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Abstract. In the present paper, a notion of M-basis and dimension of a multi vector space is introduced and some of their properties are studied.

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1. Introduction

Theory of Multisets is an important generalization of classical set theory which has emerged by violating a basic property of classical sets that an element can belong to a set only once. Many authors like Yager [14], Blizard [1], Girish and John [5], Hickman [6] etc. have studied the properties of multisets. Multisets are very useful structures arising in many areas of mathematics and computer science [4, 8, 10, 12]. Again the theory of vector space is one of the most important algebraic structures in modern mathematics and this has been extended in different setting [3, 7, 9, 13]. In [2], we introduced a notion of multi vector space and studied some of its basic properties. As a continuation of our earlier paper [2], here we have attempted to formulate the concept of basis and dimension of multi vector space and to study their properties.

2. Preliminaries

In this section, the definition of a multiset (mset in short) and some of its properties are given. Unless otherwise stated, X will be assumed to be an initial universal set and $MS(X)$ denotes the set of all mset over X.

Definition 2.1 ([5])**.** An mset M drawn from the set X is represented by a count function $C_M : X \to N$, where N represents the set of non negative integers. For any positive integer ω, $[X]^\omega$ denotes the mset spaces.

The algebraic operations of msets, level sets and operations on level sets are considered as in [5, 11]. Throughout the rest of the paper X, Y will denote vector spaces over K (where K is the field of real or complex numbers) and msets are taken from $[X]^\omega$, $[Y]^\omega$.

Definition 2.2 ([2])**.** Let $A_1, A_2, ..., A_n, B \in [X]^\omega$ and $\lambda \in K$, then $A_1 + A_2 + ... + A_n$ and λB are defined as follows:

$$C_{A_1+A_2+...+A_n}(x) = \vee\{C_{A_1}(x_1) \wedge C_{A_2}(x_2) \wedge ... \wedge C_{A_n}(x_n) : x_1, x_2, ..., x_n \in X \text{ and } x_1 + x_2 + ... + x_n = x\}$$

and

$$C_{\lambda B}(y) = \vee\{C_B(x) : \lambda x = y\}.$$

Lemma 2.3 ([2])**.** *Let $\lambda \in K$ and $B \in [X]^\omega$. Then for $\lambda \neq 0$, $C_{\lambda B}(y) = C_B(\lambda^{-1}y), \forall y \in X$. For $\lambda = 0$,*

$$C_{\lambda B}(y) = \begin{cases} 0, & y \neq \theta, \\ \sup\limits_{x \in X} C_B(x), & y = \theta. \end{cases}$$

Definition 2.4 ([2])**.** A multiset V in $[X]^\omega$ is said to be a multi vector space or multi linear space(in short mvector space) over the linear space X, if

(i) $V + V \subseteq V$,

(ii) $\lambda V \subseteq V$, for every scalar λ.

We denote the set of all multi vector space over X by $MV(X)$.

Remark 2.5. For $V \in MV(X)$, $V + V + \cdots + V(n \text{ times}) = V$, i.e., $nV = V$.

Remark 2.6 ([2])**.** If $V \in MV(X)$ with $dim\, X = m$, then $\mid C_V(X) \mid \leq m+1$, where $\mid C_V(X) \mid$ represents the cardinality of $C_V(X)$.

Theorem 2.7 ([2])**.** (Representation theorem) *Let $V \in MV(X)$ with $dim\, X = m$ and range of $C_V = \{n_0, n_1, ..., n_k\} \subseteq \{0, 1, 2, ..., \omega\}$, $k \leq m$, $n_0 = C_V(\theta)$ and $\omega \geq n_0 > n_1 > ... > n_k \geq 0$. Then there exists a nested collection of subspaces of X as*

$\{\theta\} \subseteq V_{n_0} \subsetneqq V_{n_1} \subsetneqq V_{n_2} \subsetneqq .... \subsetneqq V_{n_k} = X$ such that $V = n_0V_{n_0} \cup n_1V_{n_1} \cup ..... \cup n_kV_{n_k}$. Also

(1) *If $n, m \in (n_{i+1}, n_i]$, then $V_n = V_m = V_{n_i}$.*

(2) *If $n \in (n_{i+1}, n_i]$ and $m \in (n_i, n_{i-1}]$, then $V_n \supsetneqq V_m$.*

