcal{P}([0,1])$ represent the set of degrees of membership (validity, etc.) and the set of degrees of non-membership (non-validity, etc.) of element $x \in E$ to a fixed set $A \subseteq E$, where $\mathcal{P}(Z) = \{Y | Y \subseteq Z\}$ for an arbitrary set Z. Here, both intervals have the forms: $M_A(x) = [\inf M_A(x), \sup M_A(x)]$ and $N_A(x) = [\inf N_A(x), \sup N_A(x)]$. Hence, when

$\inf M_A(x) = \sup M_A(x) = \mu_A(x)$and $\inf N_A(x) = \sup N_A(x) = \nu_A(x)$, the IVIFS A is transformed to an IFS, that is defined by

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\},$$

where

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1,$$

and functions $\mu_A : E \to [0, 1]$ and $\nu_A : E \to [0, 1]$ represent the degree of membership (validity, etc.) and non-membership (non-validity, etc.) of element $x \in E$ to a fixed set $A \subseteq E$.

Obviously, each IVFS A can be represented by an IVIFS as:

$$\begin{aligned} A &= \{\langle x, M_A(x), N_A(x)\rangle \mid x \in E\} \\ &= \{\langle x, M_A(x), [1 - \sup M_A(x), 1 - \inf M_A(x)]\rangle \mid x \in E\}. \end{aligned}$$

IVIFSs have geometrical interpretations similar to, but more complex than these of the IFSs. For example, the analogue of the standard (second) geometrical interpretation of the IFS is shown on Fig. 1

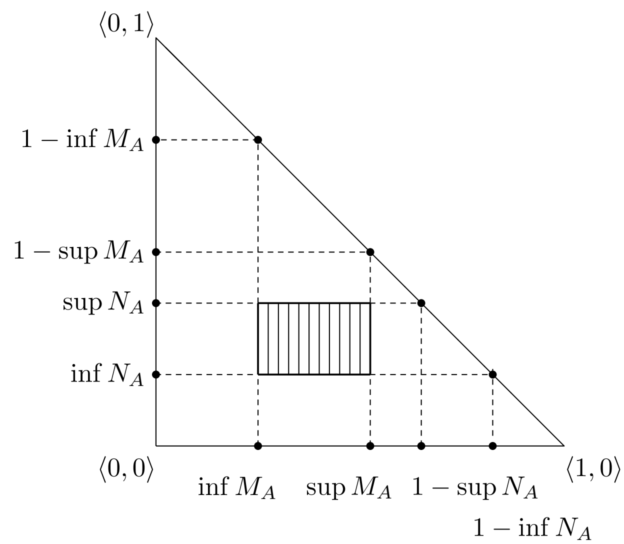

Figure 1. Geometrical interpretation of IVIFS

The geometrical interpretation of the IVFS A is shown on Fig. 2.

In the present paper, we describe the different modal type of intuitionistic fuzzy operators. Four of them are introduced here for the first time. The two most extended modal operators defined over interval valued intuitionistic fuzzy sets are given and a Theorem for the equivalence of these two most extended modal operators is proved.

2. Intuitionistic fuzzy modal operators of the first type

In a series of papers of the author (see, e.g. [4, 7, 8, 9]), different modal type of operators are defined over IVIFSs. These operators are of the two basic modal types

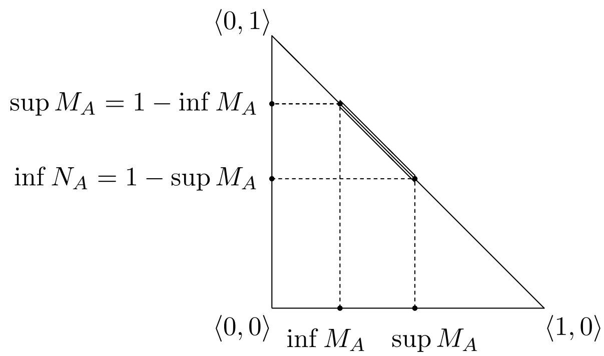

FIGURE 2. Geometrical interpretation of IVFS

of IVIFS-operators. The simplest operators of the first type are direct analogous of the standard modal operators $\Box$ (necessity) and $\Diamond$ (possibility) from modal logic (see, e.g., [11]).

$$\Box A = \{\langle x, M_A(x), [\inf N_A(x), 1 - \sup M_A(x)]\rangle \mid x \in E\},$$
$$\Diamond A = \{\langle x, [\inf M_A(x), 1 - \sup N_A(x)], N_A(x)\rangle \mid x \in E\}.$$

These two operators are extended by the operator

$$D_\alpha(A) = \{\langle x, [\inf M_A(x), \sup M_A(x) + \alpha.(1 - \sup M_A(x) - \sup N_A(x))], [\inf N_A(x), \sup N_A(x) + (1-\alpha).(1 - \sup M_A(x) - \sup N_A(x))]\rangle \mid x \in E\},$$

$$D_\alpha(A) = \{\langle x, [\inf M_A(x), \sup M_A(x) + \alpha.(1 - \sup M_A(x) - \sup N_A(x))], [\inf N_A(x), \sup N_A(x) + (1-\alpha).(1 - \sup M_A(x) - \sup N_A(x))]\rangle \mid x \in E\},$$

where $\alpha \in [0, 1]$, because $\Box A = D_0(A)$ and $\Diamond A = D_1(A)$.

