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Abstract. In this paper we give several characterizations for fuzzy ideals, fuzzy homomorphisms, and fuzzy congruences of an almost distributive lattice. In addition, the quotient of an almost distributive lattice induced by a fuzzy congruence is also presented in the paper. Furthermore, we obtain a kind of fuzzy congruences for which their quotient is a distributive lattice and for which it is not. Mainly, we construct a monomorphism of the lattice of fuzzy ideals into the lattice of fuzzy congruences of almost Boolean rings, and we give a necessary and sufficient condition for this monomorphism to become a lattice isomorphism.

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Keywords: ADLs, Fuzzy ideals, Fuzzy homomorphisms, Fuzzy congruences.

1. Introduction

The concept of an almost distributive lattice(ADL) was first introduced by U.M. Swamy and G. C. Rao [9] in 1980 as a common abstraction to most of the existing ring theoretic and lattice theoretic generalization of Boolean algebras. An ADL is an algebra with two binary operations $\vee$ and $\wedge$ which satisfies almost all the properties of a distributive lattice with smallest element 0 except possibly the commutativity of $\vee$, the commutativity of $\wedge$ and the right distributivity of $\vee$ over $\wedge$. It was also observed that any one of these three properties converts an ADL into a distributive lattice. The study of ideals, and congruence relations on ADLs was initiated in [9] and later studied by many authors. In most of algebraic structures the concept of congruences is closely related with structures such as; normal subgroups (in the case of groups), ideals (in the case of rings), and quotient algebras. This makes the study of congruences more important both from theoretical stand point and for its applications in many fields. In this view, the concept of filter congruences and factor congruences was introduced in an ADL analogous to that in a distributive lattice by

U.M. Swamy et al. [8]. Following this, Y. S. Pawar et al. [4] further studied on the class of congruences on ADLs induced by multiplicatively closed sets.

On the other hand, the study of fuzzy sets was done in 1965 by L. A. Zadeh [11]. Since then many authors have been studying fuzzy subalgebras of several algebraic structures. Rosenfeld [5] in 1971 developed the concept of fuzzy subgroup. W. J. Liu [2] in 1982 initiated the study of fuzzy subrings, and fuzzy ideals of a ring. D.S. Malik et al. [3] studied fuzzy homomorphisms of rings. In 1990, Y. Bo et al. [10] introduced the concept of fuzzy ideals and fuzzy congruences of distributive lattices and showed that if L is relatively complemented distributive lattice with zero, then there is a one-to-one correspondence between the lattice of fuzzy ideals and the lattice of fuzzy congruences of L. Later in 1998 U. M. Swamy et al. [7] studied properties of L-fuzzy ideals and L-fuzzy congruences of lattices.

More recently, U. M. Swamy et al. [6] initiated the study of L-fuzzy ideals of ADLs. They particularly proved that the class of L-fuzzy ideals of an ADL forms a complete distributive lattice. In this paper, we extend the results in [6] and give several characterizations for fuzzy ideals, fuzzy homomorphisms, and fuzzy congruences of ADLs. Quotient ADLs induced by fuzzy congruences are also presented in the paper. In addition, we obtain a kind of fuzzy congruences for which their quotient is a distributive lattice and for which it is not. Furthermore, we give the smallest fuzzy congruence on an ADL A such that its quotient is a distributive lattice. Finally, we construct a monomorphism of the lattice of fuzzy ideals and the lattice of fuzzy congruences of almost Boolean rings and we give a necessary and sufficient condition for this monomorphism to become an order isomorphism.

Most of the results in the paper seem analogous to those results in distributive lattices, though the proofs are different due to the absence of the commutativity of $\vee$ and $\wedge$.

2. Preliminaries

In this section we recall some definitions and basic results on almost distributive lattices.

Definition 2.1 ([9])**.** An algebra $(A, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an almost distributive lattice, abbreviated as ADL, if it satisfies the following axioms:

(i) $a \vee 0 = a$,
(ii) $0 \wedge a = 0$,
(iii) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$,
(iv) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,
(v) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
(v) $(a \vee b) \wedge b = b$, for all $a, b, c \in A$.

Lemma 2.2 ([9])**.** *For any $a \in A$, we have*

(1) $a \wedge 0 = 0$,
(2) $0 \vee a = a$,
(3) $a \wedge a = a$,
(4) $a \vee a = a$.

Lemma 2.3 ([9])**.** *For any $a, b \in A$, we have*

(1) $(a \wedge b) \vee b = b$,
(2) $a \vee (a \wedge b) = a = a \wedge (a \vee b)$,
(3) $a \vee (b \wedge a) = a = (a \vee b) \wedge a$,
(4) $\wedge$ *is associative and* $a \wedge b \wedge c = b \wedge a \wedge c$.

Corollary 2.4 ([9]). *For any* $a, b \in A$*, we have*
(1) $a \vee b = a$ *if and only if* $a \wedge b = b$,
(2) $a \vee b = b$ *if and only if* $a \wedge b = a$.

Definition 2.5 ([9]). For any $a, b \in A$, we say that a is less than or equals to b and we write $a \leq b$, if $a \wedge b = a$ or equivalently $a \vee b = b$.

Theorem 2.6 ([9]). *For any* $a, b \in A$*, the following are equivalent:*
(1) $(a \wedge b) \vee a = a$,
(2) $a \wedge (b \vee a) = a$,
(3) $(b \wedge a) \vee b = b$,
(4) $b \wedge (a \vee b) = b$,
(5) $a \wedge b = b \wedge a$,
(6) $a \vee b = b \vee a$,
(7) *the supremum of* a *and* b *exists in* A *and equals to* $a \vee b$,
(8) *there exists* $x \in A$ *such that* $a \leq x$ *and* $b \leq x$,
(9) *the infimum of* a *and* b *exists in* A *and equals to* $a \wedge b$.

Definition 2.7 ([9]). A nonempty subset I of an ADL A is called an ideal of A, if $a \vee b, a \wedge x \in I$, for all $a, b \in I$ and for all $x \in A$.

It can be observed that $x \wedge a \in I$ for all $a \in I$ and all $x \in A$. For any subset $S \subseteq A$, the smallest ideal of A containing S is called the ideal of A generated by S and is denoted by $\langle S]$. Note that:

$$\langle S] = \{(\bigvee x_i) \wedge a : a \in A, x_i \in S, i = 1, ..., n \text{ for some } n \in Z_+\}.$$

If $S = \{a\}$, then we write $\langle a]$, for $\langle S]$. In this case, $\langle a] = \{a \wedge x : x \in A\}$.

