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Abstract. In this paper, we establish some fixed point theorems for φ-contraction in Menger probabilistic generalized metric spaces by introducing a new type of gauge function. With this introduced guage function, we discuss several important lemmas to prove our main results. An example is given in the support of obtained results.

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1. Introduction

In 1942, the concept of probabilistic metric space was initiated by Menger [13]. Probabilistic metric space (briefly, PM-space) is a generalization of metric space in which distance between two points x and y, $d(x, y)$ is assigned by a distribution function $F_{x,y}$. Since then, many researchers extensively developed and expanded the study of PM-spaces in their pioneering works e.g., [2, 10, 18, 19, 20, 21, 22, 23, 25].

To prove existence and uniqueness of fixed point theorems in PM-spaces, contraction is one of the basic tools. Sehgal and Bharucha-Reid [20] introduced probabilistic k-contraction and proved probabilistic version of classical Banach fixed point principle. After that Ciric [5] generalizes the k-contraction and introduced the concept of φ-contraction in PM-space. In spite of the fact that probabilistic φ-contractions are a natural generalization of probabilistic k-contractions, the techniques used to prove the existence and uniqueness of fixed point results for probabilistic k-contractions are no longer usable for probabilistic φ-contractions. In 2010, Ciric [5] presented a fixed point theorem for probabilistic φ-contractions. Soon after the publication of Ciric's paper, Jachymski [11] found a counterexample to the key lemma in [5], and established a modified version of Ciric's φ-function. Recently, Fang [9] further weakened the conditions on φ-function.

In 2006, Mustafa and Sims[16] introduced the notion of a generalized metric space. After that many authors obtained several fixed point theorems for mappings satisfying different contractive conditions in generalized metric spaces (see, [7, 14, 15, 17]). In 2014, Zhou et al.[26] introduced the concept of a generalized probabilistic metric space (briefly, a PGM-space). After that Zhu et al.[27] obtained some fixed point theorems in PGM-spaces. For some very recent results in PGM-spaces, we refer [1, 3, 4, 6, 8, 12, 24].

The purpose of this work is to introduce a new class of φ-function and to establish several important results with the help of this function. We prove the existence and uniqueness of a fixed point for φ-contraction in Menger PGM-space. Finally an example is given to illustrate our main results.

2. Preliminaries

Throughout this paper, let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{N}$ be the set of all natural numbers.

Definition 2.1 ([13])**.** A mapping $F : \mathbb{R} \to [0, 1]$ is called a distribution function if it is non-decreasing and left-continuous with $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$.

Definition 2.2 ([18])**.** A continuous t-norm T is a binary operation on $[0, 1]$ which satisfies the following conditions:

(T-1) T is associative and commutative,
(T-2) T is continuous,
(T-3) $T(a, 1) = a$, for all $a \in [0, 1]$,
(T-4) $T(a, b) \le T(c, d)$, whenever $a \le c$ and $b \le d$, for each $a, b, c, d \in [0, 1]$.

Definition 2.3 ([13])**.** A Menger PM-space is a triplet (X, F, T), where X is a non-empty set, T is a t-norm and $F : X \times X \to D^+$ be a mapping satisfying the following conditions (for $x, y \in X$, we denote $F(x, y)$ by $F_{x,y}$):

(PM-1) $F_{x,y}(t) = H(t)$, for all $x, y \in X$ and $t > 0$ if and only $x = y$,
(PM-2) $F_{x,y}(t) = F_{y,x}(t)$, for all $x, y \in X$ and $t > 0$,
(PM-3) $F_{x,y}(s + t) \ge T(F_{x,z}(s), F_{z,y}(t))$, for all $x, y, z \in X$ and $s, t > 0$.

Definition 2.4 ([26])**.** The triplet (X, G, T) is called Menger probabilistic generalized metric space (briefly, a Menger PGM-space) if X is a non-empty set, T is a continuous t-norm and $G : X \times X \times X \to D^+$ be a mapping satisfying the following conditions:

(PGM-1) $G_{x,y,z}(t) = 1$, for all $x, y, z \in X$ and $t > 0$ if and only if $x = y = z$,
(PGM-2) $G_{x,x,y}(t) \ge G_{x,y,z}(t)$, for all $x, y \in X$ with $z \ne y$ and $t > 0$,
(PGM-3) $G_{x,y,z}(t) = G_{p(x,y,z)}(t)$, where p is a permutation function,
(PGM-4) $G_{x,y,z}(t+s) \ge T(G_{x,a,a}(s), G_{a,y,z}(t))$, for all $x, y, z, a \in X$ and $s, t > 0$.

Example 2.5 ([26])**.** Let (X, F, T) be a PM-space. Define a function $G : X \times X \times X \to D^+$ by $G_{x,y,z}(t) = \min\{F_{x,y}(t), F_{y,z}(t), F_{x,z}(t)\}$, for all $x, y, z \in X$ and $t > 0$. Then (X, G, T) is a PGM-space.

