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Abstract. The notion of hyperfuzzy set was introduced by Ghosh and Samanta as a generalization of fuzzy sets and interval-valued fuzzy sets. In this article, this is applied to BCK/BCI-algebras. The notions of hyperfuzzy substructures/subalgebras of type (i, j) for $i, j \in \{1, 2, 3, 4\}$ are introduced, and related properties are investigated. Relations between hyperfuzzy substructure/subalgebra and its length are discussed, and the properties of hyperfuzzy subalgebras related to upper and lower level subsets are investigated.

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1. Introduction and Preliminaries

BCK-algebras entered into Mathematics in 1966 through the work of Imai and Iséki [3], and have been applied to many branches of Mathematics, such as group theory, functional analysis, probability theory and topology. Such algebras generalize Boolean rings as well as Boolean D-posets ($= MV$-algebras). Also, Iséki introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra (see [4]). The fundamental concept of fuzzy sets was initiated by Zadeh [6] in 1965. The fuzzy algebraic structure plays an important role in Mathematics with wide applications in many other branches, for example, theoretical physics, control engineering, computer and information science, coding theory, graph theory and decision making problem etc. Ghosh and Samanta [1] introduced the notion of hyperfuzzy sets which is a generalization of fuzzy sets and interval-valued fuzzy sets.

In this paper, we apply the notion of hyperfuzzy set to BCK/BCI-algebras. We first introduce the notion of k-fuzzy substructure for $k \in \{1, 2, 3, 4\}$ and then we introduce the concepts of hyperfuzzy substructures of several types by using k-fuzzy substructures, and investigate their basic properties. We introduce the notion of hyperfuzzy subalgebras of type (i, j) for $i, j \in \{1, 2, 3, 4\}$. We discuss relations between hyperfuzzy substructure/subalgebra and its length. We investigate the properties of hyperfuzzy subalgebras related to upper and lower level subsets.

By a *BCI-algebra* we mean a system $X := (X, *, 0) \in K(\tau)$ in which the following axioms hold:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $x * y = y * x = 0 \ \Rightarrow \ x = y$

for all $x, y, z \in X$. If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK-algebra*. We can define a partial ordering $\leq$ by

$$(\forall x, y \in X)\, (x \leq y \iff x * y = 0).$$

In a BCK/BCI-algebra X, the following hold:

$$(\forall x \in X)\ (x * 0 = x), \tag{1.1}$$

$$(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y). \tag{1.2}$$

A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

We refer the reader to the books [2] and [5] for further information regarding BCK/BCI-algebras.

By a *fuzzy structure* over a nonempty set X we mean an ordered pair (X, ρ) of X and a fuzzy set ρ on X. Let X be a nonempty set. A mapping $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0, 1])$ is called a *hyperfuzzy set* over X (see [1]), where $\tilde{\mathcal{P}}([0, 1])$ is the family of all nonempty subsets of $[0, 1]$.

2. Hyperfuzzy substructures

Definition 2.1. For any nonempty set X with a binary operation $*$, a fuzzy set $\mu : X \to [0, 1]$ is called a

- *fuzzy substructure* of X with type 1 (briefly, 1-*fuzzy substructure* of X) if

$$(\forall x, y \in X)\, (\mu(x * y) \geq \min\{\mu(x), \mu(y)\})\,, \tag{2.1}$$

- *fuzzy substructure* of X with type 2 (briefly, 2-*fuzzy substructure* of X) if

$$(\forall x, y \in X)\, (\mu(x * y) \leq \min\{\mu(x), \mu(y)\})\,, \tag{2.2}$$

- *fuzzy substructure* of X with type 3 (briefly, 3-*fuzzy substructure* of X) if

$$(\forall x, y \in X)\, (\mu(x * y) \geq \max\{\mu(x), \mu(y)\})\,, \tag{2.3}$$

- *fuzzy substructure* of X with type 4 (briefly, 4-*fuzzy substructure* of X) if

$$(\forall x, y \in X)\, (\mu(x * y) \leq \max\{\mu(x), \mu(y)\})\,. \tag{2.4}$$

It is clear that every 3-fuzzy substructure is a 1-fuzzy substructure and every 2-fuzzy substructure is a 4-fuzzy substructure.

Let X be a nonempty set with a binary operation $*$. Given a hyperfuzzy set $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0, 1])$, we consider two fuzzy structures $(X, \tilde{\mu}_{\inf})$ and $(X, \tilde{\mu}_{\sup})$ over X in which

$$\tilde{\mu}_{\inf} : X \to [0,1], \quad x \mapsto \inf\{\tilde{\mu}(x)\},$$
$$\tilde{\mu}_{\sup} : X \to [0,1], \quad x \mapsto \sup\{\tilde{\mu}(x)\},$$

Let X be a nonempty set with a binary operation $*$. Given a fuzzy structure (X, ρ) over X, we can induce a hyperfuzzy set over $\tilde{\mathcal{P}}(X)$ as follows:

$$\tilde{\rho} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([0,1]), \quad A \mapsto \{\rho(a) \mid a \in A\}$$

where $\tilde{\mathcal{P}}(X)$ is the family of all nonempty subsets of X. For any $A, B \in \tilde{\mathcal{P}}(X)$, we define

$$A \,\tilde{*}\, B = \{a * b \mid a \in A, b \in B\}.$$

Theorem 2.2. *Let X be a nonempty set with a binary operation $*$. If $\mu : X \to [0,1]$ is a 1-fuzzy substructure of X, then $\tilde{\mu}_{\sup} : \tilde{\mathcal{P}}(X) \to [0,1]$ and $\tilde{\mu}_{\inf} : \tilde{\mathcal{P}}(X) \to [0,1]$ are 1-fuzzy substructures of $\tilde{\mathcal{P}}(X)$.*