3. Fuzzy ideals and Fuzzy homomorphisms on ADLs

In this section, we give several characterizations for fuzzy ideals and fuzzy homomorphisms of ADLs. Some of the results on fuzzy ideals are due to [6]. Remember that, for any set A, a function $\mu : A \to [0, 1]$ is called a fuzzy subset of A. For each $t \in [0, 1]$, the set

$$\mu_t = \{x \in A : \mu(x) \geq t\}$$

is called the level subset of μ at t [11].

Definition 3.1. A fuzzy subset μ of A is called a fuzzy subADL of A, if:

$$\mu(x \vee y) \wedge \mu(x \wedge y) \geq \mu(x) \wedge \mu(y), \quad \text{for all } x, y \in A.$$

Definition 3.2 ([6]). A fuzzy subset μ of A is called a fuzzy ideal of A, if:

$$\mu(0) = 1 \text{ and } \mu(x \vee y) = \mu(x) \wedge \mu(y), \text{ for all } x, y \in A.$$

We denote the class of all fuzzy ideals of A by $FI(A)$.

Example 3.3. Let $A = \{0, a, b, c\}$ and let $\vee$ and $\wedge$ be binary operations on A defined by:

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	a	b	c

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	c	c	c

Then $(A, \vee, \wedge, 0)$ is an ADL(a discrete ADL) [9]. Now define a fuzzy subset μ of A by:

$$\mu(0) = 1,\ \mu(a) = 0.6 = \mu(b) \text{ and } \mu(c) = 0.8.$$

Thus μ is a fuzzy ideal of A.

Lemma 3.4. [6] *A fuzzy subset μ of A is a fuzzy ideal of A if and only if*

(1) $\mu(0) = 1$,
(2) $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$,
(3) $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$, *for all* $x, y \in A$,

Lemma 3.5. *Let μ be fuzzy subADL of A. Then μ is a fuzzy ideal of A if and only if*

$$\mu(0) = 1 \text{ and } a \wedge b = b \Rightarrow \mu(a) \leq \mu(b), \text{ for all } a, b \in A$$

Lemma 3.6 ([6])**.** *Let μ be fuzzy subset of A. Then μ is a fuzzy ideal of A if and only if μ_t is an ideal of A, for all $t \in [0, 1]$.*

Lemma 3.7. *The intersection of any family of fuzzy ideals of A is a fuzzy ideal.*

Remark 3.8. Note that the union of a family of fuzzy ideals of A is not in general a fuzzy ideal of A. We verify this in the following example:

Example 3.9. Let $A = \{0, a, b, c\}$ and let $\vee$ and $\wedge$ be binary operations on A defined by:

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	a	b	a
c	c	a	a	c

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	b	0
c	0	c	0	c

Then it is clear that $(A, \vee, \wedge, 0)$ is an ADL. Now define fuzzy subsets μ and σ of A by:

$$\mu(0) = 1,\ \mu(a) = 0.5 = \mu(b) \text{ and } \mu(c) = 0.7,$$
$$\sigma(0) = 1,\ \sigma(a) = 0.6 = \sigma(c) \text{ and } \sigma(b) = 0.8.$$

Thus both μ and σ are fuzzy ideals of A but $\mu \cup \sigma$ fails to be a fuzzy ideal of A.

Lemma 3.10 ([6])**.** *A nonempty subset I of A is an ideal of A if and only if the characteristic function χ_I of I is a fuzzy ideal of A.*

Definition 3.11. Let μ be fuzzy subset of A. The smallest fuzzy ideal of A containing μ is called a fuzzy ideal of A induced by μ and is denoted by $\langle\mu]$.

Lemma 3.12. *For any fuzzy subset μ of A,*

$$\langle\mu] = \bigcap\{\sigma \in FI(A) : \mu \subseteq \sigma\}.$$

Lemma 3.13. *Let S be any subset of A and χ_S its characteristic function. Then* $\langle \chi_S] = \chi_{\langle S]}$.

Proof. To prove that $\langle \chi_S] = \chi_{\langle S]}$, we show that $\chi_{\langle S]}$ is the smallest fuzzy ideal of A containing χ_S. Since $\langle S]$ is an ideal of A containing S, it is clear that $\chi_{\langle S]}$ is a fuzzy ideal of A containing χ_S. It remains to show that it is the smallest fuzzy ideal of A containing χ_S. Let μ be any fuzzy ideal of A containing χ_S, that is, $\chi_S(x) \leq \mu(x)$, for all $x \in S$, then $\mu(x) = 1$, for all $x \in S$.

Now consider $y \in \langle S]$. Then $y = (\bigvee x_i) \wedge a$, for some $a \in A, x_i \in S, i = 1, ..., n; n \in Z_+$. Then, for each $y \in \langle S]$, we have

$$\mu(y) = \mu((\bigvee x_i) \wedge a) \geq \mu(\bigvee x_i) \vee \mu(a) \geq \mu(\bigvee x_i) \geq \bigwedge \mu(x_i) = 1.$$

Thus $\chi_{\langle S]}(y) \leq \mu(y)$, for all $y \in A$. So $\chi_{\langle S]} \subseteq \mu$. Hence the result holds. □

For any fuzzy subset μ of A, it is clear that

$$\mu(x) = Sup\{\alpha \in [0, 1] : x \in \mu_\alpha\}, \text{ for all } x \in A.$$

In the following theorem, we characterize a fuzzy ideal induced by fuzzy sets.

Theorem 3.14. *Let μ be a fuzzy subset of A. Then a fuzzy subset $\widehat{\mu}$ of A defined by:*

$$\widehat{\mu}(x) = Sup\{\alpha \in [0, 1] : x \in \langle \mu_\alpha]\}, \text{ for all } x \in A$$

is a fuzzy ideal of A induced by μ.

Proof. It is enough if we show that $\widehat{\mu}$ is the smallest fuzzy ideal of A containing μ. Clearly $\widehat{\mu}$ is a fuzzy subset of A. Also $\widehat{\mu}(0) = Sup\{\alpha \in [0, 1] : 0 \in \langle \mu_\alpha]\}$. Since $\langle \mu_\alpha]$ is an ideal of A, for all $\alpha \in [0, 1]$, $0 \in \langle \mu_\alpha]$, for all $\alpha \in [0, 1]$. Then it follows that $\widehat{\mu}(0) = Sup\{\alpha \in [0, 1]\} = 1$.

