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Abstract. The first aim of this study is to define soft sequential compact metric spaces and to investigate some important theorems on soft sequential compact metric space. Second is to introduce $\tilde{\varepsilon}-$net and totally bounded soft metric space and study properties of this space. Third is to define Lebesque number for soft sets and soft uniformly continuous mapping and investigate some theorems in detail.

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Corresponding Author: Murat Ibrahim YAZAR (myazar@kmu.edu.tr)

1. Introduction

The soft set theory,initiated by Molodtsov [10] in 1999, is one of the branches of mathematics, which aims to describe phenomena and concepts of an ambiguous, vague, undefined and imprecise meaning. Soft set theory is applicable where there is no clearly defined mathematical model. After Molodtsov's work, many papers concerning soft sets have been published [1, 3, 6, 8, 9].

M. Shabir and M. Naz [13] presented the notion of soft topology on a soft set and proved basic properties concerning soft topological spaces. Later, many researches about soft topological spaces were studied in [2, 7, 14, 19].

Soft mapping space and soft uniform space were studied in [11, 12]. Zahedi et al. studied fuzzy soft product topology and fuzzy soft boundary in [17, 18]. Xie introduced soft point and gave the relationship between soft points and soft sets in [15]. Since soft sets are defined by mappings we can not handle them like an ordinary set. In this point of view, defining a soft point rely both on the structure of the soft set and the mapping. This fact give rise to different opportunities for defining a soft point.

There exists several different approaches for introducing a soft point for a soft set. Das and Samanta introduced the concept of soft element by using a function.

Then by using soft element they introduced soft real number in [4]. In the studies [2, 5] the soft point is defined by setting some conditions on parameters. Also Xie in [15] introduced the concept of soft point in a different approach. Naturally, the different approaches on the concept of soft point makes this issue controversial. In this study, we use the concept of soft point that is defined in [2, 5] just because of better understanding the nature of this concept of soft point and we progress on using this concept.

Recently Das and Samanta introduced a different notion of soft metric space by using a different concept of soft point and investigated some basic properties of these spaces in [4, 5].

The purpose of this paper is to make contribution for investigating on soft metric spaces and we focus on soft compact sets in soft metric spaces and explore the differences and similarities between the point set topology and soft topology. In the present paper, firstly we define soft sequential compact metric spaces and totally bounded soft metric spaces. Further,we investigate some important theorems on these spaces. We introduced the Lebesque number for a soft open cover in soft metric spaces. Secondly, we show that soft compact metric space and soft sequential metric space are equivalent structures. Finally, we introduced soft uniformly continuous mapping and show that every continuous mapping in a soft sequential compact metric space is a soft uniformly continuous mapping.

2. Preliminaries

In this section, we recall some basic notions in soft set theory. Throughout this paper, X refers to an initial universe, E is the set of all parameters for X.

Definition 2.1 ([10])**.** A pair (F, E) is called a soft set over X, where F is a mapping given by $F : E \to P(X)$. In other words, a soft set over X is a parameterized family of subsets of the universe X.

Definition 2.2 ([9])**.** A soft set (F, E) over X is said to be a null soft set denoted by Φ, if for all $e \in E$, $F(e) = \varnothing$. A soft set (F, E) over X is said to be an absolute soft set denoted by $\tilde{X}$, if for all $e \in E$, $F(e) = X$.

Definition 2.3 ([9])**.** For two soft sets (F, A) and (G, B) over X, (F, A) is called a soft subset of (G, B), if

(i) $A \subseteq B$, and

(ii) $\forall e \in A$, $F(e) \subseteq G(e)$ are identical approximations.

This relationship is denoted by $(F, A)\tilde{\subset}(G, B)$. Similarly, (F, A) is said to be a soft subset of (G, B), if (G, B) is a soft superset of (F, A). This relationship is denoted by $(F, A)\tilde{\supset}(G, B)$.

Definition 2.4 ([9])**.** The intersection of two soft sets (F, A) and (G, B) over a common universe X is the soft set (H, C), where $C = A \cap B$ and $H(e) = F(e) \cap G(e)$ for all $e \in C$. This is denoted by $(F, A)\tilde{\cap}\ (G, B) = (H, C)$.

Definition 2.5 ([9])**.** The union of two soft sets (F, A) and (G, B) over X is the soft set, where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e) & ,\text{if } e \in A - B \\ G(e) & ,\text{if } e \in B - A \\ F(e) \cup G(e) & ,\text{if } e \in A \cap B \end{cases}$$

This relationship is denoted by $(F, A)\tilde{\cup}\,(G, B) = (H, C)$.

Definition 2.6 ([9])**.** The complement of a soft set (F, E) is denoted by $(F, E)^c$ and is defined by $(F, E)^c = (F^c, E)$, where $F^c : E \to P(X)$ is a mapping given by $F^c(e) = X - F(e)$ for all $e \in E$.

Definition 2.7 ([4])**.** Let $\mathbb{R}$ be the set of all real numbers, $B(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$ and E be taken as a set of parameters. Then a mapping $F : E \to B(\mathbb{R})$ is called a soft real set. If specifically a soft real set is a singleton soft set, then it will be called a soft real number and denoted $\tilde{r}, \tilde{s}, \tilde{t}$ etc.

$\bar{0}, \bar{1}$ are the soft real numbers where $\bar{0}(e) = 0$, $\bar{1}(e) = 1$ for all $e \in E$ respectively.