It is extended also to the form

$$F_{\alpha,\beta}(A) = \{\langle x, [\inf M_A(x), \sup M_A(x) + \alpha.(1 - \sup M_A(x) - \sup N_A(x))], [\inf N_A(x), \sup N_A(x) + \beta.(1 - \sup M_A(x) - \sup N_A(x))]\rangle \mid x \in E\},$$

where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$.

Obviously, for each $\alpha \in [0, 1]$:

$$D_\alpha(A) = F_{\alpha,1-\alpha}(A).$$

Seven other operators have been introduced. In some sense, they are on the level of operator $F_{\alpha,\beta}$. The first five of them are described in [4], while the last two operators are introduced here for the first time:

$$\begin{aligned}
G_{\alpha,\beta}(A) = \ & \{\langle x, [\alpha.\inf M_A(x), \alpha.\sup M_A(x)], [\beta.\inf N_A(x), \beta.\sup N_A(x)]\rangle \\
& \mid x \in E\}, \\
H_{\alpha,\beta}(A) = \ & \{\langle x, [\alpha.\inf M(x), \alpha.\sup M_A(x)], [\inf N_A(x), \sup N_A(x) \\
& +\beta.(1 - \sup M_A(x) - \sup N_A(x))]\rangle \mid x \in E\}, \\
H^*_{\alpha,\beta}(A) = \ & \{\langle x, [\alpha.\inf M_A(x), \alpha.\sup M_A(x)], [\inf N_A(x), \sup N_A(x) \\
& +\beta.(1 - \alpha.\sup M_A(x) - \sup N_A(x))]\rangle \mid x \in E\}, \\
J_{\alpha,\beta}(A) = \ & \{\langle x, [\inf M_A(x), \sup M_A(x) + \alpha.(1 - \sup M_A(x) \\
& - \sup N_A(x))], [\beta.\inf N_A(x), \beta.\sup N_A(x)]\rangle \mid x \in E\}, \\
J^*_{\alpha,\beta}(A) = \ & \{\langle x, [\inf M_A(x), \sup M_A(x) + \alpha.(1 - \sup M_A(x) \\
& -\beta.\sup N_A(x))], [\beta.\inf N_A(x), \beta.\sup N_A(x)]\rangle \mid x \in E\}, \\
H^{\#}_{\alpha,\beta}(A) = \ & \{\langle x, [\alpha.\inf M(x), \alpha.\sup M_A(x)], [\inf N_A(x) + \beta - \beta \inf N_A(x), \\
& \sup N_A(x) + \beta - \beta \sup N_A(x))]\rangle \mid x \in E\}, \\
J^{\#}_{\alpha,\beta}(A) = \ & \{\langle x, [\inf M_A(x) + \alpha - \alpha \inf M_A(x), \sup M_A + \alpha - \alpha \sup M_A], \\
& [\beta.\inf N_A(x), \beta.\sup N_A(x)]\rangle \mid x \in E\},
\end{aligned}$$

where $\alpha, \beta \in [0, 1]$.

The larger form of these operators (operators $\Box$, $\Diamond$ and D_α do not have two forms – only the above one) is (see [4]):

$$\begin{aligned}
\overline{F}_{\left(\begin{smallmatrix}\alpha & \gamma \\ \beta & \delta\end{smallmatrix}\right)}(A) = \ & \{\langle x, [\inf M_A(x) + \alpha.(1 - \sup M_A(x) - \sup N_A(x)), \\
& \sup M_A(x) + \beta.(1 - \sup M_A(x) - \sup N_A(x))], \\
& [\inf N_A(x) + \gamma.(1 - \sup M_A(x) - \sup N_A(x)), \\
& \sup N_A(x) + \delta.(1 - \sup M_A(x) - \sup N_A(x))]\rangle \mid x \in E\} \\
& \text{where } \beta + \delta \leq 1; \\
\overline{G}_{\left(\begin{smallmatrix}\alpha & \gamma \\ \beta & \delta\end{smallmatrix}\right)}(A) = \ & \{\langle x, [\alpha.\inf M_A(x), \beta.\sup M_A(x)], \\
& [\gamma.\inf N_A(x), \delta.\sup N_A(x)]\rangle \mid x \in E\}; \\
\overline{H}_{\left(\begin{smallmatrix}\alpha & \gamma \\ \beta & \delta\end{smallmatrix}\right)}(A) = \ & \{\langle x, [\alpha.\inf M_A(x), \beta.\sup M_A(x)], \\
& [\inf N_A(x) + \gamma.(1 - \sup M_A(x) - \sup N_A(x)), \\
& \sup N_A(x) + \delta.(1 - \sup M_A(x) - \sup N_A(x))]\rangle \mid x \in E\}; \\
\overline{H}^*_{\left(\begin{smallmatrix}\alpha & \gamma \\ \beta & \delta\end{smallmatrix}\right)}(A) = \ & \{\langle x, [\alpha.\inf M_A(x), \beta.\sup M_A(x)], \\
& [\inf N_A(x) + \gamma.(1 - \beta.\sup M_A(x) - \sup N_A(x)), \\
& \sup N_A(x) + \delta.(1 - \beta.\sup M_A(x) - \sup N_A(x))]\rangle \mid x \in E\};
\end{aligned}$$

