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A novel generalization of fuzzy subsemigroups

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Abstract. The notion of $(\widetilde{\alpha}, \widetilde{\beta})$-fuzzy subsemigroups in semigroups is introduced, and related properties are investigated. Characterizations of $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroups are provided. Using a collection of subsemigroups with some conditions, an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup is constructed. An $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup generated by the given fuzzy set with finite images is established.

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Corresponding Author: Y. B. Jun (skywine@gmail.com)

1. Introduction

Since the inception of the notion of a fuzzy set in 1965 which laid the foundations of fuzzy set theory, the literature on fuzzy set theory and its applications has been growing rapidly amounting by now to several papers (refer to References). These are widely scattered over many disciplines such as artificial intelligence, computer science, control engineering, expert systems, management science, operations research, pattern recognition, robotics, and others. Due to the rapid development in this new emerging line, many researchers are motivated to study fuzzy set theory in diverse branches. Of course, fuzzy algebraic structures also play a remarkable role in mathematics with wide ranging applications in automata theory, computer science, coding theory, formal language and decision making problem etc. Rosenfeld's idea [11] of fuzzy subgroup has played a sparking role for many researchers to investigate the fuzzy set theory in many algebraic structures with more general type of existing fuzzy subsystems. In [13], Yuan et al. introduced the definition of a fuzzy subgroup with thresholds which is a generalization of Rosenfeld's fuzzy subgroup and Bhakat and Das's fuzzy subgroup. Murali [9] proposed a definition of a fuzzy point belonging to fuzzy subset under a natural equivalence on fuzzy subset. The idea of quasicoincidence of a fuzzy point with a fuzzy set, which is mentioned in [10], played a vital role to generate some different types of fuzzy subgroups. It is worth pointing out that Bhakat and Das [2, 3] gave the concepts of (α, β)-fuzzy subgroups by using the "belongs to" relation $(\in)$ and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in\vee q)$-fuzzy subgroup. In particular, $(\in, \in\vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. Bhakat et al. applied the $(\in, \in\vee q)$-fuzzy type to group and ring (see [1, 2, 3, 4]). Dudek et al. [5] characterized different types of (α, β)-fuzzy ideals of hemirings. In [7] Jun and Song initiated the study of (α, β)-fuzzy interior ideals of a semigroup. In [8] Kazanci and Yamak study $(\in, \in\vee q)$-fuzzy bi-ideals of a semigroup. Shabir et al. [12] introduced the concept of (α, β)-fuzzy ideal, (α, β)-fuzzy generalized bi-ideal, and characterized regular semigroups by the properties of these ideals. Jun et al. [6] considered more general form of quasi-coincident fuzzy point.

In this paper, we deal with more general form of the fuzzy subsemigroups using Jun et al.'s concept. We introduce the notions of $(\widetilde{\alpha}, \widetilde{\beta})$-fuzzy subsemigroups in semigroups, and investigate related properties. We provide characterizations of $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroups. We provide a condition for an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup to be an $(\in, \in)$-fuzzy subsemigroup. Using a collection of subsemigroups with some conditions, we make an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup of S. Given a fuzzy set with finite images, we establish an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup generated by the given fuzzy set.

2. Preliminaries

For two fuzzy set λ and ν in S, we say $\lambda \leq \nu$ if $\lambda(x) \leq \nu(x)$ for all $x \in S$. We define $\lambda \wedge \nu$ and $\lambda \vee \nu$ as follows:

$$\lambda \wedge \nu : S \to [0,1],\ x \mapsto \min\{\lambda(x), \nu(x)\}$$

and

$$\lambda \vee \nu : S \to [0,1],\ x \mapsto \max\{\lambda(x), \nu(x)\}$$

respectively. The product $\lambda \circ \nu$ of λ and ν is defined to be fuzzy set in S as follows:

$$(\lambda \circ \nu)(x) := \begin{cases} \bigvee_{x=yz} \min\{\lambda(y), \nu(z)\} & \text{if } \exists y, z \in S \text{ such that } x = yz, \\ 0 & \text{otherwise.} \end{cases}$$

A fuzzy set λ in a set S of the form

$$\lambda(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases} \tag{2.1}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t. It is clear that $x_t \circ y_r = (xy)_{\min\{t,r\}}$ for all fuzzy points x_t and y_r in a set S.

For a fuzzy point x_t and a fuzzy set λ in a set S, we say that

- $x_t \in \lambda$ (resp. $x_t\, q\, \lambda$) (see [10]) if $\lambda(x) \geq t$ (resp. $\lambda(x) + t > 1$). In this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set λ.
- $x_t \in\vee q\, \lambda$ (resp. $x_t \in\wedge q\, \lambda$) (see [10]) if $x_t \in \lambda$ or $x_t\, q\, \lambda$ (resp. $x_t \in \lambda$ and $x_t\, q\, \lambda$).

Let $\delta \in (0, 1]$. For a fuzzy point x_t and a fuzzy set λ in a set X, we say that

- x_t is a *δ-quasi-coincident* with λ, written $x_t\, q_0^\delta\, \lambda$, (see [6]) if $\lambda(x) + t > \delta$,

- $x_t \in\vee\, q_0^\delta\, \lambda$ (resp. $x_t \in\wedge\, q_0^\delta \lambda$) (see [6]) if $x_t \in \lambda$ or $x_t\, q_0^\delta\, \lambda$ (resp. $x_t \in \lambda$ and $x_t\, q_0^\delta\, \lambda$).

Obviously, $x_t\, q\, \lambda$ implies $x_t\, q_0^\delta\, \lambda$. If $\delta = 1$, then the δ-quasi-coincident with λ is the quasi-coincident with λ, that is, $x_t\, q_0^1\, \lambda = x_t\, q\, \lambda$.

For $\alpha \in \{\in, q, \in\vee\, q, \in\wedge\, q, \in\vee\, q_0^\delta, \in\wedge\, q_0^\delta\}$, we say that $x_t\, \overline{\alpha}\, \lambda$ if $x_t\, \alpha\, \lambda$ does not hold.