Definition 2.8 ([4])**.** Let $\tilde{r}, \tilde{s}$ be two soft real numbers, then the following statement are hold:

(i) $\tilde{r}\tilde{\leq}\tilde{s}$, if $\tilde{r}(e)\leq\tilde{s}(e)$, for all $e \in E$,
(ii) $\tilde{r}\tilde{\geq}\tilde{s}$ if $\tilde{r}(e)\geq\tilde{s}(e)$, for all $e \in E$,
(iii) $\tilde{r}\tilde{<}\tilde{s}$ if $\tilde{r}(e)<\tilde{s}(e)$, for all $e \in E$,
(iv) $\tilde{r}\tilde{>}\tilde{s}$ if $\tilde{r}(e)>\tilde{s}(e)$, for all $e \in E$.

Definition 2.9 ([5, 2])**.** A soft set (F, E) over X is said to be a soft point, if there is exactly one $e \in E$, such that $F(e) = \{x\}$ for some $x \in X$ and $F(\acute{e}) = \varnothing$ for all $\acute{e} \in E/\{e\}$. It will be denoted by $\tilde{x}_e$.

Definition 2.10 ([5, 2])**.** Two soft points $\tilde{x}_e, \tilde{y}_{\acute{e}}$ are said to be soft equal, if $e = \acute{e}$ and $F(e) = F(\acute{e})$, i.e., $x = y$. Thus $\tilde{x}_e \neq \tilde{y}_{\acute{e}} \Leftrightarrow x \neq y$ or $e \neq \acute{e}$.

Proposition 2.11 ([5, 2])**.** *The union of any collection of soft points can be considered as a soft set and every soft set can be expressed as a union of all soft points belonging to it;*

$$(F, E) = \bigcup_{\tilde{x}_e \in (F,E)} \tilde{x}_e.$$

Definition 2.12 ([13])**.** Let τ be a collection of soft sets over X. Then τ is said to be a soft topology on X, if

(i) $\Phi, \tilde{X}$ belong to τ,
(ii) The union of any number of soft sets in τ belongs to τ,
(iii) The intersection of any two soft sets in τ belongs to τ.

The triplet (X, τ, E) is called a soft topological space over X. Every member of τ is called soft open set.

Definition 2.13 ([6])**.** Let (X, τ, E) be a soft topological space over X. A soft set (F, E) in (X, τ, E) is called a soft neighborhood of the soft point $\tilde{x}_e \in (F, E)$, if there exists a soft open set (G, E) such that $\tilde{x}_e \in (G, E) \subset (F, E)$.

Definition 2.14 ([6])**.** Let (X, τ, E) and $(Y, \acute{\tau}, E)$ be two soft topological spaces, $f : (X, \tau, E) \to (Y, \acute{\tau}, E)$ be a mapping. For each soft neighborhood (H, E) of $f(\tilde{x}_e)$

, if there exists a soft neighborhood (F, E) of $\tilde{x}_e$ such that $f((F, E)) \subset (H, E)$, then f is said to be a soft continuous mapping at $\tilde{x}_e$.

If f is a soft continuous mapping for all $\tilde{x}_e$, then it is called a soft continuous mapping.

Let $\tilde{X}$ be the absolute soft set and $SP(\tilde{X})$ be the collection of all soft points of $\tilde{X}$ and $\mathbb{R}(E)^*$ denote the set of all non-negative soft real numbers.

Definition 2.15 ([5])**.** A mapping $\tilde{d} : SP(\tilde{X}) \times SP(\tilde{X}) \to \mathbb{R}(E)^*$ is said to be a soft metric on the soft set $\tilde{X}$ if $\tilde{d}$ satisfies the following conditions:

(M1) $\tilde{d}(\tilde{x}_{e_1}, \tilde{y}_{e_2}) \tilde{\geq} \bar{0}$, for all $\tilde{x}_{e_1}, \tilde{y}_{e_2} \in SP(\tilde{X})$,
(M2) $\tilde{d}(\tilde{x}_{e_1}, \tilde{y}_{e_2}) = \bar{0}$ if and only if $\tilde{x}_{e_1} = \tilde{y}_{e_2}$,
(M3) $\tilde{d}(\tilde{x}_{e_1}, \tilde{y}_{e_2}) = \tilde{d}(\tilde{y}_{e_2}, \tilde{x}_{e_1})$, for all $\tilde{x}_{e_1}, \tilde{y}_{e_2} \in SP(\tilde{X})$,
(M4) For all $\tilde{x}_{e_1}, \tilde{y}_{e_2}, \tilde{z}_{e_3} \in SP(\tilde{X})$, $\tilde{d}(\tilde{x}_{e_1}, \tilde{z}_{e_3}) \tilde{\leq} \tilde{d}(\tilde{x}_{e_1}, \tilde{y}_{e_2}) + \tilde{d}(\tilde{y}_{e_2}, \tilde{z}_{e_3})$.

The soft set $\tilde{X}$ with a soft metric $\tilde{d}$ on $\tilde{X}$ is called a soft metric space and denoted by $(\tilde{X}, \tilde{d}, E)$.

