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Abstract. In this paper, the concept of generalized rough set theory is applied to the theory of ordered LA-semigroups by using pseudoorder and some of their combined results have been shown. Properties of rough two-sided ideals, rough bi-ideals and rough prime ideals in ordered LA-semigroups have been studied and discussed on the basis of pseudoorder of relations.

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1. Introduction

The notion of rough sets was introduced by Pawlak in his paper [10]. The rough set theory has emerged as another major mathematical approach for managing uncertainty that arises from inexact, noisy or incomplete information. In connection with algebraic structures, Biswas and Nanda [2], introduced the notion of rough subgroups, and Kuroki [7], introduced rough ideals in semigroups. Yaqoob et al. [1, 13, 14, 15, 16, 17] presented some results on roughness in semigroups and Γ-semihypergroups. Xiao and Zhang [12], introduced rough prime ideals and rough fuzzy prime ideals in semigroups.

The concept of an AG-groupoid was first given by Kazim and Naseeruddin in 1972 and they called it left almost semigroup (LA-semigroup), see [4]. Holgate called it left invertive groupoid [3]. An LA-semigroup is a groupoid having the left invertive law

$$(ab)c = (cb)a, \text{ for all } a,\, b,\, c \in S.$$

In an LA-semigroup [4], the medial law holds

$$(ab)(cd) = (ac)(bd), \text{ for all } a,\, b,\, c,\, d \in S.$$

An LA-semigroup with right identity becomes a commutative monoid [8]. The connection of a commutative inverse semigroup with an LA-semigroup has been given in [9] as, a commutative inverse semigroup $(S, \circ)$ becomes an LA-semigroup $(S, \cdot)$ under $a \cdot b = b \circ a^{-1}$, for all $a, b \in S$. A commutative semigroup with identity comes from LA-semigroup by the use of a right identity.

The concept of an ordered LA-semigroup was first given by Shah et al. [11] and then Khan and Faisal in [5], applied theory of fuzzy sets to ordered LA-semigroups.

In this paper we used pseudoorder to define lower and upper approximations of an ordered LA-semigroup. Use of pseudoorder to define lower and upper approximations and related term of an algebraic structure e.g. LA-semigroup makes it more strongre than a linear order in the sense of bringing it more closer to an interval rather than a line which tends to its fuzzyfication. We proved that the lower and upper approximations of an LA-subsemigroup (resp., left ideal, right ideal, two-sided ideal, bi-ideal, prime ideal) in an ordered LA-semigroup is an LA-subsemigroup (resp., left ideal, right ideal, two-sided ideal, bi-ideal, prime ideal).

2. Preliminaries and basic definitions

Definition 2.1 ([5]). An ordered LA-semigroup (po-LA-semigroup) is a structure $(S, ., \leq)$ in which the following conditions hold:

(i) $(S, .)$ is an LA-semigroup,

(ii) $(S, \leq)$ is a poset (reflexive, anti-symmetric and transitive),

(iii) for all a, b and $x \in S$, $a \leq b$ implies $ax \leq bx$ and $xa \leq xb$.

Example 2.2 ([5]). Consider an open interval $\mathbb{R}_{\mathbb{O}} = (0, 1)$ of real numbers under the binary operation of multiplication. Define $a * b = ba^{-1}r^{-1}$, for all $a, b, r \in \mathbb{R}_{\mathbb{O}}$, then it is easy to see that $(\mathbb{R}_{\mathbb{O}}, *, \leq)$ is an ordered LA-semigroup under the usual order "$\leq$" and we have called it a real ordered LA-semigroup.

For a non-empty subset A of an ordered LA-semigroup S, we define

$$(A] = \{t \in S \mid t \leq a, \text{ for some } a \in A\}.$$

For $A = \{a\}$, we usually write it as $(a]$.

Definition 2.3 ([5]). A non-empty subset A of an ordered LA-semigroup S, is called an LA-subsemigroup of S, if $A^2 \subseteq A$.

Definition 2.4 ([5]). A non-empty subset A of an ordered LA-semigroup S is called a left (right) ideal of S, if

(i) $SA \subseteq A$ $(AS \subseteq A)$,

(ii) If $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$.

Equivalently, a non-empty subset A of an ordered LA-semigroup S is called a left (right) ideal of S if $(SA] \subseteq A$ $((AS] \subseteq A)$.