Next we show that $\widehat{\mu}(x \vee y) \geq \widehat{\mu}(x) \wedge \widehat{\mu}(y)$, for all $x, y \in A$. For;

$$\begin{aligned} \widehat{\mu}(x) \wedge \widehat{\mu}(y) &= Sup\{\alpha \in [0, 1] : x \in \langle \mu_\alpha]\} \wedge Sup\{\beta \in [0, 1] : y \in \langle \mu_\beta]\} \\ &= Sup\{min\{\alpha, \beta\} : x \in \langle \mu_\alpha], y \in \langle \mu_\beta]\}. \end{aligned}$$

If we put $\lambda = min\{\alpha, \beta\}$, then $\lambda \leq \alpha$ and $\lambda \leq \beta$, which implies that $\langle \mu_\alpha] \subseteq \langle \mu_\lambda]$ and $\langle \mu_\beta] \subseteq \langle \mu_\lambda]$. That is, if $x \in \langle \mu_\alpha]$ and $y \in \langle \mu_\beta]$, then $x, y \in \langle \mu_\lambda]$. Thus $x \vee y \in \langle \mu_\lambda]$. So

$$\begin{aligned} \widehat{\mu}(x) \wedge \widehat{\mu}(y) &= Sup\{min\{\alpha, \beta\} : x \in \langle \mu_\alpha], y \in \langle \mu_\beta]\} \\ &\leq Sup\{\lambda \in [0, 1] : x \vee y \in \langle \mu_\lambda]\} \\ &= \widehat{\mu}(x \vee y). \end{aligned}$$

Next we show that $\widehat{\mu}(x \wedge y) \geq \widehat{\mu}(x) \vee \widehat{\mu}(y)$. It follows from the definition of $\widehat{\mu}$ that $\widehat{\mu}(b) \geq \widehat{\mu}(a)$, whenever $a \wedge b = b$, for all $a, b \in A$. Using this fact, Since $x \wedge (x \wedge y) = x \wedge y$ and $y \wedge (x \wedge y) = x \wedge y$, for all $x, y \in A$, we get that $\widehat{\mu}(x \wedge y) \geq \widehat{\mu}(x)$ and $\widehat{\mu}(x \wedge y) \geq \widehat{\mu}(y)$ which implies that $\widehat{\mu}(x \wedge y) \geq \widehat{\mu}(x) \vee \widehat{\mu}(y)$. Then $\widehat{\mu}$ is a fuzzy ideal of A.

Next we show that $\mu \subseteq \widehat{\mu}$. For any $x \in A$, put $\mu(x) = \lambda$. Then $x \in \mu_\lambda \subseteq \langle \mu_\lambda]$. Thus $x \in \langle \mu_\lambda]$. So $\lambda \in \{\alpha \in [0, 1] : x \in \langle \mu_\alpha]\}$, that is,

$$\widehat{\mu}(x) = Sup\{\alpha \in [0, 1] : x \in \langle \mu_\alpha]\} \geq \lambda = \mu(x).$$

Hence $\mu \subseteq \widehat{\mu}$.

Now it remains to show that $\widehat{\mu}$ is the smallest fuzzy ideal such that $\mu \subseteq \widehat{\mu}$. Let γ be any fuzzy ideal of A such that $\mu \subseteq \gamma$. Then $\mu_\alpha \subseteq \gamma_\alpha$, for all $\alpha \in [0, 1]$. For;

$x \in \mu_\alpha \Rightarrow \mu(x) \geq \alpha \Rightarrow \gamma(x) \geq \alpha \Rightarrow x \in \gamma_\alpha$. Since γ is a fuzzy ideal of A, we have γ_α is an ideal of A, for all $\alpha \in [0,1]$. That is, γ_α is an ideal of A containing μ_α. Thus $\langle \mu_\alpha] \subseteq \gamma_\alpha$.

Now for any $x \in A$, consider

$$\widehat{\mu}(x) = Sup\{\alpha \in [0,1] : x \in \langle \mu_\alpha]\} \leq Sup\{\alpha \in [0,1] : x \in \gamma_\alpha\} = \gamma(x).$$

Hence the result holds. □

Theorem 3.15 ([6]). *The class $FI(A)$ of all fuzzy ideals of A forms a complete lattice where the infimum and supremum of any family $\{\mu_\alpha : \alpha \in \Delta\}$ of fuzzy ideals is given by:*

$$\bigwedge \mu_\alpha = \cap \mu_\alpha \text{ and } \bigvee \mu_\alpha = \langle \cup \mu_\alpha] .$$

In the remaining part of this section we define fuzzy homomorphisms on ADLs and we present some results on fuzzy homomorphisms in connection with fuzzy ideals.

Recall from [1] that, for any sets A and B a mapping $f : A \times B \to [0,1]$ is called a fuzzy relation of A into B. A fuzzy relation f of A into B is called a fuzzy mapping if for each $x \in A$ there exists a unique element $y_x \in B$ such that $f(x, y_x) = 1$ in this case we call this unique element y_x a fuzzy image of x under f. We write $f : A \dashrightarrow B$, for a fuzzy mapping f of A into B. Image of f is the set $\{y_x : x \in A\} = \{y \in B : f(x,y) = 1\}$. Moreover, for any $y \in B$,

$$f^{-1}(y) = \{x \in A : y_x = y\} = \{x \in A : f(x,y) = 1\}.$$

As usual, f is said to be onto, if for each $y \in B$, there exists $x \in A$ such that $y_x = y$ and f is said to be one-one, if for each $a, b \in A$, $y_a = y_b \Longrightarrow a = b$.

Definition 3.16. Let A and B be ADLs. A fuzzy mapping $f : A \dashrightarrow B$ is called a fuzzy homomorphism of ADLs, if the following are satisfied, for all $a, b \in A$:
(i) $y_0 = 0$ (a zero element in B),
(ii) $f(x_1 \vee x_2, y) \geq sup\{f(x_1, y_1) \wedge f(x_2, y_2) : y = y_1 \vee y_2, y_1, y_2 \in B\}$,
(iii) $f(x_1 \wedge x_2, y) \geq sup\{f(x_1, y_1) \wedge f(x_2, y_2) : y = y_1 \wedge y_2, y_1, y_2 \in B\}$.

Lemma 3.17. *Let $f : A \dashrightarrow B$ be a fuzzy homomorphism of ADLs. Then we have the following:*
(1) $y_{(a\vee b)} = y_a \vee y_b$,
(2) $y_{(a\wedge b)} = y_a \wedge y_b$,
for all $a, b \in A$.