Example 2.6. Let (X, d) be a metric space. If we define

$$G_{x,y,z}(t) = \left(\frac{t}{1+t}\right)^{d(x,y)+d(y,z)+d(z,x)}$$

and we choose t-norm as product t-norm defined by

$$T_p(a,b) = a.b, \forall a,b \in [0,1].$$

Then (X, G, T_p) is a Menger PGM-space. In fact, $G_{x,y,z}(0) = 0$. Also, $\sup_{t>0} G_{x,y,z}(t) = 1$, and $G_{x,y,z}(t)$ is non-decreasing and continuous in t. Therefore, $G_{x,y,z}(t)$ is a distribution function. By the definition of $G_{x,y,z}(t)$, it is obvious that (PGM-1) and (PGM-3) in Definition 2.5 hold.

Next we will show that (PGM-2) and (PGM-4) also hold. Since $d(x,y) \leq d(x,z) + d(z,y)$, $\forall x,y,z \in X$, with $y \neq z$, we have that

$$d(x,y) + d(x,y) \leq d(x,y) + d(x,z) + d(z,y).$$

Then

$$\left(\frac{t}{1+t}\right)^{d(x,y)+d(x,y)} \geq \left(\frac{t}{1+t}\right)^{d(x,y)+d(y,z)+d(z,x)}.$$

Thus, $G_{x,x,y}(t) \geq G_{x,y,z}(t)$, for all $x,y,z \in X$ with $y \neq z$, and $t > 0$. By the definition of $G_{x,y,z}(t)$, we get

$$G_{x,y,z}(t+s) = \left(\frac{t+s}{1+t+s}\right)^{d(x,y)+d(y,z)+d(z,x)}.$$

Since, $\frac{t}{1+t}$ is strictly increasing on $[0,1)$, we have

$$\begin{aligned}\left(\frac{t+s}{1+t+s}\right)^{d(x,y)+d(y,z)+d(z,x)} &\geq \left(\frac{t+s}{1+t+s}\right)^{d(x,a)+d(a,y)+d(y,z,)+d(z,a)+d(a,x)} \\ &= \left(\frac{t+s}{1+t+s}\right)^{d(x,a)+d(a,x)} \left(\frac{t+s}{1+t+s}\right)^{d(a,y)+d(y,z)+d(z,a)} \\ &\geq \left(\frac{t}{1+t}\right)^{d(x,a)+d(x,a)} \left(\frac{s}{1+s}\right)^{d(a,y)+d(y,z)+d(z,a)} \\ &= T_P\left(\left(\frac{t}{1+t}\right)^{d(x,a)+d(x,a)}, \left(\frac{s}{1+s}\right)^{d(a,y)+d(y,z)+d(z,a)}\right).\end{aligned}$$

This implies that $G_{x,y,z}(t+s) \geq T_P(G_{x,a,a}(t), G_{a,y,z}(s))$. So, (X, G, T_P) is a Menger PGM-space.

Definition 2.7 ([26])**.** Let (X, G, T) be a Menger PGM-space.

(i) A sequence (x_n) in (X, G, T) is said to be convergent to a point $x \in X$, written as $x_n \to x$, if given $\epsilon > 0$, $\lambda > 0$ we can find $N_{\epsilon,\lambda} \in \mathbb{N}$ such that for all $n \geq N_{\epsilon,\lambda}$, $G_{x,x_n,x_n}(\epsilon) \geq 1 - \lambda$ holds.

(ii) A sequence (x_n) in (X, G, T) is called a Cauchy sequence, if for any given $\varepsilon > 0$ and $\lambda \in (0,1]$ there exists $N_{\epsilon,\lambda} \in \mathbb{N}$ such that $G_{x_n,x_m,x_l}(\varepsilon) \geq 1 - \lambda$, whenever $m,n,l \geq N_{\epsilon,\lambda}$.

(iii) A Menger PGM-space (X, G, T) is said to be complete, if every Cauchy sequence (x_n) in X is convergent to some point $x \in X$.

Definition 2.8 ([1])**.** Let (X, G, T) be a Menger PGM-space with a continuous t-norm T. A mapping $f : X \to X$ is said to be a probabilistic φ-contraction, if there exists a function $\varphi \in \Phi$ such that $G_{fx,fy,fz}(\varphi(t)) \geq G_{x,y,z}(t)$, for all $x, y, z \in X$ and $t > 0$.

Definition 2.9 ([10]). A t-norm T is said to be of H-type, if the family $\{T^p\}_{p\in\mathbb{N}}$ of its iterates defined for each $t \in (0,1)$ by $T^0(t) = 1,\ \ T^m(t) = T(t, T^{m-1}(t))$ for all $m \in \mathbb{N}$ is equi continuous at $t = 1$.