Definition 2.16 ([5])**.** Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space and $\tilde{\varepsilon}$ be a non-negative soft real number. $B(\tilde{x}_e, \tilde{\varepsilon}) = \left\{\tilde{y}_{\acute{e}} \in \tilde{X} : \tilde{d}(\tilde{x}_e, \tilde{y}_{\acute{e}}) \tilde{<} \tilde{\varepsilon}\right\} \subset SP(\tilde{X})$ is called a soft open ball with center $\tilde{x}_e$ and radius $\tilde{\varepsilon}$ and $B[\tilde{x}_e, \tilde{\varepsilon}] = \left\{\tilde{y}_{\acute{e}} \in \tilde{X} : \tilde{d}(\tilde{x}_e, \tilde{y}_{\acute{e}}) \tilde{\leq} \tilde{\varepsilon}\right\} \subset SP(\tilde{X})$ is called a soft closed ball with center $\tilde{x}_e$ and radius $\tilde{\varepsilon}$.

Definition 2.17 ([5])**.** Let $\left\{\tilde{x}^n_{e_n}\right\}$ be a sequence of soft points in a soft metric space $(\tilde{X}, \tilde{d}, E)$. The sequence $\left\{\tilde{x}^n_{e_n}\right\}$ is said to be convergent in $(\tilde{X}, \tilde{d}, E)$ if there is a soft point $\tilde{y}_{e_*} \in \tilde{X}$ such that $\tilde{d}(\tilde{x}^n_{e_n}, \tilde{y}_{e_*}) \to \bar{0}$ as $n \to \infty$. (The necessary and sufficient condition for convergence is $x^n \to y$ and $e_n \to e_*$ as $n \to \infty$,where x^n is a sequence in X, e_n is a sequence of parameters and $y \in X$, $e_* \in E$.)

Theorem 2.18 ([5])**.** *Limit of a sequence in a soft metric space, if exists, is unique.*

Definition 2.19 ([5])**.** (Cauchy Sequence). A sequence $\left\{\tilde{x}^n_{e_n}\right\}$ of soft points in $(\tilde{X}, \tilde{d}, E)$ is considered as a Cauchy sequence in $\tilde{X}$ if corresponding to every $\tilde{\varepsilon} \tilde{>} \bar{0}$, $\exists m \in \mathbb{N}$ such that $\tilde{d}(\tilde{x}^i_{e_i}, \tilde{y}^j_{e_j}) \tilde{<} \tilde{\varepsilon}$ for all $i, j \geq m$, i.e. $\tilde{d}(\tilde{x}^i_{e_i}, \tilde{y}^j_{e_j}) \to \bar{0}$ as $i, j \to \infty$.

Definition 2.20 ([5])**.** (Complete Metric Space). A soft metric space $(\tilde{X}, \tilde{d}, E)$ is called complete soft metric space, if every Cauchy sequence in $\tilde{X}$ converges to some point of $\tilde{X}$. The soft metric space $(\tilde{X}, \tilde{d}, E)$ is called incomplete soft metric space if it is not complete.

Definition 2.21 ([16])**.** Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space. A function (f, φ) : $(\tilde{X}, \tilde{d}, E) \to (\tilde{X}, \tilde{d}, E)$ is called a soft contraction mapping, if there exists a soft real number $\tilde{\alpha} \in \mathbb{R}(E)$, $\tilde{0} \tilde{\leq} \tilde{\alpha} \tilde{<} \tilde{1}$ ($\mathbb{R}(E)$, denotes the soft real numbers set) such that for every soft points $\tilde{x}_e, \tilde{y}_{e'} \tilde{\in} SP(\tilde{X})$, we have $\tilde{d}((f, \varphi)(\tilde{x}_e), (f, \varphi)(\tilde{y}_{e'})) \tilde{\leq} \tilde{\alpha} \tilde{d}(\tilde{x}_e, \tilde{y}_{e'})$.

3. Compact sets in soft metric spaces

In this section we introduce soft sequential compact metric space and totally bounded soft metric space. Furthermore, we present some important properties about these spaces.

Definition 3.1. Let $\{\tilde{x}^n_{e_n}\}$ be a soft sequence of soft points in a soft metric space $(\tilde{X}, \tilde{d}, E)$. A subsequence of the soft sequence $\{\tilde{x}^n_{e_n}\}$ is a soft sequence of the form $\left\{\tilde{x}^{n_k}_{e_{n_k}}\right\}$, where $\{\tilde{x}^k_{e_k}\}$ is a strictly increasing soft sequence of natural numbers and $k \in \mathbb{N}$.

Definition 3.2. Let $\{\tilde{x}^n_{e_n}\}$ be a sequence of soft points in a soft metric space $(\tilde{X}, \tilde{d}, E)$. A soft point $\tilde{x}_e \in \tilde{X}$ is a cluster soft point of the soft sequence $\{\tilde{x}^n_{e_n}\}$ if for every soft neighborhood (F, E) of $\tilde{x}_e$ and for every $n \in \mathbb{N}$ there exists a $m \geq n$ such that $\tilde{x}^m_{e_m} \in (F, E)$, where $m \in \mathbb{N}$.

Definition 3.3. Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space. $(\tilde{X}, \tilde{d}, E)$ is called a soft sequential compact metric space, if every soft sequence has a soft subsequence that converges in $\tilde{X}$.

