Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 14, No. 3, (September 2017), pp. 279–302
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

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Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 14, No. 3, September 2017

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Received 7 January 2017; Accepted 8 February 2017

Abstract. In this paper, level subsets of a hesitant fuzzy set on UP-algebras are introduced and proved some results. Further, we discuss the relationships among (prime, weakly prime) hesitant fuzzy UP-subalgebras (resp. hesitant fuzzy UP-filters, hesitant fuzzy UP-ideals and hesitant fuzzy strongly UP-ideals) and some level subsets of a hesitant fuzzy set on UP-algebras.

2010 AMS Classification: 03G25

Keywords: UP-algebra, Hesitant fuzzy set, Level subset.

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1. Introduction and Preliminaries

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [5], BCI-algebras [6], BCH-algebras [3], KU-algebras [13], SU-algebras [11] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [6] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [5, 6] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

A fuzzy subset f of a set S is a function from S to a closed interval $[0, 1]$. The concept of a fuzzy subset of a set was first considered by Zadeh [17] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere.

In 2009 - 2010, Torra and Narukawa [15, 16] introduced the notion of hesitant fuzzy sets, that is a function from a reference set to a power set of the unit interval. The notion of hesitant fuzzy sets is the other generalization of the notion fuzzy sets.

The hesitant fuzzy set theories developed by Torra and others have found many applications in the domain of mathematics and elsewhere.

After the introduction of the notion of hesitant fuzzy sets by Torra and Narukawa [15, 16], several researches were conducted on the generalizations of the notion of hesitant fuzzy sets and application to many logical algebras such as: In 2012, Zhu, Xu and Xia [18] introduced the notion of dual hesitant fuzzy sets, which is a new extension of fuzzy sets. In 2014, Jun, Ahn and Muhiuddin [8] introduced the notions of hesitant fuzzy soft subalgebras and (closed) hesitant fuzzy soft ideals in BCK/BCI-algebras. Jun and Song [9] introduced the notions of (Boolean, prime, ultra, good) hesitant fuzzy filters and hesitant fuzzy MV-filters of MTL-algebras. In 2015, Jun and Song [10] introduced the notions of hesitant fuzzy prefilters (resp. filters) and positive implicative hesitant fuzzy prefilters (resp. filters) of EQ-algebras. In 2016, Jun and Ahn [7] introduced the notions of hesitant fuzzy subalgebras and hesitant fuzzy ideals of BCK/BCI-algebras.

Iampan [4] introduced a new algebraic structure, called a UP-algebra, and Mosrijai, Kamti, Satirad and Iampan [12] introduced the notion of hesitant fuzzy sets on UP-algebras. The notions of hesitant fuzzy subalgebras, hesitant fuzzy filters and hesitant fuzzy ideals play an important role in studying the many logical algebras. In this paper, level subsets of a hesitant fuzzy set on UP-algebras are introduced and proved some results. Further, we discuss the relationships among (prime, weakly prime) hesitant fuzzy UP-subalgebras (resp. hesitant fuzzy UP-filters, hesitant fuzzy UP-ideals and hesitant fuzzy strongly UP-ideals) and some level subsets of a hesitant fuzzy set on UP-algebras.

Before we begin our study, we will introduce the definition of a UP-algebra.

Definition 1.1 ([4])**.** An algebra $A = (A; \cdot, 0)$ of type $(2, 0)$ is called a UP-algebra, if it satisfies the following axioms: for any $x, y, z \in A$,

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,
(UP-2) $0 \cdot x = x$,
(UP-3) $x \cdot 0 = 0$,
(UP-4) $x \cdot y = y \cdot x = 0$ implies $x = y$.

From [4], we know that the notion of UP-algebras is a generalization of KU-algebras.

Example 1.2 ([4])**.** Let X be a universal set. Define two binary operations $\cdot$ and $*$ on the power set of X by putting $A \cdot B = B \cap A' = A' \cap B = B - A$ and $A * B = B \cup A' = A' \cup B$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X); \cdot, \varnothing)$ is a UP-algebra and we shall call it the power UP-algebra of type 1, and $(\mathcal{P}(X); *, X)$ is a UP-algebra and we shall call it the power UP-algebra of type 2.

Example 1.3 ([4])**.** Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	0	0
2	0	1	0	3
3	0	1	2	0

Then $(A;\cdot,0)$ is a UP-algebra.

In what follows, let A and B denote UP-algebras unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 1.4 ([4])**.** *In a UP-algebra A, the following properties hold: for any $x,y,z \in A$,*

(1) $x \cdot x = 0$,
(2) $x \cdot y = 0$ *and* $y \cdot z = 0$ *imply* $x \cdot z = 0$,
(3) $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0$,
(4) $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$,
(5) $x \cdot (y \cdot x) = 0$,
(6) $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$,
(7) $x \cdot (y \cdot y) = 0$.

Definition 1.5 ([4])**.** A subset S of A is called a UP-subalgebra of A, if the constant 0 of A is in S, and $(S;\cdot,0)$ itself forms a UP-algebra.

Definition 1.6 ([4])**.** A subset B of A is called a UP-ideal of A if it satisfies the following properties:
(i) the constant 0 of A is in B,
(ii) for any $x,y,z \in A, x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.

Definition 1.7 ([14])**.** A subset F of A is called a UP-filter of A, if it satisfies the following properties:
(i) the constant 0 of A is in F,
(ii) for any $x,y \in A, x \cdot y \in F$ and $x \in F$ imply $y \in F$.

Definition 1.8 ([2])**.** A subset C of A is called a strongly UP-ideal of A, if it satisfies the following properties:
(i) the constant 0 of A is in C,
(ii) for any $x,y,z \in A, (z \cdot y) \cdot (z \cdot x) \in C$ and $y \in C$ imply $x \in C$.

From [2], we know that the notion of fuzzy UP-subalgebras is a generalization of fuzzy UP-filters, the notion of fuzzy UP-filters is a generalization of fuzzy UP-ideals, and the notion of fuzzy UP-ideals is a generalization of fuzzy strongly UP-ideals.

Definition 1.9 ([14])**.** A nonempty subset B of A is called a prime subset of A, if it satisfies the following property: for any $x,y \in A$,

$$x \cdot y \in B \text{ implies } x \in B \text{ or } y \in B.$$

Definition 1.10 ([14])**.** A UP-subalgebra (resp. UP-filter, UP-ideal, strongly UP-ideal) B of A is called a prime UP-subalgebra (resp. prime UP-filter, prime UP-ideal, prime strongly UP-ideal) of A if B is a prime subset of A.

Theorem 1.11 ([2])**.** *Let S be a subset of A. Then the following statements are equivalent:*

(1) *S is a prime UP-subalgebra (resp. prime UP-filter, prime UP-ideal, prime strongly UP-ideal) of A,*
(2) $S = A$,
(3) *S is a strongly UP-ideal of A.*

Definition 1.12 ([2]). A nonempty subset B of A is called a weakly prime subset of A, if it satisfies the following property: for any $x, y \in A$ and $x \neq y$,

$$x \cdot y \in B \text{ implies } x \in B \text{ or } y \in B.$$

Definition 1.13 ([2]). A UP-subalgebra (resp. UP-filter, UP-ideal, strongly UP-ideal) B of A is called a weakly prime UP-subalgebra (resp. weakly prime UP-filter, weakly prime UP-ideal, weakly prime strongly UP-ideal) of A, if B is a weakly prime subset of A.

Definition 1.14 ([1, 15]). Let X be a reference set. A hesitant fuzzy set on X is defined in term of a function h that when applied to X return a subset of $[0, 1]$, that is, $\mathrm{h}\colon X \to \mathcal{P}([0, 1])$.

Definition 1.15 ([12]). Let h be a hesitant fuzzy set on A. The hesitant fuzzy set $\overline{\mathrm{h}}$ defined by $\overline{\mathrm{h}}(x) = [0, 1] - \mathrm{h}(x)$, for all $x \in A$ is said to be the complement of h on A.

For all hesitant fuzzy set h on A, we have $\mathrm{h} = \overline{\overline{\mathrm{h}}}$.

Definition 1.16 ([12]). A hesitant fuzzy set h on A is called a hesitant fuzzy UP-subalgebra of A, if it satisfies the following property: for any $x, y \in A$,

$$\mathrm{h}(x \cdot y) \supseteq \mathrm{h}(x) \cap \mathrm{h}(y).$$

By Proposition 1.4 (1), we have $\mathrm{h}(0) = \mathrm{h}(x \cdot x) \supseteq \mathrm{h}(x) \cap \mathrm{h}(x) = \mathrm{h}(x)$, for all $x \in A$.

Definition 1.17 ([12]). A hesitant fuzzy set h on A is called a hesitant fuzzy UP-filter of A, if it satisfies the following properties: for any $x, y \in A$,
(i) $\mathrm{h}(0) \supseteq \mathrm{h}(x)$,
(ii) $\mathrm{h}(y) \supseteq \mathrm{h}(x \cdot y) \cap \mathrm{h}(x)$.

Definition 1.18 ([12]). A hesitant fuzzy set h on A is called a hesitant fuzzy UP-ideal of A if it satisfies the following properties: for any $x, y, z \in A$,
(i) $\mathrm{h}(0) \supseteq \mathrm{h}(x)$,
(ii) $\mathrm{h}(x \cdot z) \supseteq \mathrm{h}(x \cdot (y \cdot z)) \cap \mathrm{h}(y)$.

Definition 1.19 ([12]). A hesitant fuzzy set h on A is called a hesitant fuzzy strongly UP-ideal of A if it satisfies the following properties: for any $x, y, z \in A$,
(i)$\mathrm{h}(0) \supseteq \mathrm{h}(x)$,
(ii) $\mathrm{h}(x) \supseteq \mathrm{h}((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h}(y)$.

From [12], we know that the notion of hesitant fuzzy UP-ideals of UP-algebras is the generalization of the notion of hesitant fuzzy strongly UP-ideals, the notion of hesitant fuzzy UP-filters of UP-algebras is the generalization of the notion of hesitant fuzzy UP-ideals, and the notion of hesitant fuzzy UP-subalgebras of UP-algebras is the generalization of the notion of hesitant fuzzy UP-filters.

Definition 1.20 ([12]). A hesitant fuzzy set h on A is called a prime hesitant fuzzy set on A, if it satisfies the following property: for any $x, y \in A$,

$$\mathrm{h}(x \cdot y) \subseteq \mathrm{h}(x) \cup \mathrm{h}(y).$$

Definition 1.21 ([12]). A hesitant fuzzy UP-subalgebra (resp. hesitant fuzzy UP-filter, hesitant fuzzy UP-ideal, hesitant fuzzy strongly UP-ideal) h of A is called a prime hesitant fuzzy UP-subalgebra (resp. prime hesitant fuzzy UP-filter, prime hesitant fuzzy UP-ideal, prime hesitant fuzzy strongly UP-ideal), if h is a prime hesitant fuzzy set on A.

Theorem 1.22 ([12]). *Let* h *be a hesitant fuzzy set on A. Then the following statements are equivalent:*

(1) h *is a prime hesitant fuzzy UP-subalgebra (resp. prime hesitant fuzzy UP-filter, prime hesitant fuzzy UP-ideal, prime hesitant fuzzy strongly UP-ideal) of A,*
(2) h *is a constant hesitant fuzzy set on A,*
(3) h *is a hesitant fuzzy strongly UP-ideal of A.*

Definition 1.23 ([12]). A hesitant fuzzy set h on A is called a weakly prime hesitant fuzzy set on A, if it satisfies the following property: for any $x, y \in A$ and $x \neq y$,

$$\mathrm{h}(x \cdot y) \subseteq \mathrm{h}(x) \cup \mathrm{h}(y).$$

Definition 1.24 ([12]). A hesitant fuzzy UP-subalgebra (resp. hesitant fuzzy UP-filter, hesitant fuzzy UP-ideal, hesitant fuzzy strongly UP-ideal) h of A is called a weakly prime hesitant fuzzy UP-subalgebra (resp. weakly prime hesitant fuzzy UP-filter, weakly prime hesitant fuzzy UP-ideal, weakly prime hesitant fuzzy strongly UP-ideal), if h is a weakly prime hesitant fuzzy set on A.

From [12], we know that the notion of weakly prime hesitant fuzzy UP-subalgebras (resp. weakly prime hesitant fuzzy UP-filters, weakly hesitant fuzzy UP-ideals) is a generalization of prime hesitant fuzzy UP-subalgebras (resp. prime hesitant fuzzy UP-filters, prime hesitant fuzzy UP-ideals), and the notions of weakly prime hesitant fuzzy strongly UP-ideals and prime hesitant fuzzy strongly UP-ideals coincide.

2. Main Results

In this section, we discuss the relationships among (prime, weakly prime) hesitant fuzzy UP-subalgebras (resp. hesitant fuzzy UP-filters, hesitant fuzzy UP-ideals and hesitant fuzzy strongly UP-ideals) and some level subsets of a hesitant fuzzy set on UP-algebras.

Definition 2.1. Let h be a hesitant fuzzy set on A. For any $\varepsilon \in \mathcal{P}([0,1])$, the sets

$$U(\mathrm{h};\varepsilon) = \{x \in A \mid \mathrm{h}(x) \supseteq \varepsilon\} \text{ and } U^{+}(\mathrm{h};\varepsilon) = \{x \in A \mid \mathrm{h}(x) \supset \varepsilon\}$$

are called an upper ε-level subset and an upper ε-strong level subset of h, respectively.

The sets

$$L(\mathrm{h};\varepsilon) = \{x \in A \mid \mathrm{h}(x) \subseteq \varepsilon\} \text{ and } L^{-}(\mathrm{h};\varepsilon) = \{x \in A \mid \mathrm{h}(x) \subset \varepsilon\}$$

are called a lower ε-level subset and a lower ε-strong level subset of h, respectively.