Definition 2.10. We define the class of a function Φ as follows: Φ contains all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $t > 0$ there exists $r > t$ such that $\varphi(r) \leq t$. An example of this type of function is $\varphi : [0,\infty) \to [0,\infty)$ defined as:

$$\varphi(t) = \begin{cases} 0 & \text{if } t = 0 \\ \frac{t}{4^n} & \text{if } \frac{1}{4^n} \leq t < \frac{1}{4^{n-1}} \\ kt & \text{if } t \geq 1, \text{where } 0 < k < 1. \end{cases}$$

3. Major Section

Lemma 3.1. *Suppose that the sequence $\{G_{x_n,x_{n+1},x_{n+1}}(t_m)\}$ is non-decreasing in both the variables m and n, i.e., $G_{x_n,x_{n+1},x_{n+1}}(t_m) \geq G_{x_{n-1},x_n,x_n}(t_m)$ and $G_{x_n,x_{n+1},x_{n+1}}(t_{m+1}) \geq G_{x_n,x_{n+1},x_{n+1}}(t_m)$, for each $m, n \in \mathbb{N}$. Then*

$$\mathit{lim}_{n\to\infty}\left(\mathit{lim}_{m\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_m)\right) = \mathit{lim}_{m\to\infty}\left(\mathit{lim}_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_m)\right).$$

Proof. Denote $a_n = \mathit{lim}_{m\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_m)$ and $b_m = \mathit{lim}_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_m)$. Then the existence and finiteness of both the limits can be obtained by monotonicity and boundedness of distribution functions, respectively.

Since $G_{x_n,x_{n+1},x_{n+1}}(t_m) \geq G_{x_{n-1},x_n,x_n}(t_m)$, we have

$$b_m = \lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_m) \geq \lim_{n\to\infty} G_{x_{n-1},x_n,x_n}(t_m) = b_{m-1},$$

i.e., the sequence $\{b_m\}$ is non-decreasing.

Similarly, we can show that the sequence $\{a_n\}$ is non-decreasing.

Put $a = \lim_{n\to\infty} a_n$ and $b = \lim_{m\to\infty} b_m$. Now,

$G_{x_n,x_{n+1},x_{n+1}}(t_m) \leq b_m$
$\Rightarrow \lim_{m\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_m) \leq \lim_{m\to\infty} b_m$
$\Rightarrow a_n \leq b$
$\Rightarrow \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b$
$\Rightarrow a \leq b$.

Similarly, $G_{x_n,x_{n+1},x_{n+1}}(t_m) \leq a_n$
$\Rightarrow \lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_m) \leq \lim_{n\to\infty} a_n$
$\Rightarrow b_m \leq a$
$\Rightarrow \lim_{m\to\infty} b_m \leq \lim_{n\to\infty} a$
$\Rightarrow b \leq a$.

Then, $a = b$, i.e,

$$\mathit{lim}_{n\to\infty}\left(\mathit{lim}_{m\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_m)\right) = \mathit{lim}_{m\to\infty}\left(\mathit{lim}_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_m)\right).$$

□

Lemma 3.2. *Let (X, G, T) be a Menger PGM-space with a t-norm T. Let $\{x_n\}$ be a sequence in (X, G, T). If there exists a function $\varphi \in \Phi$ such that $G_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) \geq G_{x_{n-1},x_n,x_n}(t)$, for all $n \in \mathbb{N}$ and $t > 0$, then $\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t) = 1$.*

Proof. Let $t_0 > 0$ be arbitrary. Since $\varphi \in \Phi$, there exists $t_1 > t_0$ such that $\varphi(t_1) \leq t_0$. Now, since $G_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) \geq G_{x_{n-1},x_n,x_n}(t)$, by the monotonic increasing property of distribution function, we have

$$\begin{aligned} G_{x_n,x_{n+1},x_{n+1}}(t_1) &\geq G_{x_n,x_{n+1},x_{n+1}}(t_0) \\ &\geq G_{x_n,x_{n+1},x_{n+1}}(\varphi(t_1)) \\ &\geq G_{x_{n-1},x_n,x_n}(t_1) \\ &\geq G_{x_{n-1},x_n,x_n}(t_0). \end{aligned}$$

Then, the sequence $\{G_{x_n,x_{n+1},x_{n+1}}(t_0)\}$ is monotonically increasing in n and being bounded above is convergent.