A non-empty subset A of an ordered LA-semigroup S is called a two sided ideal of S if it is both a left and a right ideal of S.

Definition 2.5 ([5]). An LA-subsemigroup A of an ordered LA-semigroup S is called a bi-ideal of S, if

(i) $(AS)A \subseteq A$,

(ii) If $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$.

Definition 2.6 ([5]). An LA-subsemigroup A of an ordered LA-semigroup S is called an interior ideal of S, if
(i) $(SA)S \subseteq A$,
(ii) If $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$.

Definition 2.7 ([5]). Let S be an ordered LA-semigroup. A non-empty subset A of S is called a prime, if $xy \in A$ implies $x \in A$ or $y \in A$, for all $x, y \in S$. Let A be an ideal of S. If A is prime subset of S, then A is called prime ideal.

Definition 2.8 ([5]). A non-empty subset A of an ordered LA-semigroup S is called a quasi-ideal of S, if
(i) $AS \cap SA \subseteq A$,
(ii) If $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$.

Definition 2.9. A relation θ on an ordered LA-semigroup S is called a pseudoorder, if it satisfies the following conditions:
(i) $\leq \subseteq \theta$,
(ii) θ is transitive, that is, $(a, b), (b, c) \in \theta$ implies $(a, c) \in \theta$, for all $a, b, c \in S$,
(iii) θ is compatible, that is, if $(a, b) \in \theta$, then $(ax, bx) \in \theta$ and $(xa, xb) \in \theta$, for all $a, b, x \in S$.

An equivalence relation θ on an ordered LA-semigroup S is called a congruence relation, if $(a, b) \in \theta$, then $(ax, bx) \in \theta$ and $(xa, xb) \in \theta$, for all $a, b, x \in S$.

A congruence θ on S is called complete, if $[a]_\theta[b]_\theta = [ab]_\theta$, for all $a, b \in S$. Where $[a]_\theta$ is the congruence class containing the element $a \in S$.

3. Rough subsets in ordered LA-semigroups

Let X be a non-empty set and θ be a binary relation on X. By $\wp(X)$ we mean the power set of X. For all $A \subseteq X$, we define θ_- and $\theta_+ : \wp(X) \longrightarrow \wp(X)$ by

$$\theta_-(A) = \{x \in X : \forall\ y,\ x\theta y \Rightarrow y \in A\} = \{x \in X : \theta N(x) \subseteq A\},$$

and

$$\theta_+(A) = \{x \in X : \exists\ y \in A, \text{ such that } x\theta y\} = \{x \in X : \theta N(x) \cap A \neq \varnothing\}.$$

Where $\theta N(x) = \{y \in X : x\theta y\}$. $\theta_-(A)$ and $\theta_+(A)$ are called the lower approximation and the upper approximation operations, respectively. (cf. [6])

Example 3.1 ([1]). Let $X = \{a, b, c\}$ and $\theta = \{(a, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$. Then $\theta N(a) = \{a\}$; $\theta N(b) = \{b, c\}$; $\theta N(c) = \{a, b, c\}$; $\theta_-(\{a\}) = \{a\}$; $\theta_-(\{b\}) = \phi$; $\theta_-(\{c\}) = \phi$; $\theta_-(\{a, b\}) = \{a\}$; $\theta_-(\{a, c\}) = \{a\}$; $\theta_-(\{b, c\}) = \{b\}$; $\theta_-(\{a, b, c\}) = \{a, b, c\}$; $\theta_+(\{a\}) = \{a, c\}$; $\theta_+(\{b\}) = \{b, c\}$; $\theta_+(\{c\}) = \{b, c\}$; $\theta_+(\{a, b\}) = \{a, b, c\}$; $\theta_+(\{a, c\}) = \{a, b, c\}$; $\theta_+(\{b, c\}) = \{b, c\}$; $\theta_+(\{a, b, c\}) = \{a, b, c\}$.