Proposition 3.4. *Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space. If $(\tilde{X}, \tilde{d}, E)$ is a soft sequential compact metric space, then $(X, \tilde{d}_e)$ is a sequential compact metric space for each $e \in E$, where E is a countable set. Here $\tilde{d}_e$ stands for the metric for only parameter e and thus $(X, \tilde{d}_e)$ is a crisp metric space.*

Proof. Let $\{x^n\}$ be any sequence in $(X, \tilde{d}_e)$ for each $e \in E$. Then the sequence $\{x^n\}$ is written as $\{x^n_e\}$ in soft metric space. The sequence $\{x^n_e\}$ has soft convergent subsequence as $\{x^{n_k}_e\}$. This means that the subsequence $\{x^{n_k}\}$ of $\{x^n\}$ converges in $(X, \tilde{d}_e)$. □

The converse of the Proposition 3.4 may not be true in general. This is shown by the following example.

Example 3.5. Let $E = \mathbb{N}$, $X = [0, 1]$ and let soft metric $\tilde{d}$ be defined as follows:

$$\tilde{d}(\tilde{x}_e, \tilde{y}_{\acute{e}}) = |e - \acute{e}| + |x - y|,$$

where $\mathbb{N}$ is a natural number set. It is clear that $(X, \tilde{d}_e)$ is a sequential compact metric space for each $e \in E$. However, the soft sequence $\left\{\left(\frac{1}{2^k}\right)^n_{e_n}\right\}_{k,n}$ does not have a convergent soft subsequence in $(\tilde{X}, \tilde{d}, E)$.

Proposition 3.6. *Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space. Then $(\tilde{X}, \tilde{d}, E)$ is a soft sequential compact metric space if and only if every infinite soft set has a soft cluster point.*

Proof. ($\Rightarrow$): Let (F, E) be an infinite soft set and $\{\tilde{x}^n_{e_n}\}$ be a soft sequence in (F, E). Then $\{\tilde{x}^n_{e_n}\}$ has a convergent soft subsequence $\left\{\tilde{x}^{n_k}_{e_{n_k}}\right\}$. Assume that $\left\{\tilde{x}^{n_k}_{e_{n_k}}\right\}$ converges to a soft point $\tilde{z}_{e_0}$. Since

$$\Phi \neq B(\tilde{z}_{e_0}, \tilde{\varepsilon}) \cap \left\{\tilde{x}^{n_k}_{e_{n_k}}\right\} \subset B(\tilde{z}_{e_0}, \tilde{\varepsilon}) \cap (F, E),$$

$\tilde{z}_{e_0}$ is a soft cluster point in (F, E).

($\Leftarrow$): Let $\{\tilde{x}^n_{e_n}\}$ be an arbitrary soft sequence. If $\{\tilde{x}^n_{e_n}\}$ is finite, then there exist a fixed convergent subsequence of the soft sequence. If we take $\{\tilde{x}^n_{e_n}\} = (F, E)$, then $\{\tilde{x}^n_{e_n}\}$ is a soft set. According to the condition, this soft set has a soft cluster point as $\tilde{z}_{e_0}$. Since

$$B(\tilde{z}_{e_0}, \tilde{\varepsilon}) \cap (F, E) \neq \Phi,$$

we choose $\left\{\tilde{x}^{n_k}_{e_{n_k}}\right\} \tilde{\in} B(\tilde{z}_{e_0}, \tilde{\varepsilon}) \cap (F, E)$ for each $\tilde{\varepsilon} \tilde{>} \bar{0}$. Thus the soft subsequence $\left\{\tilde{x}^{n_k}_{e_{n_k}}\right\}$ converges to $\tilde{z}_{e_0}$. □

Definition 3.7. Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space and (S, E) be a soft set of soft points. If $\tilde{X} \subset \underset{\tilde{x}_e \in (S,E)}{\cup} B(\tilde{x}_e, \tilde{\varepsilon})$ is satisfied, then (S, E) is said to be a soft $\tilde{\varepsilon}-$net in $(\tilde{X}, \tilde{d}, E)$.

Definition 3.8. Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space. If for each $\tilde{\varepsilon} \tilde{>} \bar{0}$ there exists a finite soft $\tilde{\varepsilon}-$net of $(\tilde{X}, \tilde{d}, E)$,then $(\tilde{X}, \tilde{d}, E)$ is said to be totally bounded soft metric space.

Definition 3.9. Let $(X, \tilde{d}, E)$ be a a soft metric space and (S, E) be a soft set of soft points. The diameter of a non-empty soft set in $(X, \tilde{d}, E)$ is defined by:

$$\tilde{d}((S, E)) = Sup\left\{\tilde{d}(\tilde{x}_e, \tilde{y}_{e'}) : \tilde{x}_e, \tilde{y}_{e'} \tilde{\in} (S, E)\right\}.$$

Lemma 3.10. *Let $(\tilde{X}, \tilde{d}, E)$ be a totally bounded soft metric space and (A, E) be an infinite soft set. Then for each $\tilde{\varepsilon} \tilde{>} \bar{0}$ there exists an infinite soft set $(B, E) \subset (A, E)$ such that $\tilde{d}((B, E)) \tilde{<} \tilde{\varepsilon}$.*