Let $\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_0) = l$. We shall show that $l = 1$. On contrary, suppose $l < 1$. Then $\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_1) = l$ (by the above inequality). Thus by squeeze lemma,

$$\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t) = l < 1, \forall\, t \in [t_0, t_1].$$

Let $\bar{t} = \sup A$, where

$$A = \left\{ t : \lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t) = l \right\}. \tag{3.1}$$

If $\bar{t}$ is finite, then there exists a monotonically increasing sequence $\{t_m\}$ such that for all $m \in \mathbb{N}$, $\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_m) = l$ and $t_m \to \bar{t}$ as $m \to \infty$. Since $G_{x_n,x_{n+1},x_{n+1}}$ is left continuous,

$$G_{x_n,x_{n+1},x_{n+1}}(\bar{t}) = \lim_{m\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_m).$$

Thus, by using Lemma 3.1 and $\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_m) = l$, we have

$$\begin{aligned} \lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(\bar{t}) &= \lim_{n\to\infty} \left(\lim_{m\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_m) \right) \\ &= \lim_{m\to\infty} \left(\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t_m) \right) = l. \end{aligned}$$

So, $\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(\bar{t}) = l$. Hence proceeding as above, there exists $\bar{t_1}$ such that $\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(\bar{t_1}) = l$ and $\bar{t_1} > \bar{t}$, which is a contradiction with Equation (3.1). Therefore, for all $t > t_0$,

$$\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t) = l. \tag{3.2}$$

Since $G_{x,y,y}(t) \to 1$ as $t \to \infty$, there exists $s > t_0$ such that $G_{x_k,x_{k+1},x_{k+1}}(s) > l$, for given k.

Now, since $\{G_{x_n,x_{n+1},x_{n+1}}(t_0)\}$ is monotonically increasing in n, and, as $t_0 > 0$ is arbitrary, we have that $\{G_{x_n,x_{n+1},x_{n+1}}(t)\}$ is monotonically increasing in n for all $t > 0$. Then, the sequence $\{G_{x_n,x_{n+1},x_{n+1}}(s)\}$ is monotonically increasing in n, we have that $G_{x_n,x_{n+1},x_{n+1}}(s) > l$. But this is a contradiction as $l < 1$. Thus,

$\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t) = 1$, for all $t > t_0$. Since $t_0 > 0$ is arbitrary, we conclude that $\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t) = 1$, for all $t > 0$. □

Lemma 3.3. *Let (X, G, T) be a Menger PGM-space with a t-norm T of H-type. Let $\{x_n\}$ be a sequence in (X, G, T). If there exists a function $\varphi \in \Phi$ such that*

$$G_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) \geq G_{x_{n-1},x_n,x_n}(t), \tag{3.3}$$

for all $n \in \mathbb{N}$ and $t > 0$ then $\{x_n\}$ is a Cauchy sequence in X.

Proof. Let $\beta > 0$ be arbitrary. Since $\varphi \in \Phi$, for each t_1 such that $0 < t_1 < \beta$ there exists $r_1 > t_1$ such that $\varphi(r_1) \leq t_1$.

Now, if $\varphi(r_1) < t_1$, then we take $t = t_1$ and $r = r_1$.

If $\varphi(r_1) = t_1$, then choose t as $\min\{r_1, \beta\} > t > t_1$ and $r = r_1$. Thus in each case, we have $\beta > t > \varphi(r)$ and $r > t$.

Let $n \geq 1$ be given. Then for each t choosen in this way, we prove by induction that for any $k \in \mathbb{N}$,

$$G_{x_n,x_{n+k},x_{n+k}}(t) \geq T^{k-1}\left(G_{x_n,x_{n+1},x_{n+1}}(t - \varphi(r))\right) \tag{3.4}$$

For $k = 1$, from Equation (3.4), we have

$$G_{x_n,x_{n+1},x_{n+1}}(t) \geq T^0\left(G_{x_n,x_{n+1},x_{n+1}}(t - \varphi(r))\right) = G_{x_n,x_{n+1},x_{n+1}}(t - \varphi(r)).$$

Thus (4) holds for $k = 1$.

Assume that (4) holds for some k. Since T is monotone, from (PGM-4) and (3), we have

$$\begin{aligned}
G_{x_n,x_{n+k+1},x_{n+k+1}}(t) &= G_{x_n,x_{n+k+1},x_{n+k+1}}(t - \varphi(r) + \varphi(r)) \\
&\geq T\left(G_{x_n,x_{n+1},x_{n+1}}(t - \varphi(r)), G_{x_{n+1},x_{n+k+1},x_{n+k+1}}(\varphi(r))\right) \\
&\geq T\left(G_{x_n,x_{n+1},x_{n+1}}(t - \varphi(r)), G_{x_n,x_{n+k},x_{n+k}}(r)\right) \\
&\geq T\left(G_{x_n,x_{n+1},x_{n+1}}(t - \varphi(r)), G_{x_n,x_{n+k},x_{n+k}}(t)\right) \\
&= T^k\left(G_{x_n,x_{n+1},x_{n+1}}(t - \varphi(r))\right),
\end{aligned}$$

which completes the induction steps. Then (4) holds, for all $k \in \mathbb{N}$ and for any $t < \beta$.