3. Generalized fuzzy subsemigroups

In what follows, let δ be an element of $(0, 1]$ and let S be a semigroup and $\widetilde{\alpha}$ and $\widetilde{\beta}$ denote any one of $\in$, q_0^δ, $\in\vee\, q_0^\delta$ and $\in\wedge\, q_0^\delta$ unless otherwise specified.

Definition 3.1. A fuzzy set λ in S is called an $(\widetilde{\alpha},\, \widetilde{\beta})$-*fuzzy subsemigroup* of S, where $\widetilde{\alpha} \neq\in\wedge\, q_0^\delta$, if

$$(\forall x, y \in S)\, (\forall t, r \in (0, \delta])\, \Big(x_t\, \widetilde{\alpha}\, \lambda,\ y_r\, \widetilde{\alpha}\, \lambda\ \Rightarrow\ x_t \circ y_r\, \widetilde{\beta}\, \lambda\Big). \tag{3.1}$$

Let λ be a fuzzy set in S such that $\lambda(x) \leq \frac{\delta}{2}$ for all $x \in S$. Let $x \in S$ and $t \in (0, \delta]$ be such that $x_t \in\wedge\, q_0^\delta\, \lambda$. Then $\lambda(x) \geq t$ and $\lambda(x) + t > \delta$. It follows that $\delta < \lambda(x) + t \leq 2\lambda(x)$, so that $\lambda(x) \geq \frac{\delta}{2}$. This means that $\{x_t \mid x_t \in\wedge\, q_0^\delta\, \lambda\} = \varnothing$. Hence the case $\widetilde{\alpha} =\in\wedge\, q_0^\delta$ should be omitted.

Theorem 3.2. *If λ is a nonzero $(q_0^\delta,\, \in)$-fuzzy subsemigroup of S, then the set*

$$S_0 := \{x \in S \mid \lambda(x) > 0\}$$

is a subsemigroup of S.

Proof. Let $x, y \in S_0$. Then $\lambda(x) > 0$ and $\lambda(y) > 0$. Hence $\lambda(x) + \delta > \delta$ and $\lambda(y) + \delta > \delta$, that is, $x_\delta\, q_0^\delta\, \lambda$ and $y_\delta\, q_0^\delta\, \lambda$. It follows from (3.1) that $x_\delta \circ y_\delta \in \lambda$, i.e., $\lambda(xy) \geq \delta > 0$. Thus $xy \in S_0$. This completes the proof. □

Corollary 3.3 ([12])**.** *If λ is a nonzero $(q,\, \in)$-fuzzy subsemigroup of S, then the set*

$$S_0 := \{x \in S \mid \lambda(x) > 0\}$$

is a subsemigroup of S.

Theorem 3.4. *If λ is a nonzero $(q_0^\delta,\, q_0^\delta)$-fuzzy subsemigroup of S, then the set*

$$S_0 := \{x \in S \mid \lambda(x) > 0\}$$

is a subsemigroup of S.

Proof. Let $x, y \in S_0$. Then $\lambda(x) > 0$ and $\lambda(y) > 0$. Hence $\lambda(x) + \delta > \delta$ and $\lambda(y) + \delta > \delta$, that is, $x_\delta\, q_0^\delta\, \lambda$ and $y_\delta\, q_0^\delta\, \lambda$. It follows from (3.1) that $x_\delta \circ y_\delta\, q_0^\delta\, \lambda$, i.e., $\lambda(xy) + \delta > \delta$. Thus $\lambda(xy) \neq 0$, and so $xy \in S_0$. This completes the proof. □

Corollary 3.5 ([12])**.** *If λ is a nonzero $(q,\, q)$-fuzzy subsemigroup of S, then the set*

$$S_0 := \{x \in S \mid \lambda(x) > 0\}$$

is a subsemigroup of S.

Theorem 3.6. *For a subsemigroup L of S, consider a fuzzy set λ in S as follows:*

$$\lambda : S \to [0,1],\ x \mapsto \begin{cases} \varepsilon & \text{if } x \in L, \\ 0 & \text{if } x \notin L, \end{cases}$$

where $\varepsilon \geq \frac{\delta}{2}$. Then λ is both an $(\in, \in\vee\, q_0^\delta)$-fuzzy subsemigroup and a $(q_0^\delta, \in\vee\, q_0^\delta)$-fuzzy subsemigroup of S.

Proof. Let $x, y \in S$ and $t, r \in (0, \delta]$ such that $x_t \in \lambda$ and $y_r \in \lambda$. Then $\lambda(x) = \varepsilon = \lambda(y)$, and so $x, y \in L$. Hence $xy \in L$, and thus $\lambda(xy) = \varepsilon$. If $t \leq \frac{\delta}{2}$ or $r \leq \frac{\delta}{2}$, then

$$\lambda(xy) = \varepsilon \geq \tfrac{\delta}{2} \geq \min\{t, r\}$$

and so $x_t \circ y_r \in \lambda$. If $t > \frac{\delta}{2}$ and $r > \frac{\delta}{2}$, then

$$\lambda(xy) + \min\{t, r\} = \varepsilon + \min\{t, r\} > \tfrac{\delta}{2} + \tfrac{\delta}{2} = \delta,$$

and so $x_t \circ y_r\, q_0^\delta\, \lambda$. Therefore $x_t \circ y_r \in\vee\, q_0^\delta\, \lambda$ and λ is an $(\in, \in\vee\, q_0^\delta)$-fuzzy subsemigroup of S. Now, let $x, y \in S$ and $t, r \in (0, \delta]$ such that $x_t\, q_0^\delta\, \lambda$ and $y_r\, q_0^\delta\, \lambda$. Then $\lambda(x) + t > \delta$ and $\lambda(y) + r > \delta$, which imply that $x, y \in L$. Hence $xy \in L$ and $\lambda(xy) = \varepsilon$. If $t \leq \frac{\delta}{2}$ or $r \leq \frac{\delta}{2}$, then

$$\lambda(xy) = \varepsilon \geq \tfrac{\delta}{2} \geq \min\{t, r\}$$

and so $x_t \circ y_r \in \lambda$. If $t > \frac{\delta}{2}$ and $r > \frac{\delta}{2}$, then

$$\lambda(xy) + \min\{t, r\} = \varepsilon + \min\{t, r\} > \tfrac{\delta}{2} + \tfrac{\delta}{2} = \delta,$$

and so $x_t \circ y_r\, q_0^\delta\, \lambda$. Therefore $x_t \circ y_r \in\vee\, q_0^\delta\, \lambda$, and λ is a $(q_0^\delta, \in\vee\, q_0^\delta)$-fuzzy subsemigroup of S. □

Theorem 3.6 is a generalization of Theorem 8 in [12], that is, we have the following corollary.