Proof. Let $\tilde{\varepsilon} \tilde{>} \bar{0}$ be arbitrary. Since $(\tilde{X}, \tilde{d}, E)$ is a totally bounded soft metric space, there exists a finite soft $\tilde{\varepsilon}-$net as $\tilde{H} = \{\tilde{x}^1_{e_1}, \tilde{x}^2_{e_2}, ..., \tilde{x}^n_{e_n}\}$. Hence, we have $\tilde{X} = \overset{n}{\underset{i=1}{\cup}} B(\tilde{x}^i_{e_i}, \tilde{\varepsilon})$ and $\tilde{A} = \overset{n}{\underset{i=1}{\cup}} B(\tilde{x}^i_{e_i}, \tilde{\varepsilon}) \cap (A, E)$ is obtained. For $1 \leq i \leq n$, at least one of the soft sets $\tilde{A} = \overset{n}{\underset{i=1}{\cup}} B(\tilde{x}^i_{e_i}, \tilde{\varepsilon}) \cap (A, E)$ must have infinite elements. If we denote this set by (B, E) , it is clear that $\tilde{d}((B, E)) \tilde{<} \tilde{\varepsilon}$. □

Theorem 3.11. *Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space. $(\tilde{X}, \tilde{d}, E)$ is a totally bounded soft metric space if and only if every soft sequence has a soft Cauchy subsequence in $\tilde{X}$.*

Proof. ($\Rightarrow$): Let $(\tilde{X}, \tilde{d}, E)$ be a totally bounded soft metric space and $\{\tilde{x}^n_{e_n}\}$ be any soft sequence. If $(A, E) = \{\tilde{x}^n_{e_n}\}$ is finite, the proof is completed. Assume that $(A, E) = \{\tilde{x}^n_{e_n}\}$ is infinite. From the Lemma 3.10, there exists an infinite soft set $(B_1, E) \subset (A, E)$ such that $\tilde{d}((B_1, E)) \tilde{<} \bar{1}$.We choose the number $n_1 \in \mathbb{N}$ such that $\tilde{x}^{n_1}_{e_{n_1}} \in (B_1, E)$. If we apply the Lemma 3.10 to (B_1, E), we obtain an infinite $(B_2, E) \subset (B_1, E)$ such that $\tilde{d}((B_1, E)) \tilde{<} \widetilde{\frac{1}{2}}$. Here we take the number $n_2 > n_1$ such that $\tilde{x}^{n_2}_{e_{n_2}} \in (B_2, E)$. Thus we get the soft subsequence $\left\{\tilde{x}^{n_k}_{e_{n_k}}\right\}$.

Now, let us show that the sequence $\left\{\tilde{x}^{n_k}_{e_{n_k}}\right\}$ is a soft Cauchy subsequence. For arbitrary $\tilde{\varepsilon}\tilde{>}\bar{0}$, we choose the number k_0 such that $\widetilde{\frac{1}{k_0}}\tilde{<}\tilde{\varepsilon}$. Then since $\tilde{x}^{n_k}_{e_{n_k}}, \tilde{x}^{n_m}_{e_{n_m}}$ $\tilde{\in}(B_{k_0}, E)$ for each $k, m\tilde{\geq}k_0$,

$$\tilde{d}\left(\tilde{x}^{n_k}_{e_{n_k}}, \tilde{x}^{n_m}_{e_{n_m}}\right)\tilde{<}\widetilde{\frac{1}{k_0}}\tilde{<}\tilde{\varepsilon}$$

is satisfied. This means that $\left\{\tilde{x}^{n_k}_{e_{n_k}}\right\}$ is soft Cauchy subsequence.

($\Leftarrow$): Assume that $(\tilde{X}, \tilde{d}, E)$ is not totally bounded soft metric space. In this case, $(\tilde{X}, \tilde{d}, E)$ has not a finite soft $\tilde{\varepsilon}_0-$net, for some $\tilde{\varepsilon}_0\tilde{>}\bar{0}$. Let $\tilde{x}^1_{e_1}\in\tilde{X}$ be an arbitrary soft point. Then there can be found a soft point $\tilde{x}^2_{e_2}\in\tilde{X}$ such that $\tilde{d}\left(\tilde{x}^1_{e_1}, \tilde{x}^2_{e_2}\right)\tilde{\geq}\tilde{\varepsilon}_0$. Since the soft set $\left\{\tilde{x}^1_{e_1}, \tilde{x}^2_{e_2}\right\}$ is not an $\tilde{\varepsilon}_0-$net, there is a soft point $\tilde{x}^3_{e_3}\in\tilde{X}$ such that

$$\tilde{d}\left(\tilde{x}^1_{e_1}, \tilde{x}^3_{e_3}\right)\tilde{\geq}\tilde{\varepsilon}_0,\ \tilde{d}\left(\tilde{x}^2_{e_2}, \tilde{x}^3_{e_3}\right)\tilde{\geq}\tilde{\varepsilon}_0.$$

Thus we form a soft sequence $\left\{\tilde{x}^k_{e_k}\right\}$ such that $\tilde{d}\left(\tilde{x}^i_{e_i}, \tilde{x}^j_{e_j}\right)\tilde{\geq}\tilde{\varepsilon}_0$, for all i, j. It is clear that $\left\{\tilde{x}^k_{e_k}\right\}$ has not a soft Cauchy subsequence which contradicts the fact that every soft sequence has a soft Cauchy subsequence given in the assumption. □

Theorem 3.12. *Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space. $(\tilde{X}, \tilde{d}, E)$ is a soft sequential compact metric space if and only if $(\tilde{X}, \tilde{d}, E)$ is both complete and totally bounded soft metric space.*

Proof. ($\Rightarrow$): Let $(\tilde{X}, \tilde{d}, E)$ be a soft sequential compact metric space. Then every soft sequence $\left\{\tilde{x}^n_{e_n}\right\}$ has a soft subsequence that converges in $\tilde{X}$. Since the soft subsequence is a soft Cauchy sequence in $\tilde{X}$, then by the Theorem 3.11 $(\tilde{X}, \tilde{d}, E)$ is totally bounded soft metric space. If $\left\{\tilde{x}^n_{e_n}\right\}$ is a soft Cauchy sequence in $(\tilde{X}, \tilde{d}, E)$ and $\left\{\tilde{x}^n_{e_n}\right\}$ has a convergent soft subsequence, then it is also convergent, i.e.$(\tilde{X}, \tilde{d}, E)$ is a complete soft metric space.