To prove $\{x_n\}$ is Cauchy sequence, we need to prove that $\lim_{m,n,l\to\infty} G_{x_n,x_m,x_l}(t) = 1$, for all $t > 0$. To this end, we first prove that $\lim_{m,n\to\infty} G_{x_n,x_m,x_m}(t) = 1$, for all $t > 0$.

Now, let $0 < \varepsilon < 1$. Since $\{T^n(t)\}$ is equicontinuous at $t = 1$ and $T^n(1) = 1$, there exists $\delta > 0$ such that

$$T^n(s) > 1 - \varepsilon, \text{forall } s \in (1 - \delta, 1] \text{ and } n \geq 1. \tag{3.5}$$

From Lemma 3.2, it follows that $\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t - \varphi(r)) = 1$. Thus, there exists $n_0 \in \mathbb{N}$ such that $G_{x_n,x_{n+1},x_{n+1}}(t - \varphi(r)) > 1 - \delta$, for all $n \geq n_0$. So, by (3.4) and (3.5), we have $G_{x_n,x_{n+k},x_{n+k}}(t) > 1 - \varepsilon$, for all $k \geq 0$. Hence,

$$\lim_{n,m\to\infty} G_{x_n,x_m,x_m}(t) = 1, \tag{3.6}$$

for any $0 < t < \beta$.

Now, by PGM-4 in Definition 2.5, we have, for all $t < \beta$,

$$G_{x_n,x_m,x_l}(t) \geq T\left(G_{x_n,x_m,x_m}\left(\frac{t}{2}\right), G_{x_m,x_m,x_l}\left(\frac{t}{2}\right)\right)$$
$$= T\left(G_{x_n,x_m,x_m}\left(\frac{t}{2}\right), G_{x_l,x_m,x_m}\left(\frac{t}{2}\right)\right).$$

Taking limit $m, n, l \to \infty$ in this inequality and using the continuity of T, we get

$$\lim_{m,n,l\to\infty} G_{x_n,x_m,x_l}(t) \geq T\left(\lim_{m,n\to\infty} G_{x_n,x_m,x_m}\left(\frac{t}{2}\right), \lim_{m,l\to\infty} G_{x_l,x_m,x_m}\left(\frac{t}{2}\right)\right). \tag{3.7}$$

From Equation (3.6), for all $t < \beta$, we have

$$\lim_{n,m\to\infty} G_{x_n,x_m,x_m}\left(\frac{t}{2}\right) = 1,$$

$$\lim_{l,m\to\infty} G_{x_l,x_m,x_m}\left(\frac{t}{2}\right) = 1.$$

Using these two limits in inequality (3.7), we get

$$\lim_{m,n,l\to\infty} G_{x_n,x_m,x_l}(t) \geq T(1,1) = 1,$$

That is, $\lim_{m,n,l\to\infty} G_{x_n,x_m,x_l}(t) = 1$, for all $0 < t < \beta$. Since, $\beta > 0$ is arbitrary, we have $\lim_{m,n,l\to\infty} G_{x_n,x_m,x_l}(t) = 1$, for all $0 < t$.

Hence $\{x_n\}$ is Cauchy sequence. □

Lemma 3.4. *Let (X, G, T) be a Menger PGM-space and $x, y \in X$. If there exists a function $\varphi \in \Phi$ such that*

$$G_{x,y,y}(\varphi(t)) \geq G_{x,y,y}(t), \tag{3.8}$$

for all $t > 0$, then $x = y$.

Proof. In order to prove $x = y$, we only need to prove that $G_{x,y,y}(t) = 1$, for all $t > 0$. On contrary, assume that $\exists\, t_0 \in \mathbb{R}^+$ such that $G_{x,y,y}(t_0) < 1$. Since $\varphi \in \Phi$, $\exists\, t_1 > t_0$ such that $\varphi(t_1) \leq t_0$. Then Equation (3.8) and the monotonicity of $G_{x,y,y}$ give

$$G_{x,y,y}(t_0) \geq G_{x,y,y}(\varphi(t_1)) \geq G_{x,y,y}(t_1) \geq G_{x,y,y}(t_0). \tag{3.9}$$

If the inequality holds in Equation (3.9), then we have a contradiction. Thus we assume that equality holds. Then the set $A = \{s : G_{x,y,y}(s) = G_{x,y,y}(t_0)\,;\, s > t_0\}$ is non-empty, by the above inequality.