The set

$$E(\mathrm{h};\varepsilon) = \{x \in A \mid \mathrm{h}(x) = \varepsilon\}$$

is called an equal ε-level subset of h. Then

$$U(\mathrm{h};\varepsilon) = U^{+}(\mathrm{h};\varepsilon) \cup E(\mathrm{h};\varepsilon) \text{ and } L(\mathrm{h};\varepsilon) = L^{-}(\mathrm{h};\varepsilon) \cup E(\mathrm{h};\varepsilon).$$

2.1. **Upper ε-Level Subsets.**

Theorem 2.2. *A hesitant fuzzy set* h *on* A *is a hesitant fuzzy UP-subalgebra of* A *if and only if for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $U(\mathrm{h};\varepsilon)$ *of* A *is a UP-subalgebra of* A*.*

Proof. Assume that h is a hesitant fuzzy UP-subalgebra of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $U(\mathrm{h};\varepsilon) \neq \varnothing$ and let $x, y \in U(\mathrm{h};\varepsilon)$. Then $\mathrm{h}(x) \supseteq \varepsilon$ and $\mathrm{h}(y) \supseteq \varepsilon$. Since h is a hesitant fuzzy UP-subalgebra of A, we have $\mathrm{h}(x \cdot y) \supseteq \mathrm{h}(x) \cap \mathrm{h}(y) \supseteq \varepsilon$ and thus $x \cdot y \in U(\mathrm{h};\varepsilon)$. So, $U(\mathrm{h};\varepsilon)$ is a UP-subalgebra of A.

Conversely, assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U(\mathrm{h};\varepsilon)$ of A is a UP-subalgebra of A. Let $x, y \in A$. Choose $\varepsilon = \mathrm{h}(x) \cap \mathrm{h}(y) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x) \supseteq \varepsilon$ and $\mathrm{h}(y) \supseteq \varepsilon$. Thus $x, y \in U(\mathrm{h};\varepsilon) \neq \varnothing$. By assumption, $U(\mathrm{h};\varepsilon)$ is a UP-subalgebra of A and thus $x \cdot y \in U(\mathrm{h};\varepsilon)$. So $\mathrm{h}(x \cdot y) \supseteq \varepsilon = \mathrm{h}(x) \cap \mathrm{h}(y)$. Hence, h is a hesitant fuzzy UP-subalgebra of A. □

Theorem 2.3. *A hesitant fuzzy set* h *on* A *is a hesitant fuzzy UP-filter of* A *if and only if for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $U(\mathrm{h};\varepsilon)$ *of* A *is a UP-filter of* A*.*

Proof. Assume that h is a hesitant fuzzy UP-filter of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $U(\mathrm{h};\varepsilon) \neq \varnothing$ and let $x \in U(\mathrm{h};\varepsilon)$. Then $\mathrm{h}(x) \supseteq \varepsilon$. Since h is a hesitant fuzzy UP-filter of A, we have $\mathrm{h}(0) \supseteq \mathrm{h}(x) \supseteq \varepsilon$. Thus $0 \in U(\mathrm{h};\varepsilon)$.

Next, let $x, y \in A$ be such that $x \in U(\mathrm{h};\varepsilon)$ and $x \cdot y \in U(\mathrm{h};\varepsilon)$. Then $\mathrm{h}(x) \supseteq \varepsilon$ and $\mathrm{h}(x \cdot y) \supseteq \varepsilon$. Since h is a hesitant fuzzy UP-filter of A, we have $\mathrm{h}(y) \supseteq \mathrm{h}(x \cdot y) \cap \mathrm{h}(x) \supseteq \varepsilon$. So $y \in U(\mathrm{h};\varepsilon)$. Hence, $U(\mathrm{h};\varepsilon)$ is a UP-filter of A.

Conversely, assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U(\mathrm{h};\varepsilon)$ of A is a UP-filter of A. Let $x \in A$. Then $\mathrm{h}(x) \in \mathcal{P}([0,1])$. Choose $\varepsilon = \mathrm{h}(x) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x) \supseteq \varepsilon$. Thus $x \in U(\mathrm{h};\varepsilon)$. By assumption, we have $U(\mathrm{h};\varepsilon)$ is a UP-filter of A and thus $0 \in U(\mathrm{h};\varepsilon)$. So $\mathrm{h}(0) \supseteq \varepsilon = \mathrm{h}(x)$.

Next, let $x, y \in A$. Then $\mathrm{h}(x), \mathrm{h}(y) \in \mathcal{P}([0,1])$. Choose $\varepsilon = \mathrm{h}(x \cdot y) \cap \mathrm{h}(x) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x \cdot y) \supseteq \varepsilon$ and $\mathrm{h}(x) \supseteq \varepsilon$. So $x \cdot y, x \in U(\mathrm{h};\varepsilon) \neq \varnothing$. By assumption, we have $U(\mathrm{h};\varepsilon)$ is a UP-filter of A and then $y \in U(\mathrm{h};\varepsilon)$. Thus $\mathrm{h}(y) \supseteq \varepsilon = \mathrm{h}(x \cdot y) \cap \mathrm{h}(x)$. So, h is a hesitant fuzzy UP-filter of A. □

Theorem 2.4. *A hesitant fuzzy set* h *on* A *is a hesitant fuzzy UP-ideal of* A *if and only if for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $U(\mathrm{h};\varepsilon)$ *of* A *is a UP-ideal of* A*.*

Proof. Assume that h is a hesitant fuzzy UP-ideal of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $U(\mathrm{h};\varepsilon) \neq \varnothing$ and let $y \in U(\mathrm{h};\varepsilon)$. Then $\mathrm{h}(y) \supseteq \varepsilon$. Since h is a hesitant fuzzy UP-ideal of A, we have $\mathrm{h}(0) \supseteq \mathrm{h}(y) \supseteq \varepsilon$. Thus $0 \in U(\mathrm{h};\varepsilon)$.

Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in U(\mathrm{h};\varepsilon)$ and $y \in U(\mathrm{h};\varepsilon)$. Then $\mathrm{h}(x \cdot (y \cdot z)) \supseteq \varepsilon$ and $\mathrm{h}(y) \supseteq \varepsilon$. Since h is a hesitant fuzzy UP-ideal of A, we have $\mathrm{h}(x \cdot z) \supseteq \mathrm{h}(x \cdot (y \cdot z)) \cap \mathrm{h}(y) \supseteq \varepsilon$. Thus $x \cdot z \in U(\mathrm{h};\varepsilon)$. So, $U(\mathrm{h};\varepsilon)$ is a UP-ideal of A.

Conversely, assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U(\mathrm{h};\varepsilon)$ of A is a UP-ideal of A. Let $x \in A$. Then $\mathrm{h}(x) \in \mathcal{P}([0,1])$. Choose $\varepsilon = \mathrm{h}(x) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x) \supseteq \varepsilon$. Thus $x \in U(\mathrm{h};\varepsilon) \neq \varnothing$. By assumption, we have $U(\mathrm{h};\varepsilon)$ is a UP-ideal of A. Then $0 \in U(\mathrm{h};\varepsilon)$. Thus $\mathrm{h}(0) \supseteq \varepsilon = \mathrm{h}(x)$.

Next, let $x, y, z \in A$. Then $\mathrm{h}(x \cdot (y \cdot z)), \mathrm{h}(y) \in \mathcal{P}([0,1])$. Choose $\varepsilon = \mathrm{h}(x \cdot (y \cdot z)) \cap \mathrm{h}(y) \in \mathcal{P}([0,1])$. Thus $\mathrm{h}(x \cdot (y \cdot z)) \supseteq \varepsilon$ and $\mathrm{h}(y) \supseteq \varepsilon$, so $x \cdot (y \cdot z), y \in U(\mathrm{h};\varepsilon) \neq \varnothing$.

By assumption, we have $U(\mathrm{h};\varepsilon)$ is a UP-ideal of A and thus $x \cdot z \in U(\mathrm{h};\varepsilon)$. So $\mathrm{h}(x \cdot z) \supseteq \varepsilon = \mathrm{h}(x \cdot (y \cdot z)) \cap \mathrm{h}(y)$. Hence, h is a hesitant fuzzy UP-ideal of A. □

Theorem 2.5. *A hesitant fuzzy set* h *on* A *is a hesitant fuzzy strongly UP-ideal of* A *if and only if for all* $\varepsilon \in \mathcal{P}([0,1])$, *a nonempty subset* $U(\mathrm{h};\varepsilon)$ *of* A *is a strongly UP-ideal of* A.

Proof. Assume that h is a hesitant fuzzy strongly UP-ideal of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $U(\mathrm{h};\varepsilon) \neq \varnothing$ and let $y \in U(\mathrm{h};\varepsilon)$. Then $\mathrm{h}(y) \supseteq \varepsilon$. Since h is a hesitant fuzzy strongly UP-ideal of A, we have $\mathrm{h}(0) \supseteq \mathrm{h}(y) \supseteq \varepsilon$. Thus $0 \in U(\mathrm{h};\varepsilon)$.

Next, let $x, y, z \in A$ be such that $(z \cdot y) \cdot (z \cdot x) \in U(\mathrm{h};\varepsilon)$ and $y \in U(\mathrm{h};\varepsilon)$. Then $\mathrm{h}((z \cdot y) \cdot (z \cdot x)) \supseteq \varepsilon$ and $\mathrm{h}(y) \supseteq \varepsilon$. Since h is a hesitant fuzzy strongly UP-ideal of A, we have $\mathrm{h}(x) \supseteq \mathrm{h}((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h}(y) \supseteq \varepsilon$. Thus $x \in U(\mathrm{h};\varepsilon)$. So, $U(\mathrm{h};\varepsilon)$ is a strongly UP-ideal of A.

Conversely, assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U(\mathrm{h};\varepsilon)$ of A is a strongly UP-ideal of A. Let $x \in A$. Then $\mathrm{h}(x) \in \mathcal{P}([0,1])$. Choose $\varepsilon = \mathrm{h}(x) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x) \supseteq \varepsilon$. Thus $x \in U(\mathrm{h};\varepsilon) \neq \varnothing$. By assumption, we have $U(\mathrm{h};\varepsilon)$ is a strongly UP-ideal of A. So $0 \in U(\mathrm{h};\varepsilon)$. Hence $\mathrm{h}(0) \supseteq \varepsilon = \mathrm{h}(x)$.

Next, let $x, y, z \in A$. Then $\mathrm{h}((z \cdot y) \cdot (z \cdot x)), \mathrm{h}(y) \in \mathcal{P}([0,1])$. Choose $\varepsilon = \mathrm{h}((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h}(y) \in \mathcal{P}([0,1])$. Then $\mathrm{h}((z \cdot y) \cdot (z \cdot x)) \supseteq \varepsilon$ and $\mathrm{h}(y) \supseteq \varepsilon$. Thus $(z \cdot y) \cdot (z \cdot x), y \in U(\mathrm{h};\varepsilon) \neq \varnothing$. By assumption, we have $U(\mathrm{h};\varepsilon)$ is a strongly UP-ideal of A. So $x \in U(\mathrm{h};\varepsilon)$. Thus $\mathrm{h}(x) \supseteq \varepsilon = \mathrm{h}((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h}(y)$. Hence, h is a hesitant fuzzy strongly UP-ideal of A. □

2.2. Prime and Weakly Prime Subsets and Upper ε-Level Subsets.

Theorem 2.6. *Let* h *be a hesitant fuzzy set on* A. *Then the following statements hold*:

(1) *if* h *is a prime hesitant fuzzy set on* A, *then for all* $\varepsilon \in \mathcal{P}([0,1])$, $U(\mathrm{h};\varepsilon)$ *is a prime subset of* A *if* $U(\mathrm{h};\varepsilon)$ *is nonempty and* $A - U(\mathrm{h};\varepsilon) = \{x \in A \mid \mathrm{h}(x) \cap \varepsilon = \varnothing\}$,

(2) *if for all* $\varepsilon \in \mathcal{P}([0,1])$, *a nonempty subset* $U(\mathrm{h};\varepsilon)$ *of* A *is a prime subset of* A, *then* h *is a prime hesitant fuzzy set on* A.

Proof. (1) Assume that h is a prime hesitant fuzzy set on A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $U(\mathrm{h};\varepsilon) \neq \varnothing$ and $A - U(\mathrm{h};\varepsilon) = \{x \in A \mid \mathrm{h}(x) \cap \varepsilon = \varnothing\}$. Let $x, y \in A$ be such that $x \cdot y \in U(\mathrm{h};\varepsilon)$ and assume that $x \notin U(\mathrm{h};\varepsilon)$. Then $\mathrm{h}(x \cdot y) \supseteq \varepsilon$ and $\mathrm{h}(x) \cap \varepsilon = \varnothing$. Since h is a prime hesitant fuzzy set on A, we have $\varepsilon \subseteq \mathrm{h}(x \cdot y) \subseteq \mathrm{h}(x) \cup \mathrm{h}(y)$. Thus $\mathrm{h}(y) \supseteq \varepsilon$. So $y \in U(\mathrm{h};\varepsilon)$. Hence, $U(\mathrm{h};\varepsilon)$ is a prime subset of A.

(2) Assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U(\mathrm{h};\varepsilon)$ of A is a prime subset of A. Let $x, y \in A$. Then we get $x \cdot y \in A$. Thus $\mathrm{h}(x \cdot y) \in \mathcal{P}([0,1])$. Choose $\varepsilon = \mathrm{h}(x \cdot y) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x \cdot y) \supseteq \varepsilon$. Thus $x \cdot y \in U(\mathrm{h};\varepsilon) \neq \varnothing$. By assumption, we have $U(\mathrm{h};\varepsilon)$ is a prime subset of A. So $x \in U(\mathrm{h};\varepsilon)$ or $y \in U(\mathrm{h};\varepsilon)$. Hence $\mathrm{h}(x) \supseteq \varepsilon$ or $\mathrm{h}(y) \supseteq \varepsilon$, i.e., $\mathrm{h}(x \cdot y) = \varepsilon \subseteq \mathrm{h}(x) \cup \mathrm{h}(y)$. Therefore, h is a prime hesitant fuzzy set on A. □

Theorem 2.7. *A hesitant fuzzy set* h *on* A *is a prime hesitant fuzzy UP-subalgebra (resp. prime hesitant fuzzy UP-filter, prime hesitant fuzzy UP-ideal, prime hesitant fuzzy strongly UP-ideal) of* A *if and only if for all* $\varepsilon \in \mathcal{P}([0,1])$, *a nonempty subset*

$U(\mathrm{h};\varepsilon)$ of A is a prime UP-subalgebra (resp. prime UP-filter, prime UP-ideal, prime strongly UP-ideal) of A.