($\Leftarrow$): Let $(\tilde{X}, \tilde{d}, E)$ be a complete and totally bounded soft metric space and $\left\{\tilde{x}^n_{e_n}\right\}$ be an arbitrary soft sequence. Since $(\tilde{X}, \tilde{d}, E)$ is totally bounded, $\left\{\tilde{x}^n_{e_n}\right\}$ has a soft Cauchy subsequence. Since $(\tilde{X}, \tilde{d}, E)$ is complete soft metric space, the soft Cauchy subsequence converges. Then $(\tilde{X}, \tilde{d}, E)$ is a soft sequential compact metric space. □

Definition 3.13. Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space and a family $\mathfrak{U}$ be a soft open cover of $(\tilde{X}, \tilde{d}, E)$. A number $\tilde{\varepsilon}\tilde{>}\bar{0}$ is called a Lebesque number of $\mathfrak{U}$, if for each soft point $\tilde{x}_e\in\tilde{X}$, there exists $(F, E)\in\mathfrak{U}$ such that $B(\tilde{x}_e, \tilde{\varepsilon})\tilde{\subset}(F, E)$ for all soft points $\tilde{x}_e\in\tilde{X}$.

Proposition 3.14. *If $(\tilde{X}, \tilde{d}, E)$ is a soft sequentially compact metric space, then every soft open cover in $\tilde{X}$ has a Lebesque number.*

Proof. Assume that soft open cover $\mathfrak{U}$ has not a Lebesque number. Then for any n and for each $(F, E)\in\mathfrak{U}$ there exists a soft sequence $\left\{\tilde{x}^n_{e_n}\right\}$, where $B\left(\tilde{x}^n_{e_n}, \widetilde{\frac{1}{n}}\right)\tilde{\subset}(F, E)$ is not satisfied. Thus, we obtain a soft sequence $\left\{\tilde{x}^n_{e_n}\right\}$ satisfying the above condition.

Since $(\tilde{X}, \tilde{d}, E)$ is a soft sequential compact metric space, $\left\{\tilde{x}^n_{e_n}\right\}$ has a soft subsequence $\left\{\tilde{x}^{n_k}_{e_{n_k}}\right\}$ converging to $\tilde{x}_e$. Let $\tilde{x}_e \in (F, E) \in \mathfrak{U}$. Since (F, E) is soft open set, there is soft open ball $B\left(\tilde{x}_e, \widetilde{\frac{2}{m}}\right)$ such that $B\left(\tilde{x}_e, \widetilde{\frac{2}{m}}\right) \tilde{\subset}(F, E)$. Also since $\left\{\tilde{x}^{n_k}_{e_{n_k}}\right\}$ converges to $\tilde{x}_e$, there exists a number k_0, such that $\tilde{x}^{n_k}_{e_{n_k}} \tilde{\in} B\left(\tilde{x}_e, \widetilde{\frac{2}{m}}\right)$, whenever $k \tilde{\geq} k_0$. We take the number $k \geq k_0$ as $n_k \geq m$. Then

$$B\left(\tilde{x}^{n_k}_{e_{n_k}}, \widetilde{\frac{1}{n_k}}\right) \tilde{\subset} B\left(\tilde{x}_e, \widetilde{\frac{2}{m}}\right) \tilde{\subset}(F, E) \in \mathfrak{U}$$

is obtained and this contradicts with the choice of the soft point $\tilde{x}^{n_k}_{e_{n_k}}$. □

Theorem 3.15. *Let $(\tilde{X}, \tilde{d}, E)$ be a soft metric space. Then the following statements are equivalent:*

(1) $(\tilde{X}, \tilde{d}, E)$ *is soft compact metric space,*
(2) $(\tilde{X}, \tilde{d}, E)$ *is soft sequential compact metric space.*

Proof. (1)⇒(2): Let $(\tilde{X}, \tilde{d}, E)$ be a soft compact metric space but not a soft sequential compact metric space. Then there is an infinite soft set $\tilde{A}$ which does not have a cluster point in $(\tilde{X}, \tilde{d}, E)$. Thus there is a soft number $r_{\tilde{x}_e}$ such that $B\left(\tilde{x}_e, r_{\tilde{x}_e}\right) \cap \tilde{A} = \{\tilde{x}_e\}$ for all $\tilde{x}_e \in \tilde{A}$. A family $\{B\left(\tilde{x}_e, r_{\tilde{x}_e}\right)\}_{\tilde{x}_e \in \tilde{A}} \cup \left(\tilde{A}\right)^c$ is a soft open cover in $(\tilde{X}, \tilde{d}, E)$ and this soft open cover has not a finite soft subcover. But this implies $(\tilde{X}, \tilde{d}, E)$ is not soft compact metric space, which is a contradiction.