Let $\bar{s}$ =sup$A$ be finite. Then there exists a monotonically increasing sequence $\{s_n\}$ with $s_n \in A$, for all $n \in \mathbb{N}$, such that $s_n \to \bar{s}$. Since $G_{x,y,y}$ is left continuous, it follows that

$$G_{x,y,y}(\bar{s}) = \mathit{lim}_{n\to\infty} G_{x,y,y}(s_n) = G_{x,y,y}(t_0).$$

This implies that $\bar{s} \in A$. Again treating $\bar{s}$ in the same way as t_0, we obtain either $G_{x,y,y}(\bar{s}) > G_{x,y,y}(\bar{s})$, which is a contradiction, or there exists $\bar{s_1} > \bar{s}$ such that $G_{x,y,y}(\bar{s_1}) = G_{x,y,y}(\bar{s}) = G_{x,y,y}(t_0)$, which is a contradiction with $\bar{s} = \sup A$. Thus

$\bar{s}$ is not finite, i.e, $\lim_{n\to\infty} G_{x,y,y}(s_n) = G_{x,y,y}(\bar{s}) = G_{x,y,y}(t_0) < 1$, which is again a contradiction as $\bar{s}$ is not finite. So, $G_{x,y,y}(t) = 1$, for all $t > 0$, i.e., $x = y$. □

Theorem 3.5. *Let (X, G, T) be a complete Menger space with a t-norm T of H-type. If $f : X \to X$ is a probabilistic φ-contraction, i.e., $G_{fx,fy,fy}(\varphi(t)) \geq G_{x,y,y}(t)$, $\forall$ $x, y \in X$ and $t > 0$, where $\varphi \in \Phi$, then f has a unique fixed point $x \in X$.*

Proof. We define the sequence $\{x_n\}$ as follows: Let $x_0 \in X$ and $x_n = fx_{n-1}$, for all $n \in \mathbb{N}$. Then by the given contraction condition,

$$\begin{aligned} G_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) &= G_{fx_{n-1},fx_n,fx_n}(\varphi(t)) \\ &\geq G_{x_{n-1},x_n,x_n}(t), \end{aligned}$$

for all $n \in \mathbb{N}$ and $t > 0$. Thus by Lemma 3.3, we conclude that $\{x_n\}$ is a Cauchy sequence in(X, G, T) and X is complete. So we have that $x_n \to x \in X$. Since $\varphi \in \Phi$, for each $t > 0$, there exists $r > t$ such that $\varphi(r) \leq t$. Now

$$\begin{aligned} G_{fx_n,fx,fx}(t) &\geq G_{fx_n,fx,fx}(\varphi(r)) \\ &\geq G_{x_n,x,x}(r) \\ &\geq G_{x_n,x,x}(t). \end{aligned}$$

Taking limit $n \to \infty$ in this inequality and keeping in mind that $x_n \to x$ for each $t > 0$, we get

$$\lim_{n\to\infty} G_{fx_n,fx,fx}(t) = 1. \tag{3.10}$$

Now, using (PM-3) and the continuity of T, we get

$$\begin{aligned} G_{x_*,fx_*,fx_*}(t) &\geq T\left(G_{x,x_{n+1},x_{n+1}}\left(\frac{t}{2}\right), G_{x_{n+1},fx,fx}\left(\frac{t}{2}\right)\right) \\ &= T\left(G_{x,x_{n+1},x_{n+1}}\left(\frac{t}{2}\right), G_{fx_n,fx,fx}\left(\frac{t}{2}\right)\right). \end{aligned}$$

Taking limit $n \to \infty$ in this inequality and using Equation (3.10) and the continuity of T, we get

$$G_{x,fx,fx}(t) \geq T(1,1) = 1.$$

This implies that $G_{x,fx,fx}(t) = 1$, for all $t > 0$. Thus $fx = x$ which proves that x is a fixed point of f.

To show the uniqueness of fixed point of f, we suppose that y is another fixed point of f then by contractivity condition, we have

$$G_{x,y,y}(\varphi(t)) = G_{fx,fy,fy}(\varphi(t)) \geq G_{x,y,y}(t),$$

for all $t > 0$. So by Lemma 3.4, we get $x = y$. □

Corollary 3.6. *Let (X, G, T) be a complete Menger PGM-space with a t-norm T of H-type. Let $f_0, f_1 : X \to X$ be two mappings such that $G_{f_0x,f_0y,f_0y}(\varphi(t)) \geq G_{x,y,y}(t)$ and $G_{f_1x,f_1y,f_1y}(\varphi(t)) \geq G_{x,y,y}(t)$ hold for all $x, y \in X$ and $t > 0$, where $\varphi \in \Phi$. If $f_0f_1 = f_1f_0$ then there exists a unique common fixed point of f_0 and f_1.*

Proof. Let $f = f_0 f_1$. Since $\varphi \in \Phi$, for each $t > 0$, there exists $r > t$ such that $\varphi(r) \le t$.

$$\begin{aligned} G_{fx,fy,fy}(\varphi(t)) &= G_{(f_0f_1)x,(f_0f_1)y,(f_0f_1)y}(\varphi(t)) \\ &= G_{f_0(f_1x),f_0(f_1y),f_0(f_1y)}(\varphi(t)) \\ &\ge G_{f_1x,f_1y,f_1y}(t) \\ &\ge G_{f_1x,f_1y,f_1y}(\varphi(r)) \\ &\ge G_{x,y,y}(r) \\ &\ge G_{x,y,y}(t). \end{aligned}$$

This implies that f is a probabilistic φ-contraction. Then by the Theorem 3.5, we conclude that f has a unique fixed point x_* in X. Since $f_0f_1 = f_1f_0$, we have

$$f(f_0z) = f_0f_1(f_0z) = f_0(f_1f_0z) = f_0z$$

and

$$f(f_1z) = f_1f_0(f_1z) = f_1(f_0f_1z) = f_1z.$$

This gives that f_0z and f_1z are also fixed points of f. By the uniqueness of fixed point of f , we have $f_0z = f_1z = z$, i.e., z is a common fixed point of f_0 and f_1. It is clear that z is a unique common fixed point of f_0 and f_1. □

Theorem 3.7. *Let (X, G, T) be a complete Menger PGM-space with a t-norm T of H-type. Let $f : X \to X$ be a mapping satisfying*

$$G_{fx,fy,fz}(\varphi(t)) \ge \frac{1}{3}\left(G_{x,fx,fx}(t) + G_{y,fy,fy}(t) + G_{z,fz,fz}(t)\right), \tag{3.11}$$

for all $x, y, z \in X$, where $\varphi \in \Phi$. Then, for any $x_0 \in X$ the sequence $\{f^n(x_0)\}$ converges to a unique fixed point of f.

Proof. Take an arbitrary point $x_0 \in X$. Construct a sequence $\{x_n\}$ by $x_{n+1} = f^n(x_0)$ for all $n \ge 0$. Since $\varphi \in \Phi$, for each $t > 0$ there exists $r > t$ such that $\varphi(r) \le t$. Then

$$\begin{aligned} G_{x_n,x_{n+1},x_{n+1}}(t) &\ge G_{x_n,x_{n+1},x_{n+1}}(\varphi(r)) \\ &= G_{fx_{n-1},fx_n,fx_n}(\varphi(r)) \\ &\ge \frac{1}{3}\left(G_{x_{n-1},fx_{n-1},fx_{n-1}}(t) + 2G_{x_n,fx_n,fx_n}(r)\right) \\ &\ge \frac{1}{3}\left(G_{x_{n-1},fx_{n-1},fx_{n-1}}(t) + 2G_{x_n,fx_n,fx_n}(t)\right) \\ &= \frac{1}{3}\left(G_{x_{n-1},x_n,x_n}(t) + 2G_{x_n,x_{n+1},x_{n+1}}(t)\right), \end{aligned}$$

That is, for all $t > 0$,

$$G_{x_n,x_{n+1},x_{n+1}}(t) \ge G_{x_{n-1},x_n,x_n}(t). \tag{3.12}$$

Now, we prove that f is a φ-contraction. For this, we have

$$\begin{aligned}
G_{x_n,x_{n+1},x_{n+1}}(\varphi(t)) &= G_{fx_{n-1},fx_n,fx_n}(\varphi(t)) \\
&\geq \frac{1}{3}\left(G_{x_{n-1},fx_{n-1},fx_{n-1}}(t) + 2G_{x_n,fx_n,fx_n}(t)\right) \\
&= \frac{1}{3}\left(G_{x_{n-1},x_n,x_n}(t) + 2G_{x_n,fx_n,fx_n}(t)\right) \\
&\geq \frac{1}{3}\left(G_{x_{n-1},x_n,x_n}(t) + 2G_{fx_{n-1},fx_n,fx_n}(\varphi(r))\right) \\
&\geq \frac{1}{3}\left(G_{x_{n-1},x_n,x_n}(t) + \frac{2}{3}\left(G_{x_{n-1},fx_{n-1},fx_{n-1}}(r) + 2G_{x_n,fx_n,fx_n}(r)\right)\right) \\
&= \frac{1}{3}\left(G_{x_{n-1},x_n,x_n}(t) + \frac{2}{3}G_{x_{n-1},x_n,x_n}(r) + \frac{4}{3}G_{x_n,x_{n+1},x_{n+1}}(r)\right) \\
&\geq \frac{1}{3}\left(G_{x_{n-1},x_n,x_n}(t) + \frac{2}{3}G_{x_{n-1},x_n,x_n}(t) + \frac{4}{3}G_{x_n,x_{n+1},x_{n+1}}(t)\right) \\
&= \frac{1}{3}\left(\frac{5}{3}G_{x_{n-1},x_n,x_n}(t) + \frac{4}{3}G_{x_n,x_{n+1},x_{n+1}}(t)\right) \\
&\geq \frac{1}{3}\left(\frac{5}{3}G_{x_{n-1},x_n,x_n}(t) + \frac{4}{3}G_{x_{n-1},x_n,x_n}(t)\right) \\
&= G_{x_{n-1},x_n,x_n}(t).
\end{aligned}$$

Here, first and third inequalities are due to Equation (3.11), second and fourth are due to the monotonic increasing property of distribution function while the last one is due to Equation (3.12). Thus, f is φ-contraction, and Lemma 3.3 shows that $\{x_n\}$ is Cauchy sequence. Since X is complete Menger PGM-space, there exists a point $x \in X$ such that $x_n \to x$ as $n \to \infty$. Now, since $\varphi(r) \leq t$ and $G_{fx_n,fx,fx}$ is monotonically increasing, from Equation (3.11), we have

$$G_{fx_n,fx,fx}(t) \geq G_{fx_n,fx,fx}(\varphi(r)) \geq \frac{1}{3}\left(G_{x_n,fx_n,fx_n}(r) + 2G_{x,fx,fx}(r)\right).$$

Letting limit $n \to \infty$ in this inequality, we get

$$G_{x,fx,fx}(t) \geq \frac{1}{3}\left(G_{x,x,x}(r) + 2G_{x,fx,fx}(r)\right) \geq \frac{1}{3}\left(G_{x,x,x}(t) + 2G_{x,fx,fx}(t)\right),$$

which gives $G_{x,fx,fx}(t) \geq G_{x,x,x}(t) = 1$, for all $t > 0$. So, we have proved that $fx = x$.

To show the uniqueness of the fixed point of f, we suppose that y is another fixed point of f. Then, for all $t > 0$

$$\begin{aligned}
G_{x,y,y}(t) &\geq G_{x,y,y}(\varphi(r)) \\
&= G_{fx,fy,fy}(\varphi(r)) \\
&\geq \frac{1}{3}\left(G_{x,fx,fx}(r) + 2G_{y,fy,fy}(r)\right) \\
&\geq \frac{1}{3}\left(G_{x,fx,fx}(t) + 2G_{y,fy,fy}(t)\right) \\
&= 1.
\end{aligned}$$

Here, first and third inequality is due to the monotonic increasing property of distribution function and the second one is due to the Equation(3.11). This shows that $x = y$. Thus, f has a unique fixed point. □

Example 3.8. Let $X = [0, \infty)$ and $T(a, b) = \min(a, b)$, for all $a, b \in X$. Define a function $G : X^3 \times [0, \infty) \to [0, \infty)$ by:

$$G_{x,y,z}(t) = \frac{t}{t + (|x-y| + |y-z| + |z-x|)}.$$

Then (X, G, T) is a complete Menger PGM-space. Define $f : X \to X$ by $f(x) = \frac{x}{4}$, for each $x \in X$ and $\varphi : [0, \infty) \to [0, \infty)$ by:

$$\varphi(t) = \begin{cases} 0 & \text{if } t = 0 \\ \frac{1}{4^n} & \text{if } \frac{1}{4^n} \leq t < \frac{1}{4^{n-1}} \\ kt & \text{if } t \geq 1, \text{where } \frac{1}{4} \leq k < 1. \end{cases}.$$

Obviously, $\varphi \in \Phi$.

Now, we want to show that f is φ-contraction.

Case 1: Suppose $\frac{1}{4^n} \leq t < \frac{1}{4^{n-1}}$. Since $\frac{1}{4^{n-1}} > t$, we have $\frac{1}{4^n} > \frac{t}{4}$, i.e., $\varphi(t) \geq \frac{t}{4}$.

Case 2: Suppose $t \geq 1$. Since $k \geq \frac{1}{4}$, we have $kt \geq \frac{t}{4}$, i.e., $\varphi(t) \geq \frac{t}{4}$. Then, we have $\varphi(t) \geq \frac{t}{4}$, for each $t > 0$. Since the function $\frac{t}{t+1}$ is strictly increasing on $[0, \infty)$, we have

$$\begin{aligned} G_{fx,fy,fz}(\varphi(t)) &= \frac{\varphi(t)}{\varphi(t) + (|fx - fy| + |fy - fz| + |fz - fx|)} \\ &= \frac{\varphi(t)}{\varphi(t) + \frac{1}{4}(|x-y| + |y-z| + |z-x|)} \\ &\geq \frac{\frac{t}{4}}{\frac{t}{4} + \frac{1}{4}(|x-y| + |y-z| + |z-x|)} \\ &= \frac{t}{t + (|x-y| + |y-z| + |z-x|)} \\ &= G_{x,y,z}(t). \end{aligned}$$

Thus, from the Theorem 3.5 f has unique fixed point. In fact, the fixed point is $x = 0$.

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