Definition 2.8 ([2])**.** Let X be a finite dimensional vector space with $dim\, X = m$ and $V \in MV(X)$. Consider Proposition 2.7. Let B_{n_i} be a basis on V_{n_i}, $i = 0, 1, ..., k$ such that

$$B_{n_0} \subsetneqq B_{n_1} \subsetneqq B_{n_2} \subsetneqq ... \subsetneqq B_{n_k} \tag{2.8.1}$$

Define a multi subset β of X by:

$$C_\beta(x) = \begin{cases} \vee\{n_i : x \in B_{n_i}\} \\ 0, \; otherwise. \end{cases}$$

Then β is called a multi basis of V corresponding to (2.8.1). We denote the set of all multi bases of V by $\mathcal{B}_M(V)$.

Lemma 2.9. *Let $s, t \in \mathbb{R}$ and A, A_1 and A_2 be multisets on a vector space X. Then*
(1) $s.(t.A) = t.(s.A) = (st).A$
(2) $A_1 \leq A_2 \Rightarrow t.A_1 \leq t.A_2$.

Proposition 2.10. *Let $V \in MV(X)$. Then $x \in X,\ a \neq 0 \Rightarrow C_V(ax) = C_V(x)$.*

Proposition 2.11. *Let $V \in MV(X)$ and $u, v \in X$ such that $C_V(u) > C_V(v)$. Then*

$$C_V(u+v) = C_V(v).$$

Proposition 2.12. *Let $V \in MV(X)$ and $v, w \in X$ with $C_V(v) \neq C_V(w)$. Then*

$$C_V(v+w) = C_V(v) \wedge C_V(w).$$

3. Multi linear independence and M-basis

Definition 3.1. Let $V \in MV(X)$ and $dim\, X = m$. A finite set of vectors $\{x_i\}_{i=1}^n$ is called multi linearly independent in V, if $\{x_i\}_{i=1}^n$ is linearly independent in X and for all $\{a_i\}_{i=1}^n \subset \mathbb{R}$ with $a_i \neq 0$, $C_V(\sum_{i=1}^n a_i x_i) = \wedge_{i=1}^n C_V(a_i x_i)$.

Proposition 3.2. *Let $V \in MV(X)$ and $dimX = m$. Then any set of vectors $\{x_i\}_{i=1}^N$ $(N \leq m)$, which have distinct counts is linearly and multi linearly independent.*

Remark 3.3. Converse of the above proposition is not true. Let $X = \mathbb{R}^2$ and $\omega = 6$. We define a multi vector space $C_V : X \to N$ by:

$$C_V(x) = \begin{cases} 6,\ if\ x = (0,0) \\ 1,\ otherwise. \end{cases}$$

Then $\{(1,0), (0,1)\}$ is multi linearly independent, but $C_V((1,0)) = C_V((0,1))$.

Definition 3.4. A M-basis for a multi vector space $V \in MV(X)$ is a basis of X which is multi linearly independent in V.

We denote the set of all M-bases of V by $\mathscr{B}(V)$.

Proposition 3.5. *Let X be a vector space with dim $X = m$, $B = \{e_i\}_{i=1}^m$ be a basis of X and $0 \neq n_0 \geq n_1 \geq n_2 \geq ... \geq n_m$ be a finite sequence of number from $\{0, 1, 2, ..., \omega\}$. Define a multiset V drawn from X as follows:*
(i) $C_V(\theta) = n_0$.
(ii) $C_V(e_i) = n_i,\ 1 \leq i \leq m$
(iii) for $x(\neq \theta) = \sum_{i=1}^m a_i e_i,\ C_V(x) = \wedge_{i \in J(x)} C_V(e_i)$,
where $J(x) = \{i, 1 \leq i \leq m,\ a_i \neq 0\}$.
Then V is multi vector space over X with M-basis B.

Proof. Let $x, y \in X \setminus \{\theta\}$. Then x and y can be uniquely written in the following way:

$x = \sum_{i \in E \cup D_x} x_i e_i,\ y = \sum_{i \in E \cup D_y} y_i e_i$ such that $E \cap D_x = \phi$, $E \cap D_y = \phi$, $D_x \cap D_y = \phi$, $E \cup D_x$ and $E \cup D_y$ are finite, non-empty and for all $i \in E \cup D_x$, $x_i \neq 0$ and for all $i \in E \cup D_y$, $y_i \neq 0$.