Proof. Assume that $\mu : X \to [0,1]$ is a 1-fuzzy substructure of X, and let $A, B \in \tilde{\mathcal{P}}(X)$. Then

$$\tilde{\mu}_{\sup}(A \,\tilde{*}\, B) = \sup\{\mu(a * b) \mid a \in A, b \in B\},$$
$$\tilde{\mu}_{\inf}(A \,\tilde{*}\, B) = \inf\{\mu(a * b) \mid a \in A, b \in B\}.$$

Note that for any $\varepsilon > 0$ there exist $a_0 \in A$ and $b_0 \in B$ such that

$$\mu(a_0) > \sup\{\mu(a) \mid a \in A\} - \varepsilon,$$
$$\mu(b_0) > \sup\{\mu(b) \mid b \in B\} - \varepsilon,$$

respectively. Hence

$$\begin{aligned}
\tilde{\mu}_{\sup}(A \,\tilde{*}\, B) &= \sup\{\mu(a * b) \mid a \in A, b \in B\} \\
&\geq \mu(a_0 * b_0) \geq \min\{\mu(a_0), \mu(b_0)\} \\
&\geq \min\{\sup\{\mu(a) \mid a \in A\} - \varepsilon,\ \sup\{\mu(b) \mid b \in B\} - \varepsilon\} \\
&\geq \min\{\sup\{\mu(a) \mid a \in A\},\ \sup\{\mu(b) \mid b \in B\}\} - \varepsilon \\
&= \min\{\tilde{\mu}_{\sup}(A),\ \tilde{\mu}_{\sup}(B)\} - \varepsilon.
\end{aligned}$$

Since ε is arbitrary, it follows that $\tilde{\mu}_{\sup}(A \,\tilde{*}\, B) \geq \min\{\tilde{\mu}_{\sup}(A),\ \tilde{\mu}_{\sup}(B)\}$. Hence $\tilde{\mu}_{\sup} : \tilde{\mathcal{P}}(X) \to [0,1]$ is a 1-fuzzy substructure of $\tilde{\mathcal{P}}(X)$. Also, we have

$$\begin{aligned}
\tilde{\mu}_{\inf}(A \,\tilde{*}\, B) &= \inf\{\mu(a * b) \mid a \in A, b \in B\} \\
&\geq \inf\{\min\{\mu(a), \mu(b)\} \mid a \in A, b \in B\} \\
&\geq \min\{\inf\{\mu(a) \mid a \in A\},\ \inf\{\mu(b) \mid b \in B\}\} \\
&= \min\{\tilde{\mu}_{\inf}(A),\ \tilde{\mu}_{\inf}(B)\}.
\end{aligned}$$

Therefore $\tilde{\mu}_{\inf} : \tilde{\mathcal{P}}(X) \to [0,1]$ is a 1-fuzzy substructure of $\tilde{\mathcal{P}}(X)$. □

Theorem 2.3. *Let X be a nonempty set with a binary operation $*$. If $\mu : X \to [0,1]$ is a 4-fuzzy substructure of X, then $\tilde{\mu}_{\sup} : \tilde{\mathcal{P}}(X) \to [0,1]$ and $\tilde{\mu}_{\inf} : \tilde{\mathcal{P}}(X) \to [0,1]$ are 4-fuzzy substructures of $\tilde{\mathcal{P}}(X)$.*

Proof. Assume that $\mu : X \to [0,1]$ is a 4-fuzzy substructure of X. If $\tilde{\mu}_{\sup} : \tilde{\mathcal{P}}(X) \to [0,1]$ is not a 4-fuzzy substructures of $\tilde{\mathcal{P}}(X)$, then

$$\tilde{\mu}_{\sup}(A \tilde{*} B) > \max\{\tilde{\mu}_{\sup}(A), \tilde{\mu}_{\sup}(B)\}$$

for some $A, B \in \tilde{\mathcal{P}}(X)$, and so there exist $a_0 \in A$ and $b_0 \in B$ such that

$$\mu(a_0 * b_0) > \max\{\tilde{\mu}_{\sup}(A), \tilde{\mu}_{\sup}(B)\}.$$

Since μ is a 4-fuzzy substructure of X, it follows that

$$\mu(a_0) > \max\{\tilde{\mu}_{\sup}(A), \tilde{\mu}_{\sup}(B)\} \geq \tilde{\mu}_{\sup}(A)$$

or

$$\mu(b_0) > \max\{\tilde{\mu}_{\sup}(A), \tilde{\mu}_{\sup}(B)\} \geq \tilde{\mu}_{\sup}(B).$$