Corollary 3.7 ([12])**.** *For a subsemigroup L of S, consider a fuzzy set λ in S as follows:*

$$\lambda : S \to [0,1],\ x \mapsto \begin{cases} \varepsilon & \text{if } x \in L, \\ 0 & \text{if } x \notin L, \end{cases}$$

where $\varepsilon \geq 0.5$. Then λ is both an $(\in, \in\vee\, q)$-fuzzy subsemigroup and a $(q, \in\vee\, q)$-fuzzy subsemigroup of S.

We provide characterizations of $(\in, \in\vee\, q_0^\delta)$-fuzzy subsemigroups.

Theorem 3.8. *For any fuzzy set λ in S, the following are equivalent.*

(i) *λ is an $(\in, \in\vee\, q_0^\delta)$-fuzzy subsemigroup of S.*
(ii) $(\forall x, y \in S)\left(\lambda(xy) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\}\right)$.

Proof. Assume that λ is an $(\in, \in\vee\, q_0^\delta)$-fuzzy subsemigroup of S. For any $x, y \in S$, we consider two cases:

$$(1)\text{: } \min\{\lambda(x), \lambda(y)\} < \tfrac{\delta}{2} \text{ and } (2)\text{: } \min\{\lambda(x), \lambda(y)\} \geq \tfrac{\delta}{2}.$$

For the first case, suppose that $\lambda(xy) < \min\{\lambda(x), \lambda(y)\}$ and take $t \in (0, \frac{\delta}{2}]$ such that $\lambda(xy) < t \leq \min\{\lambda(x), \lambda(y)\}$. Then $x_t \in \lambda$ and $y_t \in \lambda$ but $x_t \circ y_t \overline{\in \vee\, q_0^\delta}\, \lambda$ since $x_t \circ y_t \overline{\in}\, \lambda$ and $\lambda(xy) + t < 2t < \delta$, that is, $x_t \circ y_t\, \overline{q_0^\delta}\, \lambda$. This is a contradiction. Hence $\lambda(xy) \geq \min\{\lambda(x), \lambda(y)\}$ whenever $\min\{\lambda(x), \lambda(y)\} < \frac{\delta}{2}$. Now assume that the case (2) is valid. Then $x_{\frac{\delta}{2}} \in \lambda$ and $y_{\frac{\delta}{2}} \in \lambda$, which imply that $x_{\frac{\delta}{2}} \circ y_{\frac{\delta}{2}} \in \vee\, q_0^\delta\, \lambda$. If $\lambda(xy) < \frac{\delta}{2}$, then $\lambda(xy) + \frac{\delta}{2} < \frac{\delta}{2} + \frac{\delta}{2} = \delta$ and so $\lambda(xy) < \frac{\delta}{2}$ which shows that $x_{\frac{\delta}{2}} \circ y_{\frac{\delta}{2}} \overline{\in}\, \lambda$ and $x_{\frac{\delta}{2}} \circ y_{\frac{\delta}{2}}\, \overline{q_0^\delta}\, \lambda$. This is a contradiction, and thus $\lambda(xy) \geq \frac{\delta}{2}$. Therefore $\lambda(xy) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\}$ for all $x, y \in S$.

Conversely assume that (ii) is valid. Let $x, y \in S$ and $t, r \in (0, \delta]$ such that $x_t \in \lambda$ and $y_r \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq r$. Suppose that $x_t \circ y_r \overline{\in}\, \lambda$. Then $\lambda(xy) < \min\{t, r\}$. If $\min\{\lambda(x), \lambda(y)\} < \frac{\delta}{2}$, then

$$\lambda(xy) \geq \min\{\lambda(x), \lambda(y), \tfrac{\delta}{2}\} \geq \min\{\lambda(x), \lambda(y)\} \geq \min\{t, r\},$$

a contradiction. Hence $\min\{\lambda(x), \lambda(y)\} \geq \frac{\delta}{2}$, and so

$$\lambda(xy) + \min\{t, r\} > 2\lambda(xy) \geq 2\min\{\lambda(x), \lambda(y), \tfrac{\delta}{2}\} = \delta.$$

This shows that $x_t \circ y_r\, q_0^\delta\, \lambda$. Therefore λ is an $(\in, \in \vee\, q_0^\delta)$-fuzzy subsemigroup of S. □

Theorem 12 in [12] is a special case of Theorem 3.8, that is,

Corollary 3.9 ([12]). *A fuzzy set λ in S is an $(\in, \in \vee\, q)$-fuzzy subsemigroup of S if and only if it satisfies:*

$$(\forall x, y \in S)\, (\lambda(xy) \geq \min\{\lambda(x), \lambda(y), 0.5\})\,. \tag{3.2}$$

.

Obviously, every $(\in, \in)$-fuzzy subsemigroup is an $(\in, \in \vee\, q_0^\delta)$-fuzzy subsemigroup, but the converse is not true in general as seen in the following example.

Example 3.10. Let $S = \{a, b, c\}$ be a semigroup in which the multiplication $\cdot$ is described by Table 1.

TABLE 1. Cayley table of the operation ·

$\cdot$	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

Define a fuzzy set λ in S as follows:

$$\lambda : S \to [0, 1],\ x \mapsto \begin{cases} 0.4 & \text{if } x = a, \\ 0.6 & \text{if } x = b, \\ 0.7 & \text{if } x = c. \end{cases}$$

Then λ is an $(\in, \in \vee q_0^{0.8})$-fuzzy subsemigroup of S. But it is not an $(\in, \in)$-fuzzy subsemigroup of S since $\lambda(bc) = \lambda(a) = 0.4 < 0.6 = \min\{\lambda(b), \lambda(c)\}$.