Suppose $a, b \neq 0$ and $a, b \in \mathbb{R}$ and $ax + by \neq \theta$. Let $Z = \{i \in E : ax_i + by_i = 0\}$ and $N = E \setminus Z$. At this stage, suppose that $E,\ D_x,\ D_y,\ Z$ and N are all non-empty. In case, some of these sets are empty the proof is almost similar. Now,

$$C_V(ax+by) = C_V(\textstyle\sum_{i \in E}(ax_i + by_i)e_i + \sum_{i \in D_x}(ax_i)e_i + \sum_{i \in D_y}(by_i)e_i)$$

$= C_V(\sum_{i\in N}(ax_i + by_i)e_i + \sum_{i\in D_x}(ax_i)e_i + \sum_{i\in D_y}(by_i)e_i)$.
All coefficient in the above linear combination are non-zero and thus by definition of C_V, we have,

$$\begin{aligned} C_V(ax+by) &= (\wedge_{i\in N}C_V(e_i)) \wedge (\wedge_{i\in D_x}C_V(e_i)) \wedge (\wedge_{i\in D_x}C_V(e_i)) \\ &= (\wedge_{i\in N}n_i) \wedge (\wedge_{i\in D_x}n_i) \wedge \left(\wedge_{i\in D_y}n_i\right) \\ &= \wedge_{i\in N\cup D_x\cup D_y}(n_i) \geq \wedge_{i\in E\cup D_x\cup D_y}(n_i) \\ &= (\wedge_{i\in E\cup D_x}n_i) \wedge \left(\wedge_{i\in E\cup D_y}n_i\right) = C_V(x) \wedge C_V(y). \end{aligned}$$

If $a, b \neq 0$ and $a, b \in \mathbb{R}$ and $ax + by \neq \theta$, then $C_V(ax + by) \geq C_V(x) \wedge C_V(y)$.
In the case where $ax + by = \theta$, we must have $C_V(ax + by) \geq C_V(x) \wedge C_V(y)$.
In the case where a or b is zero, without loss of generality we may say $a = 0$, then

$$C_V(0x + by) = C_V(by) \geq C_V(x) \wedge C_V(by) = C_V(x) \wedge C_V(y).$$

□

Lemma 3.6. *If $V \in MV(X)$ and Y is a proper subspace of X, then for any $t \in X\setminus Y$ with $C_V(t) = sup[C_V(X \setminus Y)]$, $C_V(t + y) = C_V(t) \wedge C_V(y)$, for all $y \in Y$.*

Proof. Since ω is finite, such a t exists. Let $y \in Y$. If $C_V(y) \neq C_V(t)$, then by Proposition 2.12, $C_V(t + y) = C_V(t) \wedge C_V(y)$. If $C_V(y) = C_V(t)$, then by Definition 2.4, $C_V(t + y) \geq C_V(t) \wedge C_V(y)$. Since $t + y \in X \setminus Y$ and $C_V(t) = sup[C_V(X \setminus Y)]$, we must have $C_V(t+y) \leq C_V(t) = C_V(y)$ and thus $C_V(t+y) = C_V(t) \wedge C_V(y)$. □

Lemma 3.7. *Let $V \in MV(X)$, Y be a proper subspace of X and $C_V \mid_Y = C_{V'}$. If B^* is a M-basis for V', then there exists $t \in X \setminus Y$ such that $B^+ = B^* \cup \{t\}$ is a M-basis for W, where $C_W = C_V \mid_{\prec B^+ \succ}$ and $\prec B^+ \succ$ is the vector space spanned by B^+.*

Proof. Pick $t \in X \setminus Y$ such that $C_V(t) = sup[C_V(X \setminus Y)]$. Then by Lemma 3.6, $B^+ = B^* \cup \{t\}$ is a multi linearly independent and hence a M-basis for W, where $C_W = C_V \mid_{\prec B^+ \succ}$. □

Proposition 3.8. *All multi vector spaces $V \in MV(X)$ with dim $X = m$ have M-basis.*

Proposition 3.9. *Let $V \in MV(X)$ where dim $X = m$ and $C_V(X \setminus \{\theta\}) = \{n_0, n_1, n_2, ..., n_k\}$, $k \leq m$. Then a basis $B = \{e_1, e_2, ..., e_m\}$ of X is a M-basis for V if and only if $B \cap V_{n_i}$ is a basis of V_{n_i} for any $i = 0, 1, ..., k$.*