$$\overline{J}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)} = \{\langle x, [\inf M_A(x) + \alpha.(1 - \sup M_A(x) - \sup N_A(x)),$$
$$\sup M_A(x) + \beta.(1 - \sup M_A(x) - \sup N_A(x))],$$
$$[\gamma. \inf N_A(x), \delta. \sup N_A(x)]\rangle \mid x \in E\};$$
$$\overline{J}^*_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(A) = \{\langle x, [\inf M_A(x) + \alpha.(1 - \gamma. \sup M_A(x) - \sup N_A(x)),$$

$$\sup M_A(x) + \beta.(1 - \sup M_A(x) - \delta. \sup N_A(x))],$$
$$[\gamma. \inf N_A(x), \delta. \sup N_A(x)]\rangle \mid x \in E\},$$
$$\overline{H}^{\#}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(A) = \{\langle x, [\alpha. \inf M(x), \beta. \sup M_A(x)], [\inf N_A(x) + \gamma$$
$$-\gamma \inf N_A(x), \sup N_A(x) + \delta - \delta \sup N_A(x))]\rangle \mid x \in E\},$$
$$\overline{J}^{\#}_{\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)}(A) = \{\langle x, [\inf M_A(x) + \alpha - \alpha \inf M_A(x), \sup M_A + \beta$$
$$-\beta \sup M_A], [\gamma \inf N_A(x), \delta \sup N_A(x)]\rangle \mid x \in E\},$$

where $\alpha, \beta, \gamma, \delta \in [0, 1]$ such that $\alpha \leq \beta$ and $\gamma \leq \delta$.

We must mention that the last two operators, that obviously are extension of the operators $H^{\#}_{\alpha,\beta}$ and $J^{\#}_{\alpha,\beta}$, respectively, are introduced here for the first time.

In [7], a new operator was introduced, that includes as partial cases all the above intuitionistic fuzzy operators. It has the form:

$$X_{\left(\begin{smallmatrix}a_1 & b_1 & c_1 & d_1 & e_1 & f_1\\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2\end{smallmatrix}\right)}(A)$$
$$= \{\langle x, [a_1 \inf M_A(x) + b_1(1 - \inf M_A(x) - c_1 \inf N_A(x)),$$
$$a_2 \sup M_A(x) + b_2(1 - \sup M_A(x) - c_2 \sup N_A(x))],$$
$$[d_1 \inf N_A(x) + e_1(1 - f_1 \inf M_A(x) - \inf N_A(x)),$$
$$d_2 \sup N_A(x) + e_2(1 - f_2 \sup M_A(x) - \sup N_A(x))]\rangle | x \in E\},$$

where $a_1, b_1, c_1, d_1, e_1, f_1, a_2, b_2, c_2, d_2, e_2, f_2 \in [0, 1]$, the following three conditions are valid for $i = 1, 2$:

$$a_i + e_i - e_i f_i \leq 1, \tag{2.1}$$

$$b_i + d_i - b_i.c_i \leq 1, \tag{2.2}$$

$$b_i + e_i \leq 1, \tag{2.3}$$

and

$$a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2, d_1 \leq d_2, e_1 \leq e_2, f_1 \leq f_2. \tag{2.4}$$

3. Intuitionistic fuzzy modal operators of the second type

In [8], ten intuitionistic fuzzy modal operators of the second type were introduced. They have the forms:

$$\boxplus A = \left\{\left\langle x, \left[\frac{\inf M_A(x)}{2}, \frac{\sup M_A(x)}{2}\right], \left[\frac{\inf N_A(x)+1}{2}, \frac{\sup N_A(x)+1}{2}\right]\right\rangle \right. \\ \left. |x \in E\right\},$$

$$\boxtimes A = \left\{\left\langle x, \left[\frac{\inf M_A(x)+1}{2}, \frac{\sup M_A(x)+1}{2}\right], \left[\frac{\inf N_A(x)}{2}, \frac{\sup N_A(x)}{2}\right]\right\rangle \right. \\ \left. |x \in E\right\},$$

$$\boxplus_\alpha A = \{\langle x, [\alpha \inf M_A(x), \alpha \sup M_A(x)], \\ [\alpha \inf N_A(x) + 1 - \alpha, \alpha \sup N_A(x) + 1 - \alpha]\rangle | x \in E\},$$

$$\boxtimes_\alpha A = \{\langle x, [\alpha \inf M_A(x) + 1 - \alpha, \alpha \sup M_A(x) + 1 - \alpha], \\ [\alpha \inf N_A(x), \alpha \sup N_A(x)]\rangle | x \in E\},$$

$$\boxplus_{\alpha,\beta} A = \{\langle x, [\alpha \inf M_A(x), \alpha \sup M_A(x)], \\ [\alpha \inf N_A(x) + \beta, \alpha \sup N_A(x) + \beta]\rangle | x \in E\},$$

$$\boxtimes_{\alpha,\beta} A = \{\langle x, [\alpha \inf M_A(x) + \beta, \alpha \sup M_A(x) + \beta], \\ [\alpha \inf N_A(x), \alpha \sup N_A(x)]\rangle | x \in E\},$$

where $\alpha, \beta, \alpha + \beta \in [0, 1]$.

$$\boxplus_{\alpha,\beta,\gamma} A = \{\langle x, [\alpha \inf M_A(x), \alpha \sup M_A(x)], \\ [\beta \inf N_A(x) + \gamma, \beta \sup N_A(x) + \gamma]\rangle | x \in E\},$$

$$\boxtimes_{\alpha,\beta,\gamma} A = \{\langle x, [\alpha \inf M_A(x) + \gamma, \alpha \sup M_A(x) + \gamma], \\ [\beta \inf N_A(x), \beta \sup N_A(x)]\rangle | x \in E\},$$

where $\alpha, \beta, \gamma \in [0, 1]$ and $\max(\alpha, \beta) + \gamma \leq 1$.

A natural extension of the last two operators ($\boxplus_{\alpha,\beta,\gamma}$ and $\boxtimes_{\alpha,\beta,\gamma}$) is the operator

$$\boxdot_{\alpha,\beta,\gamma,\delta} A = \{\langle x, [\alpha \inf M_A(x) + \gamma, \alpha \sup M_A(x) + \gamma], \\ [\beta \inf N_A(x) + \delta, \beta \sup N_A(x) + \delta]\rangle | x \in E\},$$

where $\alpha, \beta, \gamma, \delta \in [0, 1]$ and $\max(\alpha, \beta) + \gamma + \delta \leq 1$.

The extended form of all above operators is the operator

$$\boxcircle_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta} A \\ = \{\langle x, [\alpha \inf M_A(x) - \varepsilon \inf N_A(x) + \gamma, \alpha \sup M_A(x) - \varepsilon \inf N_A(x) + \gamma], \\ [\beta \inf N_A(x) - \zeta \inf M_A(x) + \delta, \beta \sup N_A(x) - \zeta \inf M_A(x) + \delta]\rangle | x \in E\},$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in [0, 1]$, and

$$\max(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \leq 1,$$

$$\min(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \geq 0.$$

All these operators, excluding the first two of them, are extended in [9] in the following forms:

$$\begin{aligned}
\boxplus_{\left(\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right)}A &= \{\langle x, [\alpha_1 \inf M_A(x), \alpha_2 \sup M_A(x)]\,,\\
&\quad [\alpha_1 \inf N_A(x) + 1 - \alpha_1, \alpha_2 \sup N_A(x) + 1 - \alpha_2]\rangle | x \in E\},\\
\boxtimes_{\left(\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right)}A &= \{\langle x, [\alpha_1 \inf M_A(x) + 1 - \alpha_1, \alpha_2 \sup M_A(x) + 1 - \alpha_2]\,,\\
&\quad [\alpha_1 \inf N_A(x), \alpha_2 \sup N_A(x)]\rangle | x \in E\},\\
\boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{smallmatrix}\right)}A &= \{\langle x, [\alpha_1 \inf M_A(x), \alpha_2 \sup M_A(x)]\,,\\
&\quad [\alpha_1 \inf N_A(x) + \beta_1, \alpha_2 \sup N_A(x) + \beta_2]\rangle | x \in E\},\\
\boxtimes_{\left(\begin{smallmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{smallmatrix}\right)}A &= \{\langle x, [\alpha_1 \inf M_A(x) + \beta_1, \alpha_2 \sup M_A(x) + \beta_2]\,,\\
&\quad [\alpha_1 \inf N_A(x), \alpha_2 \sup N_A(x)]\rangle | x \in E\},
\end{aligned}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \alpha_2 + \beta_2 \in [0, 1]$ and $\alpha_1 \leq \alpha_2$, $\beta_1 \leq \beta_2$.