This is a contradiction, and thus

$$\tilde{\mu}_{\sup}(A \tilde{*} B) \leq \max\{\tilde{\mu}_{\sup}(A), \tilde{\mu}_{\sup}(B)\}$$

for all $A, B \in \tilde{\mathcal{P}}(X)$. Therefore $\tilde{\mu}_{\sup} : \tilde{\mathcal{P}}(X) \to [0,1]$ is a 4-fuzzy substructure of $\tilde{\mathcal{P}}(X)$. Now, note that for any positive number δ there exist $a_0 \in A$ and $b_0 \in B$ such that

$$\inf\{\mu(a) \mid a \in A\} > \mu(a_0) - \delta,$$
$$\inf\{\mu(b) \mid b \in B\} > \mu(b_0) - \delta,$$

respectively. It follows that

$$\begin{aligned}\max\{\tilde{\mu}_{\inf}(A), \tilde{\mu}_{\inf}(B)\} &= \max\{\inf\{\mu(a) \mid a \in A\},\ \inf\{\mu(b) \mid b \in B\}\}\\ &> \max\{\mu(a_0) - \delta,\ \mu(b_0) - \delta\}\\ &= \max\{\mu(a_0),\ \mu(b_0)\} - \delta\\ &\geq \mu(a_0 * b_0) - \delta\\ &\geq \tilde{\mu}_{\inf}(A \tilde{*} B) - \delta\end{aligned}$$

which shows that $\tilde{\mu}_{\inf}(A \tilde{*} B) \leq \max\{\tilde{\mu}_{\inf}(A), \tilde{\mu}_{\inf}(B)\}$ since δ is arbitrary. Thus $\tilde{\mu}_{\inf}$ is a 4-fuzzy substructures of $\tilde{\mathcal{P}}(X)$. □

Question 2.4. Let X be a nonempty set with a binary operation $*$.

(1) If $\mu : X \to [0,1]$ is a 2-fuzzy substructure of X, then are $\tilde{\mu}_{\sup} : \tilde{\mathcal{P}}(X) \to [0,1]$ and $\tilde{\mu}_{\inf} : \tilde{\mathcal{P}}(X) \to [0,1]$ 2-fuzzy substructures of $\tilde{\mathcal{P}}(X)$?

(2) If $\mu : X \to [0,1]$ is a 3-fuzzy substructure of X, then are $\tilde{\mu}_{\sup} : \tilde{\mathcal{P}}(X) \to [0,1]$ and $\tilde{\mu}_{\inf} : \tilde{\mathcal{P}}(X) \to [0,1]$ 3-fuzzy substructures of $\tilde{\mathcal{P}}(X)$?

Definition 2.5. For any nonempty set X with a binary operation $*$, a hyperfuzzy set $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1])$ is called a *hyperfuzzy substructure* of X with type (i,j) for $i,j \in \{1,2,3,4\}$ (briefly, (i,j)-*hyperfuzzy substructure* of X) if $\tilde{\mu}_{\inf}$ is a i-fuzzy substructure of X and $\tilde{\mu}_{\sup}$ is a j-fuzzy substructure of X.

Given a hyperfuzzy set $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1])$ over a nonempty set X, we define

$$\tilde{\mu}_{\ell} : X \to [0,1], \quad x \mapsto \tilde{\mu}_{\sup}(x) - \tilde{\mu}_{\inf}(x) \tag{2.5}$$

which is called the *length* of $\tilde{\mu}$.

Theorem 2.6. *Let X be a nonempty set with a binary operation $*$. If a hyperfuzzy set $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1])$ over X is a $(4,3)$-hyperfuzzy substructure of X, then $\tilde{\mu}_\ell$ is a 3-fuzzy substructure of X.*

Proof. Assume that $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1])$ is a $(4,3)$-hyperfuzzy substructure of X. Then $\tilde{\mu}_{\inf}$ is a 4-fuzzy substructure of X and $\tilde{\mu}_{\sup}$ is a 3-fuzzy substructure of X. Hence

$$\tilde{\mu}_{\inf}(x * y) \leq \max\{\tilde{\mu}_{\inf}(x), \tilde{\mu}_{\inf}(y)\} \tag{2.6}$$

and

$$\tilde{\mu}_{\sup}(x * y) \geq \max\{\tilde{\mu}_{\sup}(x), \tilde{\mu}_{\sup}(y)\}$$

for all $x, y \in X$. Then $\tilde{\mu}_{\inf}(x * y) \leq \tilde{\mu}_{\inf}(x)$ or $\tilde{\mu}_{\inf}(x * y) \leq \tilde{\mu}_{\inf}(y)$ by (2.6). If $\tilde{\mu}_{\inf}(x * y) \leq \tilde{\mu}_{\inf}(x)$, then

$$\tilde{\mu}_\ell(x * y) = \tilde{\mu}_{\sup}(x * y) - \tilde{\mu}_{\inf}(x * y) \geq \tilde{\mu}_{\sup}(x) - \tilde{\mu}_{\inf}(x) = \tilde{\mu}_\ell(x).$$

If $\tilde{\mu}_{\inf}(x * y) \leq \tilde{\mu}_{\inf}(y)$, then

$$\tilde{\mu}_\ell(x * y) = \tilde{\mu}_{\sup}(x * y) - \tilde{\mu}_{\inf}(x * y) \geq \tilde{\mu}_{\sup}(y) - \tilde{\mu}_{\inf}(y) = \tilde{\mu}_\ell(y).$$

It follows that $\tilde{\mu}_\ell(x*y) \geq \max\{\tilde{\mu}_\ell(x), \tilde{\mu}_\ell(y)\}$. Therefore $\tilde{\mu}_\ell$ is a 3-fuzzy substructure of X. □

Corollary 2.7. *Let X be a nonempty set with a binary operation $*$. If a hyperfuzzy set $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1])$ over X is a $(4,3)$-hyperfuzzy substructure of X, then $\tilde{\mu}_\ell$ is a 1-fuzzy substructure of X.*

Theorem 2.8. *Let X be a nonempty set with a binary operation $*$. If a hyperfuzzy set $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1])$ over X is a $(3,4)$-hyperfuzzy substructure of X, then $\tilde{\mu}_\ell$ is a 4-fuzzy substructure of X.*