Proposition 3.10. *Let V be a multi vector space over X where dim $X = m$. Then there is an one-to-one correspondence between $\mathcal{B}_M(V)$ and $\mathscr{B}(V)$.*

Proposition 3.11. *Let $V \in MV(X)$ with dim $X = m$ and range of $C_V(X \setminus \{\theta\}) = \{n_0, n_1, ..., n_k\} \subseteq \{0, 1, 2, ..., \omega\}$, $k \leq m$. If a basis $B = \{e_1, e_2, ..., e_m\}$ of X is a M-basis, then $C_V(B) = \{n_0, n_1, ..., n_k\}$.*

Remark 3.12. Converse of the above proposition is not true. For example, suppose $X = \mathbb{R}^4$, $\omega = 5$. Define multi vector space V with C_V as follows:
$C_V((0, 0, 0, 0)) = 5$, $C_V((0, 0, 0, \mathbb{R} \setminus \{0\})) = 5$, $C_V((0, 0, \mathbb{R} \setminus \{0\}, \mathbb{R})) = 5$,
$C_V((0, \mathbb{R} \setminus \{0\}, \mathbb{R}, \mathbb{R})) = 2, C_V(\mathbb{R}^4 \setminus (0, \mathbb{R}, \mathbb{R}, \mathbb{R})) = 2$.
Then $B = \{(0, 0, 0, 1), (-1, 1, 1, 1), (1, -1, 1, 1), (1, 1, -1, 1)\}$ is a basis of $\mathbb{R}^4$ and

$C_V(B) = \{2, 5\} = C_V(\mathbb{R}^4)$. But B is not a M-basis as B is not multi linearly independent.

Definition 3.13. Let $V \in MV(X)$ with $dim\ X = m$, range of $C_V(X \setminus \{\theta\}) = \{n_0, n_1, ..., n_k\} \subseteq \{0, 1, 2, ..., \omega\}$, $k \leq m$ and B_0 be any M-basis for V. Then

$$C_V(B_0) = \{n_0, n_1, ..., n_k\}.$$

We define multi index of a multi M-basis B_0 with respect to V by:

$$[B_0]_M = \{r_i : r_i \text{ is the number of element of } B_0 \text{ taking the value } n_i\}.$$

Proposition 3.14. *For a multi vector space V, multi index of M-basis with respect to V is independent of M-basis.*

Proof. Let $V \in MV(X)$ with $dim\ X = m$, range of $C_V(X \setminus \{\theta\}) = \{n_0, n_1, ..., n_k\} \subseteq \{0, 1, 2, ..., \omega\}$, $k \leq m$ and $\omega \geq n_0 > n_1 > ... > n_k \geq 0$. Then for any two M-bases B_0, B_0' of V, $C_V(B_0) = C_V(B_0') = \{n_0, n_1, ..., n_k\}$. Let $[B_0]_M = \{r_i\}$ and $[B_0']_M = \{r_i'\}$. Now, $\mid B_0 \cap V_{n_i} \mid = \sum_{j=0}^{i} r_j$ and $\mid B_0' \cap V_{n_i} \mid = \sum_{j=0}^{i} r_j'$, for $i = 0, 1, 2, ..., k$. As $B_0 \cap V_{n_i}$ and $B_0' \cap V_{n_i}$ are both basis of V_{n_i}, $\mid B_0 \cap V_{n_i} \mid = \mid B_0' \cap V_{n_i} \mid$, for all $i = 0, 1, 2, ..., k$. Thus $[B_0]_M = [B_0']_M$. □

Remark 3.15. As multi index of M-basis with respect to a multi vector space V is independent of M-basis, we can use only the term multi index of V.

Definition 3.16. Let $V \in MV(X)$ with $dim\ X = m$, $C_V(X) = \{n_0, n_1, ..., n_k\} \subseteq \{0, 1, 2, ..., \omega\}$, $k \leq m$ and B be any basis for X.