$$\boxplus_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)}A$$

$$= \{\langle x, [\alpha_1 \inf M_A(x), \alpha_2 \sup M_A(x)],$$

$$[\beta_1 \inf N_A(x) + \gamma_1, \beta_2 \sup N_A(x) + \gamma_2]\rangle | x \in E\},$$

$$\boxtimes_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{smallmatrix}\right)}A$$

$$= \{\langle x, [\alpha_1 \inf M_A(x) + \gamma_1, \alpha_2 \sup M_A(x) + \gamma_2],$$

$$[\beta_1 \inf N_A(x), \beta_2 \sup N_A(x)]\rangle | x \in E\},$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in [0, 1]$, $\max(\alpha_i, \beta_i) + \gamma_i \leq 1$ for $i = 1, 2$ and $\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2, \gamma_1 \leq \gamma_2$.

As above, a natural extension of the last two operators is the operator

$$\boxdot_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1\\ \alpha_2 & \beta_2 & \gamma_1 & \delta_2\end{smallmatrix}\right)}A$$

$$= \{\langle x, [\alpha_1 \inf M_A(x) + \gamma_1, \alpha_2 \sup M_A(x) + \gamma_2]\,,$$

$$[\beta_1 \inf N_A(x) + \delta_1, \beta_2 \sup N_A(x) + \delta_2]\rangle | x \in E\},$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in [0, 1]$, $\max(\alpha_i, \beta_i) + \gamma_i + \delta_i \leq 1$ for $i = 1, 2$ and $\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2, \gamma_1 \leq \gamma_2, \delta_1 \leq \delta_2$.

The extended form of all above operators is the operator

$$\boxcircle_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1\\ \alpha_2 & \beta_2 & \gamma_1 & \delta_2 & \varepsilon_1 & \zeta_1\end{smallmatrix}\right)}A$$

$$= \{\langle x, [\alpha_1 \inf M_A(x) - \varepsilon_1 \inf N_A(x) + \gamma_1, \alpha_2 \sup M_A(x) - \varepsilon_2 \sup N_A(x) + \gamma_2]\,,$$

$$[\beta_1 \inf N_A(x) - \zeta_1 \inf M_A(x) + \delta_1, \beta_2 \sup N_A(x) - \zeta_2 \sup M_A(x) + \delta_2]\rangle | x \in E\},$$

where $\alpha_1, \beta_1, \gamma_1, \delta_1, \varepsilon_1, \zeta_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \varepsilon_2, \zeta_2 \in [0, 1]$, and

$$\alpha_1 \le \alpha_2, \quad \beta_1 \le \beta_2, \quad \gamma_1 \le \gamma_2, \quad \delta_1 \le \delta_2, \quad \varepsilon_1 \ge \varepsilon_2, \quad \zeta_1 \le \zeta_2, \tag{3.1}$$

and for $i = 1, 2$:

$$\max(\alpha_i - \zeta_i, \beta_i - \varepsilon_i) + \gamma_i + \delta_i \le 1, \tag{3.2}$$

$$\min(\alpha_i - \zeta_i, \beta_i - \varepsilon_i) + \gamma_i + \delta_i \ge 0, \tag{3.3}$$

$$\gamma_i + \delta_i \ge 0. \tag{3.4}$$

We must note that the last inequality is added to the definition of the present operator for the first time. It was not necessary in the previous research, but from the present research it is clear that it must exist.

4. Theorem for equivalence of the two most extended modal operators

In this section, for a first time we introduce and prove the following

Theorem 4.1. *The two most extended modal operators* $X_{\left(\begin{smallmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{smallmatrix}\right)}$ *and* $\boxed{\circ}_{\left(\begin{smallmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_1 & \delta_2 & \varepsilon_1 & \zeta_1 \end{smallmatrix}\right)}$ *defined over IVIFSs are equivalent.*

Proof. Let $a_1, b_1, c_1, d_1, e_1, f_1, a_2, b_2, c_2, d_2, e_2, f_2 \in [0, 1]$ and satisfy (2.1)-(2.4). Let for $i = 1, 2$:

$$\alpha_i = a_i - b_i, \quad \beta_i = d_i - e_i, \quad \gamma_i = b_i, \quad \delta_i = e_i, \quad \varepsilon_i = b_i c_i, \quad \zeta_i = e_i f_i.$$

Also, let

$$\begin{aligned} X_1 &\equiv \alpha_1 \inf M_A(x) - \varepsilon_1 \inf N_A(x) + \gamma_1, \\ Y_1 &\equiv \beta_1 \inf N_A(x) - \zeta_1 \inf M_A(x) + \delta_1, \\ X_2 &\equiv \alpha_2 \sup M_A(x) - \varepsilon_2 \sup N_A(x) + \gamma_2, \\ Y_2 &\equiv \beta_2 \sup N_A(x) - \zeta_2 \sup M_A(x) + \delta_2. \end{aligned}$$

Then

$$\begin{aligned} X_1 &= (a_1 - b_1) \inf M_A(x) - b_1 c_1 \inf N_A(x) + b_1, \\ Y_1 &= (d_1 - e_1) \inf N_A(x) - e_1 f_1 \inf M_A(x) + e_1, \\ X_2 &= (a_2 - b_2) \sup M_A(x) - b_2 c_2 \sup N_A(x) + b_2, \\ Y_2 &= (d_2 - e_2) \sup N_A(x) - e_2 f_2 \sup M_A(x) + e_2. \end{aligned}$$

Thus, for the case $i = 1$, we obtain sequentially:

$$\begin{aligned} X_1 &\ge (a_1 - b_1).0 - b_1 c_1.1 + b_1 = b_1(1 - c_1) \ge 0, \\ X_1 &\le (a_1 - b_1).1 - b_1 c_1.0 + b_1 = a_1 \le 1, \\ Y_1 &\ge (d_1 - e_1).0 - e_1 f_1.1 + e_1 = e_1(1 - f_1) \ge 0, \\ Y_1 &\le (d_1 - e_1).1 - e_1 f_1.0 + e_1 = d_1 \le 1 \end{aligned}$$

and

$$\begin{aligned} & X_1 + Y_1 \\ &= (a_1 - b_1) \inf M_A(x) - b_1 c_1 \inf N_A(x) + b_1 + (d_1 - e_1) \inf N_A(x) - e_1 f_1 \inf M_A(x) + e_1 \\ &= (a_1 - b_1 - e_1 f_1) \inf M_A(x) + (d_1 - e_1 - b_1 c_1) \inf N_A(x) + b_1 + e_1 \end{aligned}$$

$\leq (a_1 - b_1 - e_1 f_1)\inf M_A(x) + (d_1 - e_1 - b_1 c_1)(1 - \inf M_A(x)) + b_1 + e_1$
$= d_1 - e_1 - b_1 c_1 + b_1 + e_1 + (a_1 - b_1 - e_1 f_1 - d_1 + e_1 + b_1 c_1)\inf M_A(x)$
$\leq d_1 - b_1 c_1 + b_1 + a_1 - b_1 - e_1 f_1 - d_1 + e_1 + b_1 c_1$
$= a_1 - e_1 f_1 + e_1$
$\leq 1.$ [from (2.1)]

Also, from (2.1) and (2.2), we obtain that

$$\begin{aligned}\alpha_1 + \gamma_1 - \zeta_1 + \delta_1 &= (a_1 - b_1) + b_1 - e_1 f_1 + e_1\\ &= a_1 - e_1 f_1 + e_1 \leq 1\end{aligned}$$

and

$$\begin{aligned}\beta_1 + \gamma_1 - \varepsilon_1 + \delta_1 &= (d_1 - e_1) + b_1 - b_1 c_1 + e_1\\ &= b_1 + d_1 - b_1 c_1 \leq 1.\end{aligned}$$

So, $\max(\alpha_1 - \zeta_1, \beta_1 - \varepsilon_1) + \gamma_1 - \varepsilon_1 + \delta_1 \leq 1$, i.e., (3.2) is valid.

On the other hand,

$$\alpha_1 + \gamma_1 - \zeta_1 + \delta_1 = a_1 - e_1 f_1 + e_1 \geq e_1 - e_1 f_1 \geq 0$$

and

$$\beta_1 + \gamma_1 - \varepsilon_1 + \delta_1 = b_1 + d_1 - b_1 c_1 \geq b_1 - b_1 c_1 \geq 0.$$

Hence, $\max(\alpha_1 - \zeta_1, \beta_1 - \varepsilon_1) + \gamma_1 - \varepsilon_1 + \delta_1 \geq 0$, i.e., (3.3) is valid.