We provide a condition for an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup to be an $(\in,\in)$-fuzzy subsemigroup.

Theorem 3.11. *Let λ be an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup of S such that $\lambda(x) < \frac{\delta}{2}$ for all $x \in S$. Then λ is an $(\in,\in)$-fuzzy subsemigroup of S.*

Proof. Let $x, y \in S$ and $t, r \in (0, \delta]$ be such that $x_t \in \lambda$ and $y_r \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq r$. It follows from the hypothesis and Theorem 3.8 that

$$\lambda(xy) \geq \min\{\lambda(x),\, \lambda(y),\, \tfrac{\delta}{2}\} = \min\{\lambda(x),\, \lambda(y)\} \geq \min\{t,\, r\}$$

so that $x_t \circ y_r \in \lambda$. Hence λ is an $(\in,\in)$-fuzzy subsemigroup of S. □

Corollary 3.12. *Let λ be an $(\in, \in\vee q)$-fuzzy subsemigroup of S such that $\lambda(x) < 0.5$ for all $x \in S$. Then λ is an $(\in,\in)$-fuzzy subsemigroup of S.*

For a subset L of S, a fuzzy set χ_L^δ in S defined by

$$\chi_L^\delta : S \to [0,1],\ x \mapsto \begin{cases} \delta & \text{if } x \in L, \\ 0 & \text{otherwise,} \end{cases}$$

is called a δ-*characteristic fuzzy set* of L in S (see [6]).

Theorem 3.13. *For any subset L of S and the δ-characteristic fuzzy set χ_L^δ of L in S, the following are equivalent:*

(i) *L is a subsemigroup of S.*
(ii) *χ_L^δ is an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup of S.*

Proof. (i) ⇒ (ii) is straightforward.

Assume that χ_L^δ is an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup of S. Let $x, y \in L$. Then $\chi_L^\delta(x) = \delta = \chi_L^\delta(y)$, and so $x_\delta \in \chi_L^\delta$ and $y_\delta \in \chi_L^\delta$. It follows that $x_\delta \circ y_\delta \in\vee q_0^\delta \lambda$, which yields $\chi_L^\delta(xy) > 0$. Hence $xy \in L$ and L is a subsemigroup of S. □

Theorem 3.14. *A fuzzy set λ in S is an $(\in, \in)$-fuzzy subsemigroup of S if and only if the set*

$$Q_0^\delta(\lambda; t) := \{x \in S \mid x_t\, q_0^\delta\, \lambda\}$$

is a subsemigroup of S when it is nonempty for all $t \in (0, \delta]$.

Proof. Assume that λ is an $(\in, \in)$-fuzzy subsemigroup of S and let $x, y \in Q_0^\delta(\lambda; t)$. Then $x_t\, q_0^\delta\, \lambda$ and $y_t\, q_0^\delta\, \lambda$, i.e., $\lambda(x) + t > \delta$ and $\lambda(y) + t > \delta$. Hence $\lambda(xy) \geq \min\{\lambda(x), \lambda(y)\} > \delta - t$, and so $x_t \circ y_t\, q_0^\delta\, \lambda$. Hence $xy \in Q_0^\delta(\lambda; t)$, and therefore $Q_0^\delta(\lambda; t)$ is a subsemigroup of S.

Conversely, suppose that $Q_0^\delta(\lambda; t)$ is a subsemigroup of S and take $a, b \in S$ and $t \in (0, \delta]$ such that $\lambda(ab) + t \leq \delta < \min\{\lambda(a), \lambda(b)\} + t$. Then $a, b \in Q_0^\delta(\lambda; t)$, and so $ab \in Q_0^\delta(\lambda; t)$. Thus $\lambda(ab) + t > \delta$, a contradiction. Hence $\lambda(xy) \geq \min\{\lambda(x), \lambda(y)\}$ for all $x, y \in S$, that is, λ is an $(\in, \in)$-fuzzy subsemigroup of S. □

Theorem 3.15. *If λ is an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup of S, then the set $Q_0^\delta(\lambda; t)$ is a subsemigroup of S when it is nonempty for all $t \in (\frac{\delta}{2}, 1]$.*

Proof. Assume that λ is an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup of S and let $t \in (\frac{\delta}{2}, 1]$ such that $Q_0^\delta(\lambda; t) \neq \varnothing$. Let $x, y \in Q_0^\delta(\lambda; t)$. Then $x_t\, q_0^\delta\, \lambda$ and $y_t\, q_0^\delta\, \lambda$, i.e., $\lambda(x)+t > \delta$ and $\lambda(y) + t > \delta$. Using Theorem 3.8(ii), we have

$$\lambda(xy) \geq \min\{\lambda(x), \lambda(y), \tfrac{\delta}{2}\}. \tag{3.3}$$

If $\min\{\lambda(x), \lambda(y)\} \geq \frac{\delta}{2}$, then $\lambda(xy) \geq \frac{\delta}{2} > \delta - t$ by (3.3). If $\min\{\lambda(x), \lambda(y)\} < \frac{\delta}{2}$, then $\lambda(xy) \geq \min\{\lambda(x), \lambda(y)\} > \delta - t$ by (3.3). Hence $xy \in Q_0^\delta(\lambda; t)$, and $Q_0^\delta(\lambda; t)$ is a subsemigroup of S for all $t \in (\frac{\delta}{2}, 1]$. □

Corollary 3.16. *If λ is an $(\in, \in\vee q)$-fuzzy subsemigroup of S, then the set*

$$Q(\lambda; t) := \{x \in S \mid x_t\, q\, \lambda\}$$

is a subsemigroup of S when it is nonempty for all $t \in (0.5, 1]$.

Theorem 3.17. *A fuzzy set λ in S is an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup of S if and only if the set*

$$U(\lambda; t) := \{x \in S \mid \lambda(x) \geq t\}$$

is a subsemigroup of S for all $t \in (0, \frac{\delta}{2}]$.