Proof. We have y_a and y_b are the unique elements in B such that $f(a, y_a) = 1$ and $f(b, y_b) = 1$. We show that $f(a \vee b, y_a \vee y_b) = 1$. Put $z = y_a \vee y_b$ for simplicity. Then

$$\begin{aligned} f(a \vee b, z) &= Sup\{f(a, z_1) \wedge f(b, z_2) : z = z_1 \vee z_2, z_1, z_2 \in B\} \\ &\geq f(a, y_a) \wedge f(b, y_b) \\ &= 1. \end{aligned}$$

Since $y_{(a\vee b)}$ is the unique element in B such that $f(a \vee b, y_{(a\vee b)}) = 1$, we get that $y_{(a\vee b)} = y_a \vee y_b$. Similarly, it can be verified that $y_{(a\wedge b)} = y_a \wedge y_b$. □

Lemma 3.18. *Let A and B be ADLs and f a fuzzy homomorphism of A onto B. Let μ be a fuzzy subADL(respectively a fuzzy ideal) of A and σ be a fuzzy subADL(respectively a fuzzy ideal) of B . Then*

(1) *$f(\mu)$ is a fuzzy subADL(respectively a fuzzy ideal) of B,*
(2) *$f^{-1}(\sigma)$ is a fuzzy subADL(respectively a fuzzy ideal) of A.*

Theorem 3.19. *Let A and B be ADLs and $f : A \to B$ a mapping. Then f is a homomorphism if and only if its characteristic mapping χ_f is a fuzzy homomorphism of A into B, where $\chi_f : A \times B \to [0, 1]$ is defined as:*

$$\chi_f(a, b) = \begin{cases} 1 & \text{if} \quad f(a) = b \\ 0 & \text{otherwise,} \end{cases}$$

for all $(a, b) \in A \times B$.

Theorem 3.20. *Let f be a fuzzy homomorphism of A into B. Then a subset f^* of A defined by*

$$f^* = \{x \in A : f(x, 0) = 1\}$$

is an ideal of A.

Proof. Clearly, $f(0, 0) = 1$. then $0 \in f^*$. Let $a, b \in f^*$. Then $f(a, 0) = 1 = f(b, 0)$. We show that $f(a \vee b, 0) = 1$, for;

$$\begin{aligned} f(a \vee b, 0) &= sup\{f(a, y_1) \wedge f(b, y_2) : 0 = y_1 \vee y_2 \text{ and } y_1, y_2 \in B\} \\ &\geq f(a, 0) \wedge f(b, 0) = 1. \end{aligned}$$

That is, $f(a \vee b, 0) = 1$. Thus $a \vee b \in f^*$. Also let $a \in f^*$ and $x \in A$. Then $f(a, 0) = 1$. Now consider

$$f(a \wedge x, 0) = Sup\{f(a, y_1) \wedge f(x, y_2) : 0 = y_1 \wedge y_2\} \geq f(a, 0) \wedge f(x, y_x) = 1,$$

that is, $f(a \wedge x, 0) = 1$. Thus $a \wedge x \in f^*$. So f^* is an ideal of A. □

Theorem 3.21. *Let f be a fuzzy homomorphism of A into B. Then a fuzzy subset μ_f of A defined by*

$$\mu_f(x) = f(x, 0), \textit{ for all } x \in A$$

is a fuzzy ideal of A.

Proof. Clearly $\mu_f(0) = 1$. For any $a, b \in A$, consider the following:

$$\begin{aligned} \mu_f(a \vee b) &= f(a \vee b, 0) \\ &= sup\{f(a, y_1) \wedge f(b, y_2) : y_1 \vee y_2 = 0 \textit{ and } y_1, y_2 \in B\} \\ &\geq f(a, y_1) \wedge f(b, y_2) \ \forall\, y_1, y_2 \in B, \textit{ with } y_1 \vee y_2 = 0. \end{aligned}$$

In particular, for $y_1 = 0$ and $y_2 = 0$. That is, $\mu_f(a \vee b) \geq f(a, 0) \wedge f(b, 0) = \mu_f(a) \wedge \mu_f(b)$.
Also,

$$\begin{aligned} \mu_f(a \wedge b) &= f(a \wedge b, 0) \\ &= sup\{f(a, y_1) \wedge f(b, y_2) : 0 = y_1 \wedge y_2 and y_1, y_2 \in B\} \\ &\geq f(a, y_1) \wedge f(b, y_2) \ \forall y_1, y_2 \in B, \textit{ with } y_1 \wedge y_2 = 0. \end{aligned}$$

In particular, for $y_1 = 0$ and $y_2 = y_b$. That is, $\mu_f(a \wedge b) \geq f(a, 0) \wedge f(b, y_b) = f(a, 0) \wedge 1 = f(a, 0) = \mu_f(a)$. Similarly, doing we get $\mu_f(a \wedge b) \geq \mu_f(b)$ which implies that $\mu_f(a \wedge b) \geq \mu_f(a) \vee \mu_f(b)$. Thus μ_f is a fuzzy ideal of A. □

4. Fuzzy congruences on ADLs

In this section we define fuzzy congruence relations on ADLs and we give several characterizations for fuzzy congruences in terms of fuzzy ideals and fuzzy homomorphisms.
Recall that for any set A a fuzzy subset Θ of $A \times A$ is called a fuzzy relation on A.

Definition 4.1. A fuzzy relation Θ on an ADL A is a called fuzzy congruence relation on A, if the following are satisfied:
(i) $\Theta(a, a) = 1$, for all $a \in A$,
(ii) $\Theta(a, b) = \Theta(b, a)$, for all $a, b \in A$,
(iii) $\Theta(a, c) \geq \Theta(a, b) \wedge \Theta(b, c)$, for all $a, b, c \in A$,
(iv) $\Theta(a \vee c, b \vee d) \wedge \Theta(a \wedge c, b \wedge d) \geq \Theta(a, b) \wedge \Theta(c, d)$, for all $a, b, c, d \in A$.
We denote the set of all fuzzy congruence relations on A by $FC(A)$.

Example 4.2. Let A be an ADL as in Example 3.3. Define a fuzzy relation Θ on A as follows:

$$\begin{aligned}
&\Theta(0,0) = \Theta(a,a) = \Theta(b,b) = \Theta(c,c) = 1,\\
&\Theta(0,c) = \Theta(c,0) = 0.8,\\
&\Theta(a,b) = \Theta(b,a) = \Theta(a,c) = \Theta(c,a) = \Theta(b,c) = \Theta(c,b)\\
&\qquad\quad = \Theta(0,a) = \Theta(a,0) = \Theta(0,b) = \Theta(b,0) = 0.7.
\end{aligned}$$

Then Θ is a fuzzy congruence relation on A.

Lemma 4.3. *Let θ be an equivalence relation on A. Then θ is a congruence relation on A if and only if its characteristic function χ_θ is a fuzzy congruence on A.*

Lemma 4.4. *A fuzzy relation Θ on A is a fuzzy congruence on A if and only if every level subset Θ_t of Θ at $t \in [0, 1]$ is a congruence relation on A.*

Theorem 4.5. *Let Θ be a fuzzy congruence relation on A. A fuzzy subset μ_Θ defined by $\mu_\Theta(x) = \Theta(x, 0)$ for all $x \in A$ is a fuzzy ideal of A.*

Proof. The proof is analogous to that of Theorem 3.21. □

Theorem 4.6. *Let Θ be a fuzzy congruence relation on A. A fuzzy subset ν_Θ defined by $\nu_\Theta(x) = Inf\{\Theta(a \wedge x, x) : a \in A\}$, for all $x \in A$ is a fuzzy ideal of A.*

Proof. $\nu_\Theta(0) = Inf\{\Theta(a \wedge 0, 0) : a \in A\} = Inf\{\Theta(0, 0) : a \in A\} = \Theta(0, 0) = 1$. For any $x, y \in A$, consider

$$\begin{aligned}
\nu_\Theta(x \vee y) &= Inf\{\Theta(a \wedge (x \vee y), x \vee y) : a \in A\}\\
&= Inf\{\Theta[(a \wedge x) \vee (a \wedge y), x \vee y] : a \in A\}\\
&\geq Inf\{\Theta(a \wedge x, x) \wedge \Theta(a \wedge y, y) : a \in A\}\\
&= Inf\{\Theta(a \wedge x, x) : a \in A\} \wedge Inf\{\Theta(a \wedge y, y) : a \in A\}\\
&= \nu_\Theta(x) \wedge \nu_\Theta(y).
\end{aligned}$$

Also consider

$$\begin{aligned}
\nu_\Theta(x \wedge y) &= Inf\{\Theta(a \wedge (x \wedge y), x \wedge y) : a \in A\}\\
&= Inf\{\Theta[(a \wedge x) \wedge y], x \wedge y) : a \in A\}
\end{aligned}$$

$\geq Inf\{\Theta(a \wedge x, x) \wedge \Theta(y, y) : a \in A\}$
$= Inf\{\Theta(a \wedge x, x) : a \in A\}$
$= \nu_\Theta(x)$.

In the similar fashion, we get $\nu_\Theta(x \wedge y) \geq \nu_\Theta(y)$. Then $\nu_\Theta(x \wedge y) \geq \nu_\Theta(x) \vee \nu_\Theta(y)$. Thus ν_Θ is a fuzzy ideal of A. □

Theorem 4.7. *Let Θ be a fuzzy congruence relation on A. Then $\mu_\Theta = \nu_\Theta$.*

Proof. For any fuzzy congruence relation on A, we claim to show that $\mu_\Theta = \nu_\Theta$. For any $x \in A$, we have $\nu_\Theta(x) = Inf\{\Theta(a \wedge x, x) : a \in A\}$. Then $\nu_\Theta(x) \leq \Theta(a \wedge x, x)$, for all $a \in A$. In particular, for $a = 0$,

$$\nu_\Theta(x) \leq \Theta(0 \wedge x, x) = \Theta(0, x) = \Theta(x, 0) = \mu_\Theta(x).$$

On the other hand, for any $a \in A$, consider

$$\begin{aligned}\Theta(a \wedge x, x) &= \Theta[a \wedge x, (a \wedge x) \vee x] \\ &\geq \Theta(a \wedge x, a \wedge x) \wedge \Theta(0, x) \\ &= \Theta(x, 0) = \mu_\Theta(x).\end{aligned}$$

Thus $\Theta(a \wedge x, x) \geq \mu_\Theta(x)$, for all $a \in A$. So

$$\nu_\Theta(x) = Inf\{\Theta(a \wedge x, x) : a \in A\} \geq \mu_\Theta(x).$$

Hence $\mu_\Theta = \nu_\Theta$. □

Theorem 4.8. *Let $f : A \dashrightarrow B$ be a fuzzy homomorphism. Define a fuzzy kernel of f denoted by $K_f : A \times A :\rightarrow [0, 1]$ as follows:*

$$K_f(a, b) = \begin{cases} 1 & if \quad y_a = y_b \\ 0 & otherwise, \end{cases}$$

for all $a, b \in A$. Then this K_f is a fuzzy congruence relation on A.

Corollary 4.9. *f is a fuzzy monomorphism if and only if its kernel K_f is the characteristic function of the diagonal of A.*

Theorem 4.10. *Let Θ be a fuzzy congruence relation on A. For any $x \in A$, define a subset Θ_x of A by*

$$\Theta_x = \{y \in A : \Theta(x, y) = 1\}.$$

Then Θ_0 is an ideal of A.

Remark 4.11. This Θ_0 is a level subset of a fuzzy ideal μ_Θ (given in Theorem 4.5) at $t = 1$. Let Θ be a fuzzy congruence on A. For any $x \in A$ consider a subset Θ_x of A given by $\Theta_x = \{y \in A : \Theta(x, y) = 1\}$. Then we have the following properties:

(1) For any $x, y \in A$ either $\Theta_x \cap \Theta_y = \varnothing$ or $\Theta_x = \Theta_y$,
(2) $x \in \Theta_y$ if and only if $\Theta_x = \Theta_y$ or equivalently if $\Theta(x, y) = 1$.

Put $\dfrac{A}{\Theta} = \{\Theta_x : x \in A\}$ and define operations $\wedge$ and $\vee$ on $\dfrac{A}{\Theta}$ as follows:

$$\Theta_x \wedge \Theta_y = \Theta_{x \wedge y} \text{ and } \Theta_x \vee \Theta_y = \Theta_{x \vee y} \qquad (*)$$

Then $(\dfrac{A}{\Theta}, \wedge, \vee, \Theta_0)$ becomes an ADL with Θ_0 as its zero element and it is called a quotient ADL induced by the fuzzy congruence Θ on A.

Definition 4.12. A fuzzy subset μ of A is said to be multiplicatively closed, if

$$\mu(x \wedge y) \geq \mu(x) \wedge \mu(y), \text{ for all } x, y \in A.$$

Let λ be a multiplicatively closed fuzzy subset of A with $Sup\{\lambda(x) : x \in A\} = 1$. Define fuzzy relations Ψ^λ and Φ^λ on A induced by λ as follows:

$$\Psi^\lambda(x, y) = Sup\{\lambda(a) : x \wedge a = y \wedge a, a \in A\} \text{ and}$$
$$\Phi^\lambda(x, y) = Sup\{\lambda(b) : b \wedge x = b \wedge y, b \in A\}. \text{ for all } x, y \in A.$$

Then we have the following results.

Theorem 4.13. *Ψ^λ is a fuzzy congruence relation on A and the quotient $\dfrac{A}{\Psi^\lambda}$ is a distributive lattice. Moreover if A has maximal elements then the quotient $\dfrac{A}{\Psi^\lambda}$ becomes bounded with the class of all maximal elements, its unit element and $\{0\}$ its least element.*

Proof. We first show that Ψ^λ is a fuzzy congruence on A. For; for any $x, y, z \in A$, consider the following:

(1) $\Psi^\lambda(x, x) = Sup\{\lambda(a) : x \wedge a = x \wedge a, a \in A\} = Sup\{\lambda(a) : a \in A\} = 1$,

(2) $\Psi^\lambda(x, y) = Sup\{\lambda(a) : x \wedge a = y \wedge a, a \in A\} = \Psi^\lambda(y, x)$,

(3) for any $a, b \in A$, if $x \wedge a = y \wedge a$ and $y \wedge b = z \wedge b$, then we get $x \wedge (a \wedge b) = y \wedge (a \wedge b)$ and $y \wedge (a \wedge b) = z \wedge (a \wedge b)$ which implies that $x \wedge (a \wedge b) = z \wedge (a \wedge b)$.

Now consider

$$\begin{aligned}
&\Psi^\lambda(x, y) \wedge \Psi^\lambda(y, z)\\
&= Sup\{\lambda(a) : x \wedge a = y \wedge a, a \in A\} \wedge Sup\{\lambda(b) : y \wedge b = z \wedge b, b \in A\}\\
&= Sup\{\lambda(a) \wedge \lambda(b) : x \wedge a = y \wedge a \textit{ and } y \wedge b = z \wedge b, a, b \in A\}\\
&\leq Sup\{\lambda(a \wedge b) : x \wedge a = y \wedge a \textit{ and } y \wedge b = z \wedge b,\ a, b \in A\}\\
&\leq Sup\{\lambda(c) : x \wedge c = y \wedge c,\ c \in A\}\\
&= \Psi^\lambda(x, z).
\end{aligned}$$

(4) similarly, it can be verified that

$$\Psi^\lambda(x_1 \vee x_2, y_1 \vee y_2) \geq \Psi^\lambda(x_1, y_1) \wedge \Psi^\lambda(x_2, y_2)$$

and

$$\Psi^\lambda(x_1 \wedge x_2, y_1 \wedge y_2) \geq \Psi^\lambda(x_1, y_1) \wedge \Psi^\lambda(x_2, y_2).$$

Thus Ψ^λ is a fuzzy congruence relation on A.

Next we show that the quotient $\dfrac{A}{\Psi^\lambda}$ is a distributive lattice. Clearly, it is an ADL together with binary operations $\vee$ and $\wedge$ defined as in (*). It suffices to show that either $\wedge$ or $\vee$ is commutative on $\dfrac{A}{\Psi^\lambda}$. For any $x, y \in A$, consider

$$\Psi^\lambda(x \wedge y, y \wedge x) = sup\{\lambda(a) : (x \wedge y) \wedge a = (y \wedge x) \wedge a, a \in A\} = sup\{\lambda(a) : a \in A\} = 1.$$

Then $\Psi^\lambda_{x \wedge y} = \Psi^\lambda_{y \wedge x}$ which says that $\wedge$ is commutative. Thus the quotient $\dfrac{A}{\Psi^\lambda}$ is a distributive lattice. □

Remark 4.14. It is clear that Φ^λ is a fuzzy congruence on A. But the quotient $\dfrac{A}{\Phi^\lambda}$ is not in general a distributive lattice. We verify this by giving the following example.

Example 4.15. Let A be a discrete ADL with $|A| \geq 3$ (see [9] and [4]). Let λ be a fuzzy subset of A defined by:

$$\lambda(x) = \begin{cases} 0 & \texttt{if} \quad x = 0 \\ 1 & \texttt{otherwise.} \end{cases}$$

Then λ is a multiplicatively closed fuzzy subset of A and Φ^λ is a fuzzy congruence on A. We show that $\dfrac{A}{\Phi^\lambda} \cong A$. For; consider the canonical map $f : A \to \frac{A}{\Phi^\lambda}$ defined by:

$$f(x) = \Phi_x^\lambda, \text{ for all } x \in A.$$

Then it is clear that f is an epimorphism. It remains to show that f is one-one. For any $x, y \in A$

$$\begin{aligned} f(x) = f(y) \quad &\Rightarrow \quad \Phi_x^\lambda = \Phi_y^\lambda \\ &\Rightarrow \quad \Phi^\lambda(x, y) = 1 \\ &\Rightarrow \quad Sup\{\lambda(a) : a \wedge x = a \wedge y\} = 1. \end{aligned}$$

Since $Img\lambda = \{0, 1\}$, there exists a nonzero $a \in A$ such that $a \wedge x = a \wedge y$. By the fact that every nonzero element in A is maximal, it follows that $x = y$. Thus f is one-one and hence an isomorphism. Since A is a discrete ADL with at least 3 elements, it is not a lattice. So the quotient $\dfrac{A}{\Phi^\lambda}$ is not a distributive lattice.

In the next theorem, we give the smallest fuzzy congruence on A for which its quotient is a distributive lattice.

Theorem 4.16. *A fuzzy relation η on A defined by*

$$\eta(a, b) = \begin{cases} 1 & \textit{if} \quad \langle a] = \langle b] \\ 0 & \textit{otherwise,} \end{cases}$$

for all $a, b \in A$, is a fuzzy congruence relation on A and it is the smallest such that the quotient $\dfrac{A}{\eta}$ is a distributive lattice.

Proof. Clearly, η is a fuzzy congruence on A. It is also clear that $\langle a \wedge b] = \langle b \wedge a]$, for all $a, b \in A$. Then $\eta(a \wedge b, b \wedge a) = 1$. Thus $\eta_{(a \wedge b)} = \eta_{(b \wedge a)}$ which implies that the quotient $\frac{A}{\eta}$ is a distributive lattice. Now let Θ be any fuzzy congruence on A such that the quotient $\frac{A}{\Theta}$ is a distributive lattice. We claim to show that $\eta \subseteq \Theta$. For any $a, b \in A$, consider the following.

If $\langle a] \neq \langle b]$, then $\eta(a, b) = 0 \leq \Theta(a, b)$. Otherwise,

$$\begin{aligned} \langle a] = \langle b] \quad &\Rightarrow \quad a \wedge b = b, \ b \wedge a = a \\ &\Rightarrow \quad \Theta_{(a \wedge b)} = \Theta_b, \ \Theta_{(b \wedge a)} = \Theta_a \\ &\Rightarrow \quad \Theta_a = \Theta_b \ \ (\textit{since } \frac{A}{\Theta} \textit{ is a lattice}) \\ &\Rightarrow \quad \Theta(a, b) = 1 = \eta(a, b). \end{aligned}$$

Thus $\eta \subseteq \Theta$. □

Definition 4.17. An ADL A is said to be associative, if the binary operation $\vee$ in A is associative.

Theorem 4.18. *Let A be an associative ADL and μ a fuzzy ideal of A. Let us define fuzzy relation ϕ_μ on A by:*

$$\phi_\mu(x,y) = Sup\{\mu(a) : a \vee x = a \vee y, a \in A\}, \text{ for all } x,y \in A.$$

Then ϕ_μ is a fuzzy congruence relation on A.

Proof. For any $x,y,z \in A$, consider

(1) $\phi_\mu(x,x) = Sup\{\mu(a) : a \vee x = a \vee x, a \in A\} = Sup\{\mu(a) : a \in A\} = 1$,

(2) $\phi_\mu(x,y) = Sup\{\mu(a) : a \vee x = a \vee y, a \in A\} = \phi_\mu(y,x)$,

(3) if $a \vee x = a \vee y$ and $b \vee y = b \vee z$, for $a,b \in A$, then as A is an associative ADL, we get $(a \vee b) \vee x = (a \vee b) \vee y$ and $(a \vee b) \vee y = (a \vee b) \vee z$ which implies that $(a \vee b) \vee x = (a \vee b) \vee z$.

Now consider

$$\begin{aligned}
&\phi_\mu(x,y) \wedge \phi_\mu(y,z)\\
&= Sup\{\mu(a) : a \vee x = a \vee y, a \in A\} \wedge Sup\{\mu(b) : b \vee y = b \vee z, b \in A\}\\
&= Sup\{\mu(a) \wedge \mu(b) : a \vee x = a \vee y \text{ and } b \vee y = b \vee z,\ a,b \in A\}\\
&= Sup\{\mu(a) \wedge \mu(b) : (a \vee b) \vee x = (a \vee b) \vee z,\ a,b \in A\}\\
&\leq Sup\{\mu(a \vee b) : (a \vee b) \vee x = (a \vee b) \vee z,\ a,b \in A\}\\
&\leq Sup\{\mu(c) : c \vee x = c \vee z,\ c \in A\}\\
&= \phi_\mu(x,z).
\end{aligned}$$

(4) Similar to (3), we can verify that

$$\phi_\mu(x_1 \vee x_2, y_1 \vee y_2) \geq \phi_\mu(x_1,y_1) \wedge \phi_\mu(x_2,y_2)$$

and

$$\phi_\mu(x_1 \wedge x_2, y_1 \wedge y_2) \geq \phi_\mu(x_1,y_1) \wedge \phi_\mu(x_2,y_2).$$

Thus ϕ_μ is a fuzzy congruence relation on A. □

Theorem 4.19. *ϕ_μ is the smallest fuzzy congruence on A containing the product fuzzy ideal $\mu \times \mu$, of $A \times A$, where the product of any two fuzzy subsets μ and ν of A and B respectively is defined as:*

$$(\mu \times \nu)(x,y) = \mu(x) \wedge \nu(y), \text{ for all } (x,y) \in A \times B.$$

Proof. We see in the above theorem that ϕ_μ is a fuzzy congruence on A. We first show that $\mu \times \mu \subseteq \phi_\mu$. For; for any $x,y \in A$, we have $(\mu \times \mu)(x,y) = \mu(x) \wedge \mu(y) = \mu(x \vee y)$. Put $B = \{\mu(a) : a \vee x = a \vee y,\ a \in A\}$. Since $(x \vee y) \vee x = x \vee y = (x \vee y) \vee y$, $\mu(x \vee y) \in B$. Then $\mu(x \vee y) \leq Sup\ B = \phi_\mu(x,y)$. That is, $\mu \times \mu \subseteq \phi_\mu$. Let Γ be any fuzzy congruence on A such that $\mu \times \mu \subseteq \Gamma$. For any $x,y \in A$, let $a \in A$ such that $a \vee x = a \vee y$. Since Γ is a fuzzy congruence on A, we have $\Gamma(x,y) \geq \Gamma(x,z) \wedge \Gamma(z,y)$, for all $z \in A$. In particular, for $z = a \vee x = a \vee y$, $\Gamma(x,y) \geq \Gamma(x, a \vee x) \wedge \Gamma(a \vee y, y)$. But

$$\begin{aligned}
\Gamma(x, a \vee x) &= \Gamma((a \wedge x) \vee x, a \vee x)\\
&\geq \Gamma(a \wedge x, a) \wedge \Gamma(x,x)\\
&= \Gamma(a \wedge x, a)\\
&\geq (\mu \times \mu)(a \wedge x, a)\\
&= \mu(a \wedge x) \wedge \mu(a)\\
&\geq \mu(a).
\end{aligned}$$

Similarly, we have $\Gamma(a \vee y, y) \geq \mu(a)$. Then $\Gamma(x, y) \geq \mu(a)$, for all $a \in A$ with $\mu(a) \in B$. Thus Sup $B \leq \Gamma(x, y)$. So $\phi_\mu(x, y) \leq \Gamma(x, y)$, for all $x, y \in A$. Hence the result holds. □

5. Fuzzy Ideals and fuzzy congruenes in almost Boolean rings

Definition 5.1 ([9]). An algebra $(A, +, \cdot, 0)$ is called an almost Boolean ring abbreviated as ABR, if for any $a, b, c, d \in R$, it satisfies the following:
(i) $a + 0 = a$,
(ii) $a + a = 0$,
(iii) $(ab)c = a(bc)$,
(iv) $a(b + c) = ab + ac$,
(v) $(a + b)c = ac + bc$,
(vi) $\{a + (b + c)\}d = \{(a + b) + c\}d$.

Definition 5.2 ([9]). An ADL $(A, \vee, \wedge, 0)$ is said to be relatively complemented, if every interval is a Boolean algebra.

Lemma 5.3 ([9]). *An ADL A is relatively complemented if and only if for any $a, b \in A$, there exists $x \in A$ such that $a \vee b = a \vee x$ and $a \wedge x = 0$. In this case, x is unique which we denote by a^b.*

Theorem 5.4 ([9]). *Let $(A, \vee, \wedge, 0)$ be a relatively complemented ADL. Define binary operations "$\cdot$" and "$+$" on A by $a \cdot b = a \wedge b$ and $a + b = a^b \vee b^a$. Then $(A, +, \cdot, 0)$ is an almost Boolean ring. Furthermore, $a \wedge b = a \cdot b$ and $a \vee b = a + (b + a \cdot b)$.*

Theorem 5.5 ([9]). *Let $(A, +, \cdot, 0)$ be an almost Boolean ring. Define binary operations $\wedge$ and $\vee$ by $a \wedge b = a \cdot b$ and $a \vee b = a + (b + a \cdot b)$. Then $(A, \vee, \wedge, 0)$ is a relatively complemented ADL. Furthermore, we get $a \cdot b = a \wedge b$ and $a + b = a^b \vee b^a$.*

The above two theorems give us a duality between the class of relatively complemented ADLs and the class of almost Boolean rings analogous to the well known Stone's duality between the class of relatively complemented lattices with 0 and the class of Boolean Rings.

Definition 5.6. Let A be an almost Boolean ring. A fuzzy subset μ of A is called a fuzzy ideal of A, if the following are satisfied:

$$\mu(0) = 1,\ \mu(a + b) \geq \mu(a) \wedge \mu(b) \text{ and } \mu(a \cdot b) \geq \mu(a) \vee \mu(b), \text{ for all } a, b \in A.$$

Moreover, a fuzzy relation Θ on an almost Boolean ring A is said to be a fuzzy congruence relation on A, if

$$\Theta(a + c, b + d) \wedge \Theta(a \cdot c, b \cdot d) \geq \Theta(a, b) \wedge \Theta(c, d), \text{ for all } a, b, c, d \in A.$$

As a result of the duality in [9] between the class of relatively complemented ADLs and the class of almost Boolean rings, one can easily verify that a fuzzy subset μ of A is a fuzzy ideal of A as an ADL (i.e., considering A as a relatively complemented ADL) if and only if it is a fuzzy ideal of A as an almost Boolean ring (i.e., considering A as an almost Boolean ring). Similarly, a fuzzy equivalence relation θ on A is a fuzzy congruence on A as an ADL if and only if it is a fuzzy congruence on A as an almost Boolean ring.

Theorem 5.7. *Let A be an almost Boolean ring and μ be a fuzzy ideal of A. Then a fuzzy relation Θ_μ defined by:*

$$\Theta_\mu(a,b) = \mu(a+b), \text{ for all } a,b \in A$$

is a fuzzy congruence relation on A.

Proof. For any $a,b,c \in A$, consider the following.
(1) $\Theta_\mu(a,a) = \mu(a+a) = \mu(0) = 1$.
(2) $\Theta_\mu(a,b) = \mu(a+b) = \mu(b+a) = \Theta_\mu(b,a)$.
(3) For any $a,b,c \in A$, let us first see that $a+c = ((a+b)+(b+c))(a+c)$. For;

$$a+c = (a+c)(a+c) = (a+b+b+c)(a+c) = ((a+b)+(b+c))(a+c).$$

Then

$$\begin{aligned}\Theta_\mu(a,c) = \mu(a+c) &= \mu\{((a+b)+(b+c))(a+c)\}\\ &\geq \mu((a+b)+(b+c)) \vee \mu(a+c)\\ &\geq \mu((a+b)+(b+c))\\ &\geq \mu(a+b) \wedge \mu(b+c)\\ &= \Theta_\mu(a,b) \wedge \Theta_\mu(b,c).\end{aligned}$$

(4) Using similar techniques as in (3), one can verify that

$$\Theta_\mu(a \vee c, b \vee d) \geq \Theta_\mu(a,b) \wedge \Theta_\mu(c,d)$$

and

$$\Theta_\mu(a \wedge c, b \wedge d) \geq \Theta_\mu(a,b) \wedge \Theta_\mu(c,d).$$

Then Θ_μ is a fuzzy congruence relation on A. □

Now let us denote Θ_μ by $C(\mu)$ to say that it is induced by the fuzzy ideal μ. On the other hand, for any given fuzzy congruence Θ on an almost Boolean ring A, we can define a fuzzy ideal μ_Θ on A by $\mu_\Theta(x) = \Theta(x,0)$, for all $x \in A$ (see Theorem 4.5). Let us denote μ_Θ by $I(\Theta)$ to say that it is induced by Θ. Then we have the following results.

Lemma 5.8. *Let A be an almost Boolean ring. If μ is any fuzzy ideal of A, then $I(C(\mu)) = \mu$.*

Proof. For any $x \in A$, consider $I(C(\mu))(x) = C(\mu)(x,0) = \mu(x+0) = \mu(x)$. Then $I(C(\mu)) = \mu$. □

Theorem 5.9. *There is a monomorphism of the lattice $FI(A)$ of all fuzzy ideals of an almost Boolean ring A into the lattice $FC(A)$ of all fuzzy congruences on A.*

Proof. Consider a mapping $\mu \mapsto C(\mu)$ of $FI(A)$ into $FC(A)$. It follows from the above lemma that this mapping is a lattice monomorphism. □

Theorem 5.10. *The monomorphism $\mu \mapsto C(\mu)$ of the lattice $FI(A)$ into the lattice $FC(A)$ is an isomorphism if and only if A is a generalized Boolean algebra (or simply a Boolean Ring).*

Proof. It is observed in [10] that if A is a generalized Boolean algebra, then the mapping $\mu \mapsto C(\mu)$ is a lattice isomorphism.

Conversely, suppose that the mapping $\mu \mapsto C(\mu)$ is a lattice isomorphism of $FI(A)$ into $FC(A)$. We claim to show that A is a distributive lattice. Now it suffices to show that the binary operation "$\cdot$" is commutative on A. Let Θ and Φ be fuzzy relations on A defined by:

$$\Theta(x,y) = \begin{cases} 1 & \text{if} \quad x = y \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Phi(x,y) = \begin{cases} 1 & \text{if} \quad \langle x] = \langle y] \\ 0 & \text{otherwise,} \end{cases}$$

for all $x, y \in A$. Then it can be easily verified that both $\Theta, \Phi \in FC(A)$. Since the mapping $\mu \mapsto C(\mu)$ is an isomorphism, there exists $\mu, \nu \in FI(A)$ such that $C(\mu) = \Theta$ and $C(\nu) = \Phi$ which will give us that both μ and ν are the characteristic function of $\{0\}$. That is, $\mu = \nu$ which implies that $\Theta = \Phi$. Thus $\langle x] = \langle y]$ if and only if $x = y$, for all $x, y \in A$. It follows from the fact $\langle x \cdot y] = \langle y \cdot x]$ that $x \cdot y = y \cdot x$, for all $x, y \in A$. This says that A is a generalized Boolean algebra (or simply a Boolean ring). □

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