From (2.3), it follows that

$$1 \geq b_1 + e_1 = \gamma_1 + \delta_1, \text{ i.e., } (3.4) \text{ is valid.}$$

The case $i = 2$ is checked in the same manner. Then, we obtain

$$\boxed{\circ}_{\left(\begin{smallmatrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1\\ \alpha_2 & \beta_2 & \gamma_1 & \delta_2 & \varepsilon_1 & \zeta_1\end{smallmatrix}\right)}(A)$$

$$\begin{gathered}
= \{\langle x, [\alpha_1 \inf M_A(x) - \varepsilon_1 \inf N_A(x) + \gamma_1, \alpha_2 \sup M_A(x) - \varepsilon_2 \sup N_A(x) + \gamma_2],\\
[\beta_1 \inf N_A(x) - \zeta_1 \inf M_A(x) + \delta_1, \beta_2 \sup N_A(x) - \zeta_2 \sup M_A(x) + \delta_2]\rangle | x \in E\},\\
= \{\langle x, [(a_1 - b_1)\inf M_A(x) - b_1 c_1 \inf N_A(x) + b_1,\\
(a_2 - b_2)\sup M_A(x) - b_2 c_2 \sup N_A(x) + b_2],\\
[(d_1 - e_1)\inf N_A(x) - e_1 f_1 \inf M_A(x) + e_2, b\\
(d_2 - e_2)\sup N_A(x) - e_2 f_2 \sup M_A(x) + e_2]\rangle | x \in E\}\\
= \{\langle x, [a_1 \inf M_A(x) + b_1(1 - \inf M_A(x) - c_1 \inf N_A(x)),\\
a_2 \sup M_A(x) + b_2(1 - \sup M_A(x) - c_2 \sup N_A(x))],\\
[d_1 \inf N_A(x) + e_1(1 - f_1 \inf M_A(x) - \inf N_A(x)),\\
d_2 \sup N_A(x) + e_2(1 - f_2 \sup M_A(x) - \sup N_A(x))]\rangle | x \in E\},\\
X_{\left(\begin{smallmatrix}a_1 & b_1 & c_1 & d_1 & e_1 & f_1\\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2\end{smallmatrix}\right)}(A).
\end{gathered}$$

Conversely, let $\alpha_1, \beta_1, \gamma_1, \delta_1, \varepsilon_1, \zeta_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \varepsilon_2, \zeta_2 \in [0, 1]$, and let them satisfy (3.1) – (3.3).

From (3.3), for $i = 1, 2$ it follows that for $\alpha_i = \beta_i = \delta_i = \zeta_i = 0:\ \varepsilon_i \leq \gamma_i$, while for $\alpha_i = \beta_i = \gamma_i = \varepsilon_i = 0:\ \zeta_i \leq \delta_i$; from (3.2) it follows that for $\beta_i = \delta_i = \varepsilon_i = \zeta_i = 0:\ \alpha_i + \gamma_i \leq 1$, while for $\alpha_i = \gamma_i = \varepsilon_i = \zeta_i = 0:\ \beta_i + \delta_i \leq 1$. Then, let

$$a_i = \alpha_i + \gamma_i\ (\leq 1),$$

$$b_i = \gamma_i,$$

$$c_i = \frac{\varepsilon_i}{\gamma_i} \ (\leq 1),$$
$$d_i = \beta_i + \delta_i \ (\leq 1),$$
$$e_i = \delta_i,$$
$$f_i = \frac{\zeta_i}{\delta_i} \ (\leq 1).$$

Let

$$X_1 \equiv a_1 \inf M_A(x) + b_1(1 - \inf M_A(x) - c_1 \inf N_A(x)),$$
$$Y_1 \equiv d_1 \inf N_A(x) + e_1(1 - f_1 \inf M_A(x) - \inf N_A(x)),$$
$$X_2 \equiv a_2 \sup M_A(x) + b_2(1 - \sup M_A(x) - c_2 \sup N_A(x)),$$
$$Y_2 \equiv d_2 \sup N_A(x) + e_2(1 - f_2 \sup M_A(x) - \sup N_A(x)).$$

Then

$$X_1 = (\alpha_1 + \gamma_1) \inf M_A(x) + \gamma_1(1 - \inf M_A(x) - \frac{\varepsilon_1}{\gamma_1} \inf N_A(x)),$$
$$Y_1 = (\beta_1 + \delta_1) \inf N_A(x) + \delta_1(1 - \frac{\zeta_1}{\delta_1} \inf M_A(x) - \inf N_A(x))$$
$$X_2 = (\alpha_2 + \gamma_2) \sup M_A(x) + \gamma_2(1 - \sup M_A(x) - \frac{\varepsilon_2}{\gamma_2} \sup N_A(x)),$$
$$Y_2 = (\beta_2 + \delta_2) \sup N_A(x) + \delta_2(1 - \frac{\zeta_2}{\delta_2} \sup M_A(x) - \sup N_A(x)).$$