Proof. Assume that λ is an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup of S. Let $t \in (0, \frac{\delta}{2}]$ and $x, y \in U(\lambda; t)$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq t$. It follows from Theorem 3.8 that

$$\lambda(xy) \geq \min\{\lambda(x), \lambda(y), \tfrac{\delta}{2}\} \geq \min\{t, \tfrac{\delta}{2}\} = t$$

and so that $xy \in U(\lambda; t)$. Therefore $U(\lambda; t)$ is a subsemigroup of S.

Conversely, let λ be a fuzzy set in S such that $U(\lambda; t)$ is a subsemigroup of S for all $t \in (0, \frac{\delta}{2}]$. Suppose that there are elements a and b of S such that

$$\lambda(ab) < \min\{\lambda(a), \lambda(b), \tfrac{\delta}{2}\},$$

and take $t \in (0, \delta]$ such that $\lambda(ab) < t \leq \min\{\lambda(a), \lambda(b), \frac{\delta}{2}\}$. Then $a, b \in U(\lambda; t)$ and $t \leq \frac{\delta}{2}$, which implies that $ab \in U(\lambda; t)$ since $U(\lambda; t)$ is a subsemigroup of S. This induces $\lambda(ab) \geq t$, and this is a contradiction. Hence $\lambda(xy) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\}$ for all $x, y \in S$, and therefore λ is an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup of S by Theorem 3.8. □

We say that the subsemigroup $U(\lambda; t)$ in Theorem 3.17 is a *level subsemigroup* of S.

Corollary 3.18. *A fuzzy set λ in S is an $(\in, \in\vee q)$-fuzzy subsemigroup of S if and only if the set*

$$U(\lambda; t) := \{x \in S \mid \lambda(x) \geq t\}$$

is a subsemigroup of S for all $t \in (0, 0.5]$.

Theorem 3.19. *For any fuzzy set λ in S, the following are equivalent.*

(i) *λ is an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup of S.*
(ii) *The set $U_0^\delta(\lambda; t) := \{x \in S \mid x_t \in\vee q_0^\delta\, \lambda\}$ is a subsemigroup of S for all $t \in (0, \delta]$ with $U_0^\delta(\lambda; t) \neq \varnothing$.*

Proof. Assume that λ is an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup of S. Let $t \in (0, \delta]$ be such that $U_0^\delta(\lambda; t) \neq \varnothing$. Note that $U_0^\delta(\lambda; t) = U(\lambda; t) \cup Q_0^\delta(\lambda; t)$. Let $x, y \in U_0^\delta(\lambda; t)$. Then $x_t \in\vee q_0^\delta \lambda$ and $y_t \in\vee q_0^\delta \lambda$, that is, $\lambda(x) \geq t$ or $\lambda(x) + t > \delta$, and $\lambda(y) \geq t$ or $\lambda(y) + t > \delta$. We consider four cases:

(i) $\lambda(x) \geq t$ and $\lambda(y) \geq t$,
(ii) $\lambda(x) \geq t$ and $\lambda(y) + t > \delta$,
(iii) $\lambda(x) + t > \delta$ and $\lambda(y) \geq t$,
(iv) $\lambda(x) + t > \delta$ and $\lambda(y) + t > \delta$.

For the first case, Theorem 3.8 implies that

$$\lambda(xy) \geq \min\{\lambda(x), \lambda(y), \tfrac{\delta}{2}\} \geq \min\{t, \tfrac{\delta}{2}\} = \begin{cases} \frac{\delta}{2} & \text{if } t > \frac{\delta}{2}, \\ t & \text{if } t \leq \frac{\delta}{2}. \end{cases}$$

Hence $(xy)_t \in \lambda$ or $\lambda(xy) + t = \frac{\delta}{2} + t > \frac{\delta}{2} + \frac{\delta}{2} = \delta$. It follows that $xy \in U(\lambda; t) \cup Q_0^\delta(\lambda; t) = U_0^\delta(\lambda; t)$. For the second case, if $t > \frac{\delta}{2}$ then $\delta - t < \frac{\delta}{2}$. Hence

$$\lambda(xy) \geq \min\{\lambda(y), \tfrac{\delta}{2}\} > \delta - t$$

whenever $\min\{\lambda(y), \frac{\delta}{2}\} \leq \lambda(x)$, and $\lambda(xy) \geq \lambda(x) \geq t$ whenever $\min\{\lambda(y), \frac{\delta}{2}\} > \lambda(x)$. Thus $xy \in U(\lambda; t) \cup Q_0^\delta(\lambda; t) = U_0^\delta(\lambda; t)$. If $t \leq \frac{\delta}{2}$, then $\delta - t \geq \frac{\delta}{2}$ and so

$$\lambda(xy) \geq \min\{\lambda(x), \tfrac{\delta}{2}\} \geq t$$

whenever $\min\{\lambda(x), \frac{\delta}{2}\} \leq \lambda(y)$, and $\lambda(xy) \geq \lambda(y) > \delta - t$ whenever $\min\{\lambda(x), \frac{\delta}{2}\} > \lambda(y)$. Hence $xy \in U(\lambda; t) \cup Q_0^\delta(\lambda; t) = U_0^\delta(\lambda; t)$. We have similar result for the third case. For the final case, if $t > \frac{\delta}{2}$ then $\delta - t < \frac{\delta}{2}$. It follows that

$$\begin{aligned} \lambda(xy) &\geq \min\{\lambda(x), \lambda(y), \tfrac{\delta}{2}\} \\ &= \begin{cases} \frac{\delta}{2} > \delta - t & \text{if } \min\{\lambda(x), \lambda(y)\} \geq \frac{\delta}{2}, \\ \min\{\lambda(x), \lambda(y)\} > \delta - t & \text{if } \min\{\lambda(x), \lambda(y)\} < \frac{\delta}{2}, \end{cases} \end{aligned}$$

and so that $xy \in Q_0^\delta(\lambda; t) \subseteq U_0^\delta(\lambda; t)$. If $t \leq \frac{\delta}{2}$, then $\delta - t \geq \frac{\delta}{2}$ and thus

$$\begin{aligned} \lambda(xy) &\geq \min\{\lambda(x), \lambda(y), \tfrac{\delta}{2}\} \\ &= \begin{cases} \frac{\delta}{2} \geq t & \text{if } \min\{\lambda(x), \lambda(y)\} \geq \frac{\delta}{2}, \\ \min\{\lambda(x), \lambda(y)\} > \delta - t & \text{if } \min\{\lambda(x), \lambda(y)\} < \frac{\delta}{2}. \end{cases} \end{aligned}$$

Hence $xy \in U(\lambda; t) \cup Q_0^\delta(\lambda; t) = U_0^\delta(\lambda; t)$. Consequently, $U_0^\delta(\lambda; t)$ is a subsemigroup of S for all $t \in (0, \delta]$ with $U_0^\delta(\lambda; t) \neq \varnothing$.