(2)⇒(1): Let $(\tilde{X}, \tilde{d}, E)$ be a soft sequential compact metric space and $\mathfrak{U}$ be any soft open cover in $(\tilde{X}, \tilde{d}, E)$. Then by Proposition 3.14, $\mathfrak{U}$ has a Lebesque number $\tilde{\varepsilon} \tilde{>} \bar{0}$. Since $(\tilde{X}, \tilde{d}, E)$ is totally bounded soft metric space, then $(\tilde{X}, \tilde{d}, E)$ has finite $\widetilde{\frac{\varepsilon}{3}}$−net as $\left\{\tilde{x}^1_{e_1}, \tilde{x}^2_{e_2}, ..., \tilde{x}^n_{e_n}\right\}$. For each $k = 1, 2, ..., n$

$$\tilde{d}\left(B\left(\tilde{x}^k_{e_k}, \widetilde{\frac{\varepsilon}{3}}\right)\right) \tilde{\leq} \widetilde{\frac{2\varepsilon}{3}} \tilde{<} \tilde{\varepsilon}$$

is satisfied. Then we obtain a soft set $B\left(\tilde{x}^k_{e_k}, \widetilde{\frac{\varepsilon}{3}}\right) \tilde{\subset}(F_k, E) \in \mathfrak{U}$. Since

$$\tilde{X} = \bigcup_{k=1}^{n} B\left(\tilde{x}^k_{e_k}, \widetilde{\frac{\varepsilon}{3}}\right) \tilde{\subset} \bigcup_{k=1}^{n} (F_k, E),$$

$(\tilde{X}, \tilde{d}, E)$ is a soft compact metric space. □

Definition 3.16. Let $(\tilde{X}, \tilde{d}_1, E_1)$ and $(\tilde{Y}, \tilde{d}_2, E_2)$ be two soft metric spaces. The mapping $(f, \varphi) : (\tilde{X}, \tilde{d}_1, E_1) \to (\tilde{Y}, \tilde{d}_2, E_2)$ is called soft uniformly continuous mapping, if given any $\tilde{\varepsilon} \tilde{>} \bar{0}$, there exists a $\tilde{\delta} \tilde{>} \bar{0}$ ($\tilde{\delta}$ depending only on $\tilde{\varepsilon}$) such that for any soft point $\tilde{x}_e, \tilde{y}_{\acute{e}} \tilde{\in} \tilde{X}$ when $\tilde{d}_1\left(\tilde{x}_e, \tilde{y}_{\acute{e}}\right) \tilde{<} \tilde{\delta}$ satisfied then $\tilde{d}_2\left(\widetilde{f(x)}_{\varphi(e)}, \widetilde{f(y)}_{\varphi(\acute{e})}\right) \tilde{<} \tilde{\varepsilon}$.

Proposition 3.17. *If $(f, \varphi) : (\tilde{X}, \tilde{d}_1, E_1) \to (\tilde{Y}, \tilde{d}_2, E_2)$ is a soft uniformly continuous mapping, then $f : \left(X, \tilde{d}_{1_e}\right) \to \left(Y, \tilde{d}_{2_{\varphi(e)}}\right)$ is also a uniformly continuous mapping for each $e \in E$.*

Proof. The proof is straightforward. □

The converse of the Proposition 3.17 may not be true in general. This is shown by the following example.

Example 3.18. Let $E = \mathbb{R}$ be a parameter set and $X = \mathbb{R}^2$. Consider usual metrics on this sets and define soft metric on $\tilde{X}$ by $\tilde{d}(\tilde{x}_e, \tilde{y}_{e'}) = |e - e'| + d(x, y)$. Then if we define the soft mapping $(f, \varphi) : (\tilde{X}, \tilde{d}, E) \to (\tilde{X}, \tilde{d}, E)$ as follows:

$$(f,\varphi)(\tilde{x}_e) = \left(\frac{1}{2}x\right)_{3e},$$

then

$$\tilde{d}\left((f,\varphi)(\widetilde{0,1})_2, (f,\varphi)(\widetilde{1,0})_1\right) = \tilde{d}\left(\left(\widetilde{0,\frac{1}{2}}\right)_6, \left(\widetilde{\frac{1}{2},0}\right)_3\right) = 3 + \frac{\sqrt{2}}{2},$$

$$\tilde{d}\left(\left(\widetilde{0,1}\right)_2, \left(\widetilde{1,0}\right)_1\right) = 1 + \sqrt{2}.$$

Since $3 + \frac{\sqrt{2}}{2} > 1 + \sqrt{2}$, we see that the soft mapping (f, φ) is not a soft contraction mapping and thus it is not uniformly continuous. On the other hand, the mapping $f_e : (X, d_e) \to (X, d_{3e})$ is a uniformly continuous mapping for all $e \in E$.