Thus, for the case $i = 1$, we obtain sequentially:

$$\begin{aligned}
& 0 \leq \gamma_1 - \varepsilon_1 \leq X = \alpha_1 \inf M_A(x) + \gamma_1 - \varepsilon_1 \inf N_A(x) \leq \alpha_1 + \gamma_1 \leq 1, \\
& 0 \leq \delta_1 - \zeta_1 \leq Y = \beta_1 \inf N_A(x) + \delta_1 - \zeta_1 \inf M_A(x) \leq \beta_1 + \delta_1 \leq 1, \\
& X_1 + Y_1 \\
&= \alpha_1 \inf M_A(x) + \gamma_1 - \varepsilon_1 \inf N_A(x) + \beta_1 \inf N_A(x) + \delta_1 - \zeta_1 \inf M_A(x) \\
&= (\alpha_1 - \zeta_1) \inf M_A(x) - (\beta_1 - \varepsilon_1) \inf N_A(x) + \gamma_1 + \delta_1 \\
&\leq (\alpha_1 - \zeta_1) \inf M_A(x) - (\beta_1 - \varepsilon_1).(1 - \inf M_A(x)) + \gamma_1 + \delta_1 \\
&= (\alpha_1 - \zeta_1 + \beta_1 - \varepsilon_1) \inf M_A(x) - \beta_1 + \gamma_1 + \delta_1 + \varepsilon_1 \\
&\leq \alpha_1 - \zeta_1 + \beta_1 - \varepsilon_1 - \beta_1 + \gamma_1 + \delta_1 + \varepsilon_1 \\
&= \alpha_1 - \zeta_1 + \gamma_1 + \delta_1 \\
&\leq \max(\alpha_1 - \zeta_1, \beta_1 - \varepsilon_1) + \gamma_1 + \delta_1 \\
&\leq 1. \text{ [from (3.2)]}
\end{aligned}$$

Also,

$$\begin{aligned}
a_1 + e_1 - e_1 f_1 &= \alpha_1 + \gamma_1 + \delta_1 - \delta_1 \tfrac{\zeta_1}{\delta_1} = \alpha_1 + \gamma_1 + \delta_1 - \zeta_1 \\
&\leq \max(\alpha_1 - \zeta_1, \beta_1 - \varepsilon_1) + \gamma_1 + \delta_1 \\
&\leq 1 \text{ [from (3.2)]},
\end{aligned}$$

i.e., (2.1) is valid.

On the other hand,

$$\begin{aligned}
b_1 + d_1 - b_1.c_1 &= \gamma_1 + \beta_1 + \delta_1 - \gamma_1 \tfrac{\varepsilon_1}{\gamma_1} = \gamma_1 + \beta_1 + \delta_1 - \varepsilon_1 \\
&\leq \max(\alpha_1 - \zeta_1, \beta_1 - \varepsilon_1) + \gamma_1 + \delta_1 \\
&\leq 1 \text{ [from (3.2)]},
\end{aligned}$$

i.e., (2.2) is valid.

From (3.4), we obtain that

$$b_1 + e_1 = \gamma_1 + \delta_1 \leq 1,$$

i.e., (2.3) is valid.
So, we obtain

$$X_{\begin{pmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{pmatrix}}(A)$$

$$= \{\langle x, [a_1 \inf M_A(x) + b_1(1 - \inf M_A(x) - c_1 \inf N_A(x)),$$

$$a_2 \sup M_A(x) + b_2(1 - \sup M_A(x) - c_2 \sup N_A(x))],$$

$$[d_1 \inf N_A(x) + e_1(1 - f_1 \inf M_A(x) - \inf N_A(x)),$$

$$d_2 \sup N_A(x) + e_2(1 - f_2 \sup M_A(x) - \sup N_A(x))]\rangle | x \in E\},$$

$$= \{\langle x, [(a_1 - b_1) \inf M_A(x) - b_1 c_1 \inf N_A(x) + b_1,$$

$$(a_2 - b_2) \sup M_A(x) - b_2 c_2 \sup N_A(x) + b_2],$$

$$[(d_1 - e_1) \inf N_A(x) - e_1 f_1 \inf M_A(x) + e_2, b$$

$$(d_2 - e_2) \sup N_A(x) - e_2 f_2 \sup M_A(x) + e_2]\rangle | x \in E\}$$

$$= \{\langle x, [\alpha_1 \inf M_A(x) - \varepsilon_1 \inf N_A(x) + \gamma_1, \alpha_2 \sup M_A(x) - \varepsilon_2 \sup N_A(x) + \gamma_2],$$

$$[\beta_1 \inf N_A(x) - \zeta_1 \inf M_A(x) + \delta_1, \beta_2 \sup N_A(x) - \zeta_2 \sup M_A(x) + \delta_2]\rangle | x \in E\},$$

$$\boxed{\circ}_{\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \zeta_1 \\ \alpha_2 & \beta_2 & \gamma_1 & \delta_2 & \varepsilon_1 & \zeta_1 \end{pmatrix}}(A).$$

Hence, the two operators are equivalent. □

5. Conclusion

In near future, the author plans to study some other properties of the two most extended modal types of operators defined over IVIFSs.

In [6], it is shown that the IFSs are a suitable tool for evaluation of Data mining processes and objects. We plan to discuss the possibilities to use IVIFSs as a similar tool.

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