Proof. It is straightforward by Theorem 1.22, 2.5 and 1.11. □

Theorem 2.8. *Let* h *be a hesitant fuzzy set on A. Then the following statements hold*:

(1) *if* h *is a weakly prime hesitant fuzzy set on A, then for all $\varepsilon \in \mathcal{P}([0,1])$, $U(\mathrm{h};\varepsilon)$ is a weakly prime subset of A if $U(\mathrm{h};\varepsilon)$ is nonempty and $A - U(\mathrm{h};\varepsilon) = \{x \in A \mid \mathrm{h}(x) \cap \varepsilon = \varnothing\}$,*

(2) *if for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U(\mathrm{h};\varepsilon)$ of A is a weakly prime subset of A, then* h *is a weakly prime hesitant fuzzy set on A.*

Proof. (1) Assume that h is a weakly prime hesitant fuzzy set on A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $U(\mathrm{h};\varepsilon) \neq \varnothing$ and $A - U(\mathrm{h};\varepsilon) = \{x \in A \mid \mathrm{h}(x) \cap \varepsilon = \varnothing\}$. Let $x, y \in A$ and $x \neq y$ be such that $x \cdot y \in U(\mathrm{h};\varepsilon)$ and assume that $x \notin U(\mathrm{h};\varepsilon)$. Then $\mathrm{h}(x \cdot y) \supseteq \varepsilon$ and $\mathrm{h}(x) \cap \varepsilon = \varnothing$. Since h is a prime hesitant fuzzy set on A, we have $\varepsilon \subseteq \mathrm{h}(x \cdot y) \subseteq \mathrm{h}(x) \cup \mathrm{h}(y)$. Thus $\mathrm{h}(y) \supseteq \varepsilon$. So $y \in U(\mathrm{h};\varepsilon)$. Hence, $U(\mathrm{h};\varepsilon)$ is a weakly prime subset of A.

(2) Assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U(\mathrm{h};\varepsilon)$ of A is a weakly prime subset of A. Let $x, y \in A$ and $x \neq y$. Choose $\varepsilon = \mathrm{h}(x \cdot y) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x \cdot y) \supseteq \varepsilon$. Thus $x \cdot y \in U(\mathrm{h};\varepsilon) \neq \varnothing$. By assumption, we have $U(\mathrm{h};\varepsilon)$ is a weakly prime subset of A and then $x \in U(\mathrm{h};\varepsilon)$ or $y \in U(\mathrm{h};\varepsilon)$. Thus $\mathrm{h}(x) \supseteq \varepsilon$ or $\mathrm{h}(y) \supseteq \varepsilon$. So $\mathrm{h}(x \cdot y) = \varepsilon \subseteq \mathrm{h}(x) \cup \mathrm{h}(y)$. Hence, h is a weakly prime hesitant fuzzy set on A. □

Theorem 2.9. *Let* h *be a hesitant fuzzy set on A. Then the following statements hold*:

(1) *if* h *is a weakly prime hesitant fuzzy UP-subalgebra (resp. weakly prime hesitant fuzzy UP-filter, weakly prime hesitant fuzzy UP-ideal, weakly prime hesitant fuzzy strongly UP-ideal) of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, $U(\mathrm{h};\varepsilon)$ is a weakly prime UP-subalgebra (resp. weakly prime UP-filter, weakly prime UP-ideal, weakly prime strongly UP-ideal) of A if $U(\mathrm{h};\varepsilon)$ is nonempty and $A - U(\mathrm{h};\varepsilon) = \{x \in A \mid \mathrm{h}(x) \cap \varepsilon = \varnothing\}$,*

(2) *if for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U(\mathrm{h};\varepsilon)$ of A is a weakly prime UP-subalgebra (resp. weakly prime UP-filter, weakly prime UP-ideal, weakly prime strongly UP-ideal) of A, then* h *is a weakly prime hesitant fuzzy UP-subalgebra (resp. weakly prime hesitant fuzzy UP-filter, weakly prime hesitant fuzzy UP-ideal, weakly prime hesitant fuzzy strongly UP-ideal) of A.*

Proof. It is straightforward by Theorem 2.2 (resp. Theorem 2.3, Theorem 2.4, Theorem 2.5) and 2.8. □

2.3. Upper ε-Strong Level Subsets.

Theorem 2.10. *Let* h *be a hesitant fuzzy set on A. Then the following statements hold*:

(1) *if* h *is a hesitant fuzzy UP-subalgebra of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, $U^+(\mathrm{h};\varepsilon)$ is a UP-subalgebra of A if $U^+(\mathrm{h};\varepsilon)$ is nonempty and $E(\mathrm{h};\varepsilon)$ is empty,*

(2) *if* $\mathrm{Im(h)}$ *is a chain and for all* $\varepsilon \in \mathcal{P}([0,1])$, *a nonempty subset* $U^+(\mathrm{h};\varepsilon)$ *of* A *is a UP-subalgebra of* A, *then* h *is a hesitant fuzzy UP-subalgebra of* A.

Proof. (1) It is straightforward by Theorem 2.2.

(2) Assume that Im(h) is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U^+(\mathrm{h};\varepsilon)$ of A is a UP-subalgebra of A. Assume that there exist $x, y \in A$ such that $\mathrm{h}(x \cdot y) \not\supseteq \mathrm{h}(x) \cap \mathrm{h}(y)$. Since Im(h) is a chain, we have $\mathrm{h}(x \cdot y) \subset \mathrm{h}(x) \cap \mathrm{h}(y)$. Then $\mathrm{h}(x \cdot y) \in \mathcal{P}([0,1])$. Choose $\varepsilon = \mathrm{h}(x \cdot y) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x) \supset \varepsilon$ and $\mathrm{h}(y) \supset \varepsilon$. Thus $x, y \in U^+(\mathrm{h};\varepsilon) \neq \varnothing$. By assumption, we have $U^+(\mathrm{h};\varepsilon)$ is a UP-subalgebra of A and thus $x \cdot y \in U^+(\mathrm{h};\varepsilon)$. So $\mathrm{h}(x \cdot y) \supset \varepsilon = \mathrm{h}(x \cdot y)$, a contradiction. Hence, $\mathrm{h}(x \cdot y) \supseteq \mathrm{h}(x) \cap \mathrm{h}(y)$ for all $x, y \in A$. Therefore, h is a hesitant fuzzy UP-subalgebra of A. □

Example 2.11. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set h on A as follows:

$$\mathrm{h}(0) = \{0.1\}, \mathrm{h}(1) = \{0.1\}, \mathrm{h}(2) = \varnothing, \text{ and } \mathrm{h}(3) = \{0.5\}.$$

Then Im(h) is not a chain. If $\varepsilon \neq \varnothing$, then $U^+(\mathrm{h};\varepsilon) = \varnothing$. If $\varepsilon = \varnothing$, then $U^+(\mathrm{h};\varepsilon) = \{0, 1, 3\}$. Using this data, we can show that $U^+(\mathrm{h};\varepsilon)$ is a UP-subalgebra of A. Since $\mathrm{h}(3 \cdot 3) = \mathrm{h}(0) = \{0.1\} \not\supseteq \{0.5\} = \mathrm{h}(3) \cap \mathrm{h}(3)$, we have h is not a hesitant fuzzy UP-subalgebra of A.

Theorem 2.12. *Let* h *be a hesitant fuzzy set on* A. *Then the following statements hold*:

(1) *if* h *is a hesitant fuzzy UP-filter of* A, *then for all* $\varepsilon \in \mathcal{P}([0,1])$, $U^+(\mathrm{h};\varepsilon)$ *is a UP-filter of* A *if* $U^+(\mathrm{h};\varepsilon)$ *is nonempty and* $E(\mathrm{h};\varepsilon)$ *is empty,*

(2) *if* $\mathrm{Im(h)}$ *is a chain and for all* $\varepsilon \in \mathcal{P}([0,1])$, *a nonempty subset* $U^+(\mathrm{h};\varepsilon)$ *of* A *is a UP-filter of* A, *then* h *is a hesitant fuzzy UP-filter of* A.

Proof. (1) It is straightforward by Theorem 2.3.

(2) Assume that Im(h) is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U^+(\mathrm{h};\varepsilon)$ of A is a UP-filter of A. Assume that there exists $x \in A$ such that $\mathrm{h}(0) \not\supseteq \mathrm{h}(x)$. Since Im(h) is a chain, we have $\mathrm{h}(0) \subset \mathrm{h}(x)$. Choose $\varepsilon = \mathrm{h}(0) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x) \supset \mathrm{h}(0) = \varepsilon$. Thus $x \in U^+(\mathrm{h};\varepsilon) \neq \varnothing$. By assumption, we have $U^+(\mathrm{h};\varepsilon)$ is a UP-filter of A. and thus $0 \in U^+(\mathrm{h};\varepsilon)$. So $\mathrm{h}(0) \supset \varepsilon = \mathrm{h}(0)$, a contradiction. Hence, $\mathrm{h}(0) \supseteq \mathrm{h}(x)$, for all $x \in A$.

Assume that there exist $x, y \in A$ such that $\mathrm{h}(y) \not\supseteq \mathrm{h}(x \cdot y) \cap \mathrm{h}(x)$. Since Im(h) is a chain, we have $\mathrm{h}(y) \subset \mathrm{h}(x \cdot y) \cap \mathrm{h}(x)$. Choose $\varepsilon = \mathrm{h}(y) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x \cdot y) \supset \varepsilon$ and $\mathrm{h}(x) \supset \varepsilon$. Thus $x \cdot y, x \in U^+(\mathrm{h};\varepsilon) \neq \varnothing$. By assumption, we have $U^+(\mathrm{h};\varepsilon)$ is a UP-filter of A and thus $y \in U^+(\mathrm{h};\varepsilon)$. So $\mathrm{h}(y) \supset \varepsilon = \mathrm{h}(y)$, a contradiction. Hence, $\mathrm{h}(y) \supseteq \mathrm{h}(x \cdot y) \cap \mathrm{h}(x)$, for all $x, y \in A$. Therefore, h is a hesitant fuzzy UP-filter of A. □

Example 2.13. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	0
2	0	1	0	3
3	0	1	2	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set h on A as follows:

$$\mathrm{h}(0) = \{0.3\}, \mathrm{h}(1) = \{0.3\}, \mathrm{h}(2) = \varnothing, \text{ and } \mathrm{h}(3) = \{0.4\}.$$

Then Im(h) is not a chain. If $\varepsilon \neq \varnothing$, then $U^+(\mathrm{h}; \varepsilon) = \varnothing$. If $\varepsilon = \varnothing$, then $U^+(\mathrm{h}; \varepsilon) = \{0, 1, 3\}$. Using this data, we can show that $U^+(\mathrm{h}; \varepsilon)$ is a UP-filter of A. Since $\mathrm{h}(0) = \{0.3\} \not\supseteq \{0.4\} = \mathrm{h}(3)$ and $\mathrm{h}(3) = \{0.5\} \not\supseteq \{0.3\} = \mathrm{h}(1) \cap \mathrm{h}(0) = \mathrm{h}(1) \cap \mathrm{h}(1 \cdot 3)$, we have h is not a hesitant fuzzy UP-filter of A.

Theorem 2.14. *Let* h *be a hesitant fuzzy set on A. Then the following statements hold*:

if h *is a hesitant fuzzy UP-ideal of A, then for all $\varepsilon \in \mathcal{P}([0, 1])$, $U^+(\mathrm{h}; \varepsilon)$ is a UP-ideal of A if $U^+(\mathrm{h}; \varepsilon)$ is nonempty and $E(\mathrm{h}; \varepsilon)$ is empty,*

(2) *if* Im(h) *is a chain and for all $\varepsilon \in \mathcal{P}([0, 1])$, a nonempty subset $U^+(\mathrm{h}; \varepsilon)$ of A is a UP-ideal of A, then* h *is a hesitant fuzzy UP-ideal of A.*

Proof. (1) It is straightforward by Theorem 2.4.

(2) Assume that Im(h) is a chain and for all $\varepsilon \in \mathcal{P}([0, 1])$, a nonempty subset $U^+(\mathrm{h}; \varepsilon)$ of A is a UP-ideal of A. Assume that there exists $x \in A$ such that $\mathrm{h}(0) \not\supseteq \mathrm{h}(x)$. Since Im(h) is a chain, we have $\mathrm{h}(0) \subset \mathrm{h}(x)$. Choose $\varepsilon = \mathrm{h}(0) \in \mathcal{P}([0, 1])$. Then $\mathrm{h}(x) \supset \mathrm{h}(0) = \varepsilon$. Thus $x \in U^+(\mathrm{h}; \varepsilon) \neq \varnothing$. By assumption, we have $U^+(\mathrm{h}; \varepsilon)$ is a UP-ideal of A and thus $0 \in U^+(\mathrm{h}; \varepsilon)$. So $\mathrm{h}(0) \supset \varepsilon = \mathrm{h}(0)$, a contradiction. Hence, $\mathrm{h}(0) \supseteq \mathrm{h}(x)$, for all $x \in A$.