Conversely, suppose that (ii) is valid and there exist $a, b \in S$ such that

$$\lambda(ab) < \min\{\lambda(a), \lambda(b), \tfrac{\delta}{2}\}.$$

Then $\lambda(ab) < r \leq \min\{\lambda(a), \lambda(b), \frac{\delta}{2}\}$ for some $r \in (0, \frac{\delta}{2}]$. It follows that $a, b \in U(\lambda; r) \subseteq U_0^\delta(\lambda; r)$ so that $ab \in U_0^\delta(\lambda; r)$. Hence $\lambda(ab) \geq r$ or $\lambda(ab) + r > \delta$, which is a contradiction. Therefore $\lambda(xy) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\}$ for all $x, y \in S$. Using Theorem 3.8, we know that λ is an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup of S. □

Definition 3.20. For a fuzzy set λ in S, we define two fuzzy sets λ^+ and λ^- in S as follows:

$$\lambda^+ : S \to [0, 1], \; x \mapsto \max\{\lambda(x), \tfrac{\delta}{2}\}$$

and

$$\lambda^- : S \to [0,1],\ x \mapsto \min\{\lambda(x), \tfrac{\delta}{2}\}.$$

Proposition 3.21. *For two fuzzy sets λ and ν in S, we have*

$$(\lambda \wedge \nu)^- = \lambda^- \wedge \nu^-,\ (\lambda \vee \nu)^- = \lambda^- \vee \nu^-,\ (\lambda \circ \nu)^- = \lambda^- \circ \nu^-.$$

$$(\lambda \wedge \nu)^+ = \lambda^+ \wedge \nu^+,\ (\lambda \vee \nu)^+ = \lambda^+ \vee \nu^+ \text{ and } (\lambda \circ \nu)^+ \geq \lambda^+ \circ \nu^+.$$

If every element x of S is expressible as $x = yz$, then $(\lambda \circ \nu)^+ = \lambda^+ \circ \nu^+$.

Proof. First parts are straightforward. Assume that every element x of S is expressible as $x = yz$. Then

$$\begin{aligned}
(\lambda \circ \nu)^+(x) &= \max\{(\lambda \circ \nu)(x), \tfrac{\delta}{2}\} \\
&= \max\left\{\bigvee_{x=yz} \min\{\lambda(y), \nu(z)\}, \tfrac{\delta}{2}\right\} \\
&= \bigvee_{x=yz} \max\left\{\min\{\lambda(y), \nu(z)\}, \tfrac{\delta}{2}\right\} \\
&= \bigvee_{x=yz} \min\left\{\max\{\lambda(y), \tfrac{\delta}{2}\}, \max\{\lambda(z), \tfrac{\delta}{2}\}\right\} \\
&= \bigvee_{x=yz} \min\left\{\lambda^+(y), \lambda^+(z)\right\} \\
&= (\lambda^+ \circ \nu^+)(x).
\end{aligned}$$

This completes the proof. □

Theorem 3.22. *If λ is an $(\in, \in\vee\, q_0^\delta)$-fuzzy subsemigroup of S, then λ^- is an $(\in, \in)$-fuzzy subsemigroup of S.*

Proof. For any $x, y \in S$, we have

$$\begin{aligned}
\lambda^-(xy) = \min\{\lambda(xy), \tfrac{\delta}{2}\} &\geq \min\{\min\{\lambda(x), \lambda(y), \tfrac{\delta}{2}\}, \tfrac{\delta}{2}\} \\
&= \min\{\min\{\lambda(x), \tfrac{\delta}{2}\}, \min\{\lambda(y), \tfrac{\delta}{2}\}\} \\
&= \min\{\lambda^-(x), \lambda^-(y)\}.
\end{aligned}$$

Thus λ^- is an $(\in, \in)$-fuzzy subsemigroup of S. □

Using Theorems 3.22 and 3.14, we have the following corollary.

Corollary 3.23. *If λ is an $(\in, \in\vee\, q_0^\delta)$-fuzzy subsemigroup of S, the set*

$$Q_0^\delta(\lambda^-; t) := \{x \in S \mid x_t\, q_0^\delta\, \lambda^-\}$$

is a subsemigroup of S when it is nonempty for all $t \in (0, \delta]$.

Theorem 3.24. *The intersection of any family of $(\in, \in\vee\, q_0^\delta)$-fuzzy subsemigroups of S is an $(\in, \in\vee\, q_0^\delta)$-fuzzy subsemigroups of S.*

Proof. Let $\{\lambda_i \mid i \in \Lambda\}$ be a family of $(\in, \in\vee\, q_0^\delta)$-fuzzy subsemigroups of S and let λ be the intersection of this family. We verify that $x_t \circ y_r \in\vee\, q_0^\delta\, \lambda$ for all $x, y \in S$ and $t, r \in (0, \delta]$ with $x_t \in \lambda$ and $y_r \in \lambda$. Assume that $a_t \in \lambda$, $b_r \in \lambda$ and $a_t \circ b_r\, \overline{\in\vee\, q_0^\delta}\, \lambda$

for some $a, b \in S$ and $t, r \in (0, \delta]$. Then $\lambda(ab) < \min\{t, r\}$ and $\lambda(ab) + \min\{t, r\} \leq \delta$. Hence $\lambda(ab) < \frac{\delta}{2}$. Since each λ_i is an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup of S, the family $\{\lambda_i \mid i \in \Lambda\}$ can be divided into two disjoint parts:

$$\Lambda_1 := \{\lambda_i \mid \lambda_i(ab) \geq \min\{t, r\}\}$$

and

$$\Lambda_2 := \{\lambda_i \mid \lambda_i(ab) < \min\{t, r\} \text{ and } \lambda_i(ab) + \min\{t, r\} > \delta\}.$$