Proof. Assume that $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1])$ is a $(3,4)$-hyperfuzzy substructure of X. Then $\tilde{\mu}_{\inf}$ is a 3-fuzzy substructure of X and $\tilde{\mu}_{\sup}$ is a 4-fuzzy substructure of X. Hence

$$\tilde{\mu}_{\inf}(x * y) \geq \max\{\tilde{\mu}_{\inf}(x), \tilde{\mu}_{\inf}(y)\}$$

and

$$\tilde{\mu}_{\sup}(x * y) \leq \max\{\tilde{\mu}_{\sup}(x), \tilde{\mu}_{\sup}(y)\} \tag{2.7}$$

for all $x, y \in X$. Then $\tilde{\mu}_{\sup}(x * y) \leq \tilde{\mu}_{\sup}(x)$ or $\tilde{\mu}_{\sup}(x * y) \leq \tilde{\mu}_{\sup}(y)$ by (2.7). If $\tilde{\mu}_{\sup}(x * y) \leq \tilde{\mu}_{\sup}(x)$, then

$$\tilde{\mu}_\ell(x * y) = \tilde{\mu}_{\sup}(x * y) - \tilde{\mu}_{\inf}(x * y) \leq \tilde{\mu}_{\sup}(x) - \tilde{\mu}_{\inf}(x) = \tilde{\mu}_\ell(x).$$

If $\tilde{\mu}_{\sup}(x * y) \leq \tilde{\mu}_{\sup}(y)$, then

$$\tilde{\mu}_\ell(x * y) = \tilde{\mu}_{\sup}(x * y) - \tilde{\mu}_{\inf}(x * y) \leq \tilde{\mu}_{\sup}(y) - \tilde{\mu}_{\inf}(y) = \tilde{\mu}_\ell(y).$$

Thus $\tilde{\mu}_\ell(x * y) \leq \max\{\tilde{\mu}_\ell(x), \tilde{\mu}_\ell(y)\}$, and therefore $\tilde{\mu}_\ell$ is a 4-fuzzy substructure of X. □

Theorem 2.9. *Let X be a nonempty set with a binary operation $*$. If a hyperfuzzy set $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1])$ over X is a $(3,2)$-hyperfuzzy substructure of X, then $\tilde{\mu}_\ell$ is a 2-fuzzy substructure of X.*

Proof. Assume that $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1])$ is a $(3,2)$-hyperfuzzy substructure of X. Then $\tilde{\mu}_{\inf}$ is a 3-fuzzy substructure of X and $\tilde{\mu}_{\sup}$ is a 2-fuzzy substructure of X. Hence

$$\tilde{\mu}_{\inf}(x * y) \geq \max\{\tilde{\mu}_{\inf}(x), \tilde{\mu}_{\inf}(y)\}$$

and

$$\tilde{\mu}_{\sup}(x * y) \leq \min\{\tilde{\mu}_{\sup}(x), \tilde{\mu}_{\sup}(y)\}$$

for all $x, y \in X$. It follows that

$$\tilde{\mu}_{\ell}(x * y) = \tilde{\mu}_{\sup}(x * y) - \tilde{\mu}_{\inf}(x * y) \leq \tilde{\mu}_{\sup}(x) - \tilde{\mu}_{\inf}(x) = \tilde{\mu}_{\ell}(x)$$

and

$$\tilde{\mu}_{\ell}(x * y) = \tilde{\mu}_{\sup}(x * y) - \tilde{\mu}_{\inf}(x * y) \leq \tilde{\mu}_{\sup}(y) - \tilde{\mu}_{\inf}(y) = \tilde{\mu}_{\ell}(y).$$

Hence $\tilde{\mu}_{\ell}(x * y) \leq \min\{\tilde{\mu}_{\ell}(x), \tilde{\mu}_{\ell}(y)\}$ for all $x, y \in X$. Therefore $\tilde{\mu}_{\ell}$ is a 2-fuzzy substructure of X. □

Corollary 2.10. *Let X be a nonempty set with a binary operation $*$. If a hyperfuzzy set $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1])$ over X is a $(3,2)$-hyperfuzzy substructure of X, then $\tilde{\mu}_{\ell}$ is a 4-fuzzy substructure of X.*

Theorem 2.11. *Let X be a nonempty set with a binary operation $*$. If a hyperfuzzy set $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1])$ over X is a $(2,3)$-hyperfuzzy substructure of X, then $\tilde{\mu}_{\ell}$ is a 3-fuzzy substructure of X.*

Proof. Assume that $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1])$ is a $(2,3)$-hyperfuzzy substructure of X. Then $\tilde{\mu}_{\inf}$ is a 2-fuzzy substructure of X and $\tilde{\mu}_{\sup}$ is a 3-fuzzy substructure of X. Hence

$$\tilde{\mu}_{\inf}(x * y) \leq \min\{\tilde{\mu}_{\inf}(x), \tilde{\mu}_{\inf}(y)\}$$

and

$$\tilde{\mu}_{\sup}(x * y) \geq \max\{\tilde{\mu}_{\sup}(x), \tilde{\mu}_{\sup}(y)\}$$

for all $x, y \in X$. Then

$$\tilde{\mu}_{\ell}(x * y) = \tilde{\mu}_{\sup}(x * y) - \tilde{\mu}_{\inf}(x * y) \geq \tilde{\mu}_{\sup}(x) - \tilde{\mu}_{\inf}(x) = \tilde{\mu}_{\ell}(x)$$

and

$$\tilde{\mu}_{\ell}(x * y) = \tilde{\mu}_{\sup}(x * y) - \tilde{\mu}_{\inf}(x * y) \geq \tilde{\mu}_{\sup}(y) - \tilde{\mu}_{\inf}(y) = \tilde{\mu}_{\ell}(y).$$