Theorem 3.19. *If $(f, \varphi) : (\tilde{X}, \tilde{d}_1, E_1) \to (\tilde{Y}, \tilde{d}_2, E_2)$ is a soft continuous mapping and $(\tilde{X}, \tilde{d}_1, E_1)$ is soft sequentially compact metric space, then (f, φ) is a soft uniformly continuous mapping.*

Proof. Since $(\tilde{X}, \tilde{d}_1, E_1)$ is a soft sequentially compact metric space, it is also soft compact metric space. Since (f, φ) is a soft continuous mapping, for any $\tilde{\varepsilon}\tilde{>}\bar{0}$ and for any soft point $\tilde{x}_e$ there exists a soft number $\tilde{\delta}\left(\tilde{x}_e\right)\tilde{>}\bar{0}$ such that for every soft point $\tilde{y}_{\acute{e}}$ that satisfies the condition $\tilde{d}_1\left(\tilde{x}_e, \tilde{y}_{\acute{e}}\right)\tilde{<}2\tilde{\delta}(\tilde{x}_e)$, where we have $\tilde{d}_2\left(\widetilde{f(x)}_{\varphi(e)}, \widetilde{f(y)}_{\varphi(\acute{e})}\right)\tilde{<}\frac{\tilde{\varepsilon}}{2}$. Then the family $\mathfrak{U} = \left\{B\left(\tilde{x}_e, \tilde{\delta}(\tilde{x}_e)\right)\right\}_{\tilde{x}_e\tilde{\in}\tilde{X}}$ is a soft open cover in $(\tilde{X}, \tilde{d}_1, E_1)$. Since $(\tilde{X}, \tilde{d}_1, E_1)$ is a soft compact metric space, this soft open cover has a finite soft subcover such as $\left\{B\left(\tilde{x}^1_{e_1}, \tilde{\delta}\left(\tilde{x}^1_{e_1}\right)\right), ..., B\left(\tilde{x}^n_{e_n}, \tilde{\delta}\left(\tilde{x}^n_{e_n}\right)\right)\right\}$. Let us take

$$\tilde{\delta} = \min\left\{\tilde{\delta}\left(\tilde{x}^1_{e_1}\right), ..., \tilde{\delta}\left(\tilde{x}^n_{e_n}\right)\right\}.$$

Now consider only two soft points $\tilde{x}_e, \tilde{z}_{\acute{e}}\tilde{\in}\tilde{X}$ such that $\tilde{d}_1\left(\tilde{x}_e, \tilde{z}_{\acute{e}}\right)\tilde{<}\tilde{\delta}$. Assume that $\tilde{y}_e\tilde{\in}B\left(\tilde{x}^i_{e_i}, \tilde{\delta}\left(\tilde{x}^i_{e_i}\right)\right), 1 \le i \le n$. Then $\tilde{d}_1\left(\tilde{y}_e, \tilde{x}^i_{e_i}\right)\tilde{<}\tilde{\delta}(\tilde{x}^i_{e_i})$ and

$$\tilde{d}_1\left(\tilde{z}_{\acute{e}}, \tilde{x}^i_{e_i}\right)\tilde{\le}\tilde{d}_1\left(\tilde{z}_{\acute{e}}, \tilde{y}_e\right) + \tilde{d}_1\left(\tilde{y}_e, \tilde{x}^i_{e_i}\right)\tilde{<}\tilde{\delta} + \tilde{\delta}(\tilde{x}^i_{e_i})\tilde{\le}2\tilde{\delta}\left(\tilde{x}^i_{e_i}\right)$$

is satisfied. Since (f, φ) is a soft continuous mapping at the soft point $\tilde{x}^i_{e_i}$,

$$\tilde{d}_2\left(\widetilde{f(y)}_{\varphi(e)}, \widetilde{f(x^i)}_{\varphi(e_i)}\right)\tilde{<}\frac{\tilde{\varepsilon}}{2} \text{ and } \tilde{d}_2\left(\widetilde{f(z)}_{\varphi(\acute{e})}, \widetilde{f(x^i)}_{\varphi(e_i)}\right)\tilde{<}\frac{\tilde{\varepsilon}}{2}.$$

Thus,

$$\tilde{d}_2\left(\widetilde{f(y)}_{\varphi(e)}, \widetilde{f(z)}_{\varphi(\acute{e})}\right)\tilde{\le}\tilde{d}_2\left(\widetilde{f(y)}_{\varphi(e)}, \widetilde{f(x^i)}_{\varphi(e_i)}\right) + \tilde{d}_2\left(\widetilde{f(x^i)}_{\varphi(e_i)}, \widetilde{f(z)}_{\varphi(\acute{e})}\right)\tilde{<}\tilde{\varepsilon}.$$

Consequently, (f, φ) is a soft uniformly continuous mapping. □

4. Conclusion

The soft set theory that was proposed by Molodtsov offers a general mathematical tool for dealing with uncertain and vague objects. Many researchers have contributed towards the topologization of soft set theory. This study contributes some important theorems on soft sequential compact metric spaces. Later we give the concepts of $\tilde{\varepsilon}-$net and totally bounded soft metric spaces. We continue to investigate some important properties of totally bounded soft metric spaces. Finally we introduce the concepts of Lebesque number for soft sets and soft uniformly continuous mapping and investigate some theorems in detail. Throughout our investigations we see that there is no meaningful difference between the crisp case and soft topology in terms of soft compact sets in soft metric spaces.

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Cigdem Gunduz Aras (carasgunduz@gmail.com)
Department of Mathematics, Kocaeli University, Kocaeli, 41380-Turkey

Murat Ibrahim Yazar (myazar@kmu.edu.tr)
Department of Mathematics and Science Education, Karamanoglu Mehmetbey University, Karaman, 75100-Turkey

Sadi Bayramov (baysadi@gmail.com)
Department of Algebra and Geometry, Baku State University, Baku Az, 1148-Azerbaican