Theorem 3.2 ([10]). *Let θ and λ be relations on X. If A and B are non-empty subsets of S. Then the following hold:*
(1) $\theta_-(X) = X = \theta_+(X)$,
(2) $\theta_-(\varnothing) = \varnothing = \theta_+(\varnothing)$,
(3) $\theta_-(A) \subseteq A \subseteq\ \theta_+(A)$,
(4) $\theta_+(A \cup B) = \theta_+(A) \cup \theta_+(B)$,
(5) $\theta_-(A \cap B) = \theta_-(A) \cap \theta_-(B)$,

(6) $A \subseteq B$ *implies* $\theta_-(A) \subseteq \theta_-(B)$,
(7) $A \subseteq B$ *implies* $\theta_+(A) \subseteq \theta_+(B)$,
(8) $\theta_-(A \cup B) \supseteq \theta_-(A) \cup \theta_-(B)$,
(9) $\theta_+(A \cap B) \subseteq \theta_+(A) \cap \theta_+(B)$.

Definition 3.3. Let θ be a pseudoorder on an ordered LA-semigroup S and A be a non-empty subset of S. Then the sets

$$\theta_-(A) = \{x \in S : \forall\ y,\ x\theta y \Rightarrow y \in A\} = \{x \in S : \theta N(x) \subseteq A\}$$

and

$$\theta_+(A) = \{x \in S : \exists\ y \in A,\ \text{such that } x\theta y\} = \{x \in S : \theta N(x) \cap A \neq \varnothing\}$$

are called the θ-lower approximation and the θ-upper approximation of A.

Example 3.4. We consider a set $S = \{a, b, c, d, e\}$ with the following operation "." and the order " $\leq$ " :

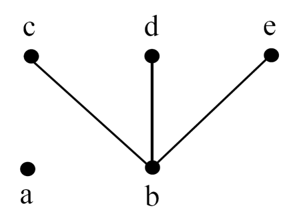
.	a	b	c	d	e
a	a	a	a	a	a
b	a	b	b	b	b
c	a	b	d	e	c
d	a	b	e	c	d
e	a	b	c	d	e

$$\leq := \{(a, a), (b, b), (b, c), (b, d), (b, e), (c, c), (d, d), (e, e)\}.$$

We give the covering relation " $\prec$ " and the figure of S as follows:

$$\prec := \{(b, c), (b, d), (b, e)\}$$

Then S is an ordered LA-semigroup because the elements of S satisfies left invertive



law. Now let

$$\theta = \{(a, a), (a, d), (b, b), (b, c), (b, d), (b, e), (c, c), (d, d), (e, c), (e, d), (e, e)\}$$

be a complete pseudoorder on S, such that

$\theta N(a) = \{a, d\},\ \theta N(b) = \{b, c, d, e\}$ and $\theta N(c) = \{c\},\ \theta N(d) = \{d\},\ \theta N(e) = \{c, d, e\}$.

Now for $A = \{a, b, d\} \subseteq S$,

$$\theta_-(\{a, b, d\}) = \{a, d\} \quad \text{and} \quad \theta_+(\{a, b, d\}) = \{a, b, c, d, e\}.$$

Thus, $\theta_-(\{a, b, d\})$ is θ-lower approximation of A and $\theta_+(\{a, b, d\})$ is θ-upper approximation of A.

For a non-empty subset A of S, $\theta(A) = (\theta_-(A), \theta_+(A))$ is called a rough set with respect to θ if $\theta_-(A) \neq \theta_+(A)$.

Theorem 3.5. *Let θ be a pseudoorder on an ordered LA-semigroup S. If A and B are non-empty subsets of S. Then*

$$\theta_+(A)\theta_+(B) \subseteq \theta_+(AB).$$

Proof. Let c be any element of $\theta_+(A)\theta_+(B)$. Then $c = ab$ where $a \in \theta_+(A)$ and $b \in \theta_+(B)$. Thus there exist elements $x, y \in S$ such that

$$x \in A \text{ and } a\theta x \text{ ; } y \in B \text{ and } b\theta y.$$

Since θ is a pseudoorder on S, $ab\theta xy$. As $xy \in AB$, we have

$$c = ab \in \theta_+(AB).$$

So $\theta_+(A)\theta_+(B) \subseteq \theta_+(AB)$. □

Definition 3.6. Let θ be a pseudoorder on an ordered LA-semigroup S, then for each $a, b \in S$, $\theta N(a)\theta N(b) \subseteq \theta N(ab)$. If

$$\theta N(a)\theta N(b) = \theta N(ab),$$

then θ is called complete pseudoorder.