Assume that there exist $x, y, z \in A$ such that $\mathrm{h}(x \cdot z) \not\supseteq \mathrm{h}(x \cdot (y \cdot z)) \cap \mathrm{h}(y)$. Since Im(h) is a chain, we have $\mathrm{h}(x \cdot z) \subset \mathrm{h}(x \cdot (y \cdot z)) \cap \mathrm{h}(y)$. Choose $\varepsilon = \mathrm{h}(x \cdot z) \in \mathcal{P}([0, 1])$. Then $\mathrm{h}(x \cdot (y \cdot z)) \supset \varepsilon$ and $\mathrm{h}(y) \supset \varepsilon$. Thus $x \cdot (y \cdot z), y \in U^+(\mathrm{h}; \varepsilon) \neq \varnothing$. By assumption, we have $U^+(\mathrm{h}; \varepsilon)$ is a UP-ideal of A and thus $x \cdot z \in U^+(\mathrm{h}; \varepsilon)$. So $\mathrm{h}(x \cdot z) \supset \varepsilon = \mathrm{h}(x \cdot z)$, a contradiction. Hence, $\mathrm{h}(x \cdot z) \supseteq \mathrm{h}(x \cdot (y \cdot z)) \cap \mathrm{h}(y)$, for all $x, y, z \in A$. Therefore, h is a hesitant fuzzy UP-ideal of A. □

Example 2.15. Let $A = \{0, 1, 2\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2
0	0	1	2
1	0	0	2
2	0	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set h on A as follows:

$$\mathrm{h}(0) = \{0.5\}, \mathrm{h}(1) = \{0.6\}, \text{ and } \mathrm{h}(2) = \{0.5\}.$$

Then Im(h) is not a chain. If $\varepsilon \neq \varnothing$, then $U^+(\mathrm{h}; \varepsilon) = \varnothing$. If $\varepsilon = \varnothing$, then $U^+(\mathrm{h}; \varepsilon) = A$. Using this data, we can show that $U^+(\mathrm{h}; \varepsilon)$ is a UP-ideal of A. Since $\mathrm{h}(0) = \{0.5\} \not\supseteq \{0.6\} = \mathrm{h}(1)$ and $\mathrm{h}(0 \cdot 1) = \mathrm{h}(1) = \{0.6\} \not\supseteq \{0.5\} = \mathrm{h}(0) \cap \mathrm{h}(2) = \mathrm{h}(0 \cdot (2 \cdot 1)) \cap \mathrm{h}(2)$, we have h is not a hesitant fuzzy UP-ideal of A.

Theorem 2.16. *Let* h *be a hesitant fuzzy set on A. Then the following statements hold:*

(1) *if* h *is a hesitant fuzzy strongly UP-ideal of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, $U^+(\mathrm{h};\varepsilon)$ is a strongly UP-ideal of A if $U^+(\mathrm{h};\varepsilon)$ is nonempty and $E(\mathrm{h};\varepsilon)$ is empty,*

(2) *if* $\mathrm{Im}(\mathrm{h})$ *is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U^+(\mathrm{h};\varepsilon)$ of A is a strongly UP-ideal of A, then* h *is a hesitant fuzzy strongly UP-ideal of A.*

Proof. (1) It is straightforward by Theorem 2.5.

(2) Assume that $\mathrm{Im}(\mathrm{h})$ is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U^+(\mathrm{h};\varepsilon)$ of A is a strongly UP-ideal of A. Assume that there exists $x \in A$ such that $\mathrm{h}(0) \not\supseteq \mathrm{h}(x)$. Since $\mathrm{Im}(\mathrm{h})$ is a chain, we have $\mathrm{h}(0) \subset \mathrm{h}(x)$. Choose $\varepsilon = \mathrm{h}(0) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x) \supset \mathrm{h}(0) = \varepsilon$. Thus $x \in U^+(\mathrm{h};\varepsilon) \neq \varnothing$. By assumption, we have $U^+(\mathrm{h};\varepsilon)$ is a strongly UP-ideal of A and hus $0 \in U^+(\mathrm{h};\varepsilon)$. So $\mathrm{h}(0) \supset \varepsilon = \mathrm{h}(0)$, a contradiction. Hence, $\mathrm{h}(0) \supseteq \mathrm{h}(x)$, for all $x \in A$.

Assume that there exist $x, y, z \in A$ such that $\mathrm{h}(x) \not\supseteq \mathrm{h}((z\cdot y)\cdot(z\cdot x)) \cap \mathrm{h}(y)$. Since $\mathrm{Im}(\mathrm{h})$ is a chain, we have $\mathrm{h}(x) \subset \mathrm{h}((z\cdot y)\cdot(z\cdot x)) \cap \mathrm{h}(y)$. Choose $\varepsilon = \mathrm{h}(x) \in \mathcal{P}([0,1])$. Then $\mathrm{h}((z\cdot y)\cdot(z\cdot x)) \supset \varepsilon$ and $\mathrm{h}(y) \supset \varepsilon$, so $(z\cdot y)\cdot(z\cdot x), y \in U^+(\mathrm{h};\varepsilon) \neq \varnothing$. By assumption, we have $U^+(\mathrm{h};\varepsilon)$ is a strongly UP-ideal of A. Thus $x \in U^+(\mathrm{h};\varepsilon)$. So $\mathrm{h}(x) \supset \varepsilon = \mathrm{h}(x)$, a contradiction. Hence, $\mathrm{h}(x) \supseteq \mathrm{h}((z\cdot y)\cdot(z\cdot x)) \cap \mathrm{h}(y)$, for all $x, y, z \in A$. Therefore, h is a hesitant fuzzy strongly UP-ideal of A. □

Example 2.17. Let $A = \{0, 1, 2\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2
0	0	1	2
1	0	0	2
2	0	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set h on A as follows:

$$\mathrm{h}(0) = \{0.6\}, \mathrm{h}(1) = \{0.6\}, \text{ and } \mathrm{h}(2) = \{0.5\}.$$

Then $\mathrm{Im}(\mathrm{h})$ is not a chain. If $\varepsilon \neq \varnothing$, then $U^+(\mathrm{h};\varepsilon) = \varnothing$. If $\varepsilon = \varnothing$, then $U^+(\mathrm{h};\varepsilon) = A$. Using this data, we can show that $U^+(\mathrm{h};\varepsilon)$ is a strongly UP-ideal of A. Since $\mathrm{h}(0) = \{0.6\} \not\supseteq \{0.5\} = \mathrm{h}(2)$ and $\mathrm{h}(2) = \{0.5\} \not\supseteq \{0.6\} = \mathrm{h}(0)\cap\mathrm{h}(0) = \mathrm{h}((2\cdot 0)\cdot(2\cdot 2))\cap\mathrm{h}(0)$, we have h is not a hesitant fuzzy strongly UP-ideal of A.

2.4. Prime and Weakly Prime Subsets and Upper ε-Strong Level Subsets.

Theorem 2.18. *Let* h *be a hesitant fuzzy set on A. Then the following statements hold*:

(1) *if* h *is a prime hesitant fuzzy set on A, then for all $\varepsilon \in \mathcal{P}([0,1])$, $U^+(\mathrm{h};\varepsilon)$ is a prime subset of A if $U^+(\mathrm{h};\varepsilon)$ is nonempty and $A - U^+(\mathrm{h};\varepsilon) = \{x \in A \mid \mathrm{h}(x) \cap \varepsilon = \varnothing\}$,*

(2) *if* $\mathrm{Im}(\mathrm{h})$ *is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U^+(\mathrm{h};\varepsilon)$ of A is prime subset of A, then* h *is a prime hesitant fuzzy set on A.*

Proof. (1) Assume that h is a prime hesitant fuzzy set on A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $U^+(\mathrm{h};\varepsilon) \neq \varnothing$ and $A - U^+(\mathrm{h};\varepsilon) = \{x \in A \mid \mathrm{h}(x) \cap \varepsilon = \varnothing\}$. Let $x, y \in A$ and $x \cdot y \in U^+(\mathrm{h};\varepsilon)$. Then $\mathrm{h}(x\cdot y) \supset \varepsilon$. Assume that $x \notin U^+(\mathrm{h};\varepsilon)$. Then $\mathrm{h}(x) \cap \varepsilon = \varnothing$. Since h is a prime hesitant fuzzy set on A, we have $\varepsilon \subset \mathrm{h}(x\cdot y) \subseteq \mathrm{h}(x) \cup \mathrm{h}(y)$. Thus $\mathrm{h}(y) \supset \varepsilon$. So $y \in U^+(\mathrm{h};\varepsilon)$. Hence, $U^+(\mathrm{h};\varepsilon)$ is a prime subset of A.

(2) Assume that $\mathrm{Im}(\mathrm{h})$ is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U^+(\mathrm{h};\varepsilon)$ of A is a prime subset of A. Assume that there exist $x, y \in A$ such that $\mathrm{h}(x \cdot y) \nsubseteq \mathrm{h}(x) \cup \mathrm{h}(y)$. Since $\mathrm{Im}(\mathrm{h})$ is a chain, we have $\mathrm{h}(x \cdot y) \supset \mathrm{h}(x) \cup \mathrm{h}(y)$. Choose $\varepsilon = \mathrm{h}(x) \cup \mathrm{h}(y) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x \cdot y) \supset \varepsilon$. Thus $x \cdot y \in U^+(\mathrm{h};\varepsilon) \neq \varnothing$. By assumption, we have $U^+(\mathrm{h};\varepsilon)$ is a prime subset of A and thus $x \in U^+(\mathrm{h};\varepsilon)$ or $y \in U^+(\mathrm{h};\varepsilon)$. So $\mathrm{h}(x) \supset \varepsilon = \mathrm{h}(x) \cup \mathrm{h}(y)$ or $\mathrm{h}(y) \supset \varepsilon = \mathrm{h}(x) \cup \mathrm{h}(y)$, a contradiction. Hence, $\mathrm{h}(x \cdot y) \subseteq \mathrm{h}(x) \cup \mathrm{h}(y)$, for all $x, y \in A$. Therefore, h is a prime hesitant fuzzy set on A. □

Theorem 2.19. *Let* h *be a hesitant fuzzy set on* A. *Then the following statements hold*:

(1) *if* h *is a prime hesitant fuzzy UP-subalgebra (resp. prime hesitant fuzzy UP-filter, prime hesitant fuzzy UP-ideal, prime hesitant fuzzy strongly UP-ideal) of* A, *then for all* $\varepsilon \in \mathcal{P}([0,1])$, $U^+(\mathrm{h};\varepsilon)$ *is a prime UP-subalgebra (resp. prime UP-filter, prime UP-ideal, prime strongly UP-ideal) of* A *if* $U^+(\mathrm{h};\varepsilon)$ *is nonempty and* $E(\mathrm{h};\varepsilon)$ *is empty,*

(2) *if* $\mathrm{Im}(\mathrm{h})$ *is a chain and for all* $\varepsilon \in \mathcal{P}([0,1])$, *a nonempty subset* $U^+(\mathrm{h};\varepsilon)$ *of* A *is a prime UP-subalgebra (resp. prime UP-filter, prime UP-ideal, prime strongly UP-ideal) of* A, *then* h *is a prime hesitant fuzzy UP-subalgebra (resp. prime hesitant fuzzy UP-filter, prime hesitant fuzzy UP-ideal, prime hesitant fuzzy strongly UP-ideal) of* A.

Proof. It is straightforward by Theorem 1.22, 2.16 and 1.11. □

Theorem 2.20. *Let* h *be a hesitant fuzzy set on* A. *Then the following statements hold:*

(1) *if* h *is a weakly prime hesitant fuzzy set on* A, *then for all* $\varepsilon \in \mathcal{P}([0,1])$, $U^+(\mathrm{h};\varepsilon)$ *is a weakly prime subset of* A *if* $U^+(\mathrm{h};\varepsilon)$ *is nonempty and* $A - U^+(\mathrm{h};\varepsilon) = \{x \in A \mid \mathrm{h}(x) \cap \varepsilon = \varnothing\}$,

(2) *if* $\mathrm{Im}(\mathrm{h})$ *is a chain and for all* $\varepsilon \in \mathcal{P}([0,1])$, *a nonempty subset* $U^+(\mathrm{h};\varepsilon)$ *of* A *is a weakly prime subset of* A, *then* h *is a weakly prime hesitant fuzzy set on* A.

Proof. (1) Assume that h is a weakly prime hesitant fuzzy set on A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $U^+(\mathrm{h};\varepsilon) \neq \varnothing$ and $A - U^+(\mathrm{h};\varepsilon) = \{x \in A \mid \mathrm{h}(x) \cap \varepsilon = \varnothing\}$. Let $x, y \in A$ be such that $x \neq y$ and $x \cdot y \in U^+(\mathrm{h};\varepsilon)$. Then $\mathrm{h}(x \cdot y) \supset \varepsilon$. Assume that $x \notin U^+(\mathrm{h};\varepsilon)$. Then $\mathrm{h}(x) \cap \varepsilon = \varnothing$. Since h is a weakly prime hesitant fuzzy set on A, we have $\varepsilon \subset \mathrm{h}(x \cdot y) \subseteq \mathrm{h}(x) \cup \mathrm{h}(y)$. Thus $\mathrm{h}(y) \supset \varepsilon$. So $y \in U^+(\mathrm{h};\varepsilon)$. Hence, $U^+(\mathrm{h};\varepsilon)$ is a weakly prime subset of A.