If $\lambda_i(ab) \geq \min\{t, r\}$ for all λ_i, then $\lambda(ab) \geq \min\{t, r\}$ which is a contradiction. Hence $\lambda_i(ab) < \min\{t, r\}$ and $\lambda_i(ab) + \min\{t, r\} > \delta$ for some λ_i. It follows that $\min\{t, r\} > \frac{\delta}{2}$, whence $\lambda_i(a) \geq \lambda(a) \geq t \geq \min\{t, r\} > \frac{\delta}{2}$ for all $\lambda_i \in \Lambda_2$. Similarly, $\lambda_i(b) > \frac{\delta}{2}$ for all $\lambda_i \in \Lambda_2$. Now suppose that $t_i := \lambda_i(ab) < \frac{\delta}{2}$ for some λ_i, and let $r_i \in (0, \frac{\delta}{2})$ such that $t_i < r_i$. Then $\lambda_i(a) > \frac{\delta}{2} > r_i$ and $\lambda_i(b) > \frac{\delta}{2} > r_i$, i.e., $a_{r_i} \in \lambda_i$ and $b_{r_i} \in \lambda_i$. But $\lambda_i(ab) = t_i < r_i$ and $\lambda_i(ab) + r_i < \delta$. Hence $a_{r_i} \circ b_{r_i} \overline{\in\vee q_0^\delta}\, \lambda_i$. This contradicts that λ_i is an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup of S. Thus $\lambda_i(ab) \geq \frac{\delta}{2}$ for all λ_i, and so $\lambda(ab) \geq \frac{\delta}{2}$. This is impossible, and therefore $x_t \circ y_r \in\vee q_0^\delta\, \lambda$ for all $x, y \in S$ and $t, r \in (0, \delta]$ with $x_t \in \lambda$ and $y_r \in \lambda$. □

Corollary 3.25. *The intersection of any family of $(\in, \in\vee q)$-fuzzy subsemigroups of S is an $(\in, \in\vee q)$-fuzzy subsemigroups of S.*

Theorem 3.26. *For any chain $L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_n = S$ of subsemigroups of S there exists an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup of S for which level subsemigroups coincide with this chain.*

Proof. Let $t_0, t_1, \cdots, t_n$ be a finite decreasing sequence in $[0, \delta]$. Consider the fuzzy set λ in S defined by $\lambda(L_0) = t_0$ and $\lambda(L_k \setminus L_{k-1}) = t_k$ for $0 < k \leq n$. Let $x, y \in S$. If $x, y \in L_k \setminus L_{k-1}$, then $xy \in L_k$ and so $\lambda(xy) \geq t_k = \min\{\lambda(x), \lambda(y)\}$. Now let $x \in L_i \setminus L_{i-1}$ and $y \in L_j \setminus L_{j-1}$ where $i \neq j$. We may assume that $i > j$ without loss of generality. Then $L_j \subset L_i$, $\lambda(x) = t_i < t_j = \lambda(y)$ and $xy \in L_j$. It follows that $\lambda(xy) \geq t_i = \min\{\lambda(x), \lambda(y)\}$. Therefore λ is an $(\in, \in)$-fuzzy subsemigroup of S, and so it is an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroups of S. Note that λ has the values $t_0, t_1, \cdots, t_n$ only, and thus the level subsemigroups form a chain: $U(\lambda; t_0) \subset U(\lambda; t_1) \subset \cdots \subset U(\lambda; t_n) = S$. We now prove that $U(\lambda; t_k) = L_k$ for $0 \leq k \leq \delta$. Indeed, $U(\lambda; t_0) = L_0$. Moreover, $L_k \subseteq U(\lambda; t_k)$ for $k \neq 0$. If $x \in U(\lambda; t_k)$, then $\lambda(x) \geq t_k$ and so $x \notin L_i$ for $i > k$. Hence $\lambda(x) \in \{t_0, t_1, \cdots, t_k\}$, which implies $x \in L_i$ for some $i \leq k$. Since $L_i \subseteq L_k$, it follows that $x \in L_k$. Consequently, $U(\lambda; t_k) = L_k$ for every $k = 1, 2, \cdots, n$. This completes the proof. □

Theorem 3.27. *Let $\{L_t \mid t \in \Lambda \subseteq (0, \frac{\delta}{2}]\}$ be a collection of subsemigroups of S for which their union is S and for any $s, t \in \Lambda$, $s < t$ if and only if $L_t \subset L_s$. Then a fuzzy set λ defined by*

$$\lambda : S \to [0, 1],\ x \mapsto \sup\{r \in \Lambda \mid x \in L_r\}$$

is an $(\in, \in\vee q_0^\delta)$-fuzzy subsemigroup of S.

Proof. According to Theorem 3.17, it is sufficient to show that the nonempty set $U(\lambda; t)$ is a subsemigroup of S for all $t \in (0, \frac{\delta}{2}]$. We consider two cases:

$$\text{(i) } t = \sup\{s \in \Lambda \mid s < t\} \text{ and (ii) } t \neq \sup\{s \in \Lambda \mid s < t\}.$$

The case (i) implies that

$$x \in U(\lambda;t) \;\Leftrightarrow\; (x \in L_s \;\forall s < t) \;\Leftrightarrow\; x \in \bigcap_{s<t} L_s.$$