Theorem 3.7. *Let θ be a complete pseudoorder on an ordered LA-semigroup S. If A and B are non-empty subsets of S. Then*

$$\theta_-(A)\theta_-(B) \subseteq \theta_-(AB).$$

Proof. Let c be any element of $\theta_-(A)\theta_-(B)$. Then $c = ab$ where $a \in \theta_-(A)$ and $b \in \theta_-(B)$. Thus we have $\theta N(a) \subseteq A$ and $\theta N(b) \subseteq B$. Since θ is complete pseudoorder on S, we have

$$\theta N(ab) = \theta N(a)\theta N(b) \subseteq AB,$$

which implies that $ab \in \theta_-(AB)$. So $\theta_-(A)\theta_-(B) \subseteq \theta_-(AB)$. □

Theorem 3.8. *Let θ and λ be pseudoorders on an ordered LA-semigroup S and A be a non-empty subset of S. Then*

$$(\theta \cap \lambda)_+(A) \subseteq \theta_+(A) \cap \lambda_+(A).$$

Proof. The proof is straightforward. □

Theorem 3.9. *Let θ and λ be pseudoorders on an ordered LA-semigroup S and A be a non-empty subset of S. Then*

$$(\theta \cap \lambda)_-(A) = \theta_-(A) \cap \lambda_-(A).$$

Proof. The proof is straightforward. □

4. Rough ideals in ordered LA-semigroups

Definition 4.1. Let θ be a pseudoorder on an ordered LA-semigroup S. Then a non-empty subset A of S is called a θ-upper (resp., θ-lower) rough LA-subsemigroup of S, if $\theta_+(A)$ (resp., $\theta_-(A)$) is an LA-subsemigroup of S.

Theorem 4.2. *Let θ be a pseudoorder on an ordered LA-semigroup S and A be an LA-subsemigroup of S. Then*

(1) *$\theta_+(A)$ is an LA-subsemigroup of S,*

(2) *If θ is complete, then $\theta_-(A)$ is, if it is non-empty, an LA-subsemigroup of S.*

Proof. (1) Let A be an LA-subsemigroup of S. Then by Theorem 3.2(3),

$$\varnothing \neq A \subseteq \theta_+(A).$$

By Theorem 3.2(7) and Theorem 3.5, we have

$$\theta_+(A)\theta_+(A) \subseteq \theta_+(AA) \subseteq \theta_+(A).$$

Thus $\theta_+(A)$ is an LA-subsemigroup of S, that is, A is a θ-upper rough LA-subsemigroup of S.

(2) Let A be an LA-subsemigroup of S. Then by Theorem 3.2(6) and Theorem 3.7, we have

$$\theta_-(A)\theta_-(A) \subseteq \theta_-(AA) \subseteq \theta_-(A).$$

Thus $\theta_-(A)$ is, if it is non-empty, an LA-subsemigroup of S, that is, A is a θ-lower rough LA-subsemigroup of S. □

The following example shows that the converse of above theorem does not hold.

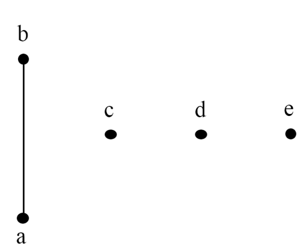
Example 4.3. We consider a set $S = \{a, b, c, d, e\}$ with the following operation "." and the order " $\leq$ " :

.	a	b	c	d	e
a	a	a	a	a	a
b	a	b	b	b	b
c	a	b	d	e	c
d	a	b	c	d	e
e	a	b	e	c	d

$$\leq := \{(a,a),(a,b),(b,b),(c,c),(d,d),(e,e)\}$$

We give the covering relation " $\prec$ " and the figure of S as follows:

$$\prec := \{(a,b)\}$$



Here S is not an ordered semigroup because $c = c \cdot (d \cdot e) \neq (c \cdot d) \cdot e = d$. Then S is an ordered LA-semigroup because the elements of S satisfies left invertive law. Now let

$$\theta = \{(a,a),(a,b),(b,b),(c,c),(c,d),(c,e),(d,c),(d,d),(d,e),(e,c),(e,d),(e,e)\}$$

be a complete pseudoorder on S, such that

$$\theta N(a) = \{a,b\},\ \theta N(b) = \{b\} \text{ and } \theta N(c) = \theta N(d) = \theta N(e) = \{c,d,e\}.$$

Now for $\{a,b,c\} \subseteq S$,

$$\theta_-(\{a,b,c\}) = \{a,b\} \text{ and } \theta_+(\{a,b,c\}) = \{a,b,c,d,e\}.$$

It is clear that $\theta_-(\{a,b,c\})$ and $\theta_+(\{a,b,c\})$ are both LA-subsemigroups of S but $\{a,b,c\}$ is not an LA-subsemigroup of S.

Definition 4.4. Let θ be a pseudoorder on an ordered LA-semigroup S. Then a non-empty subset A of S is called a θ-upper (resp., θ-lower) rough left ideal of S, if $\theta_+(A)$ (resp., $\theta_-(A)$) is a left ideal of S.

Similarly, we can define θ-upper, θ-lower rough right ideal and θ-upper, θ-lower rough two-sided ideals of S.

Theorem 4.5. *Let θ be a pseudoorder on an ordered LA-semigroup S and A be a left (right, two-sided) ideal of S. Then*

(1) $\theta_+(A)$ is a left (right, two-sided) of S,

(2) If θ is complete, then $\theta_-(A)$ is, if it is non-empty, a left (right, two-sided) of S.

Proof. (1) Let A be a left ideal of S. By Theorem 3.2(1), $\theta_+(S) = S$.

(i) Now by Theorem 3.5, we have

$$S\theta_+(A) = \theta_+(S)\theta_+(A) \subseteq \theta_+(SA) \subseteq \theta_+(A).$$

(ii) Let $a \in \theta_+(A)$ and $b \in S$ such that $b \leq a$. Then there exist $y \in A$, such that $a\theta y$ and $b\theta a$. Since θ is transitive, $b\theta y$ implies $b \in \theta_+(A)$. Thus $\theta_+(A)$ is a left ideal of S, that is, A is a θ-upper rough left ideal of S.

(2) Let A be a left ideal of S. By Theorem 3.2(1), $\theta_-(S) = S$.

(i) Now by Theorem 3.7, we have

$$S\theta_-(A) = \theta_-(S)\theta_-(A) \subseteq \theta_-(SA) \subseteq \theta_-(A).$$

(ii) Let $a \in \theta_-(A)$ and $b \in S$ such that $b \leq a$. Then $[a]_\theta \subseteq A$ and $b\theta a$. This implies that $[a]_\theta = [b]_\theta$. Since $[a]_\theta \subseteq A$, $[b]_\theta \subseteq A$. Thus $b \in \theta_-(A)$. So $\theta_-(A)$ is, if it is non-empty, a left ideal of S, that is, A is a θ-lower rough left ideal of S. The other cases can be proved in a similar way. □

Definition 4.6. Let θ be a pseudoorder on an ordered LA-semigroup S. Then a non-empty subset A of S is called a θ-upper (resp., θ-lower) rough bi-ideal of S, if $\theta_+(A)$ (resp., $\theta_-(A)$) is a bi-ideal of S.

Theorem 4.7. *Let θ be a pseudoorder on an ordered LA-semigroup S. If A is a bi-ideal of S, then it is a θ-upper rough bi-ideal of S.*

Proof. Let A be a bi-ideal of S.

(i) By Theorem 3.5, we have

$$(\theta_+(A)S)\theta_+(A) = (\theta_+(A)\theta_+(S))\theta_+(A) \subseteq \theta_+((AS)A) \subseteq \theta_+(A).$$

(ii) Let $a \in \theta_+(A)$ and $b \in S$ such that $b \leq a$. Then there exist $y \in A$, such that $a\theta y$ and $b\theta a$. Since θ is transitive, so $b\theta y$ implies $b \in \theta_+(A)$.

From this and Theorem 4.2(1), we have $\theta_+(A)$ is a bi-ideal of S, that is, A is a θ-upper rough bi-ideal of S. □

Theorem 4.8. *Let θ be a complete pseudoorder on an ordered LA-semigroup S. If A is a bi-ideal of S, then $\theta_-(A)$ is, if it is non-empty, a bi-ideal of S.*

Proof. Let A be a bi-ideal of S.

(i) By Theorem 3.7, we have

$$(\theta_-(A)S)\theta_-(A) = (\theta_-(A)\theta_-(S))\theta_-(A) \subseteq \theta_-((AS)A) \subseteq \theta_-(A).$$

(ii) Let $a \in \theta_-(A)$ and $b \in S$ such that $b \leq a$. Then $[a]_\theta \subseteq A$ and $b\theta a$. This implies that $[a]_\theta = [b]_\theta$. Since $[a]_\theta \subseteq A$, $[b]_\theta \subseteq A$. Thus $b \in \theta_-(A)$.

From this and Theorem 4.2(2), we obtain that $\theta_-(A)$ is, if it is non-empty, a bi-ideal of S. □

Theorem 4.9. *Let θ be a pseudoorder on an ordered LA-semigroup S. If A and B are a right and a left ideal of S respectively, then*

$$\theta_+(AB) \subseteq \theta_+(A) \cap \theta_+(B).$$

Proof. The proof is straightforward. □

Theorem 4.10. *Let θ be a pseudoorder on an ordered LA-semigroup S. If A is a right and B is a left ideal of S, then*

$$\theta_-(AB) \subseteq \theta_-(A) \cap \theta_-(B).$$

Proof. The proof is straightforward. □

Definition 4.11. Let θ be a pseudoorder on an ordered LA-semigroup S. Then a non-empty subset A of S is called a θ-upper (resp., θ-lower) rough interior ideal of S, if $\theta_+(A)$ (resp., $\theta_-(A)$) is an interior ideal of S.

Theorem 4.12. *Let θ be a pseudoorder on an ordered LA-semigroup S. If A is an interior ideal of S, then A is a θ-upper rough interior ideal of S.*

Proof. The proof of this theorem is similar to the Theorem 4.7. □

Theorem 4.13. *Let θ be a pseudoorder on an ordered LA-semigroup S. If A is an interior ideal of S, then $\theta_-(A)$ is, if it is non-empty, an interior ideal of S.*

Proof. The proof of this theorem is similar to the Theorem 4.8. □

We call A a rough interior ideal of S if it is both a θ-lower and θ-upper rough interior ideal of S.

Definition 4.14. Let θ be a pseudoorder on an ordered LA-semigroup S. Then a non-empty subset Q of S is called a θ-upper (resp., θ-lower) rough quasi-ideal of S, if $\theta_+(Q)$ (resp., $\theta_-(Q)$) is a quasi-ideal of S.

Theorem 4.15. *Let θ be a complete pseudoorder on an ordered LA-semigroup S. If Q is a quasi-ideal of S, then Q is a θ-lower rough quasi-ideal of S.*

Proof. Let Q be a quasi-ideal of S.

(i) Now by Theorem 3.2(5) and Theorem 3.7, we get

$$\begin{aligned}\theta_-(Q)S\cap S\theta_-(Q) &= \theta_-(Q)\theta_-(S)\cap\theta_-(S)\theta_-(Q)\\ &\subseteq \theta_-(QS)\cap\theta_-(SQ)\\ &= \theta_-(QS\cap SQ)\\ &\subseteq \theta_-(Q).\end{aligned}$$

(ii) Let $a\in\theta_-(Q)$ and $b\in S$ such that $b\leq a$. Then $[a]_\theta\subseteq Q$ and $b\theta a$. This implies that $[a]_\theta=[b]_\theta$. Since $[a]_\theta\subseteq Q$, $[b]_\theta\subseteq Q$. Thus $b\in\theta_-(Q)$. So we obtain that $\theta_-(Q)$ is a quasi-ideal of S, that is, Q is a θ-lower rough quasi-ideal of S. □

Theorem 4.16. *Let θ be a complete pseudoorder on an ordered LA-semigroup S. Let L and R be a θ-lower rough left ideal and a θ-lower rough right ideal of S, respectively. Then $L\cap R$ is a θ-lower rough quasi-ideal of S.*

Proof. The proof is straightforward. □

Definition 4.17. Let θ be a pseudoorder on an ordered LA-semigroup S. Then a non-empty subset A of S is called a θ-upper (resp., θ-lower) rough prime ideal of S, if $\theta_+(A)$ (resp., $\theta_-(A)$) is a prime ideal of S.

Theorem 4.18. *Let θ be a complete pseudoorder on an ordered LA-semigroup S. If A is a prime ideal of S, then A is a θ-upper rough prime ideal of S.*

Proof. Since A is a prime ideal of S, it follows from Theorem 4.5(1), that $\theta_+(A)$ is an ideal of S. Let $xy\in\theta_+(A)$ for some $x,y\in S$. Then

$$\theta N(xy)\cap A=\theta N(x)\theta N(y)\cap A\neq\varnothing.$$

Thus there exist elements

$$x^{'}\in\theta N(x)\ \text{ and }\ y^{'}\in\theta N(y),\ \text{such that}\ x^{'}y^{'}\in A.$$

Since A is a prime ideal of S, we have $x^{'}\in A$ or $y^{'}\in A$. So $\theta N(x)\cap A\neq\phi$ or $\theta N(y)\cap A\neq\phi$, and thus $x\in\theta_+(A)$ or $y\in\theta_+(A)$. Hence $\theta_+(A)$ is a prime ideal of S. □

Theorem 4.19. *Let θ be a complete pseudoorder on an ordered LA-semigroup S and A be a prime ideal of S. Then $\theta_-(A)$ is, if it is non-empty, a prime ideal of S.*

Proof. Since A is an ideal of S, by Theorem 4.5(2), we have, $\theta_-(A)$ is an ideal of S. Let

$$xy\in\theta_-(A)\text{ for some }x,y\in S.$$

Then

$$\theta N(xy)\subseteq A,\ \text{which implies that}\ \theta N(x)\theta N(y)\subseteq\theta N(xy)\subseteq A.$$

We suppose that $\theta_-(A)$ is not a prime ideal of S. Then there exists $x,y\in S$ such that $xy\in\theta_-(A)$ but $x\notin\theta_-(A)$ and $y\notin\theta_-(A)$. Thus $\theta N(x)\nsubseteq A$ and $\theta N(y)\nsubseteq A$. So there exists $x^{'}\in\theta N(x)$, $x^{'}\notin A$ and $y^{'}\in\theta N(y)$, $y^{'}\notin A$. Hence

$$x^{'}y^{'}\in\theta N(x)\theta N(y)\subseteq A.$$

Since A is a prime ideal of S, we have $x' \in A$ or $y' \in A$. It contradicts our supposition. This means that $\theta_-(A)$ is, if it is non-empty, a prime ideal of S. □

We call A a rough prime ideal of S, if it is both a θ-lower and a θ-upper rough prime ideal of S.

The following example shows that the converse of Theorem 4.18 and Theorem 4.19 does not hold.

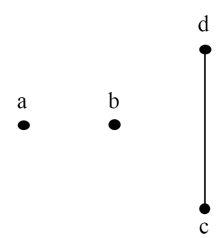
Example 4.20. We consider a set $S = \{a, b, c, d, e\}$ with the following operation "." and the order " $\leq$ " :

$\cdot$	a	b	c	d
a	c	c	c	d
b	a	c	c	d
c	c	c	c	d
d	d	d	d	d

$$\leq := \{(a, a), (b, b), (c, c), (c, d), (d, d)\}$$

We give the covering relation " $\prec$ " and the figure of S as follows:

$$\prec := \{(c, d)\}$$



Here S is not an ordered semigroup because $a = b \cdot (b \cdot a) \neq (b \cdot b) \cdot a = c$. Then S is an ordered LA-semigroup because the elements of S satisfies left invertive law. Now let

$$\theta = \{(a, a), (a, c), (a, d), (b, b), (b, c), (b, d), (c, c), (c, d), (d, d)\}$$

be a complete pseudoorder on S, such that

$$\theta N(a) = \{a, c, d\},\ \theta N(b) = \{b, c, d\},\ \theta N(c) = \{c, d\} \text{ and } \theta N(d) = \{d\}.$$

Now for $\{b, d\} \subseteq S$,

$$\theta_-(\{b, d\}) = \{d\} \quad \text{and} \quad \theta_+(\{b, d\}) = \{a, b, c, d\}.$$

It is clear that $\theta_-(\{b, d\})$ and $\theta_+(\{b, d\})$ are prime ideals of S. The subset $\{b, d\}$ is not an ideal and hence not a prime ideal.

5. Conclusions

The properties of ordered LA-semigroups in terms of rough sets have been discussed. Then through pseudoorders, it is proved that the lower and upper approximations of two-sided ideals (resp., bi-ideals and prime ideals) in ordered LA-semigroups becomes two-sided ideals (resp., bi-ideals and prime ideals).

In our future studies, following topics may be considered:

1. Rough prime bi-ideals of ordered LA-semigroups,
2. Rough fuzzy ideals in ordered LA-semigroups,
3. Rough fuzzy prime bi-ideals of ordered LA-semigroups.

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