Define index of a basis B with respect to V by:

$$[B] = \{r_i : \ r_i \text{ is the number of element of } B \text{ taking the value } n_i\}.$$

Proposition 3.17. *Let $V \in MV(X)$ with $dim\ X = m$, $C_V(X \setminus \{\theta\}) = \{n_0, n_1, ..., n_k\} \subseteq \{0, 1, 2, ..., \omega\}$, $k \leq m$ and B be any basis of X with $C_V(B) = \{n_0, n_1, ..., n_k\}$. If index $[B]$ of B with respect to V is equal to the multi index of V, then B becomes a M-basis.*

Proof. Let us assume that $\omega \geq n_0 > n_1 > ... > n_k \geq 0$. Then $\{\theta\} \subsetneqq V_{n_0} \subsetneqq V_{n_1} \subsetneqq V_{n_2} \subsetneqq .... \subsetneqq V_{n_k} = X$. Suppose that $[B]_M = \{r_i : i = 0, 1, 2, ...k\}$. Then $dim\ V_{n_i} = \sum_{j=0}^{i} r_j = \mid B \cap V_{n_i} \mid$, for all $i = 0, 1, 2, ..., k$. Thus, $B \cap V_{n_i}$ becomes a basis for V_{n_i} for each $i = 0, 1, 2, .., k$. By Proposition 3.9, B is a M-basis for V. □

4. Dimension of multi vector space

Definition 4.1. We define the dimension of a multi vector space V over X by:

$$dim(V) = \underset{B\ a\ base\ for\ X}{sup} \left(\sum_{x \in B} C_V(x) \right).$$

Clearly, dim is a function from the set of all multi vector spaces to $\mathbb{N}$.

Proposition 4.2. *Let $V \in MV(X)$ where $dim\ X = m < \infty$. If B is a M-basis for V and B^* is any basis for X, then* $\sum_{x \in B^*} C_V(x) \leq \sum_{x \in B} C_V(x)$.

Proposition 4.3. *If V is a multi vector space over a finite dimensional vector space X, then $dim(V) = \sum_{x \in B} C_V(x)$, where B is any M-basis for V.*

Remark 4.4. If V is a multi vector space over a finite dimensional vector space X, then $dim(V)$ is independent of M-basis for V. It follows from Proposition 3.9 and Proposition 3.11.

Proposition 4.5. *Let X be any finite dimensional vector space and $V, W \in MV(X)$ such that $C_V(\theta) \geq sup[C_W(X \setminus \{\theta\})]$ and $C_W(\theta) \geq sup[C_V(X \setminus \{\theta\})]$. Then there exists a basis B for X which is also a M-basis for V, W, $V \cap W$ and $V + W$.*

In addition, if $A_1 = \{x \in X : C_V(x) < C_W(x)\}$, $A_2 = X \setminus A_1$, then for all $v \in B \cap A_1$,

$$(C_{V \cap W})(v) = C_V(v) \text{ and } C_{V+W}(v) = C_W(v)$$

and for all $v \in B \cap A_2$,

$$(C_{V \cap W})(v) = C_W(v) \text{ and } C_{V+W}(v) = C_V(v).$$

Proof. We prove this by induction on $dim\ X$. In case $dim\ X = 1$ the statement is clearly true.

Now suppose that the theorem is true for all the multi vector space with dimension of the underlying vector space equal to n.

Let V and W be two multi vector spaces over X with $dim\ X = n + 1 > 1$. Let $B_1 = \{v_i\}_{i=1}^{n+1}$ be any M-basis for V. We may assume that $C_V(v_1) \leq C_V(v_i)$, for all $i = \{2, 3, ..., n + 1\}$. Let $H = \prec \{v_i\}_{i=2}^{n+1} \succ$. Since $n + 1 > 1$, $H \neq \{\theta\}$. Clearly $dim\ H = n$. Define the following two multi vector spaces: V_1 with count function $C_{V_1} = C_V \mid_H$ and W_1 with the count function $C_{W_1} = C_W \mid_H$. By inductive hypothesis there exists a basis B^*, for H which is also a M-basis for V_1, W_1, $V_1 \cap W_1$ and $V_1 + W_1$. Also for all $v \in B^* \cap A_1$,

$$(C_{V_1 \cap W_1})(v) = C_{V_1}(v) \text{ and } C_{V_1+W_1}(v) = C_{W_1}(v)$$

and for all $v \in B^* \cap A_2$,

$$(C_{V_1 \cap W_1})(v) = C_{W_1}(v) \text{ and } C_{V_1+W_1}(v) = C_{V_1}(v).$$

We shall now show that B^* can be extended to B such that B is a M-basis for V, W, $V \cap W$ and $V + W$. Furthermore, for all $v \in B \cap A_1$,

$$(C_{V \cap W})(v) = C_V(v) \text{ and } C_{V+W}(v) = C_W(v)$$

and for all $v \in B \cap A_2$,

$$(C_{V \cap W})(v) = C_W(v) \text{ and } C_{V+W}(v) = C_V(v).$$

Step - 1: First it will be shown that for all $x \in H$,

$$C_{(V+W)} \mid_H (x) = C_{V_1+W_1}(x) \tag{4.5.1}$$

Since B^* is a M-basis of $V_1 + W_1$, (4.5.1) implies that B^* is multi linearly independent in $V + W$.

Step - 2: Let $v^* \in X \setminus H$ such that $C_W(v^*) = sup[C_W(X \setminus H)]$. By Lemma 3.6 and Lemma 3.7, $B(= B^* \cup \{v^*\})$ is an extended M-basis of B^* for W.

Step - 3: Since $C_V(X \setminus H) = C_V(v_1)$, $C_V(v_1) = C_V(v^*)$ and then $B(= B^* \cup \{v^*\})$ is an extended M-basis of B^* for V.

Step - 4: Next it is shown that $B(= B^* \cup \{v^*\})$ is an extended M-basis of B^* for $V \cap W$.

Step - 5: In this step it is shown that $B(= B^* \cup \{v^*\})$ is an extended M-basis of B^* for $V + W$.

Step - 6: Finally, it is shown that if $v^* \in A_1$ then $C_{V+W}(v^*) = C_W(v^*)$ and if $v^* \in A_2$ then $C_{V+W}(v^*) = C_V(v^*)$.
Through all this step, the proof is done. □

Corollary 4.6. *If $V, W \in MV(X)$ with dim X is finite and $C_V(\theta) \geq sup[C_W(X \setminus \{\theta\})]$ and $C_W(\theta) \geq sup[C_V(X \setminus \{\theta\})]$, then $dim(V + W) = dim\ V + dim\ W - dim\ (V \cap W)$.*

Example 4.7. Suppose $X = \mathbb{R}^2$, $\omega = 6$. Define two multi vector spaces V and W with count functions C_V and C_W respectively as follows:
$C_V((0,0)) = 5$, $C_V((0, \mathbb{R} \setminus \{0\})) = 3$, $C_V(X \setminus \mathbb{R}) = 1$,
$C_W((0,0)) = 6$, $C_W(\{(x,x) : x \in \mathbb{R} \setminus \{0\}\}) = 2$, $C_W(X \setminus \{(x,x) : x \in \mathbb{R}\}) = 1$.
Then $C_V(\theta) \geq sup[C_W(X \setminus \{\theta\})]$ and $C_W(\theta) \geq sup[C_V(X \setminus \{\theta\})]$. It is also easy to check that
$C_{V \cap W}((0,0)) = 5$, $C_{V \cap W}(\{(x,x) : x \in \mathbb{R} \setminus \{0\}\}) = 1$,
$C_{V \cap W}(X \setminus \{(x,x) : x \in \mathbb{R}\}) = 1$, $C_{V+W}((0,0)) = 5$,
$C_{V+W}((0, \mathbb{R} \setminus \{0\})) = 3$; $C_{V+W}(X \setminus (0, \mathbb{R})) = 2$
and
$B = \{(0,1), (1,1)\}$ is a M-basis for V, W, $V \cap W$ and $V + W$.
Thus $dim\ (V + W) = 3 + 2 = 5$, $dim(V \cap W) = 1 + 1 = 2$,
$dim\ V = 3 + 1 = 4$, $dim\ W = 2 + 1 = 3$.
So, $dim\ V + dim\ W - dim\ (V \cap W) = 4 + 3 - 2 = 5 = dim\ (V + W)$.

Definition 4.8. Let $V \in MV(X)$ and $f : X \to Y$ be a linear map. Then we define $f(V)$ as:

$$C_{f(V)}(x) = \begin{cases} sup\{C_V(z) : z \in f^{-1}(x)\} & if\ f^{-1}(x) \neq \phi \\ 0 & otherwise. \end{cases}$$

and $\tilde{kerf} = (kerf, C_V |_{kerf})$, $\tilde{imf} = (imf, C_V |_{imf})$.

Proposition 4.9. *If $V \in MV(X)$ where dim X is finite and $f : X \to Y$ is a linear map, then $dim(\tilde{kerf}) + dim(\tilde{imf}) = dim(V)$.*

5. Conclusion

There is a future scope of study about infinite dimensional multi vector space and behavior of linear operators over multi vector spaces.

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