It follows that $\tilde{\mu}_{\ell}(x * y) \geq \max\{\tilde{\mu}_{\ell}(x), \tilde{\mu}_{\ell}(y)\}$. Hence $\tilde{\mu}_{\ell}$ is a 3-fuzzy substructure of X. □

Corollary 2.12. *Let X be a nonempty set with a binary operation $*$. If a hyperfuzzy set $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1])$ over X is a $(2,3)$-hyperfuzzy substructure of X, then $\tilde{\mu}_{\ell}$ is a 1-fuzzy substructure of X.*

Given a hyperfuzzy set $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1])$ over X, we define

$$\tilde{\mu}_{\xi} : X \to [-1, 0], \quad x \mapsto \tilde{\mu}_{\inf}(x) - \tilde{\mu}_{\sup}(x).$$

Note that $\tilde{\mu}_{\xi}(x) = -\tilde{\mu}_{\ell}(x)$ for all $x \in X$.

Theorem 2.13. *Let X be a nonempty set with a binary operation $*$. If a hyperfuzzy set $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1])$ over X is a $(2,1)$-hyperfuzzy substructure of X, then*

$$(\forall x, y \in X)\,(\tilde{\mu}_\xi(x * y) \leq \max\{\tilde{\mu}_\xi(x), \tilde{\mu}_\xi(y)\})\,. \tag{2.8}$$

Proof. If $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1])$ is a $(2,1)$-hyperfuzzy substructure of X, then $\tilde{\mu}_{\inf}$ is a 2-fuzzy substructure of X and $\tilde{\mu}_{\sup}$ is a 1-fuzzy substructure of X. Hence

$$\tilde{\mu}_{\inf}(x * y) \leq \min\{\tilde{\mu}_{\inf}(x), \tilde{\mu}_{\inf}(y)\}$$

and

$$\tilde{\mu}_{\sup}(x * y) \geq \min\{\tilde{\mu}_{\sup}(x), \tilde{\mu}_{\sup}(y)\} \tag{2.9}$$

for all $x, y \in X$. (2.9) implies that $\tilde{\mu}_{\sup}(x * y) \geq \tilde{\mu}_{\sup}(x)$ or $\tilde{\mu}_{\sup}(x * y) \geq \tilde{\mu}_{\sup}(y)$. If $\tilde{\mu}_{\sup}(x * y) \geq \tilde{\mu}_{\sup}(x)$, then

$$\tilde{\mu}_\xi(x * y) = \tilde{\mu}_{\inf}(x * y) - \tilde{\mu}_{\sup}(x * y) \leq \tilde{\mu}_{\inf}(x) - \tilde{\mu}_{\sup}(x) = \tilde{\mu}_\xi(x).$$

If $\tilde{\mu}_{\sup}(x * y) \geq \tilde{\mu}_{\sup}(y)$, then

$$\tilde{\mu}_\xi(x * y) = \tilde{\mu}_{\inf}(x * y) - \tilde{\mu}_{\sup}(x * y) \leq \tilde{\mu}_{\inf}(y) - \tilde{\mu}_{\sup}(y) = \tilde{\mu}_\xi(y).$$

It follows that $\tilde{\mu}_\xi(x * y) \leq \max\{\tilde{\mu}_\xi(x), \tilde{\mu}_\xi(y)\}$ for all $x, y \in X$. □

Theorem 2.14. *Let X be a nonempty set with a binary operation $*$. If a hyperfuzzy set $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1])$ over X is a $(1,2)$-hyperfuzzy substructure of X, then*

$$(\forall x, y \in X)\,(\tilde{\mu}_\xi(x * y) \geq \max\{\tilde{\mu}_\xi(x), \tilde{\mu}_\xi(y)\})\,. \tag{2.10}$$

Proof. If $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1])$ is a $(1,2)$-hyperfuzzy substructure of X, then $\tilde{\mu}_{\inf}$ is a 1-fuzzy substructure of X and $\tilde{\mu}_{\sup}$ is a 2-fuzzy substructure of X. Hence

$$\tilde{\mu}_{\inf}(x * y) \geq \min\{\tilde{\mu}_{\inf}(x), \tilde{\mu}_{\inf}(y)\}$$

and

$$\tilde{\mu}_{\sup}(x * y) \leq \min\{\tilde{\mu}_{\sup}(x), \tilde{\mu}_{\sup}(y)\}$$

for all $x, y \in X$. It follows that

$$\tilde{\mu}_\xi(x * y) = \tilde{\mu}_{\inf}(x * y) - \tilde{\mu}_{\sup}(x * y) \geq \tilde{\mu}_{\inf}(x) - \tilde{\mu}_{\sup}(x) = \tilde{\mu}_\xi(x)$$

or

$$\tilde{\mu}_\xi(x * y) = \tilde{\mu}_{\inf}(x * y) - \tilde{\mu}_{\sup}(x * y) \geq \tilde{\mu}_{\inf}(y) - \tilde{\mu}_{\sup}(y) = \tilde{\mu}_\xi(y).$$

Hence $\tilde{\mu}_\xi(x * y) \geq \max\{\tilde{\mu}_\xi(x), \tilde{\mu}_\xi(y)\}$ for all $x, y \in X$. □

3. Hyperfuzzy subalgebras

Given a nonempty set X, let $\mathcal{B}_K(X)$ and $\mathcal{B}_I(X)$ denote the collection of all BCK-algebras and all BCI-algebras, respectively. Also $\mathcal{B}(X) := \mathcal{B}_K(X) \cup \mathcal{B}_I(X)$.