So, $U(\lambda;t) = \bigcap_{s<t} L_s$, and it is a subsemigroup of S. For the second case, if $x \in \bigcup_{s\geq t} L_s$, then $x \in L_s$ for some $s \geq t$. Hence $\lambda(x) \geq s \geq t$, and so $x \in U(\lambda;t)$. Now let $x \notin \bigcup_{s\geq t} L_s$. Then $x \notin L_s$ for all $s \geq t$. Since $t \neq \sup\{s \in \Lambda \mid s < t\}$, there exists $\varepsilon > 0$ such that $\Lambda \cap (t-\varepsilon, t) = \varnothing$. Thus $x \notin L_s$ for all $s > t-\varepsilon$, and so if $x \in L_s$ then $s \leq t-\varepsilon$. It follows that $\lambda(x) \leq t-\varepsilon < t$, and that $x \notin U(\lambda;t)$. Therefore $U(\lambda;t) = \bigcup_{s\geq t} L_s$, which is clearly a subsemigroup of S. Consequently, λ is an $(\in, \in\vee\, q_0^\delta)$-fuzzy subsemigroup of S. □

Let λ be a fuzzy set in S. An $(\in, \in\vee\, q_0^\delta)$-fuzzy subsemigroup ν in S is said to be an $(\in, \in\vee\, q_0^\delta)$-*fuzzy subsemigroup generated* by λ in S if

(i) $\lambda \leq \nu$,
(ii) For any $(\in, \in\vee\, q_0^\delta)$-fuzzy subsemigroup γ in S, if $\lambda \leq \gamma$ then $\nu \leq \gamma$.

Theorem 3.28. *Let λ be a fuzzy set in S with finite images. Define subsemigroups L_i of S as follows:*

$L_0 = \langle\{x \in S \mid \lambda(x) \geq \frac{\delta}{2}\}\rangle$,
$L_i = \langle L_{i-1} \cup \{x \in S \mid \lambda(x) = \sup\{\lambda(z) \mid z \in S \setminus L_{i-1}\}\rangle$

for $i = 1, 2, \cdots, n$ where $n < |\mathrm{Im}(\lambda)|$ and $L_n = S$. Let λ^* be a fuzzy set in S defined by

$$\lambda^*(x) = \begin{cases} \lambda(x) & \text{if } x \in L_0, \\ \sup\{\lambda(z) \mid z \in S \setminus L_{i-1}\} & \text{if } x \in L_i \setminus L_{i-1}. \end{cases}$$

Then λ^* is the $(\in, \in\vee\, q_0^\delta)$-fuzzy subsemigroup generated by λ in S.

Proof. Note that the L_i's form a chain

$$L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n = S$$

of subsemigroups ending at S. We first show that λ^* is an $(\in, \in\vee\, q_0^\delta)$-fuzzy subsemigroup of S. Let $x, y \in S$. If $x, y \in L_0$, then $x * y \in L_0$ and so

$$\lambda^*(x*y) = \lambda(x*y) \geq \min\{\lambda(x), \lambda(y), \tfrac{\delta}{2}\} = \min\{\lambda^*(x), \lambda^*(y), \tfrac{\delta}{2}\}.$$

Let $x \in L_i \setminus L_{i-1}$ and $y \in L_j \setminus L_{j-1}$. We may assume that $i < j$ without loss of generality. Then $x, y \in L_j$ and so $x * y \in L_j$. It follows that

$$\begin{aligned} \lambda^*(x*y) &\geq \sup\{\lambda(z) \mid z \in S \setminus L_{j-1}\} \\ &\geq \min\{\sup\{\lambda(z) \mid z \in S \setminus L_{i-1}\}, \sup\{\lambda(z) \mid z \in S \setminus L_{j-1}\}, \tfrac{\delta}{2}\} \\ &= \min\{\lambda^*(x), \lambda^*(y), \tfrac{\delta}{2}\}. \end{aligned}$$

Hence λ^* is an $(\in, \in\vee\, q_0^\delta)$-fuzzy subsemigroup of S whose $\in\vee\, q_0^\delta$-level subsemigroups are precisely the members of the chain above. Obviously, $\lambda \subseteq \lambda^*$ by the construction of λ^*. Now let ν be any $(\in, \in\vee\, q_0^\delta)$-fuzzy subsemigroup of S such that $\lambda \subseteq \nu$. If $x \in L_0$, then $\lambda^*(x) = \lambda(x) \leq \nu(x)$. Let $\{B_{t_i}\}$ be the class of $\in\vee\, q_0^\delta$-level subsemigroups of ν in S. Let $x \in L_1 \setminus L_0$. Then $\lambda^*(x) = \sup\{\lambda(z) \mid z \in S \setminus L_0\}$ and $L_1 = \langle K_1 \rangle$ where

$$K_1 = L_0 \cup \{x \in S \mid \lambda(x) = \sup\{\lambda(z) \mid z \in S \setminus L_0\}\}.$$

Let $x \in K_1 \setminus L_0$. Then $\lambda(x) = \sup\{\lambda(z) \mid z \in S \setminus L_0\}$. Since $\lambda \subseteq \nu$, it follows that

$$\sup\{\lambda(z) \mid z \in S \setminus L_0\} \leq \inf\{\nu(x) \mid x \in K_1 \setminus L_0\} \leq \nu(x).$$

Putting $t_{i1} = \inf\{\nu(x) \mid x \in K_1 \setminus L_0\}$, we get $x \in B_{t_{i1}}$ and hence $K_1 \setminus L_0 \subseteq B_{t_{i1}}$. Since $L_0 \subseteq B_{t_{i1}}$, we have $L_1 = \langle K_1 \rangle \subseteq B_{t_{i1}}$. Thus $\nu(x) \geq t_{i1}$ for all $x \in L_1$. Therefore

$$\lambda^*(x) = \sup\{\lambda(z) \mid z \in S \setminus L_0\} \leq t_{i1} \leq \nu(x)$$

for all $x \in L_1 \setminus L_0$. Similarly, we can prove that $\lambda^*(x) \leq \nu(x)$ for all $x \in L_i \setminus L_{i-1}$ where $2 \leq i \leq n$. Consequently, λ is the $(\in, \in \vee\, q_0^\delta)$-fuzzy subsemigroup generated by λ in S. $\square$

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Young Bae Jun (skywine@gmail.com)
Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea

Mehmet Ali Öztürk (mehaliozturk@gmail.com)
Department of Mathematics, Faculty of Arts and Sciences, Adıyaman University 02040 Adıyaman, Turkey

G. Muhiuddin (chishtygm@gmail.com)
Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia