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$\mathscr{G}$-Proximity Spaces

A. Kandil, S. A. El-Sheikh, E. Said

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Abstract. In this paper, we introduce a new approach of proximity structure based on the grill notion. For $\mathscr{G} = P(X)\backslash\{\phi\}$, we have the Efremovič proximity structure and for the other types of $\mathscr{G}$, we have many types of proximity structures. Some results on these spaces have been obtained. Some of these results are : every $\mathscr{G}$-normal T_1space is $\mathscr{G}$-proximizable space (Theorem 3.8). Also, for such space, we show that it has a unique compatible $\mathscr{G}$-proximity under the condition that X is compact relative to τ^* (Theorem 4.10). Finally, for a surjective map $f : X \longrightarrow (Y, \delta_{f(\mathscr{G})})$ ($\mathscr{G}$ is a grill on X), we establish the largest $\mathscr{G}$-proximity $\delta_{\mathscr{G}}$ on X for which the map f is a $\mathscr{G}$-proximally continuous (Theorem 4.16).

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1. Introduction

The fundamental concepts of Efremovič proximity and generalized proximity were introduced by Efremovič, Lodato, and others [2, 3, 8, 9]. The notion of grill was initiated by choquet [1]. The grill is a powerful tool, since it related to many topics such as the theory of proximity spaces and the theory of compactifications etc,. Recently, Kandil et al. [5, 6, 7] introduced a new approaches of proximity structure based on the ideal notion. Thron [12] showed that the concept of grill plays an important role in the theory of proximities. Grills are extremely useful and convenient tool for many situations like filters and nets.

In this paper, based on any given grill $\mathscr{G}$, a new proximity structure is established namely, $\mathscr{G}$-proximity which is an Efremovič if the grill is $P(X)\backslash\{\phi\}$. Many properties of this proximity structures are studied. Some of them are: every $\mathscr{G}$-normal T_1 space is a $\mathscr{G}$-proximizable space (Theorem 3.8) and has a unique compatible $\mathscr{G}$-proximity provided that the space X is compact with respect to τ^* (Theorem 4.10). Also, for a surjective map $f : X \longrightarrow (Y, \delta_{f(\mathscr{G})})$ ($\mathscr{G}$ is a grill on X), we establish the largest $\mathscr{G}$-proximity $\delta_{\mathscr{G}}$ on X which makes f a $\mathscr{G}$-proximally continuous mapping (Theorem 4.16).

Now we recall some definitions and results defined and discussed in [1, 4, 9, 10, 11].

Definition 1.1. A nonempty collection $\mathscr{G}$ of subsets of a set X is called a grill on X, if it satisfies the following conditions:
(i) $\phi \notin \mathscr{G}$,
(ii) $A \in \mathscr{G}$ and $A \subseteq B \Rightarrow B \in \mathscr{G}$,
(iii) $A \cup B \in \mathscr{G} \Rightarrow A \in \mathscr{G}$ or $B \in \mathscr{G}$.

Definition 1.2. Let (X, τ) be a topological space and $\mathscr{G}$ be a grill on X. Then the operator

$$\Phi_{(\mathscr{G},\tau)} : P(X) \longrightarrow P(X)$$

defined by

$$\Phi_{(\mathscr{G},\tau)}(A) := \{x \in X \mid O_x \cap A \in \mathscr{G} \quad \text{for every } O_x \in \tau\}$$

is called the local function of A with respect to $\mathscr{G}$ and τ, where O_x is open set containing x. For simplicity, we will call $\Phi_{(\mathscr{G},\tau)}$ as Φ.

Proposition 1.3. *Let (X, τ) be a topological space and $\mathscr{G}$ be a grill on X. Then the operator*

$$\Psi_{(\mathscr{G},\tau)} : P(X) \longrightarrow P(X)$$

defined by

$$\Psi_{(\mathscr{G},\tau)}(A) = A \cup \Phi(A) \tag{1.1}$$

satisfies Kuratwski's axioms and induces a topology on X called τ^* given by

$$\tau^* = \{A \subseteq X \mid \Psi_{(\mathscr{G},\tau)}(A^c) = A^c\}, \tag{1.2}$$

where A^c denotes the complement of A and when there is no ambiguity, we will write $\Psi(A)$ for $\Psi_{(\mathscr{G},\tau)}(A)$.

Definition 1.4. A binary relation δ on $P(X)$ is called an (Efremovič) proximity on X if δ satisfies the following conditions:

(p_1) $A\ \delta B \Rightarrow B\ \delta A$,
(p_2) $A\ \delta(B \cup C) \Leftrightarrow A\ \delta B$ or $A\ \delta C$,
(p_3) $A\ \delta B \Rightarrow A \neq \phi$ and $B \neq \phi$,
(p_4) $A \cap B \neq \phi \Rightarrow A\ \delta B$,
(p_5) $A\ \overline{\delta} B \Rightarrow$ there exist $C, D \subseteq X$ such that $A\ \overline{\delta} C^c$, $D^c\ \overline{\delta} B$ and $C \cap D = \phi$.

A proximity space is a pair (X, δ) consisting of a set X and a proximity relation on X. We shall write $A\delta B$ if the sets $A, B \subseteq X$ are δ-related, otherwise we shall write $A\overline{\delta}B$

Lemma 1.5. *Let $\mathscr{G}$ be a grill on a nonempty set X and $f : X \longrightarrow Y$ be an onto function. Then*

$$f(\mathscr{G}) = \{f(A) \mid A \in \mathscr{G}\}$$

is a grill.

2. New structure of proximity spaces

Definition 2.1. Let $\mathscr{G}$ be a grill on a nonempty set X. A binary relation $\delta_{\mathscr{G}}$ on $P(X)$ is called a $\mathscr{G}$-proximity on X if $\delta_{\mathscr{G}}$ satisfies the following conditions:

($\mathscr{G}P_1$) $A\delta_{\mathscr{G}}B \Rightarrow B\delta_{\mathscr{G}}A$,
($\mathscr{G}P_2$) $A\delta_{\mathscr{G}}(B \cup C) \Leftrightarrow A\delta_{\mathscr{G}}B$ or $A\delta_{\mathscr{G}}C$,
($\mathscr{G}P_3$) $A\delta_{\mathscr{G}}B \Rightarrow A, B \in \mathscr{G}$,
($\mathscr{G}P_4$) $A \cap B \in \mathscr{G} \Rightarrow A\delta_{\mathscr{G}}B$,
($\mathscr{G}P_5$) $A\overline{\delta}_{\mathscr{G}}B \Rightarrow$ there exist $C, D \subseteq X$ such that $A\ \overline{\delta}_{\mathscr{G}}C^c, D^c\overline{\delta}_{\mathscr{G}}B$ and $C \cap D \notin \mathscr{G}$.

A $\mathscr{G}$-proximity space is a pair $(X, \delta_{\mathscr{G}})$ consisting of a set X and a $\mathscr{G}$-proximity relation on X. We shall write $A\delta_{\mathscr{G}}B$, if the sets $A, B \subseteq X$ are $\delta_{\mathscr{G}}$-related, otherwise we shall write $A\overline{\delta}_{\mathscr{G}}B$.

$\delta_{\mathscr{G}}$ is said to be separated, if it satisfies:

($\mathscr{G}P_6$) $x\delta_{\mathscr{G}}y \Rightarrow x = y$.

Proposition 2.2. *If $\mathscr{G} = P(X)\backslash\{\phi\}$, then the $\mathscr{G}$-proximity relation $\delta_{\mathscr{G}}$ is an Efremovič proximity relation.*

Proof. Straightforward. □

Example 2.3. Let $\mathscr{G}$ be a grill on a nonempty set X and $\delta_{\mathscr{G}}$ be a binary relation on $P(X)$ defined as:

$$A\delta_{\mathscr{G}}B \Leftrightarrow A, B \in \mathscr{G}. \tag{2.1}$$

Then $\delta_{\mathscr{G}}$ is a $\mathscr{G}$-proximity relation. Indeed, one easily sees that $\delta_{\mathscr{G}}$ satisfies conditions ($\mathscr{G}P_1$)-($\mathscr{G}P_4$), and to check that $\delta_{\mathscr{G}}$ also satisfies condition ($\mathscr{G}P_5$), let $A\overline{\delta}_{\mathscr{G}}B$. It follows that $A \notin \mathscr{G}$ or $B \notin \mathscr{G}$. If $A \notin \mathscr{G}$, by taking $C = A$ and $D = A^c$, we have the required properties. If $B \notin \mathscr{G}$, by taking $C = B^c$ and $D = B$, we obtain required properties.

Example 2.4. Let $\mathscr{G}$ be a grill on a nonempty set X. For any $A, B \subseteq X$, let us define

$$A\delta_{\mathscr{G}}B \Leftrightarrow A \cap B \in \mathscr{G}. \tag{2.2}$$

we shall show that $\delta_{\mathscr{G}}$ is a $\mathscr{G}$-proximity on X. It follows directly from the definition that $\delta_{\mathscr{G}}$ satisfies conditions ($\mathscr{G}P_1$)-($\mathscr{G}P_4$). To prove that $\delta_{\mathscr{G}}$ satisfies condition ($\mathscr{G}P_5$), let $A\overline{\delta}_{\mathscr{G}}B$. It follows that $A \cap B \notin \mathscr{G}$. If we take $C = B^c$ and $D = B$, then we obtain required properties.

Lemma 2.5. *If $A\delta_{\mathscr{G}}B$, $A \subseteq C$, and $B \subseteq D$, then $C\delta_{\mathscr{G}}D$.*

Proof. The result is a direct consequence of ($\mathscr{G}P_1$) and ($\mathscr{G}P_2$). □

Theorem 2.6. *Let $(X, \delta_{\mathscr{G}})$ be a $\mathscr{G}$-proximity space. Then the $\delta_{\mathscr{G}}$-operator*

$$\delta_{\mathscr{G}} : P(X) \longrightarrow P(X)$$

defined by

$$A^{\delta_{\mathscr{G}}} = \{x \in X \mid x\delta_{\mathscr{G}}A\} \tag{2.3}$$

satisfies the following:

(1) $A \subseteq B \Rightarrow A^{\delta_{\mathscr{G}}} \subseteq B^{\delta_{\mathscr{G}}}$,

(2) $(A \cup B)^{\delta_{\mathscr{G}}} = A^{\delta_{\mathscr{G}}} \cup B^{\delta_{\mathscr{G}}}$,
(3) $(A \cap B)^{\delta_{\mathscr{G}}} \subseteq A^{\delta_{\mathscr{G}}} \cap B^{\delta_{\mathscr{G}}}$,
(4) $A^{\delta_{\mathscr{G}}} - B^{\delta_{\mathscr{G}}} \subseteq (A - B)^{\delta_{\mathscr{G}}}$,
(5) $A \notin \mathscr{G} \Rightarrow A^{\delta_{\mathscr{G}}} = \phi$,
(6) $B \notin \mathscr{G} \Rightarrow (A \cup B)^{\delta_{\mathscr{G}}} = A^{\delta_{\mathscr{G}}} = (A - B)^{\delta_{\mathscr{G}}}$,
(7) $A \triangle B \notin \mathscr{G} \Rightarrow A^{\delta_{\mathscr{G}}} = B^{\delta_{\mathscr{G}}}$, *where* $A \triangle B = (A - B) \cup (B - A)$,
(8) $A^{\delta_{\mathscr{G}}} - (B^{\delta_{\mathscr{G}}})^{\delta_{\mathscr{G}}} \subseteq (A - B^{\delta_{\mathscr{G}}})^{\delta_{\mathscr{G}}}$,
(9) $A \nsubseteq A^{\delta_{\mathscr{G}}}$, *in general.*

Proof. (1) Let $x \in A^{\delta_{\mathscr{G}}}$. Then (2.3) implies that $x\delta_{\mathscr{G}}A$ and lemma 2.5 implies that $x\ \delta_{\mathscr{G}}B$. Thus $x \in B^{\delta_{\mathscr{G}}}$.

(2) By part (1), we get $A^{\delta_{\mathscr{G}}} \cup B^{\delta_{\mathscr{G}}} \subseteq (A \cup B)^{\delta_{\mathscr{G}}}$. To prove the other inclusion, let $x \in (A \cup B)^{\delta_{\mathscr{G}}}$. Then $x\delta_{\mathscr{G}}(A \cup B)$. Thus ($\mathscr{G}P_2$) implies that $x\delta_{\mathscr{G}}A$ or $x\delta_{\mathscr{G}}B$. So $x \in (A^{\delta_{\mathscr{G}}} \cup B^{\delta_{\mathscr{G}}})$. Hence $(A \cup B)^{\delta_{\mathscr{G}}} \subseteq A^{\delta_{\mathscr{G}}} \cup B^{\delta_{\mathscr{G}}}$. Therefore the result holds.

(3) The result is a direct consequence of part (1).

(4) For any $A, B \subseteq X$, we know that $A = (A - B) \cup (A \cap B)$. Then (2) implies that $A^{\delta_{\mathscr{G}}} = (A - B)^{\delta_{\mathscr{G}}} \cup (A \cap B)^{\delta_{\mathscr{G}}}$. Also (3) implies that $(A \cap B)^{\delta_{\mathscr{G}}} \subseteq B^{\delta_{\mathscr{G}}}$. Thus

$$A^{\delta_{\mathscr{G}}} - B^{\delta_{\mathscr{G}}} \subseteq [(A - B)^{\delta_{\mathscr{G}}} - B^{\delta_{\mathscr{G}}}] \subseteq (A - B)^{\delta_{\mathscr{G}}}.$$

(5) Let $A \notin \mathscr{G}$. Then ($\mathscr{G}P_3$) implies that $x\overline{\delta}_{\mathscr{G}}A$, for all $x \in X$. Thus $A^{\delta_{\mathscr{G}}} = \phi$.

(6) Let $B \notin \mathscr{G}$. By using (2), (5) and (4) of this theorem, then we have the required result.

(7) If $A \triangle B = (A - B) \cup (B - A) \notin \mathscr{G}$, then $(A - B), (B - A) \notin \mathscr{G}$. Since $A^{\delta_{\mathscr{G}}} = ((A-B)\cup(A\cap B))^{\delta_{\mathscr{G}}}$ and $(A-B) \notin \mathscr{G}$, by using (6), $A^{\delta_{\mathscr{G}}} = (A\cap B)^{\delta_{\mathscr{G}}} \subseteq B^{\delta_{\mathscr{G}}}$. It follows that

$$A^{\delta_{\mathscr{G}}} \subseteq B^{\delta_{\mathscr{G}}}. \tag{2.4}$$

Similarly, since $B^{\delta_{\mathscr{G}}} = ((B - A) \cup (A \cap B))^{\delta_{\mathscr{G}}}$ and $(B - A) \notin \mathscr{G}$, by using (6), $B^{\delta_{\mathscr{G}}} = (A \cap B)^{\delta_{\mathscr{G}}} \subseteq A^{\delta_{\mathscr{G}}}$. So

$$B^{\delta_{\mathscr{G}}} \subseteq A^{\delta_{\mathscr{G}}}. \tag{2.5}$$

Hence, from (2.4) and(2.5), $A^{\delta_{\mathscr{G}}} = B^{\delta_{\mathscr{G}}}$.

(8) The proof is obvious ,by using (4).

(9) Let us give an example. Let $X = \{a, b, c, d\}$, $\mathscr{G} = \{X, \{a\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, c, d\}\}$, $A = \{b, c\}$ and let $\delta_{\mathscr{G}}$ be a $\mathscr{G}$-proximity which is defined in Example 2.4. Then $A^{\delta_{\mathscr{G}}} = \phi$. □

Lemma 2.7. *Let* $(X, \delta_{\mathscr{G}})$ *be a* $\mathscr{G}$*-proximity space.*

$$\textit{If}\, B\overline{\delta}_{\mathscr{G}}A, \textit{then}\, A^{\delta_{\mathscr{G}}} \subseteq B^c. \tag{2.6}$$

Proof. Let $A^{\delta_{\mathscr{G}}} \cap B \neq \phi$. Then there exists an $x \in A^{\delta_{\mathscr{G}}}$ and $x \in B$, that is, $x\delta_{\mathscr{G}}A$ and $x \in B$. Lemma 2.5 implies $A\ \delta_{\mathscr{G}}B$ which is a contradiction. Thus the result holds. □

Theorem 2.8. *For every* $\mathscr{G}$*-proximity* $\delta_{\mathscr{G}}$ *on* X *and any sets* $A, B \subseteq X$*,*

$$B\delta_{\mathscr{G}}A^{\delta_{\mathscr{G}}} \Rightarrow B\delta_{\mathscr{G}}A. \tag{2.7}$$

Proof. Let $B\overline{\delta}_{\mathscr{G}}A$. Then ($\mathscr{G}P_5$) implies that there exist $C, D \subseteq X$ such that

$$B\overline{\delta}_{\mathscr{G}}C^c, D^c\overline{\delta}_{\mathscr{G}}A \quad \text{and} \quad C \cap D \notin \mathscr{G}. \tag{2.8}$$

This result, combined with lemma 2.7, implies

$$A^{\delta_{\mathscr{G}}} \subseteq D. \tag{2.9}$$

Now, we want to prove that $A^{\delta_{\mathscr{G}}} \subseteq C^c$. Let $x \in A^{\delta_{\mathscr{G}}}$. Then $x\delta_{\mathscr{G}}A$. If $x \in C$, then (2.9) implies that $x \in C \cap D$. By definition 1.1 part (ii), we have $\{x\} \notin \mathscr{G}$. Thus, by ($\mathscr{G}P_3$), $x\overline{\delta}_{\mathscr{G}}A$, which is a contradiction. So $x \in C^c$. Hence

$$A^{\delta_{\mathscr{G}}} \subseteq C^c. \tag{2.10}$$

From (2.8), (2.10) and lemma 2.5, we have $B\overline{\delta}_{\mathscr{G}}A^{\delta_{\mathscr{G}}}$ which is a contradiction. Therefore the result holds. □

Corollary 2.9. *For every $\mathscr{G}$-proximity $\delta_{\mathscr{G}}$ on X and any sets $A, B \subseteq X$,*

$$B^{\delta_{\mathscr{G}}}\delta_{\mathscr{G}}A^{\delta_{\mathscr{G}}} \Rightarrow B\delta_{\mathscr{G}}A. \tag{2.11}$$

Proof. ($\mathscr{G}P_1$) and Theorem 2.8 imply the result. □

Remark 2.10. The converse of Theorem 2.8 is not true. Let X be an infinite set,

$$\mathscr{G} = \mathscr{G}_{\inf} = \{A \subseteq X \mid A \,\text{is infinite}\}$$

be a grill on X and $\delta_{\mathscr{G}}$ be defined as in Example 2.3. If A, B are infinite subsets of X, then $A^{\delta_{\mathscr{G}}} = \phi$. Thus $B\ \overline{\delta}_{\mathscr{G}}A^{\delta_{\mathscr{G}}}$ but $B\delta_{\mathscr{G}}A$.

Lemma 2.11. *Let $(X, \delta_{\mathscr{G}})$ be a $\mathscr{G}$-proximity space. Then*

$$(A^{\delta_{\mathscr{G}}})^{\delta_{\mathscr{G}}} \subseteq A^{\delta_{\mathscr{G}}}. \tag{2.12}$$

Proof. Let $x \notin A^{\delta_{\mathscr{G}}}$. Then $x\overline{\delta}_{\mathscr{G}}A$. Thus, Theorem 2.8 implies that $x\overline{\delta}_{\mathscr{G}}A^{\delta_{\mathscr{G}}}$, i.e., $x \notin (A^{\delta_{\mathscr{G}}})^{\delta_{\mathscr{G}}}$. □

Proposition 2.12. *Let $(X, \delta_{\mathscr{G}})$ be a $\mathscr{G}$-proximity space, $A \subseteq X$ and $\mathscr{G} = \mathscr{G}_{\inf} \subseteq P(X)$. Then $A^{\delta_{\mathscr{G}}} = \phi$.*

Proof. Let $\mathscr{G} = \mathscr{G}_{\inf} \subseteq P(X)$. Then $\{x\} \notin \mathscr{G}$, for all $x \in X$. ($\mathscr{G}P_3$) implies that $x\overline{\delta}_{\mathscr{G}}A$. Thus it follows that $A^{\delta_{\mathscr{G}}} = \phi$. □

Theorem 2.13. *For a subset A of a space $(X, \delta_{\mathscr{G}})$, the following statements are valid:*

1. $A \cap B^{\delta_{\mathscr{G}}} = \phi$, *for every $A \notin \mathscr{G}$ and $B \subseteq X$,*
2. $x\delta_{\mathscr{G}}X$, *for all $x \in X \Leftrightarrow \mathscr{G} = P(X)\backslash\{\phi\}$.*

Proof. (1) Let $A \cap B^{\delta_{\mathscr{G}}} \neq \phi$ and $A \notin \mathscr{G}$. Then there exists an $x \in X$ such that $x \in A$ and $x\delta_{\mathscr{G}}B$. Thus lemma 2.5 implies that $A\delta_{\mathscr{G}}B$ which is a contradiction with ($\mathscr{G}P_3$). So $A \cap B^{\delta_{\mathscr{G}}} = \phi$. (2) Let $x\delta_{\mathscr{G}}X$, for all $x \in X$. Then ($\mathscr{G}P_3$) implies that $\{x\} \in \mathscr{G}$, for all $x \in X$. Thus $\mathscr{G} = P(X)\backslash\{\phi\}$. Conversely, $\mathscr{G} = P(X)\backslash\{\phi\}$ and ($\mathscr{G}P_4$) imply the result. □

3. $\mathscr{G}$-PROXIMIZABLE SPACES

Theorem 3.1. *Let $(X, \delta_{\mathscr{G}})$ be a $\mathscr{G}$-proximity space. Then the operator*
$$Cl^{\delta_{\mathscr{G}}} : P(X) \longrightarrow P(X)$$
defined by

$$Cl^{\delta_{\mathscr{G}}}(A) = A \cup A^{\delta_{\mathscr{G}}} \tag{3.1}$$

satisfies Kuratwski's axioms and induces a topology on X called $\tau_{\delta_{\mathscr{G}}}$ given by:
$$\tau_{\delta_{\mathscr{G}}} = \{A \subseteq X \mid Cl^{\delta_{\mathscr{G}}}(A^c) = A^c\}.$$

Proof. (1) By $(\mathscr{G}P_3)$ $\phi^{\delta_{\mathscr{G}}} = \phi$. Then $Cl^{\delta_{\mathscr{G}}}(\phi) = \phi$.
(2) (3.1) implies that $A \subseteq Cl^{\delta_{\mathscr{G}}}(A)$.
(3) By Theorem 2.6 part (2), we have $Cl^{\delta_{\mathscr{G}}}(A \cup B) = Cl^{\delta_{\mathscr{G}}}(A) \cup Cl^{\delta_{\mathscr{G}}}(B)$.
(4) By Theorem 2.6 part (1), we have

$$Cl^{\delta_{\mathscr{G}}}(A) \subseteq Cl^{\delta_{\mathscr{G}}}(Cl^{\delta_{\mathscr{G}}}(A)). \tag{3.2}$$

Then, it suffices to show that for every $A \subseteq X$, we have $Cl^{\delta_{\mathscr{G}}}(Cl^{\delta_{\mathscr{G}}}(A)) \subseteq Cl^{\delta_{\mathscr{G}}}(A)$ or equivalently that

$$\text{If}\, x \notin Cl^{\delta_{\mathscr{G}}}(A), \text{ then}\, x \notin Cl^{\delta_{\mathscr{G}}}(Cl^{\delta_{\mathscr{G}}}(A)). \tag{3.3}$$

Let $x \notin Cl^{\delta_{\mathscr{G}}}(A)$. Then $x \notin A$ and $x\overline{\delta}_{\mathscr{G}}A$. Theorem 2.8 implies that $x\overline{\delta}_{\mathscr{G}}A^{\delta_{\mathscr{G}}}$ and $(\mathscr{G}P_2)$ implies that $x\overline{\delta}_{\mathscr{G}}(A \cup A^{\delta_{\mathscr{G}}})$, i.e. , $x\overline{\delta}_{\mathscr{G}}Cl^{\delta_{\mathscr{G}}}(A)$. This result, combined with $x\overline{\delta}_{\mathscr{G}}A$ and (3.2), completes the proof. □

Theorem 3.2. *Let $(X, \delta_{\mathscr{G}})$ be a $\mathscr{G}$-proximity space. Then the closure operator defined in (3.1) has the following property:*

$$B\delta_{\mathscr{G}}A \Leftrightarrow B\delta_{\mathscr{G}}Cl^{\delta_{\mathscr{G}}}(A). \tag{3.4}$$

Proof. The result follows immediately by Theorem 2.8 and $(\mathscr{G}P_2)$. □

Theorem 3.3. *Let $(X, \delta_{\mathscr{G}})$ be a $\mathscr{G}$-proximity space. Then*

$$Cl^{\delta_{\mathscr{G}}}(A^{\delta_{\mathscr{G}}}) = A^{\delta_{\mathscr{G}}}, \tag{3.5}$$

i.e. $A^{\delta_{\mathscr{G}}}$ is $\tau_{\delta_{\mathscr{G}}}$-closed set.

Proof. We want to prove that $Cl^{\delta_{\mathscr{G}}}(A^{\delta_{\mathscr{G}}}) \subseteq A^{\delta_{\mathscr{G}}}$. Let $x \in Cl^{\delta_{\mathscr{G}}}(A^{\delta_{\mathscr{G}}})$. Then $x \in A^{\delta_{\mathscr{G}}}$ or $x\delta_{\mathscr{G}}A^{\delta_{\mathscr{G}}}$. It follows that $x \in (A^{\delta_{\mathscr{G}}})^{\delta_{\mathscr{G}}}$. Thus by lemma 2.11, we get $x \in A^{\delta_{\mathscr{G}}}$. □

Proposition 3.4. *Let $(X, \delta_{\mathscr{G}})$ be a $\mathscr{G}$-proximity space, $A \subseteq X$ and $\mathscr{G} = \mathscr{G}_{\inf} \subseteq P(X)$. Then $\tau_{\delta_{\mathscr{G}}} = P(X)$.*

Proof. The result follows immediately by proposition 2.12. □

Definition 3.5. A topological space (X, τ) is called a $\mathscr{G}$-normal space, if for every $F_1, F_2 \in \tau^{*c}$ such that $F_1 \cap F_2 \notin \mathscr{G}$, there exist $H, G \in \tau$ such that
$$F_1 \subseteq H,\ F_2 \subseteq G \text{ and } H \cap G \notin \mathscr{G},$$
where τ^{*c} is the family of all τ^*-closed sets.

Example 3.6. Let (X, τ) be a $\mathscr{G}$-normal space and $\delta_{\mathscr{G}}$ be a relation on $P(X)$ defined as:

$$A\delta_{\mathscr{G}}B \Leftrightarrow \Psi(A) \cap \Psi(B) \in \mathscr{G}, \quad \text{for every} \quad A, B \subseteq X. \tag{3.6}$$

Then $\delta_{\mathscr{G}}$ is a $\mathscr{G}$-proximity relation on X. It follows directly from (3.6) that $\delta_{\mathscr{G}}$ satisfies conditions ($\mathscr{G}P_1$)-($\mathscr{G}P_4$). To prove that $\delta_{\mathscr{G}}$ satisfies condition ($\mathscr{G}P_5$), let $A\overline{\delta}_{\mathscr{G}}B$. Then $\Psi(A) \cap \Psi(B) \notin \mathscr{G}$. Since $\Psi(A)$ satisfies Kuratwski's axioms, $\Psi(\Psi(A)) = \Psi(A)$, i.e. , $\Psi(A) \in \tau^{*c}$. Similarly, $\Psi(B) \in \tau^{*c}$. Since (X, τ) is a $\mathscr{G}$-normal space, it follows that there exist $H, G \in \tau$ such that $\Psi(A) \subseteq H$, $\Psi(B) \subseteq G$ and $H \cap G \notin \mathscr{G}$. Thus there exist $H, G \subseteq X$ such that $A\overline{\delta}_{\mathscr{G}}H^c$, $G^c\ \overline{\delta}_{\mathscr{G}}B$ and $H \cap G \notin \mathscr{G}$.

Definition 3.7. A topological space(X, τ) is called a $\mathscr{G}$-proximizable space, if there exists $\mathscr{G}$-proximity relation $\delta_{\mathscr{G}}$ such that $\tau_{\delta_{\mathscr{G}}} = \tau^*$. Moreover, $\delta_{\mathscr{G}}$ is said to be a compatible $\mathscr{G}$-proximity with τ^*.

Theorem 3.8. *Let $\mathscr{G}$ be a grill on a nonempty set X, (X, τ) be a $\mathscr{G}$-normal T_1 space and $\delta_{\mathscr{G}}$ be defined as in Example 3.6. Then (X, τ) is a $\mathscr{G}$-proximizable space.*

Proof. To prove the theorem, it suffices to show that the topology generated by the closure operator Ψ coincide with the topology generated by $Cl^{\delta_{\mathscr{G}}}$. In other words, we show that for every $A \subseteq X$,

$$\Psi(A) = Cl^{\delta_{\mathscr{G}}}(A). \tag{3.7}$$

Let $x \in Cl^{\delta_{\mathscr{G}}}(A)$. Then $x \in A$ or $x \in A^{\delta_{\mathscr{G}}}$.

If $x \in A$, then the result holds.

Now, if $x \in A^{\delta_{\mathscr{G}}}$, then $x\delta_{\mathscr{G}}A$. Thus $\Psi(\{x\}) \cap \Psi(A) \in \mathscr{G}$. Since (X, τ) is T_1 space and $\tau^c \subseteq \tau^{*c}$, $\{x\} \cap \Psi(A) \in \mathscr{G}$. So $x \in \Psi(A)$. Hence

$$Cl^{\delta_{\mathscr{G}}}(A) \subseteq \Psi(A). \tag{3.8}$$

Now, we want to prove that $\Psi(A) \subseteq Cl^{\delta_{\mathscr{G}}}(A)$ or equivalently, if $x \notin Cl^{\delta_{\mathscr{G}}}(A)$, then $x \notin \Psi(A)$. Let $x \notin Cl^{\delta_{\mathscr{G}}}(A)$. Then $x \notin A$ and $x \notin A^{\delta_{\mathscr{G}}}$. It follows that $x\overline{\delta}_{\mathscr{G}}A$. Thus (3.6) implies that $\Psi(\{x\}) \cap \Psi(A) \notin \mathscr{G}$. Since (X, τ) is $\mathscr{G}$-normal T_1 space and $\tau^c \subseteq \tau^{*c}$, there exist $H, G \in \tau$ such that

$$\{x\} \subseteq H, \Psi(A) \subseteq G \text{ and } H \cap G \notin \mathscr{G}. \tag{3.9}$$

By definition 1.1 part (ii) and (3.9), we get $H \cap A \notin \mathscr{G}$, i.e. , there exists an $H \in \tau$ such that $x \in H$ and $H \cap A \notin \mathscr{G}$. So $x \notin \Phi(A)$ and we have $x \notin A$. Hence $x \notin \Psi(A)$. It follows that

$$\Psi(A) \subseteq Cl^{\delta_{\mathscr{G}}}(A).$$

This result, combined with (3.8) and Definition 3.7, completes the Proof of the theorem. □

4. $\mathscr{G}$-proximal neighborhood structure and $\mathscr{G}$-proximity mapping

Definition 4.1. A subset B of a $\mathscr{G}$-proximity space $(X, \delta_{\mathscr{G}})$ is a $\delta_{\mathscr{G}}$-neighborhood of A (in symbols, $A \ll_{\mathscr{G}} B$), if $A\overline{\delta}_{\mathscr{G}}B^c$.

Theorem 4.2. *Let $(X, \delta_{\mathscr{G}})$ be a $\mathscr{G}$-proximity space. Then*
(1) *$A \ll_{\mathscr{G}} B$ implies $Cl^{\delta_{\mathscr{G}}}(A) \ll_{\mathscr{G}} B$,*
(2) *$A \ll_{\mathscr{G}} B$ implies $A \ll_{\mathscr{G}} int^{\delta_{\mathscr{G}}}(B)$,*
where $int^{\delta_{\mathscr{G}}}(B)$ is the interior of B with respect to $\tau_{\delta_{\mathscr{G}}}$.

Proof. (1) By using Theorem 3.2, $A\overline{\delta}_{\mathscr{G}}B^c$ implies $Cl^{\delta_{\mathscr{G}}}(A)\overline{\delta}_{\mathscr{G}}B^c$, i.e., $Cl^{\delta_{\mathscr{G}}}(A) \ll_{\mathscr{G}} B$.

(2) $A\overline{\delta}_{\mathscr{G}}B^c$ implies $A\overline{\delta}_{\mathscr{G}}Cl^{\delta_{\mathscr{G}}}(B^c)$. Equivalently, $A\overline{\delta}_{\mathscr{G}}(int^{\delta_{\mathscr{G}}}(B))^c$, i.e., $A \ll_{\mathscr{G}} int^{\delta_{\mathscr{G}}}(B)$. □

Theorem 4.3. *Let $(X, \delta_{\mathscr{G}})$ be a $\mathscr{G}$-proximity space. Then the relation $\ll_{\mathscr{G}}$ satisfies the following properties*
(1) *$X \ll_{\mathscr{G}} X$,*
(2) *$A \ll_{\mathscr{G}} B$ implies $A \cap B^c \notin \mathscr{G}$,*
(3) *$A \subseteq B \ll_{\mathscr{G}} C \subseteq D$ implies $A \ll_{\mathscr{G}} D$,*
(4) *$A \ll_{\mathscr{G}} B_i$, for $i = 1, 2, \dots, n$ iff $A \ll_{\mathscr{G}} \bigcap_{i=1}^{n} B$,*
(5) *$A \ll_{\mathscr{G}} B$ implies $B^c \ll_{\mathscr{G}} A^c$,*
(6) *if $A \notin \mathscr{G}$ or $B \notin \mathscr{G}$, then $A \ll_{\mathscr{G}} B^c$,*
(7) *$A \ll_{\mathscr{G}} B$ implies there exist $C, D \subseteq X$ such that*
$$A \ll_{\mathscr{G}} C,\ D^c \ll_{\mathscr{G}} B \text{ and } C \cap D \notin \mathscr{G},$$
(8) *if $\delta_{\mathscr{G}}$ is a separated $\mathscr{G}$-proximity, then $x \neq y \Rightarrow x \ll_{\mathscr{G}} \{y\}^c$.*

Proof. (1) ($\mathscr{G}P_3$) and Definition 1.1 implies that $A\overline{\delta}_{\mathscr{G}}\phi$, i.e., $X \ll_{\mathscr{G}} X$.

(2) Let $A \ll_{\mathscr{G}} B$. Then ($\mathscr{G}P_4$) implies $A \cap B^c \notin \mathscr{G}$.

(3) Suppose that $A \not\ll_{\mathscr{G}} D$. Then $A\ \delta_{\mathscr{G}}D^c$. Lemma 2.5 implies that $B\delta_{\mathscr{G}}C^c$, i.e., $B \not\ll_{\mathscr{G}} C$, which is a contradiction.

(4) It suffices to consider $n = 2$. $A \ll_{\mathscr{G}} B_1$ and $A \ll_{\mathscr{G}} B_2 \Leftrightarrow A\overline{\delta}_{\mathscr{G}}(B_1 \cap B_2)^c \Leftrightarrow A \ll_{\mathscr{G}} (B_1 \cap B_2)$.

(5) If $A \ll_{\mathscr{G}} B$, then $A\overline{\delta}_{\mathscr{G}}B^c$ and ($\mathscr{G}P_1$) implies $B^c\overline{\delta}_{\mathscr{G}}A$. Thus $B^c \ll_{\mathscr{G}} A^c$.

(6) Let $A \notin \mathscr{G}$. Then ($\mathscr{G}P_3$) implies $A\overline{\delta}_{\mathscr{G}}B$, i.e., $A \ll_{\mathscr{G}} B^c$. If $B \notin \mathscr{G}$, then similarly, $A \ll_{\mathscr{G}} B^c$.

(7) $A \ll_{\mathscr{G}} B$ implies $A\overline{\delta}_{\mathscr{G}}B^c$. ($\mathscr{G}P_5$) implies there exist $C, D \subseteq X$ such that $A\overline{\delta}_{\mathscr{G}}C^c$, $B^c\overline{\delta}_{\mathscr{G}}D^c$ and $C \cap D \notin \mathscr{G}$, i.e. $A \ll_{\mathscr{G}} C$, $D^c \ll_{\mathscr{G}} B$ and $C \cap D \notin \mathscr{G}$.

(8) $x \neq y$ implies $x\overline{\delta}_{\mathscr{G}}y$, by ($\mathscr{G}P_6$), i.e., $x \ll_{\mathscr{G}} \{y\}^c$. □

Corollary 4.4. *$A_i \ll_{\mathscr{G}} B_i$ for $i = 1, 2, \dots, n$ implies $\bigcap_{i=1}^{n} A_i \ll_{\mathscr{G}} \bigcap_{i=1}^{n} B_i$ and $\bigcup_{i=1}^{n} A_i \ll_{\mathscr{G}} \bigcup_{i=1}^{n} B_i$*

Theorem 4.5. *If $\ll_{\mathscr{G}}$ is a binary relation on X satisfying (1)-(7) in Theorem 4.3 and $\delta_{\mathscr{G}}$ is defined by*

$$A\overline{\delta}_{\mathscr{G}}B \Leftrightarrow A \ll_{\mathscr{G}} B^c, \tag{4.1}$$

then $\delta_{\mathscr{G}}$ is a $\mathscr{G}$-proximity relation on X. B is a $\delta_{\mathscr{G}}$-neighborhood of A if and only if $A \ll_{\mathscr{G}} B$. Moreover, if $\ll_{\mathscr{G}}$ also satisfies (8) in Theorem 4.3, then $\delta_{\mathscr{G}}$ is separated.

Proof. ($\mathscr{G}P_1$) $A\ \overline{\delta}_{\mathscr{G}}B$ implies $A \ll_{\mathscr{G}} B^{\ c}$. Then by Theorem 4.3 part 5), $B \ll_{\mathscr{G}} A^c$. Thus $B\ \overline{\delta}_{\mathscr{G}}A$.

($\mathscr{G}P_2$) $(A \cup B)\overline{\delta}_{\mathscr{G}}C$ implies $(A \cup B) \ll_{\mathscr{G}} C^c$. Then by Theorem 4.3) part (3), $A \ll_{\mathscr{G}} C^c$ and $B \ll_{\mathscr{G}} C^c$, i.e., $A\overline{\delta}_{\mathscr{G}}C$ and $B\overline{\delta}_{\mathscr{G}}C$.

Conversely if $A\overline{\delta}_{\mathscr{G}}C$ and $B\overline{\delta}_{\mathscr{G}}C$, then by ($\mathscr{G}P_1$), $C\overline{\delta}_{\mathscr{G}}A$ and $C\overline{\delta}_{\mathscr{G}}B$, that is, $C \ll_{\mathscr{G}} A^c$ and $C \ll_{\mathscr{G}} B^c$. Thus by Theorem 4.3 part (4), $C \ll_{\mathscr{G}} (A^c \cap B^c)$, i.e., $C \ll_{\mathscr{G}} (A \cup B)^c$. So $C\overline{\delta}_{\mathscr{G}}(A \cup B)$.

($\mathscr{G}P_3$) Let $A \notin \mathscr{G}$. Then by Theorem 4.3 part (6), $A \ll_{\mathscr{G}} B^c$, i.e., $A\overline{\delta}_{\mathscr{G}}B$. If $B \notin \mathscr{G}$, then similarly, $A\overline{\delta}_{\mathscr{G}}B$.

($\mathscr{G}P_4$) $A\overline{\delta}_{\mathscr{G}}B$ implies $A \ll_{\mathscr{G}} B^c$. By Theorem 4.3 part (2), $A \cap B \notin \mathscr{G}$.

($\mathscr{G}P_5$) Suppose $A\overline{\delta}_{\mathscr{G}}B$, i.e. $A \ll_{\mathscr{G}} B^c$. Then by Theorem 4.3 part (7), there exist $C, D \subseteq X$ such that $A \ll_{\mathscr{G}} C$, $D^c \ll_{\mathscr{G}} B^c$ and $C \cap D \notin \mathscr{G}$. Thus there exist $C, D \subseteq X$ such that $A\overline{\delta}_{\mathscr{G}}C^c$, $D^c\overline{\delta}_{\mathscr{G}}B$ and $C \cap D \notin \mathscr{G}$. □

Theorem 4.6. *If $A \ll_{\mathscr{G}} B$ and $B \notin \mathscr{G}$, then $A \notin \mathscr{G}$.*

Proof. $A \ll_{\mathscr{G}} B$ implies $A\overline{\delta}_{\mathscr{G}}B^c$. Then by ($\mathscr{G}P_4$), $B \notin \mathscr{G}$ and by Definition 1.1 part (iii), we have $(A \cap B^c) \cup B \notin \mathscr{G}$, i.e., $A \cup B \notin \mathscr{G}$. Thus by Definition 1.1 part (ii), $A \notin \mathscr{G}$. □

Theorem 4.7. *$A \ll_{\mathscr{G}} B$ for every $B \subseteq X$ if and only if $A \notin \mathscr{G}$*

Proof. Let $A \ll_{\mathscr{G}} B$, for every $B \subseteq X$. Then $A \ll_{\mathscr{G}} \phi$, i.e., $A\overline{\delta}_{\mathscr{G}}X$. Thus by ($\mathscr{G}P_4$), $A \notin \mathscr{G}$.

Conversely, if $A \notin \mathscr{G}$, then ($\mathscr{G}P_3$) implies that $A\overline{\delta}_{\mathscr{G}}B^c$, for every $B \subseteq X$. Thus $A \ll_{\mathscr{G}} B$, for every $B \subseteq X$. □

Lemma 4.8. *Let $(X, \delta_{\mathscr{G}})$ be a $\mathscr{G}$-proximity space, A, B and $C \subseteq X$ such that $A\overline{\delta}_{\mathscr{G}}B$ and $(B^c \cap C) \notin \mathscr{G}$. Then $A\overline{\delta}_{\mathscr{G}}C$.*

Proof. Since $(B^c \cap C) \notin \mathscr{G}$, ($\mathscr{G}P_3$) implies $A\overline{\delta}_{\mathscr{G}}(B^c \cap C)$ and we have $A\overline{\delta}_{\mathscr{G}}B$. Then $A\overline{\delta}_{\mathscr{G}}(B \cup C)$, by ($\mathscr{G}P_2$). Thus $A\overline{\delta}_{\mathscr{G}}C$. □

Theorem 4.9. *Let $\mathscr{G}$ be a grill on a nonempty set X, $\delta_{\mathscr{G}}$ be a $\mathscr{G}$-proximity relation on X and (X, τ) be a $\mathscr{G}$-normal T_1 space such that $\tau^* = \tau_{\delta_{\mathscr{G}}}$. If A is compact with respect to τ^*, B is closed set in τ^* and $A \cap B \notin \mathscr{G}$, then $A\overline{\delta}_{\mathscr{G}}B$.*

Proof. For all $a \in A$, if $a \in B$, then $\{a\} \notin \mathscr{G}$. Thus, ($\mathscr{G}P_3$) implies $a\overline{\delta}_{\mathscr{G}}B$. Also, if $a \notin B$ and B is closed, then $a\overline{\delta}_{\mathscr{G}}B$. This result implies that there exist $C, D \subseteq X$ such that $a\overline{\delta}_{\mathscr{G}}C^c$, $D^c\overline{\delta}_{\mathscr{G}}B$ and $C \cap D \notin \mathscr{G}$. This result and Lemma 4.8 imply $C\overline{\delta}_{\mathscr{G}}B$, i.e., $C \ll_{\mathscr{G}} B^C$. So we have $a \ll_{\mathscr{G}} C$ and $C \ll_{\mathscr{G}} B^c$. By Theorem 4.2 part (2), $a \ll_{\mathscr{G}} int^{\delta_{\mathscr{G}}}(C) \subseteq C \ll_{\mathscr{G}} B^{\ c}$. Let $N_a = int^{\delta_{\mathscr{G}}}(C)$. Then $N_a\overline{\delta}_{\mathscr{G}}B$.

On the other hand, $\{N_a : a \in A\}$ is an open cover of the compact set A. Then there is a finite subcover $\{N_{a_i} : i = 1, 2, \ldots, n\}$. Thus by ($\mathscr{G}P_2$), $N\overline{\delta}_{\mathscr{G}}B$, where $N = \bigcup_{i=1}^{n} N_{a_i}$. But $A \subset N$. So $A\ \overline{\delta}_{\mathscr{G}}B$. □

Theorem 4.10. *Let (X, τ) be a $\mathscr{G}$-normal T_1 and let X be compact with respect to τ^*. Then the space (X, τ) has a unique compatible $\mathscr{G}$-proximity defined as:*

$$A\delta_{\mathscr{G}}B \Leftrightarrow \Psi(A) \cap \Psi(B) \in \mathscr{G},\ \textit{for every } A, B \subseteq X$$

Proof. We proved that $\tau^* = \tau_{\delta_{\mathscr{G}}}$ in Theorem 3.8. Then $\delta_{\mathscr{G}}$ is a compatible $\mathscr{G}$-proximity with τ^*. Thus, it remains to show that $\delta_{\mathscr{G}}$ is unique. Let $\alpha_{\mathscr{G}}$ be any compatible $\mathscr{G}$-proximity and $A\delta_{\mathscr{G}}B$. Then $\Psi(A) \cap \Psi(B) \in \mathscr{G}$. Thus by ($\mathscr{G}P_4$), Theorem 3.2 and ($\mathscr{G}P_1$), we get $A\alpha_{\mathscr{G}}B$. To prove the other inclusion, suppose that $A\overline{\delta}_{\mathscr{G}}B$. Then $\Psi(A) \cap \Psi(B) \notin \mathscr{G}$. Since closed subsets of a compact space are compact. Then Theorem 4.9 implies $A\overline{\alpha}_{\mathscr{G}}B$. Thus the result holds. □

Definition 4.11. Let $(X, \delta_{\mathscr{G}_1})$ and $(Y, \delta_{\mathscr{G}_2})$ be two $\mathscr{G}$-proximity spaces. A function $f : X \longrightarrow Y$ is said to be a $\mathscr{G}$-proximity mapping, if

$$A\delta_{\mathscr{G}_1}B \Rightarrow f(A)\delta_{\mathscr{G}_2}f(B). \tag{4.2}$$

Equivalently f is a $\mathscr{G}$-proximity mapping iff

$$C\overline{\delta}_{\mathscr{G}_2}D \Rightarrow f^{-1}(C)\overline{\delta}_{\mathscr{G}_1}f^{-1}(D)$$

Theorem 4.12. *A $\mathscr{G}$-proximity mapping $f : (X, \delta_{\mathscr{G}_1}) \longrightarrow (Y, \delta_{\mathscr{G}_2})$ is continuous with respect to $\tau(\delta_{\mathscr{G}_1})$ and $\tau(\delta_{\mathscr{G}_2})$.*

Proof. Since f is a $\mathscr{G}$-proximity mapping, if $x\ \delta_{g_1}A$, then we have $f(x)\ \delta_{g_2}\ f(A)$, i.e., $f(A^{\delta_{\mathscr{G}_1}}) \subseteq (f(A))^{\delta_{\mathscr{G}_2}}$. Thus

$$f(Cl_{\delta_{\mathscr{G}_1}}(A)) = f(A) \cup f(A^{\delta_{\mathscr{G}_1}}) \subseteq f(A) \cup (f(A))^{\delta_{\mathscr{G}_2}} = Cl_{\delta_{\mathscr{G}_2}}(f(A)).$$

So the result holds. □

Remark 4.13. The converse of the foregoing theorem is not true in general. Let $\mathscr{G} = P(X)\backslash\{\Phi\}$, then the continuous function is not necessary to be a proximity mapping [9].

Theorem 4.14. *Let $\mathscr{G}$ be a grill on a nonempty set X, $f : X \longrightarrow Y$ be an onto function, $(X, \delta_{\mathscr{G}})$ and $(Y, \delta_{f(\mathscr{G})})$ be two proximity spaces, and (X, τ) be a $\mathscr{G}$-normal T_1 space. If X is compact with respect to τ^*, then every continuous function $f : (X, \delta_{\mathscr{G}}) \longrightarrow (Y, \delta_{f(\mathscr{G})})$ is a $\mathscr{G}$-proximity mapping.*

Proof. Let $A, B \subseteq X$ such that $A\delta_{\mathscr{G}}B$. Then $\Psi(A) \cap \Psi(B) \in \mathscr{G}$, by Theorem 4.10. Thus $f(\Psi(A)) \cap f(\Psi(B)) \in f(\mathscr{G})$. ($\mathscr{G}P_4$) implies that $f(\Psi(A))\delta_{f(\mathscr{G})}f(\Psi(B))$. Since f is continuous, $f(\Psi(A)) \subseteq Cl_{\delta_{f(\mathscr{G})}}(f(A))$ and $f(\Psi(B)) \subseteq Cl_{\delta_{f(\mathscr{G})}}(f(B))$. So $Cl_{\delta_{f(\mathscr{G})}}(f(A))\ \delta_{f(\mathscr{G})}Cl_{\delta_{f(\mathscr{G})}}(f(B))$. From Theorem 3.2 and ($\mathscr{G}P_1$), it follows that $f(A)\delta_{f(\mathscr{G})}f(B)$. Hence f is a $\mathscr{G}$-proximity mapping. □

Remark 4.15. A function f is said to be $\mathscr{G}$-proximally continuous mapping if it is $\mathscr{G}$-proximity mapping.

Theorem 4.16. *Let $\mathscr{G}$ be a grill on a set X, $f : X \longrightarrow Y$ be an onto function, and $(Y,\ \delta_{f(\mathscr{G})})$ be a $\mathscr{G}$-proximity space. The largest $\mathscr{G}$-proximity $\delta_{\mathscr{G}}$ which may be assigned to X such that $f : X \longrightarrow (Y,\ \delta_{f(\mathscr{G})})$ is a $\mathscr{G}$-proximally continuous is defined by*

$$A\overline{\delta}_{\mathscr{G}}B \Leftrightarrow \text{there exists a } C \subseteq Y \text{ such that } f(A)\overline{\delta}_{f(\mathscr{G})}Y - C \text{ and } f(B) \cap C \notin f(\mathscr{G}) \tag{4.3}$$

Proof. We first verify that $\delta_{\mathscr{G}}$ is a $\mathscr{G}$-proximity on X.

($\mathscr{G}P_1$) Suppose $A\overline{\delta}_{\mathscr{G}}B$. Then there exists a $C \subseteq Y$ such that

$$f(A)\overline{\delta}_{f(\mathscr{G})}Y - C \text{ and } f(B) \cap C \notin f(\mathscr{G}).$$

Thus by Lemma 4.8, $f(B)\overline{\delta}_{f(\mathscr{G})}f(A)$. Let $D = Y - f(A)$. Since $f(B)\overline{\delta}_{f(\mathscr{G})}f(A)$ and $f(A) \cap D = \phi \notin \mathscr{G}$, $B\overline{\delta}_{\mathscr{G}}A$.

($\mathscr{G}P_2$) $(A \cup B)\overline{\delta}_{\mathscr{G}}C$ implies there exists a $D \subseteq Y$ such that

$$(f(A) \cup f(B))\overline{\delta}_{f(\mathscr{G})}Y - D \text{ and } f(C) \cap D \notin f(\mathscr{G}).$$

Then by ($\mathscr{G}P_2$), we have $A\overline{\delta}_{\mathscr{G}}C$ and $B\overline{\delta}_{\mathscr{G}}C$.

Conversely, suppose $A\overline{\delta}_{\mathscr{G}}C$ and $B\overline{\delta}_{\mathscr{G}}C$. Then there exist $D_1, D_2 \subseteq Y$ such that $f(A)\overline{\delta}_{f(\mathscr{G})}Y - D_1$, $f(B)\overline{\delta}_{f(\mathscr{G})}Y - D_2$, $f(C) \cap D_1 \notin f(\mathscr{G})$ and $f(C) \cap D_2 \notin f(\mathscr{G})$. Thus by ($\mathscr{G}P_2$) and Definition 1.1 part (iii), we have

$$(f(A) \cup f(B))\overline{\delta}_{f(\mathscr{G})}Y - (D_1 \cup D_2) \text{ and } f(C) \cap (D_1 \cup D_2) \notin f(\mathscr{G}).$$

So $(A \cup B)\overline{\delta}_{\mathscr{G}}C$.

($\mathscr{G}P_3$) Suppose $A \notin \mathscr{G}$ and let $C = \phi \subseteq Y$. Since $f(A)\overline{\delta}_{f(\mathscr{G})}Y$ and $f(B) \cap C = \phi \notin f(\mathscr{G})$, $A\overline{\delta}_{\mathscr{G}}B$.

($\mathscr{G}P_4$) Suppose $A\overline{\delta}_{\mathscr{G}}B$. Then there exists a $C \subseteq Y$ such that

$$f(A)\overline{\delta}_{f(\mathscr{G})}Y - C \text{ and } f(B) \cap C \notin f(\mathscr{G}).$$

Thus by Lemma 4.8, $f(A)\overline{\delta}_{f(\mathscr{G})}f(B)$. So $f(A) \cap f(B) \notin f(\mathscr{G})$, by ($\mathscr{G}P_4$) . Since $f(A \cap B) \subseteq f(A) \cap f(B)$, $f(A \cap B) \notin f(\mathscr{G})$, by Definition 1.1 part (2). Hence $A \cap B \notin g$.

($\mathscr{G}P_5$) Suppose $A\overline{\delta}_{\mathscr{G}}B$. Then there exists a $C \subseteq Y$ such that

$$f(A)\overline{\delta}_{f(\mathscr{G})}Y - C \text{ and } f(B) \cap C \notin f(\mathscr{G}).$$

Thus by ($\mathscr{G}P_5$), there exist $D_1, D_2 \subseteq Y$ such that

$$f(A)\overline{\delta}_{f(\mathscr{G})}Y - D_1,\ Y - D_2\overline{\delta}_{f(\mathscr{G})}Y - C \text{ and } D_1 \cap D_2 \notin f(\mathscr{G}).$$

Let $E = f^{-1}(D_1)$ and $F = f^{-1}(D_2)$. Since $f(A)\overline{\delta}_{f(\mathscr{G})}Y - D_1$ and $f(X - E) \cap D_1 = \phi \notin f(\mathscr{G})$, $A\overline{\delta}_{\mathscr{G}}(X - E)$.

On the other hand, $f(X - F) \subseteq Y - D_2\overline{\delta}_{f(\mathscr{G})}Y - C$ and $f(B) \cap C \notin f(\mathscr{G})$. Then $(X - F)\overline{\delta}_{f(\mathscr{G})}B$. Thus there exist $E, F \subseteq X$ such that

$$A\ \overline{\delta}_{\mathscr{G}}(X - E),\ (X - F)\ \overline{\delta}_{f(\mathscr{G})}B \text{ and } E \cap F \notin \mathscr{G}.$$

To prove that $f : (X, \delta_{\mathscr{G}}) \longrightarrow (Y, \delta_{f(\mathscr{G})})$ is $\mathscr{G}$-proximally continuous, suppose that $A, B \subseteq X$ such that $f(A)\overline{\delta}_{f(\mathscr{G})}f(B)$. Then by ($\mathscr{G}P_5$), there exist $C, D \subseteq Y$ such that

$$f(A)\overline{\delta}_{f(\mathscr{G})}Y - C,\ Y - D\overline{\delta}_{f(\mathscr{G})}f(B) \text{ and } C \cap D \notin f(\mathscr{G}).$$

Thus by Lemma 4.8, we have $C\overline{\delta}_{f(\mathscr{G})}f(B)$. So $f(B) \cap C \notin f(\mathscr{G})$, by ($\mathscr{G}P_4$). This result and $f(A)\overline{\delta}_{f(\mathscr{G})}Y - C$ imply $A\overline{\delta}_{\mathscr{G}}B$.

It remains to show that $\delta_{\mathscr{G}}$ is the largest $\mathscr{G}$-proximity relation on X. Let $\alpha_{\mathscr{G}}$ be any $\mathscr{G}$-proximity such that $f : (X, \alpha_{\mathscr{G}}) \longrightarrow (Y, \delta_{f(\mathscr{G})})$ is $\mathscr{G}$-proximally continuous and $A\overline{\delta}_{\mathscr{G}}B$. Then there exists a $C \subseteq Y$ such that $f(A)\overline{\delta}_{f(\mathscr{G})}Y - C$ and $f(B) \cap C \notin f(\mathscr{G})$. Thus by Lemma 4.8, $f(A)\overline{\delta}_{f(\mathscr{G})}f(B)$. Since f is $\mathscr{G}$-proximally continuous, $A\overline{\alpha}_{\mathscr{G}}B$. So the result holds. $\square$

5. Conclusions

Proximity is a very important structure, since it related to many topics in topological spaces as compactifications and extension problems etc. In this paper we have presented a new structure of proximity spaces based on the grill notion. For $\mathscr{G} = P(X) \setminus \{\phi\}$, we have the Efremovič proximity structure and for the other types of $\mathscr{G}$, we have many types of proximity structures. Some of the important results are : every $\mathscr{G}$-normal T_1space is $\mathscr{G}$-proximizable space and has a unique compatible $\mathscr{G}$-proximity under the condition that X is compact relative to τ^*. Also, for a surjective map $f : X \longrightarrow (Y, \delta_{f(\mathscr{G})})$, we established the largest $\mathscr{G}$-proximity $\delta_{\mathscr{G}}$ on X for which f is a $\mathscr{G}$-proximally continuous. Finally, The notion of $\delta_{\mathscr{G}}$-neighborhood structure and $\mathscr{G}$-proximity mapping have been investigated.

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