Example 3.1. Given a set $X = \{0, 1, 2\}$, we have three BCK-algebras as follows:

$*_1$	0	1	2
0	0	0	0
1	1	0	0
2	2	1	0

$*_2$	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

$*_3$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Hence $\mathcal{B}_K(X) = \{(X, *_1, 0), (X, *_2, 0), (X, *_3, 0)\}$. Also, we have two BCI-algebras as follows:

$\circ_1$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

$\circ_2$	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

and so $\mathcal{B}_I(X) = \{(X, \circ_1, 0), (X, \circ_2, 0)\}$. Therefore

$$\mathcal{B}(X) = \{(X, *_1, 0), (X, *_2, 0), (X, *_3, 0), (X, \circ_1, 0), (X, \circ_2, 0)\}.$$

Definition 3.2. For any $(X, *, 0) \in \mathcal{B}(X)$, a mapping $\mu : (X, *, 0) \to [0, 1]$ is called a *fuzzy subalgebra* of X with type 1, 2, 3 and 4 (briefly, 1, 2, 3 and 4-*fuzzy subalgebra* of X) if μ satisfies the condition (2.1), (2.2), (2.3) and (2.4), respectively.

Definition 3.3. For any $(X, *, 0) \in \mathcal{B}(X)$, a hyperfuzzy set $\tilde{\mu} : (X, *, 0) \to \tilde{\mathcal{P}}([0, 1])$ is called a *hyperfuzzy subalgebra* of $(X, *, 0)$ with type (i, j) for $i, j \in \{1, 2, 3, 4\}$ (briefly, (i, j)-*hyperfuzzy subalgebra* of $(X, *, 0)$) if $\tilde{\mu}_{\inf}$ is an i-fuzzy subalgebra of X and $\tilde{\mu}_{\sup}$ is a j-fuzzy subalgebra of X.

Example 3.4. Consider a set $X = \{0, 1, 2, 3\}$ with the Cayley table which is given in Table 1.

Table 1. Cayley table for the binary operation "$*$"

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Then $(X, *, 0) \in \mathcal{B}_K(X)$.

(1) Define a hyperfuzzy set $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0, 1])$ as the Table 2.

Table 2. hyperfuzzy set

X	0	1	2	3
$\tilde{\mu}$	$[0.4, 0.5]$	$(0.3, 0.5)$	$[0.2, 0.6)$	$(0.1, 0.7]$

It is routine to verify that $\tilde{\mu}$ is a $(1, 4)$-hyperfuzzy subalgebra of X.

(2) Define a hyperfuzzy set $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0, 1])$ as the Table 3.

Table 3. hyperfuzzy set

X	0	1	2	3
$\tilde{\mu}$	$(0.4, 0.8)$	$[0.3, 0.5]$	$[0.2, 0.4)$	$(0.1, 0.7]$

It is easy to check that $\tilde{\mu}$ is a $(1, 1)$-hyperfuzzy subalgebra of X.

Table 4. Cayley table for the binary operation "$*$"

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Example 3.5. Consider a set $X = \{0, a, b, c\}$ with the Cayley table which is given in Table 4.
Then $(X, *, 0) \in \mathcal{B}_I(X)$.

(1) Define a hyperfuzzy set $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0, 1])$ as the Table 5.

Table 5. hyperfuzzy set

X	0	a	b	c
$\tilde{\mu}$	$(0.3, 0.9]$	$(0.7, 0.9)$	$[0.7, 0.8)$	$[0.5, 0.8]$

It is easy to check that $\tilde{\mu}$ is a $(4, 1)$-hyperfuzzy subalgebra of X.

(2) Define a hyperfuzzy set $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0, 1])$ as the Table 6.

Table 6. hyperfuzzy set

X	0	a	b	c
$\tilde{\mu}$	$[0.3, 0.5)$	$(0.7, 0.9]$	$(0.7, 0.8)$	$[0.5, 0.9]$

It is easy to check that $\tilde{\mu}$ is a $(4, 4)$-hyperfuzzy subalgebra of X.

By the proofs of Theorems 2.6, 2.8, 2.9 and 2.11, we have the following theorem.

Theorem 3.6. *For any $(X, *, 0) \in \mathcal{B}(X)$, let $\tilde{\mu} : (X, *, 0) \to \tilde{\mathcal{P}}([0, 1])$ be a hyperfuzzy set over X.*

(1) *If $\tilde{\mu}$ is a $(4, 3)$-hyperfuzzy subalgebra of X, then $\tilde{\mu}_\ell$ is a 3-fuzzy subalgebra of X.*
(2) *If $\tilde{\mu}$ is a $(3, 4)$-hyperfuzzy subalgebra of X, then $\tilde{\mu}_\ell$ is a 4-fuzzy subalgebra of X.*
(3) *If $\tilde{\mu}$ is a $(3, 2)$-hyperfuzzy subalgebra of X, then $\tilde{\mu}_\ell$ is a 2-fuzzy subalgebra of X.*
(4) *If $\tilde{\mu}$ is a $(2, 3)$-hyperfuzzy subalgebra of X, then $\tilde{\mu}_\ell$ is a 3-fuzzy subalgebra of X.*

Corollary 3.7. *For any $(X, *, 0) \in \mathcal{B}(X)$, let $\tilde{\mu} : (X, *, 0) \to \tilde{\mathcal{P}}([0, 1])$ be a hyperfuzzy set over X.*

(1) *If $\tilde{\mu}$ is a $(4, 3)$-hyperfuzzy subalgebra of X, then $\tilde{\mu}_\ell$ is a 1-fuzzy subalgebra of X.*
(2) *If $\tilde{\mu}$ is a $(3, 2)$-hyperfuzzy subalgebra of X, then $\tilde{\mu}_\ell$ is a 4-fuzzy subalgebra of X.*

(3) *If $\tilde{\mu}$ is a $(2,3)$-hyperfuzzy subalgebra of X, then $\tilde{\mu}_\ell$ is a 1-fuzzy subalgebra of X.*

Proof. Straightforward. □

Let $\tilde{\mu} : (X,*,0) \to \tilde{\mathcal{P}}([0,1])$ be a hyperfuzzy set over $(X,*,0) \in \mathcal{B}(X)$ and let α, $\beta \in [0,1]$. We consider the following sets.

$$A_\alpha^{\geq} := \{x \in X \mid \tilde{\mu}_{\inf}(x) \geq \alpha\},$$
$$A_\alpha^{\leq} := \{x \in X \mid \tilde{\mu}_{\inf}(x) \leq \alpha\},$$
$$A_\beta^{\geq} := \{x \in X \mid \tilde{\mu}_{\sup}(x) \geq \beta\},$$
$$A_\beta^{\leq} := \{x \in X \mid \tilde{\mu}_{\sup}(x) \leq \beta\},$$

which are called *upper α-level subset*, *lower α-level subset*, *upper β-level subset* and *lower β-level subset* of X under $\tilde{\mu}$, respectively.

Theorem 3.8. *Let $(X,*,0) \in \mathcal{B}(X)$. If a hyperfuzzy set $\tilde{\mu} : (X,*,0) \to \tilde{\mathcal{P}}([0,1])$ over X is a $(k,4)$-hyperfuzzy subalgebra of X for $k = 1,3$, then the nonempty upper α-level subset and lower β-level subset of X under $\tilde{\mu}$ are subalgebras of X for all $\alpha, \beta \in [0,1]$.*

Proof. Assume that $\tilde{\mu} : (X,*,0) \to \tilde{\mathcal{P}}([0,1])$ is a $(1,4)$-hyperfuzzy subalgebra of X. Let $\alpha, \beta \in [0,1]$ be such that $A_\alpha^{\geq} \neq \varnothing \neq A_\beta^{\leq}$. If $x, y \in A_\alpha^{\geq}$, then $\tilde{\mu}_{\inf}(x) \geq \alpha$ and $\tilde{\mu}_{\inf}(y) \geq \alpha$. It follows from (2.1) that

$$\tilde{\mu}_{\inf}(x * y) \geq \min\{\tilde{\mu}_{\inf}(x), \tilde{\mu}_{\inf}(y)\} \geq \alpha.$$

Hence $x * y \in A_\alpha^{\geq}$. If $a, b \in A_\beta^{\leq}$, then $\tilde{\mu}_{\sup}(a) \leq \beta$ and $\tilde{\mu}_{\sup}(b) \leq \beta$, which implies from (2.4) that

$$\tilde{\mu}_{\sup}(a * b) \leq \max\{\tilde{\mu}_{\sup}(a), \tilde{\mu}_{\sup}(b)\} \leq \beta.$$

Thus $a * b \in A_\beta^{\leq}$. Therefore $A_\alpha^{\geq}$ and $A_\beta^{\leq}$ are subalgebras of X. Similarly, we can prove it for $k = 3$. □

Corollary 3.9. *Let $(X,*,0) \in \mathcal{B}(X)$. If a hyperfuzzy set $\tilde{\mu} : (X,*,0) \to \tilde{\mathcal{P}}([0,1])$ over X is a $(k,2)$-hyperfuzzy subalgebra of X for $k = 1,3$, then the nonempty upper α-level subset and lower β-level subset of X under $\tilde{\mu}$ are subalgebras of X for all $\alpha, \beta \in [0,1]$.*

Similarly, we have the following theorems and corollaries.

Theorem 3.10. *Let $(X,*,0) \in \mathcal{B}(X)$. If a hyperfuzzy set $\tilde{\mu} : (X,*,0) \to \tilde{\mathcal{P}}([0,1])$ over X is a $(k,4)$-hyperfuzzy subalgebra of X for $k = 2,4$, then the nonempty lower α-level subset and lower β-level subset of X under $\tilde{\mu}$ are subalgebras of X for all $\alpha, \beta \in [0,1]$.*

Corollary 3.11. *Let $(X,*,0) \in \mathcal{B}(X)$. If a hyperfuzzy set $\tilde{\mu} : (X,*,0) \to \tilde{\mathcal{P}}([0,1])$ over X is a $(k,2)$-hyperfuzzy subalgebra of X for $k = 2,4$, then the nonempty lower α-level subset and lower β-level subset of X under $\tilde{\mu}$ are subalgebras of X for all $\alpha, \beta \in [0,1]$.*

Theorem 3.12. *Let $(X,*,0) \in \mathcal{B}(X)$. If a hyperfuzzy set $\tilde{\mu} : (X,*,0) \to \tilde{\mathcal{P}}([0,1])$ over X is a $(k,1)$-hyperfuzzy subalgebra of X for $k = 1,3$, then the nonempty upper α-level subset and upper β-level subset of X under $\tilde{\mu}$ are subalgebras of X for all $\alpha, \beta \in [0,1]$.*

Corollary 3.13. *Let $(X,*,0) \in \mathcal{B}(X)$. If a hyperfuzzy set $\tilde{\mu} : (X,*,0) \to \tilde{\mathcal{P}}([0,1])$ over X is a $(k,3)$-hyperfuzzy subalgebra of X for $k = 1,3$, then the nonempty upper α-level subset and upper β-level subset of X under $\tilde{\mu}$ are subalgebras of X for all $\alpha, \beta \in [0,1]$.*

Theorem 3.14. *Let $(X,*,0) \in \mathcal{B}(X)$. If a hyperfuzzy set $\tilde{\mu} : (X,*,0) \to \tilde{\mathcal{P}}([0,1])$ over X is a $(k,1)$-hyperfuzzy subalgebra of X for $k = 2,4$, then the nonempty lower α-level subset and upper β-level subset of X under $\tilde{\mu}$ are subalgebras of X for all $\alpha, \beta \in [0,1]$.*

Corollary 3.15. *Let $(X,*,0) \in \mathcal{B}(X)$. If a hyperfuzzy set $\tilde{\mu} : (X,*,0) \to \tilde{\mathcal{P}}([0,1])$ over X is a $(k,3)$-hyperfuzzy subalgebra of X for $k = 2,4$, then the nonempty lower α-level subset and upper β-level subset of X under $\tilde{\mu}$ are subalgebras of X for all $\alpha, \beta \in [0,1]$.*

Theorem 3.16. *Let $(X,*,0) \in \mathcal{B}(X)$ and let $\tilde{\mu} : (X,*,0) \to \tilde{\mathcal{P}}([0,1])$ be a hyperfuzzy set over X in which the nonempty upper α-level subset and lower β-level subset of X under $\tilde{\mu}$ are subalgebras of X for all $\alpha, \beta \in [0,1]$. Then $\tilde{\mu}$ is a $(1,4)$-hyperfuzzy subalgebra of X.*

Proof. Assume that $A_{\alpha}^{\geq}$ and $A_{\beta}^{\leq}$ are subalgebras of X for all $\alpha, \beta \in [0,1]$ with $A_{\alpha}^{\geq} \neq \varnothing \neq A_{\beta}^{\leq}$. If $\tilde{\mu}_{\inf}(x*y) < \min\{\tilde{\mu}_{\inf}(x), \tilde{\mu}_{\inf}(y)\}$ for some $x, y \in X$, then there exists $\alpha \in (0,1]$ such that $\tilde{\mu}_{\inf}(x*y) < \alpha \leq \min\{\tilde{\mu}_{\inf}(x), \tilde{\mu}_{\inf}(y)\}$. Thus $x, y \in A_{\alpha}^{\geq}$, but $x*y \notin A_{\alpha}^{\geq}$ which is a contradiction. Hence $\tilde{\mu}_{\inf}(x*y) \geq \min\{\tilde{\mu}_{\inf}(x), \tilde{\mu}_{\inf}(y)\}$ for all $x, y \in X$, that is, $\tilde{\mu}_{\inf}$ is a 1-fuzzy subalgebra of X. Now, if $\tilde{\mu}_{\sup}(a*b) > \max\{\tilde{\mu}_{\sup}(a), \tilde{\mu}_{\sup}(b)\}$ for some $a, b \in X$, then $\tilde{\mu}_{\sup}(a*b) > \beta \geq \max\{\tilde{\mu}_{\sup}(a), \tilde{\mu}_{\sup}(b)\}$ for some $\beta \in [0,1)$. It follows that $a, b \in A_{\beta}^{\leq}$, but $a*b \notin A_{\beta}^{\leq}$, a contradiction. Therefore $\tilde{\mu}_{\sup}(a*b) \leq \max\{\tilde{\mu}_{\sup}(a), \tilde{\mu}_{\sup}(b)\}$ for all $a, b \in X$, which shows that $\tilde{\mu}_{\sup}$ is a 4-fuzzy subalgebra of X. Therefore $\tilde{\mu}$ is a $(1,4)$-hyperfuzzy subalgebra of X. □

Similarly, we have the following theorems.

Theorem 3.17. *Let $(X,*,0) \in \mathcal{B}(X)$ and let $\tilde{\mu} : (X,*,0) \to \tilde{\mathcal{P}}([0,1])$ be a hyperfuzzy set over X in which the nonempty lower α-level subset and upper β-level subset of X under $\tilde{\mu}$ are subalgebras of X for all $\alpha, \beta \in [0,1]$. Then $\tilde{\mu}$ is a $(4,1)$-hyperfuzzy subalgebra of X.*

Theorem 3.18. *Let $(X,*,0) \in \mathcal{B}(X)$ and let $\tilde{\mu} : (X,*,0) \to \tilde{\mathcal{P}}([0,1])$ be a hyperfuzzy set over X in which the nonempty lower α-level subset and lower β-level subset of X under $\tilde{\mu}$ are subalgebras of X for all $\alpha, \beta \in [0,1]$. Then $\tilde{\mu}$ is a $(4,4)$-hyperfuzzy subalgebra of X.*

Theorem 3.19. *Let $(X,*,0) \in \mathcal{B}(X)$ and let $\tilde{\mu} : (X,*,0) \to \tilde{\mathcal{P}}([0,1])$ be a hyperfuzzy set over X in which the nonempty upper α-level subset and upper β-level subset of X under $\tilde{\mu}$ are subalgebras of X for all $\alpha, \beta \in [0,1]$. Then $\tilde{\mu}$ is a $(1,1)$-hyperfuzzy subalgebra of X.*

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