(2) Assume that $\mathrm{Im}(\mathrm{h})$ is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U^+(\mathrm{h};\varepsilon)$ of A is a weakly prime subset of A. Assume that there exist $x, y \in A$ such that $x \neq y$ and $\mathrm{h}(x \cdot y) \nsubseteq \mathrm{h}(x) \cup \mathrm{h}(y)$. Since $\mathrm{Im}(\mathrm{h})$ is a chain, we have $\mathrm{h}(x \cdot y) \supset \mathrm{h}(x) \cup \mathrm{h}(y)$. Choose $\varepsilon = \mathrm{h}(x) \cup \mathrm{h}(y) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x \cdot y) \supset \varepsilon$. Thus $x \cdot y \in U^+(\mathrm{h};\varepsilon) \neq \varnothing$. By assumption, we have $U^+(\mathrm{h};\varepsilon)$ is a weakly prime subset of A and hus $x \in U^+(\mathrm{h};\varepsilon)$ or $y \in U^+(\mathrm{h};\varepsilon)$. So $\mathrm{h}(x) \supset \varepsilon = \mathrm{h}(x) \cup \mathrm{h}(y)$ or $\mathrm{h}(y) \supset \varepsilon = \mathrm{h}(x) \cup \mathrm{h}(y)$, a contradiction. Hence, $\mathrm{h}(x \cdot y) \subseteq \mathrm{h}(x) \cup \mathrm{h}(y)$, for all $x, y \in A$ and $x \neq y$. Therefore, h is a weakly prime hesitant fuzzy set on A. □

Theorem 2.21. *Let* h *be a hesitant fuzzy set on* A. *Then the following statements hold:*

(1) *if* h *is a weakly prime hesitant fuzzy UP-subalgebra (resp. weakly prime hesitant fuzzy UP-filter, weakly prime hesitant fuzzy UP-ideal, weakly prime hesitant fuzzy strongly UP-ideal) of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, $U^+(\mathrm{h};\varepsilon)$ is a weakly prime UP-subalgebra (resp. weakly prime UP-filter, weakly prime UP-ideal, weakly prime strongly UP-ideal) of A if $U^+(\mathrm{h};\varepsilon)$ is nonempty, $E(\mathrm{h};\varepsilon)$ is empty and $A - U^+(\mathrm{h};\varepsilon) = \{x \in A \mid \mathrm{h}(x) \cap \varepsilon = \varnothing\}$,*

(2) *if* $\mathrm{Im}(\mathrm{h})$ *is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U^+(\mathrm{h};\varepsilon)$ of A is a weakly prime UP-subalgebra (resp. weakly prime UP-filter, weakly prime UP-ideal, weakly prime strongly UP-ideal) of A, then* h *is a weakly prime hesitant fuzzy UP-subalgebra (resp. weakly prime hesitant fuzzy UP-filter, weakly prime hesitant fuzzy UP-ideal, weakly prime hesitant fuzzy strongly UP-ideal) of A.*

Proof. It is straightforward by Theorem 2.10 (resp. Theorem 2.12, Theorem 2.14, Theorem 2.16) and 2.20. □

2.5. **Lower ε-Level Subsets.**

Lemma 2.22. *Let* h *be a hesitant fuzzy set on A. Then the following statements hold: for any $x, y \in A$,*

(1) $[0,1] - (\mathrm{h}(x) \cup \mathrm{h}(y)) = ([0,1] - \mathrm{h}(x)) \cap ([0,1] - \mathrm{h}(y))$,

(2) $[0,1] - (\mathrm{h}(x) \cap \mathrm{h}(y)) = ([0,1] - \mathrm{h}(x)) \cup ([0,1] - \mathrm{h}(y))$.

Proof. (1) Let $x, y \in A$. Then

$$\begin{aligned}[0,1] - (\mathrm{h}(x) \cup \mathrm{h}(y)) &= [0,1] \cap (\mathrm{h}(x) \cup \mathrm{h}(y))' \\ &= ([0,1] \cap [0,1]) \cap ((\mathrm{h}(x))' \cap (\mathrm{h}(y))') \\ &= ([0,1] \cap (\mathrm{h}(x))') \cap ([0,1] \cap (\mathrm{h}(y))') \\ &= ([0,1] - \mathrm{h}(x)) \cap ([0,1] - \mathrm{h}(y)).\end{aligned}$$

(2) Let $x, y \in A$. Then

$$\begin{aligned}[0,1] - (\mathrm{h}(x) \cap \mathrm{h}(y)) &= [0,1] \cap (\mathrm{h}(x) \cap \mathrm{h}(y))' \\ &= [0,1] \cap ((\mathrm{h}(x))' \cup (\mathrm{h}(y))') \\ &= ([0,1] \cap (\mathrm{h}(x))') \cup ([0,1] \cap (\mathrm{h}(y))') \\ &= ([0,1] - \mathrm{h}(x)) \cup ([0,1] - \mathrm{h}(y)).\end{aligned}$$

□

Theorem 2.23. *Let* h *be a hesitant fuzzy set on A. Then $\overline{\mathrm{h}}$ is a hesitant fuzzy UP-subalgebra of A if and only if for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L(\mathrm{h};\varepsilon)$ of A is a UP-subalgebra of A.*

Proof. Assume that $\overline{\mathrm{h}}$ is a hesitant fuzzy UP-subalgebra of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $L(\mathrm{h};\varepsilon) \neq \varnothing$ and let $x, y \in L(\mathrm{h};\varepsilon)$. Then $\mathrm{h}(x) \subseteq \varepsilon$ and $\mathrm{h}(y) \subseteq \varepsilon$. Since $\overline{\mathrm{h}}$ is a hesitant fuzzy UP-subalgebra of A, we have $\overline{\mathrm{h}}(x \cdot y) \supseteq \overline{\mathrm{h}}(x) \cap \overline{\mathrm{h}}(y)$. By Lemma 2.22 (1), we have $[0,1] - \mathrm{h}(x \cdot y) \supseteq ([0,1] - \mathrm{h}(x)) \cap ([0,1] - \mathrm{h}(y)) = [0,1] - (\mathrm{h}(x) \cup \mathrm{h}(y))$. Thus $\mathrm{h}(x \cdot y) \subseteq \mathrm{h}(x) \cup \mathrm{h}(y) \subseteq \varepsilon$. So $x \cdot y \in L(\mathrm{h};\varepsilon)$. Hence, $L(\mathrm{h};\varepsilon)$ is a UP-subalgebra of A.

Conversely, assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L(\mathrm{h};\varepsilon)$ of A is a UP-subalgebra of A. Let $x, y \in A$. Choose $\varepsilon = \mathrm{h}(x) \cup \mathrm{h}(y) \in \mathcal{P}([0,1])$. Then

$\mathrm{h}(x) \subseteq \varepsilon$ and $\mathrm{h}(y) \subseteq \varepsilon$. Thus $x, y \in L(\mathrm{h}; \varepsilon) \neq \varnothing$. By assumption, we have $L(\mathrm{h}; \varepsilon)$ is a UP-subalgebra of A and thus $x \cdot y \in L(\mathrm{h}; \varepsilon)$. So $\mathrm{h}(x \cdot y) \subseteq \varepsilon = \mathrm{h}(x) \cup \mathrm{h}(y)$. By Lemma 2.22 (1), we have

$$\begin{aligned}\overline{\mathrm{h}}(x \cdot y) &= [0,1] - \mathrm{h}(x \cdot y)\\ &\supseteq [0,1] - (\mathrm{h}(x) \cup \mathrm{h}(y))\\ &= ([0,1] - \mathrm{h}(x)) \cap ([0,1] - \mathrm{h}(y))\\ &= \overline{\mathrm{h}}(x) \cap \overline{\mathrm{h}}(y).\end{aligned}$$

Hence, $\overline{\mathrm{h}}$ is a hesitant fuzzy UP-subalgebra of A. □

Theorem 2.24. *Let* h *be a hesitant fuzzy set on* A. *Then* $\overline{\mathrm{h}}$ *is a hesitant fuzzy UP-filter of* A *if and only if for all* $\varepsilon \in \mathcal{P}([0,1])$, *a nonempty subset* $L(\mathrm{h}; \varepsilon)$ *of* A *is a UP-filter of* A.

Proof. Assume that $\overline{\mathrm{h}}$ is a hesitant fuzzy UP-filter of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $L(\mathrm{h}; \varepsilon) \neq \varnothing$ and let $a \in L(\mathrm{h}; \varepsilon)$. Then $\mathrm{h}(a) \subseteq \varepsilon$. Since $\overline{\mathrm{h}}$ is a hesitant fuzzy UP-filter of A, we have $\overline{\mathrm{h}}(0) \supseteq \overline{\mathrm{h}}(a)$. Thus $[0,1] - \mathrm{h}(0) \supseteq [0,1] - \mathrm{h}(a)$. So $\mathrm{h}(0) \subseteq \mathrm{h}(a) \subseteq \varepsilon$. Hence, $0 \in L(\mathrm{h}; \varepsilon)$.

Next, let $x, y \in A$ be such that $x \cdot y \in L(\mathrm{h}; \varepsilon)$ and $x \in L(\mathrm{h}; \varepsilon)$. Then $\mathrm{h}(x \cdot y) \subseteq \varepsilon$ and $\mathrm{h}(x) \subseteq \varepsilon$. Since $\overline{\mathrm{h}}$ is a hesitant fuzzy UP-filter of A, we have $\overline{\mathrm{h}}(y) \supseteq \overline{\mathrm{h}}(x \cdot y) \cap \overline{\mathrm{h}}(x)$. By Lemma 2.22 (1), we have $[0,1] - \mathrm{h}(y) \supseteq ([0,1] - \mathrm{h}(x \cdot y)) \cap ([0,1] - \mathrm{h}(x)) = [0,1] - (\mathrm{h}(x \cdot y) \cup \mathrm{h}(x))$. Thus $\mathrm{h}(y) \subseteq \mathrm{h}(x \cdot y) \cup \mathrm{h}(x) \subseteq \varepsilon$. So $y \in L(\mathrm{h}; \varepsilon)$. Hence, $L(\mathrm{h}; \varepsilon)$ is a UP-filter of A.

Conversely, assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L(\mathrm{h}; \varepsilon)$ of A is a UP-filter of A. Let $x \in A$. Choose $\varepsilon = \mathrm{h}(x) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x) \subseteq \varepsilon$, so $x \in L(\mathrm{h}; \varepsilon) \neq \varnothing$. By assumption, we have $L(\mathrm{h}; \varepsilon)$ is a UP-filter of A. Thus $0 \in L(\mathrm{h}; \varepsilon)$. So $\mathrm{h}(0) \subseteq \varepsilon = \mathrm{h}(x)$. Hence $\overline{\mathrm{h}}(0) = [0,1] - \mathrm{h}(0) \supseteq [0,1] - \mathrm{h}(x) = \overline{\mathrm{h}}(x)$.

Next, let $x, y \in A$. Choose $\varepsilon = \mathrm{h}(x \cdot y) \cup \mathrm{h}(x) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x \cdot y) \subseteq \varepsilon$ and $\mathrm{h}(x) \subseteq \varepsilon$. Thus $x \cdot y, x \in L(\mathrm{h}; \varepsilon) \neq \varnothing$. By assumption, we have $L(\mathrm{h}; \varepsilon)$ is a UP-filter of A. So $y \in L(\mathrm{h}; \varepsilon)$. Hence $\mathrm{h}(y) \subseteq \varepsilon = \mathrm{h}(x \cdot y) \cup \mathrm{h}(x)$. By Lemma 2.22 (1), we have

$$\begin{aligned}\overline{\mathrm{h}}(y) &= [0,1] - \mathrm{h}(y)\\ &\supseteq [0,1] - (\mathrm{h}(x \cdot y) \cup \mathrm{h}(x))\\ &= ([0,1] - \mathrm{h}(x \cdot y)) \cap ([0,1] - \mathrm{h}(x))\\ &= \overline{\mathrm{h}}(x \cdot y) \cap \overline{\mathrm{h}}(x).\end{aligned}$$

Therefore, $\overline{\mathrm{h}}$ is a hesitant fuzzy UP-filter of A. □

Theorem 2.25. *Let* h *be a hesitant fuzzy set on* A. *Then* $\overline{\mathrm{h}}$ *is a hesitant fuzzy UP-ideal of* A *if and only if for all* $\varepsilon \in \mathcal{P}([0,1])$, *a nonempty subset* $L(\mathrm{h}; \varepsilon)$ *of* A *is a UP-ideal of* A.

Proof. Assume that $\overline{\mathrm{h}}$ is a hesitant fuzzy UP-ideal of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $L(\mathrm{h}; \varepsilon) \neq \varnothing$ and let $b \in L(\mathrm{h}; \varepsilon)$. Then $\mathrm{h}(b) \subseteq \varepsilon$. Since $\overline{\mathrm{h}}$ is a hesitant fuzzy UP-ideal of A, we have $\overline{\mathrm{h}}(0) \supseteq \overline{\mathrm{h}}(b)$. Thus $[0,1] - \mathrm{h}(0) \supseteq [0,1] - \mathrm{h}(b)$. So $\mathrm{h}(0) \subseteq \mathrm{h}(b) \subseteq \varepsilon$. Hence, $0 \in L(\mathrm{h}; \varepsilon)$.

Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in L(\mathrm{h}; \varepsilon)$ and $y \in L(\mathrm{h}; \varepsilon)$. Then $\mathrm{h}(x \cdot (y \cdot z)) \subseteq \varepsilon$ and $\mathrm{h}(y) \subseteq \varepsilon$. Since $\overline{\mathrm{h}}$ is a hesitant fuzzy UP-ideal of A, we have $\overline{\mathrm{h}}(x \cdot z) \supseteq \overline{\mathrm{h}}(x \cdot (y \cdot z)) \cap \overline{\mathrm{h}}(y)$. By Lemma 2.22 (1), we have $[0,1] - \mathrm{h}(x \cdot z) \supseteq ([0,1] - \mathrm{h}(x \cdot (y \cdot z))) \cap ([0,1] - \mathrm{h}(y)) = [0,1] - (\mathrm{h}(x \cdot (y \cdot z)) \cup \mathrm{h}(y))$. Thus $\mathrm{h}(x \cdot z) \subseteq \mathrm{h}(x \cdot (y \cdot z)) \cup \mathrm{h}(y) \subseteq \varepsilon$. So $x \cdot z \in L(\mathrm{h}; \varepsilon)$. Hence, $L(\mathrm{h}; \varepsilon)$ is a UP-ideal of A.

Conversely, assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L(\mathrm{h}; \varepsilon)$ of A is a UP-ideal of A. Let $x \in A$. Choose $\varepsilon = \mathrm{h}(x) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x) \subseteq \varepsilon$, so $x \in L(\mathrm{h}; \varepsilon) \neq \varnothing$. By assumption, we have $L(\mathrm{h}; \varepsilon)$ is a UP-filter of A. Thus $0 \in L(\mathrm{h}; \varepsilon)$. So $\mathrm{h}(0) \subseteq \varepsilon = \mathrm{h}(x)$. Hence $\overline{\mathrm{h}}(0) = [0,1] - \mathrm{h}(0) \supseteq [0,1] - \mathrm{h}(x) = \overline{\mathrm{h}}(x)$.

Next, let $x, y, z \in A$. Choose $\varepsilon = \mathrm{h}(x \cdot (y \cdot z)) \cup \mathrm{h}(y) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x \cdot (y \cdot z)) \subseteq \varepsilon$ and $\mathrm{h}(y) \subseteq \varepsilon$. Thus $x \cdot (y \cdot z), y \in L(\mathrm{h}; \varepsilon) \neq \varnothing$. By assumption, we have $L(\mathrm{h}; \varepsilon)$ is a UP-ideal of A. So $x \cdot z \in L(\mathrm{h}; \varepsilon)$. Hence $\mathrm{h}(x \cdot z) \subseteq \varepsilon = \mathrm{h}(x \cdot (y \cdot z)) \cup \mathrm{h}(y)$. By Lemma 2.22 (1), we have

$$\begin{aligned}\overline{\mathrm{h}}(x \cdot z) &= [0,1] - \mathrm{h}(x \cdot z) \\ &\supseteq [0,1] - (\mathrm{h}(x \cdot (y \cdot z)) \cup \mathrm{h}(y)) \\ &= ([0,1] - \mathrm{h}(x \cdot (y \cdot z))) \cap ([0,1] - \mathrm{h}(y)) \\ &= \overline{\mathrm{h}}(x \cdot (y \cdot z)) \cap \overline{\mathrm{h}}(y).\end{aligned}$$

Therefore, $\overline{\mathrm{h}}$ is a hesitant fuzzy UP-ideal of A. □

Theorem 2.26. *Let* h *be a hesitant fuzzy set on* A. *Then* $\overline{\mathrm{h}}$ *is a hesitant fuzzy strongly UP-ideal of* A *if and only if for all* $\varepsilon \in \mathcal{P}([0,1])$, *a nonempty subset* $L(\mathrm{h}; \varepsilon)$ *of* A *is a strongly UP-ideal of* A.

Proof. Assume that $\overline{\mathrm{h}}$ is a hesitant fuzzy strongly UP-ideal of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $L(\mathrm{h}; \varepsilon) \neq \varnothing$ and let $a \in L(\mathrm{h}; \varepsilon)$. Then $\mathrm{h}(a) \subseteq \varepsilon$. Since $\overline{\mathrm{h}}$ is a hesitant fuzzy strongly UP-ideal of A, we have $\overline{\mathrm{h}}(0) \supseteq \overline{\mathrm{h}}(a)$. Thus $[0,1] - \mathrm{h}(0) \supseteq [0,1] - \mathrm{h}(a)$, so $\mathrm{h}(0) \subseteq \mathrm{h}(a) \subseteq \varepsilon$. So, $0 \in L(\mathrm{h}; \varepsilon)$.

Next, let $x, y, z \in A$ be such that $(z \cdot y) \cdot (z \cdot x) \in L(\mathrm{h}; \varepsilon)$ and $y \in L(\mathrm{h}; \varepsilon)$. Then $\mathrm{h}((z \cdot y) \cdot (z \cdot x)) \subseteq \varepsilon$ and $\mathrm{h}(y) \subseteq \varepsilon$. Thus $\mathrm{h}((z \cdot y) \cdot (z \cdot x)) \cup \mathrm{h}(y) \subseteq \varepsilon$. Since $\overline{\mathrm{h}}$ is a hesitant fuzzy strongly UP-ideal of A, we have $\overline{\mathrm{h}}(x) \supseteq \overline{\mathrm{h}}((z \cdot y) \cdot (z \cdot x)) \cap \overline{\mathrm{h}}(y)$. By Lemma 2.22 (1), we have $[0,1] - \mathrm{h}(x) \supseteq ([0,1] - \mathrm{h}((z \cdot y) \cdot (z \cdot x))) \cap ([0,1] - \mathrm{h}(y)) = [0,1] - (\mathrm{h}((z \cdot y) \cdot (z \cdot x)) \cup \mathrm{h}(y))$. So $\mathrm{h}(x) \subseteq \mathrm{h}((z \cdot y) \cdot (z \cdot x)) \cup \mathrm{h}(y) \subseteq \varepsilon$. Hence $x \in L(\mathrm{h}; \varepsilon)$. Therefore, $L(\mathrm{h}; \varepsilon)$ is a strongly UP-ideal of A.

Conversely, assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L(\mathrm{h}; \varepsilon)$ of A is a strongly UP-ideal of A. Let $x \in A$. Choose $\varepsilon = \mathrm{h}(x) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x) \subseteq \varepsilon$, so $x \in L(\mathrm{h}; \varepsilon) \neq \varnothing$. By assumption, we have $L(\mathrm{h}; \varepsilon)$ is a strongly UP-ideal of A and thus $0 \in L(\mathrm{h}; \varepsilon)$. So $\mathrm{h}(0) \subseteq \varepsilon = \mathrm{h}(x)$. Hence $\overline{\mathrm{h}}(0) = [0,1] - \mathrm{h}(0) \supseteq [0,1] - \mathrm{h}(x) = \overline{\mathrm{h}}(x)$.

Next, let $x, y, z \in A$. Choose $\varepsilon = \mathrm{h}((z \cdot y) \cdot (z \cdot x)) \cup \mathrm{h}(y) \in \mathcal{P}([0,1])$. Then $\mathrm{h}((z \cdot y) \cdot (z \cdot x)) \subseteq \varepsilon$ and $\mathrm{h}(y) \subseteq \varepsilon$. Thus $(z \cdot y) \cdot (z \cdot x), y \in L(\mathrm{h}; \varepsilon) \neq \varnothing$. By assumption, we have $L(\mathrm{h}; \varepsilon)$ is a strongly UP-ideal of A. So $x \in L(\mathrm{h}; \varepsilon)$. Hence

$\mathrm{h}(x) \subseteq \varepsilon = \mathrm{h}((z \cdot y) \cdot (z \cdot x)) \cup \mathrm{h}(y)$. By Lemma 2.22 (1), we have

$$\begin{aligned}\overline{\mathrm{h}}(x) &= [0,1] - \mathrm{h}(x)\\ &\supseteq [0,1] - (\mathrm{h}((z \cdot y) \cdot (z \cdot x)) \cup \mathrm{h}(y))\\ &= ([0,1] - \mathrm{h}((z \cdot y) \cdot (z \cdot x))) \cap ([0,1] - \mathrm{h}(y))\\ &= \overline{\mathrm{h}}((z \cdot y) \cdot (z \cdot x)) \cap \overline{\mathrm{h}}(y).\end{aligned}$$

Therefore, $\overline{\mathrm{h}}$ is a hesitant fuzzy strongly UP-ideal of A. □

2.6. **Prime and Weakly Prime Subsets and Lower ε-Level Subsets.**

Theorem 2.27. *Let* h *be a hesitant fuzzy set on A. Then the following statements hold*:

(1) *if* $\mathrm{Im}(\mathrm{h})$ *is a chain and* $\overline{\mathrm{h}}$ *is a prime hesitant fuzzy set on A, then for all* $\varepsilon \in \mathcal{P}([0,1])$*,* $L(\mathrm{h}; \varepsilon)$ *is a prime subset of A if* $L(\mathrm{h}; \varepsilon)$ *is nonempty,*

(2) *if for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $L(\mathrm{h}; \varepsilon)$ *of A is a prime subset of A, then* $\overline{\mathrm{h}}$ *is a prime hesitant fuzzy set on A.*

Proof. (1) Assume that $\mathrm{Im}(\mathrm{h})$ is a chain and $\overline{\mathrm{h}}$ is a prime hesitant fuzzy set on A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $L(\mathrm{h}; \varepsilon) \neq \varnothing$. Let $x, y \in A$ be such that $x \cdot y \in L(\mathrm{h}; \varepsilon)$ and assume that $x \notin L(\mathrm{h}; \varepsilon)$. Then $\mathrm{h}(x \cdot y) \subseteq \varepsilon$ and $\mathrm{h}(x) \nsubseteq \varepsilon$. Since $\overline{\mathrm{h}}$ is a prime hesitant fuzzy set on A, we have $\overline{\mathrm{h}}(x \cdot y) \subseteq \overline{\mathrm{h}}(x) \cup \overline{\mathrm{h}}(y)$. By Lemma 2.22 (2), we have $[0,1] - \mathrm{h}(x \cdot y) \subseteq ([0,1] - \mathrm{h}(x)) \cup ([0,1] - \mathrm{h}(y)) = [0,1] - (\mathrm{h}(x) \cap \mathrm{h}(y))$. Thus $\varepsilon \supseteq \mathrm{h}(x \cdot y) \supseteq \mathrm{h}(x) \cap \mathrm{h}(y)$.

If $\mathrm{h}(x) \subseteq \mathrm{h}(y)$, then $\varepsilon \supseteq \mathrm{h}(x) \cap \mathrm{h}(y) = \mathrm{h}(x)$. Thus $x \in L(\mathrm{h}; \varepsilon)$, a contradiction.

If $\mathrm{h}(x) \supseteq \mathrm{h}(y)$, then $\varepsilon \supseteq \mathrm{h}(x) \cap \mathrm{h}(y) = \mathrm{h}(y)$. Thus $y \in L(\mathrm{h}; \varepsilon)$.

So, $L(\mathrm{h}; \varepsilon)$ is a prime subset of A.

(2) Assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L(\mathrm{h}; \varepsilon)$ of A is a prime subset of A. Let $x, y \in A$. Choose $\varepsilon = \mathrm{h}(x \cdot y) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x \cdot y) \subseteq \varepsilon$. Thus $x \cdot y \in L(\mathrm{h}; \varepsilon) \neq \varnothing$. By assumption, we have $L(\mathrm{h}; \varepsilon)$ is a prime subset of A and thus $x \in L(\mathrm{h}; \varepsilon)$ or $y \in L(\mathrm{h}; \varepsilon)$. So $\varepsilon \supseteq \mathrm{h}(x)$ or $\varepsilon \supseteq \mathrm{h}(y)$. Hence $\mathrm{h}(x \cdot y) = \varepsilon \supseteq \mathrm{h}(x) \cap \mathrm{h}(y)$. By Lemma 2.22 (2), we have

$$\begin{aligned}\overline{\mathrm{h}}(x \cdot y) &= [0,1] - \mathrm{h}(x \cdot y)\\ &\subseteq [0,1] - (\mathrm{h}(x) \cap \mathrm{h}(y))\\ &= ([0,1] - \mathrm{h}(x)) \cup ([0,1] - \mathrm{h}(y))\\ &= \overline{\mathrm{h}}(x) \cup \overline{\mathrm{h}}(y).\end{aligned}$$

Therefore, $\overline{\mathrm{h}}$ is a prime hesitant fuzzy set on A. □

Theorem 2.28. *Let* h *be a hesitant fuzzy set on A. Then* $\overline{\mathrm{h}}$ *is a prime hesitant fuzzy UP-subalgebra (resp. prime hesitant fuzzy UP-filter, prime hesitant fuzzy UP-ideal, prime hesitant fuzzy strongly UP-ideal) of A if and only if for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $L(\mathrm{h}; \varepsilon)$ *of A is a prime UP-subalgebra (resp. prime UP-filter, prime UP-ideal, prime strongly UP-ideal) of A.*

Proof. It is straightforward by Theorem 1.22, 2.26 and 1.11. □

Theorem 2.29. *Let* h *be a hesitant fuzzy set on* A*. Then the following statements hold*:

(1) *if* Im(h) *is a chain and* $\overline{\mathrm{h}}$ *is a weakly prime hesitant fuzzy set on* A*, then for all* $\varepsilon \in \mathcal{P}([0,1])$*,* $L(\mathrm{h};\varepsilon)$ *is a weakly prime subset of* A *if* $L(\mathrm{h};\varepsilon)$ *is nonempty,*

(2) *if for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $L(\mathrm{h};\varepsilon)$ *of* A *is a weakly prime subset of* A*, then* $\overline{\mathrm{h}}$ *is a weakly prime hesitant fuzzy set on* A*.*

Proof. (1) Assume that Im(h) is a chain and $\overline{\mathrm{h}}$ is a weakly prime hesitant fuzzy set on A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $L(\mathrm{h};\varepsilon) \neq \varnothing$. Let $x, y \in A$ be such that $x \neq y$ and $x \cdot y \in L(\mathrm{h};\varepsilon)$. Assume that $x \notin L(\mathrm{h};\varepsilon)$. Then $\mathrm{h}(x\cdot y) \subseteq \varepsilon$ and $\mathrm{h}(x) \nsubseteq \varepsilon$. Since $\overline{\mathrm{h}}$ is a weakly prime hesitant fuzzy set on A, we have $\overline{\mathrm{h}}(x\cdot y) \subseteq \overline{\mathrm{h}}(x) \cup \overline{\mathrm{h}}(y)$. By Lemma 2.22 (2), we have $[0,1]-\mathrm{h}(x\cdot y) \subseteq ([0,1]-\mathrm{h}(x)) \cup ([0,1]-\mathrm{h}(y)) = [0,1]-(\mathrm{h}(x)\cap \mathrm{h}(y))$. Thus $\varepsilon \supseteq \mathrm{h}(x\cdot y) \supseteq \mathrm{h}(x) \cap \mathrm{h}(y)$.

If $\mathrm{h}(x) \subseteq \mathrm{h}(y)$, then $\varepsilon \supseteq \mathrm{h}(x) \cap \mathrm{h}(y) = \mathrm{h}(x)$. Thus $x \in L(\mathrm{h};\varepsilon)$, a contradiction.

If $\mathrm{h}(x) \supseteq \mathrm{h}(y)$, then $\varepsilon \supseteq \mathrm{h}(x) \cap \mathrm{h}(y) = \mathrm{h}(y)$, Thus $y \in L(\mathrm{h};\varepsilon)$.

So, $L(\mathrm{h};\varepsilon)$ is a weakly prime subset of A.

(2) Assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L(\mathrm{h};\varepsilon)$ of A is a weakly prime subset of A. Let $x, y \in A$ be such that $x \neq y$. Choose $\varepsilon = \mathrm{h}(x\cdot y) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x\cdot y) \subseteq \varepsilon$. Thus $x \cdot y \in L(\mathrm{h};\varepsilon) \neq \varnothing$. By assumption, we have $L(\mathrm{h};\varepsilon)$ is a weakly prime subset of A and thus $x \in L(\mathrm{h};\varepsilon)$ or $y \in L(\mathrm{h};\varepsilon)$. So $\varepsilon \supseteq \mathrm{h}(x)$ or $\varepsilon \supseteq \mathrm{h}(y)$, Hence $\mathrm{h}(x\cdot y) = \varepsilon \supseteq \mathrm{h}(x) \cap \mathrm{h}(y)$. By Lemma 2.22 (2), we have

$$\begin{aligned}\overline{\mathrm{h}}(x\cdot y) &= [0,1]-\mathrm{h}(x\cdot y)\\ &\subseteq [0,1]-(\mathrm{h}(x)\cap\mathrm{h}(y))\\ &= ([0,1]-\mathrm{h}(x)) \cup ([0,1]-\mathrm{h}(y))\\ &= \overline{\mathrm{h}}(x) \cup \overline{\mathrm{h}}(y).\end{aligned}$$

Therefore, $\overline{\mathrm{h}}$ is a weakly prime hesitant fuzzy set on A. □

Theorem 2.30. *Let* h *be a hesitant fuzzy set on* A*. Then the following statements hold*:

(1) *if* Im(h) *is a chain and* $\overline{\mathrm{h}}$ *is a weakly prime hesitant fuzzy UP-subalgebra (resp. weakly prime hesitant fuzzy UP-filter, weakly prime hesitant fuzzy UP-ideal, weakly prime hesitant fuzzy strongly UP-ideal) of* A*, then for all* $\varepsilon \in \mathcal{P}([0,1])$*,* $L(\mathrm{h};\varepsilon)$ *is a weakly prime UP-subalgebra (resp. weakly prime UP-filter, weakly prime UP-ideal, weakly prime strongly UP-ideal) of* A *if* $L(\mathrm{h};\varepsilon)$ *is nonempty,*

(2) *if for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $L(\mathrm{h};\varepsilon)$ *of* A *is a weakly prime UP-subalgebra (resp. weakly prime UP-filter, weakly prime UP-ideal, weakly prime strongly UP-ideal) of* A*, then* $\overline{\mathrm{h}}$ *is a weakly prime hesitant fuzzy UP-subalgebra (resp. weakly prime hesitant fuzzy UP-filter, weakly prime hesitant fuzzy UP-ideal, weakly prime hesitant fuzzy strongly UP-ideal) of* A*.*

Proof. It is straightforward by Theorem 2.23 (resp. Theorem 2.24, Theorem 2.25, 2.26) and 2.29. □

2.7. **Lower ε-Strong Level Subsets.**

Theorem 2.31. *Let* h *be a hesitant fuzzy set on* A*. Then the following statements hold*:

(1) *if* $\overline{\mathrm{h}}$ *is a hesitant fuzzy UP-subalgebra of* A*, then for all* $\varepsilon \in \mathcal{P}([0,1])$*,* $L^-(\mathrm{h};\varepsilon)$ *is a UP-subalgebra of* A *if* $L^-(\mathrm{h};\varepsilon)$ *is nonempty and* $E(\mathrm{h};\varepsilon)$ *is empty,*

(2) *if* $\mathrm{Im}(\mathrm{h})$ *is a chain and for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $L^-(\mathrm{h};\varepsilon)$ *of* A *is a UP-subalgebra of* A*, then* $\overline{\mathrm{h}}$ *is a hesitant fuzzy UP-subalgebra of* A*.*

Proof. (1) It is straightforward by Theorem 2.23.

(2) Assume that $\mathrm{Im}(\mathrm{h})$ is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L^-(\mathrm{h};\varepsilon)$ of A is a UP-subalgebra of A. Assume that there exist $x, y \in A$ such that $\overline{\mathrm{h}}(x\cdot y) \not\supseteq \overline{\mathrm{h}}(x)\cap\overline{\mathrm{h}}(y)$. Since $\mathrm{Im}(\mathrm{h})$ is a chain, we have $\overline{\mathrm{h}}(x\cdot y) \subset \overline{\mathrm{h}}(x)\cap\overline{\mathrm{h}}(y)$. By Lemma 2.22 (1), we have $[0,1]-\mathrm{h}(x\cdot y) \subset ([0,1]-\mathrm{h}(x))\cap([0,1]-\mathrm{h}(y)) = [0,1]-(\mathrm{h}(x)\cup\mathrm{h}(y))$. Then $\mathrm{h}(x\cdot y) \supset \mathrm{h}(x)\cup\mathrm{h}(y)$. Choose $\varepsilon = \mathrm{h}(x\cdot y) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x) \subset \varepsilon$ and $\mathrm{h}(y) \subset \varepsilon$. Thus $x, y \in L^-(\mathrm{h};\varepsilon) \neq \varnothing$. By assumption, we have $L^-(\mathrm{h};\varepsilon)$ is a UP-subalgebra of A and thus $x\cdot y \in L^-(\mathrm{h};\varepsilon)$. So $\mathrm{h}(x\cdot y) \subset \varepsilon = \mathrm{h}(x\cdot y)$, a contradiction. Hence, $\overline{\mathrm{h}}(x\cdot y) \supseteq \overline{\mathrm{h}}(x)\cap\overline{\mathrm{h}}(y)$, for all $x, y \in A$. Therefore, $\overline{\mathrm{h}}$ is a hesitant fuzzy UP-subalgebra of A, □

Example 2.32. Let $A = \{0,1,2,3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

Then $(A;\cdot,0)$ is a UP-algebra. We define a hesitant fuzzy set h on A as follows:

$$\mathrm{h}(0) = \{0.1\}, \mathrm{h}(1) = \{0.2\}, \mathrm{h}(2) = \varnothing, \text{ and } \mathrm{h}(3) = \{0.1\}.$$

Then $\mathrm{Im}(\mathrm{h})$ is not a chain. If $\varepsilon = \varnothing$, then $L^-(\mathrm{h};\varepsilon) = \varnothing$. If $\varepsilon \neq \varnothing$, then $L^-(\mathrm{h};\varepsilon) = \{0,1,3\}$. Using this data, we can show that $L^-(\mathrm{h};\varepsilon)$ is a UP-subalgebra of A. By Definition 1.15, we have

$$\overline{\mathrm{h}}(0) = [0,1]-\{0.1\}, \overline{\mathrm{h}}(1) = [0,1]-\{0.2\}, \overline{\mathrm{h}}(2) = [0,1] \text{ and } \overline{\mathrm{h}}(3) = [0,1]-\{0.1\}.$$

Since $\overline{\mathrm{h}}(1\cdot 1) = \overline{\mathrm{h}}(0) = [0,1]-\{0.1\} \not\supseteq [0,1]-\{0.2\} = \mathrm{h}(1)\cap\mathrm{h}(1)$, we have $\overline{\mathrm{h}}$ is not a hesitant fuzzy UP-subalgebra of A.

Theorem 2.33. *Let* h *be a hesitant fuzzy set on* A*. Then the following statements hold:*

(1) *if* $\overline{\mathrm{h}}$ *is a hesitant fuzzy UP-filter of* A*, then for all* $\varepsilon \in \mathcal{P}([0,1])$*,* $L^-(\mathrm{h};\varepsilon)$ *is a UP-filter of* A *if* $L^-(\mathrm{h};\varepsilon)$ *is nonempty and* $E(\mathrm{h};\varepsilon)$ *is empty,*

(2) *if* $\mathrm{Im}(\mathrm{h})$ *is a chain and for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $L^-(\mathrm{h};\varepsilon)$ *of* A *is a UP-filter of* A*, then* $\overline{\mathrm{h}}$ *is a hesitant fuzzy UP-filter of* A*.*

Proof. (1) It is straightforward by Theorem 2.24.

(2) Assume that $\mathrm{Im}(\mathrm{h})$ is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L^-(\mathrm{h};\varepsilon)$ of A is a UP-filter of A. Assume that there exists $x \in A$ such that $\overline{\mathrm{h}}(0) \not\supseteq \overline{\mathrm{h}}(x)$. Since $\mathrm{Im}(\mathrm{h})$ is a chain, we have $\overline{\mathrm{h}}(0) \subset \overline{\mathrm{h}}(x)$. Then $[0,1]-\mathrm{h}(0) \subset [0,1]-\mathrm{h}(x)$. Thus $\mathrm{h}(0) \supset \mathrm{h}(\mathrm{x})$. Choose $\varepsilon = \mathrm{h}(0) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x) \subset \varepsilon$. Thus $x \in L^-(\mathrm{h};\varepsilon) \neq \varnothing$. By assumption, we have $L^-(\mathrm{h};\varepsilon)$ is a UP-filter of A and thus $0 \in L^-(\mathrm{h};\varepsilon)$. So $\mathrm{h}(0) \subset \varepsilon = \mathrm{h}(0)$, a contradiction. Hence, $\overline{\mathrm{h}}(0) \supseteq \overline{\mathrm{h}}(x)$, for all $x \in A$.

Assume that there exist $x, y \in A$ such that $\overline{\mathrm{h}}(y) \not\supseteq \overline{\mathrm{h}}(x \cdot y) \cap \overline{\mathrm{h}}(x)$. Since $\mathrm{Im}(\mathrm{h})$ is a chain, we have $\overline{\mathrm{h}}(y) \subset \overline{\mathrm{h}}(x \cdot y) \cap \overline{\mathrm{h}}(x)$. By Lemma 2.22 (1), we have $[0,1] - \mathrm{h}(y) \subset ([0,1] - \mathrm{h}(x \cdot y)) \cap ([0,1] - \mathrm{h}(x)) = [0,1] - (\mathrm{h}(x \cdot y) \cup \mathrm{h}(x))$. Then $\mathrm{h}(y) \supset \mathrm{h}(x \cdot y) \cup \mathrm{h}(x)$. Choose $\varepsilon = \mathrm{h}(y) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x \cdot y) \subset \varepsilon$ and $\mathrm{h}(x) \subset \varepsilon$. Thus $x \cdot y, x \in L^-(\mathrm{h}; \varepsilon) \neq \varnothing$. By assumption, we have $L^-(\mathrm{h}; \varepsilon)$ is a UP-filter of A and thus $y \in L^-(\mathrm{h}; \varepsilon)$. So $\mathrm{h}(y) \subset \varepsilon = \mathrm{h}(y)$, a contradiction. Hence, $\overline{\mathrm{h}}(y) \supseteq \overline{\mathrm{h}}(x \cdot y) \cap \overline{\mathrm{h}}(x)$, for all $x, y \in A$. Therefore, $\overline{\mathrm{h}}$ is a hesitant fuzzy UP-filter of A. □

Example 2.34. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	0
2	0	1	0	3
3	0	1	2	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set h on A as follows:

$$\mathrm{h}(0) = \{0.1\}, \mathrm{h}(1) = \{0.1\}, \mathrm{h}(2) = \varnothing, \text{ and } \mathrm{h}(3) = \{0.2\}.$$

Then $\mathrm{Im}(\mathrm{h})$ is not a chain. If $\varepsilon = \varnothing$, then $L^-(\mathrm{h}; \varepsilon) = \varnothing$. If $\varepsilon \neq \varnothing$, then $L^-(\mathrm{h}; \varepsilon) = \{0, 1, 3\}$. Using this data, we can show that $L^-(\mathrm{h}; \varepsilon)$ is a UP-filter of A. By Definition 1.15, we have

$$\overline{\mathrm{h}}(0) = [0,1] - \{0.1\}, \overline{\mathrm{h}}(1) = [0,1] - \{0.1\}, \overline{\mathrm{h}}(2) = [0,1] \text{ and } \overline{\mathrm{h}}(3) = [0,1] - \{0.2\}.$$

Since $\overline{\mathrm{h}}(0) = [0,1] - \{0.1\} \not\supseteq [0,1] - \{0.2\} = \overline{\mathrm{h}}(3)$ and $\overline{\mathrm{h}}(3) = [0,1] - \{0.2\} \not\supseteq [0,1] - \{0.1\} = \mathrm{h}(1) \cap \mathrm{h}(0) = \overline{\mathrm{h}}(1) \cap \overline{\mathrm{h}}(1 \cdot 3)$, we have $\overline{\mathrm{h}}$ is not a hesitant fuzzy UP-filter of A.

Theorem 2.35. *Let* h *be a hesitant fuzzy set on* A*. Then the following statements hold:*

(1) *if* $\overline{\mathrm{h}}$ *is a hesitant fuzzy UP-ideal of* A*, then for all* $\varepsilon \in \mathcal{P}([0,1])$*,* $L^-(\mathrm{h}; \varepsilon)$ *is a UP-ideal of* A *if* $L^-(\mathrm{h}; \varepsilon)$ *is nonempty and* $E(\mathrm{h}; \varepsilon)$ *is empty,*

(2) *if* $\mathrm{Im}(\mathrm{h})$ *is a chain and for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $L^-(\mathrm{h}; \varepsilon)$ *of* A *is a UP-ideal of* A*, then* $\overline{\mathrm{h}}$ *is a hesitant fuzzy UP-ideal of* A*.*

Proof. (1) It is straightforward by Theorem 2.25.

(2) Assume that $\mathrm{Im}(\mathrm{h})$ is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L^-(\mathrm{h}; \varepsilon)$ of A is a UP-ideal of A. Assume that there exists $x \in A$ such that $\overline{\mathrm{h}}(0) \not\supseteq \overline{\mathrm{h}}(x)$. Since $\mathrm{Im}(\mathrm{h})$ is a chain, we have $\overline{\mathrm{h}}(0) \subset \overline{\mathrm{h}}(x)$. Then $[0,1] - \mathrm{h}(0) \subset [0,1] - \mathrm{h}(x)$. Thus $\mathrm{h}(0) \supset \mathrm{h}(\mathrm{x})$. Choose $\varepsilon = \mathrm{h}(0) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x) \subset \varepsilon$. Thus $x \in L^-(\mathrm{h}; \varepsilon) \neq \varnothing$. By assumption, we have $L^-(\mathrm{h}; \varepsilon)$ is a UP-ideal of A and thus $0 \in L^-(\mathrm{h}; \varepsilon)$. So $\mathrm{h}(0) \subset \varepsilon = \mathrm{h}(0)$, a contradiction. Hence, $\overline{\mathrm{h}}(0) \supseteq \overline{\mathrm{h}}(x)$, for all $x \in A$.

Assume that there exist $x, y, z \in A$ such that $\overline{\mathrm{h}}(x \cdot z) \not\supseteq \overline{\mathrm{h}}(x \cdot (y \cdot z)) \cap \overline{\mathrm{h}}(y)$. Since $\mathrm{Im}(\mathrm{h})$ is a chain, we have $\overline{\mathrm{h}}(x \cdot z) \subset \overline{\mathrm{h}}(x \cdot (y \cdot z)) \cap \overline{\mathrm{h}}(y)$. By Lemma 2.22 (1), we have $[0,1] - \mathrm{h}(x \cdot z) \subset ([0,1] - \mathrm{h}(x \cdot (y \cdot z))) \cap ([0,1] - \mathrm{h}(y)) = [0,1] - (\mathrm{h}(x \cdot (y \cdot z)) \cup \mathrm{h}(y))$. Then $\mathrm{h}(x \cdot z) \supset \mathrm{h}(x \cdot (y \cdot z)) \cup \mathrm{h}(y)$. Choose $\varepsilon = \mathrm{h}(x \cdot z) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x \cdot (y \cdot z)) \subset \varepsilon$ and $\mathrm{h}(y) \subset \varepsilon$. Thus $x \cdot (y \cdot z), y \in L^-(\mathrm{h}; \varepsilon) \neq \varnothing$. By assumption, we have $L^-(\mathrm{h}; \varepsilon)$ is a UP-ideal of A and thus $x \cdot z \in L^-(\mathrm{h}; \varepsilon)$. So $\mathrm{h}(x \cdot z) \subset \varepsilon = \mathrm{h}(x \cdot z)$, a contradiction.

Hence, $\overline{\mathrm{h}}(x \cdot z) \supseteq \overline{\mathrm{h}}(x \cdot (y \cdot z)) \cap \overline{\mathrm{h}}(y)$, for all $x, y, z \in A$. Therefore, $\overline{\mathrm{h}}$ is a hesitant fuzzy UP-ideal of A. □

Example 2.36. Let $A = \{0, 1, 2\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2
0	0	1	2
1	0	0	2
2	0	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set h on A as follows:

$$\mathrm{h}(0) = \{0.1, 0.2\}, \mathrm{h}(1) = \{0.3\}, \text{ and } \mathrm{h}(2) = \{0.2\}.$$

Then $\mathrm{Im}(\mathrm{h})$ is not a chain. If $\varepsilon = \varnothing$, then $L^{-}(\mathrm{h}; \varepsilon) = \varnothing$. If $\varepsilon \neq \varnothing$, then $L^{-}(\mathrm{h}; \varepsilon) = A$. Using this data, we can show that $L^{-}(\mathrm{h}; \varepsilon)$ is a UP-ideal of A. By Definition 1.15, we have

$$\overline{\mathrm{h}}(0) = [0, 1] - \{0.1, 0.2\}, \overline{\mathrm{h}}(1) = [0, 1] - \{0.3\} \text{ and } \overline{\mathrm{h}}(2) = [0, 1] - \{0.2\}.$$

Since $\overline{\mathrm{h}}(0) = [0, 1] - \{0.1, 0.2\} \not\supseteq [0, 1] - \{0.3\} = \overline{\mathrm{h}}(2)$ and $\overline{\mathrm{h}}(0 \cdot 1) = \overline{\mathrm{h}}(1) = [0, 1] - \{0.3\} \not\supseteq [0, 1] - \{0.1, 0.2\} = \mathrm{h}(0) \cap \mathrm{h}(2) = \overline{\mathrm{h}}(0 \cdot (2 \cdot 1)) \cap \overline{\mathrm{h}}(2)$, we have $\overline{\mathrm{h}}$ is not a hesitant fuzzy UP-ideal of A.

Theorem 2.37. *Let* h *be a hesitant fuzzy set on* A*. Then the following statements hold:*

(1) *if* $\overline{\mathrm{h}}$ *is a hesitant fuzzy strongly UP-ideal of* A*, then for all* $\varepsilon \in \mathcal{P}([0, 1])$, $L^{-}(\mathrm{h}; \varepsilon)$ *is a strongly UP-ideal of* A *if* $L^{-}(\mathrm{h}; \varepsilon)$ *is nonempty and* $E(\mathrm{h}; \varepsilon)$ *is empty,*

(2) *if* $\mathrm{Im}(\mathrm{h})$ *is a chain and for all* $\varepsilon \in \mathcal{P}([0, 1])$*, a nonempty subset* $L^{-}(\mathrm{h}; \varepsilon)$ *of* A *is a strongly UP-ideal of* A*, then* $\overline{\mathrm{h}}$ *is a hesitant fuzzy strongly UP-ideal of* A*.*

Proof. (1) It is straightforward by Theorem 2.26.

(2) Assume that $\mathrm{Im}(\mathrm{h})$ is a chain and for all $\varepsilon \in \mathcal{P}([0, 1])$, a nonempty subset $L^{-}(\mathrm{h}; \varepsilon)$ of A is a strongly UP-ideal of A. Assume that there exists $x \in A$ such that $\overline{\mathrm{h}}(0) \not\supseteq \overline{\mathrm{h}}(x)$. Since $\mathrm{Im}(\mathrm{h})$ is a chain, we have $\overline{\mathrm{h}}(0) \subset \overline{\mathrm{h}}(x)$. Then $[0, 1] - \mathrm{h}(0) \subset [0, 1] - \mathrm{h}(x)$. Thus $\mathrm{h}(0) \supset \mathrm{h}(x)$. Choose $\varepsilon = \mathrm{h}(0) \in \mathcal{P}([0, 1])$. Then $\mathrm{h}(x) \subset \varepsilon$, so $x \in L^{-}(\mathrm{h}; \varepsilon) \neq \varnothing$. By assumption, we have $L^{-}(\mathrm{h}; \varepsilon)$ is a strongly UP-ideal of A. Thus $0 \in L^{-}(\mathrm{h}; \varepsilon)$. So $\mathrm{h}(0) \subset \varepsilon = \mathrm{h}(0)$, a contradiction. Hence, $\overline{\mathrm{h}}(0) \supseteq \overline{\mathrm{h}}(x)$, for all $x \in A$.

Assume that there exist $x, y, z \in A$ such that $\overline{\mathrm{h}}(x) \not\supseteq \overline{\mathrm{h}}((z \cdot y) \cdot (z \cdot x)) \cap \overline{\mathrm{h}}(y)$. Since $\mathrm{Im}(\mathrm{h})$ is a chain, we have $\overline{\mathrm{h}}(x) \subset \overline{\mathrm{h}}((z \cdot y) \cdot (z \cdot x)) \cap \overline{\mathrm{h}}(y)$. By Lemma 2.22 (1), we have $[0, 1] - \mathrm{h}(x) \subset ([0, 1] - \mathrm{h}((z \cdot y) \cdot (z \cdot x))) \cap ([0, 1] - \mathrm{h}(y)) = [0, 1] - (\mathrm{h}((z \cdot y) \cdot (z \cdot x)) \cup \mathrm{h}(y))$. Then $\mathrm{h}(x) \supset \mathrm{h}((z \cdot y) \cdot (z \cdot x)) \cup \mathrm{h}(y)$. Choose $\varepsilon = \mathrm{h}(x) \in \mathcal{P}([0, 1])$. Then $\mathrm{h}((z \cdot y) \cdot (z \cdot x)) \subset \varepsilon$ and $\mathrm{h}(y) \subset \varepsilon$. Thus $(z \cdot y) \cdot (z \cdot x), y \in L^{-}(\mathrm{h}; \varepsilon) \neq \varnothing$. By assumption, we have $L^{-}(\mathrm{h}; \varepsilon)$ is a strongly UP-ideal of A and thus $x \in L^{-}(\mathrm{h}; \varepsilon)$. So $\mathrm{h}(x) \subset \varepsilon = \mathrm{h}(x)$, a contradiction. Hence, $\overline{\mathrm{h}}(x) \supseteq \overline{\mathrm{h}}((z \cdot y) \cdot (z \cdot x)) \cap \overline{\mathrm{h}}(y)$, for all $x, y, z \in A$. Therefore, $\overline{\mathrm{h}}$ is a hesitant fuzzy strongly UP-ideal of A. □

Example 2.38. Let $A = \{0, 1, 2\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2
0	0	1	2
1	0	0	2
2	0	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set h on A as follows:

$$\mathrm{h}(0) = \{0.1\}, \mathrm{h}(1) = \{0.1\}, \text{ and } \mathrm{h}(2) = \{0.2\}.$$

Then $\mathrm{Im}(\mathrm{h})$ is not a chain. If $\varepsilon = \varnothing$, then $L^-(\mathrm{h}; \varepsilon) = \varnothing$. If $\varepsilon \neq \varnothing$, then $L^-(\mathrm{h}; \varepsilon) = A$. Using this data, we can show that $L^-(\mathrm{h}; \varepsilon)$ is a strongly UP-ideal of A. By Definition 1.15, we have

$$\overline{\mathrm{h}}(0) = [0, 1] - \{0.1\}, \overline{\mathrm{h}}(1) = [0, 1] - \{0.1\} \text{ and } \overline{\mathrm{h}}(2) = [0, 1] - \{0.2\}.$$

Since $\overline{\mathrm{h}}(0) = [0, 1] - \{0.1\} \not\supseteq [0, 1] - \{0.2\} = \overline{\mathrm{h}}(2)$ and $\overline{\mathrm{h}}(2) = [0, 1] - \{0.2\} \not\supseteq [0, 1] - \{0.1\} = \mathrm{h}(0) \cap \mathrm{h}(0) = \overline{\mathrm{h}}((2 \cdot 0) \cdot (2 \cdot 2)) \cap \overline{\mathrm{h}}(0)$, we have $\overline{\mathrm{h}}$ is not a hesitant fuzzy strongly UP-ideal of A.

2.8. Prime and Weakly Prime Subsets and Lower ε-Strong Level Subsets.

Theorem 2.39. *Let* h *be a hesitant fuzzy set on* A. *Then the following statements hold:*

(1) *if* $\mathrm{Im}(\mathrm{h})$ *is a chain and* $\overline{\mathrm{h}}$ *is a prime hesitant fuzzy set on* A, *then for all* $\varepsilon \in \mathcal{P}([0, 1])$, $L^-(\mathrm{h}; \varepsilon)$ *is a prime subset of* A *if* $L^-(\mathrm{h}; \varepsilon)$ *is nonempty,*

(2) *if* $\mathrm{Im}(\mathrm{h})$ *is a chain and for all* $\varepsilon \in \mathcal{P}([0, 1])$, *a nonempty subset* $L^-(\mathrm{h}; \varepsilon)$ *of* A *is a prime subset of* A, *then* $\overline{\mathrm{h}}$ *is a prime hesitant fuzzy set on* A.

Proof. (1) Assume that $\mathrm{Im}(\mathrm{h})$ is a chain and $\overline{\mathrm{h}}$ is a prime hesitant fuzzy set on A. Let $\varepsilon \in \mathcal{P}([0, 1])$ be such that $L^-(\mathrm{h}; \varepsilon) \neq \varnothing$. Let $x, y \in A$ be such that $x \cdot y \in L^-(\mathrm{h}; \varepsilon)$ and assume that $x \notin L^-(\mathrm{h}; \varepsilon)$. Then $\mathrm{h}(x \cdot y) \subset \varepsilon$ and $\mathrm{h}(x) \not\subset \varepsilon$. Since $\overline{\mathrm{h}}$ is a prime hesitant fuzzy set on A, we have $\overline{\mathrm{h}}(x \cdot y) \subseteq \overline{\mathrm{h}}(x) \cup \overline{\mathrm{h}}(y)$. By Lemma 2.22 (2), we have $[0, 1] - \mathrm{h}(x \cdot y) \subseteq ([0, 1] - \mathrm{h}(x)) \cup ([0, 1] - \mathrm{h}(y)) = [0, 1] - (\mathrm{h}(x) \cap \mathrm{h}(y))$. Thus $\varepsilon \supset \mathrm{h}(x \cdot y) \supseteq \mathrm{h}(x) \cap \mathrm{h}(y)$.

If $\mathrm{h}(x) \subseteq \mathrm{h}(y)$, then $\varepsilon \supset \mathrm{h}(x) \cap \mathrm{h}(y) = \mathrm{h}(x)$. Thus $x \in L^-(\mathrm{h}; \varepsilon)$, a contradiction.

If $\mathrm{h}(x) \supseteq \mathrm{h}(y)$, then $\varepsilon \supset \mathrm{h}(x) \cap \mathrm{h}(y) = \mathrm{h}(y)$. Thus $y \in L^-(\mathrm{h}; \varepsilon)$.

So, $L^-(\mathrm{h}; \varepsilon)$ ia a prime subset of A.

(2) Assume that $\mathrm{Im}(\mathrm{h})$ is a chain and for all $\varepsilon \in \mathcal{P}([0, 1])$, a nonempty subset $L^-(\mathrm{h}; \varepsilon)$ of A is a prime subset of A. Assume that there exist $x, y \in A$ such that $\overline{\mathrm{h}}(x \cdot y) \not\subseteq \overline{\mathrm{h}}(x) \cup \overline{\mathrm{h}}(y)$. Since $\mathrm{Im}(\mathrm{h})$ is a chain, we have $\overline{\mathrm{h}}(x \cdot y) \supset \overline{\mathrm{h}}(x) \cup \overline{\mathrm{h}}(y)$. By Lemma 2.22 (2), we have $[0, 1] - \mathrm{h}(x \cdot y) \supset ([0, 1] - \mathrm{h}(x)) \cup ([0, 1] - \mathrm{h}(y)) = [0, 1] - (\mathrm{h}(x) \cap \mathrm{h}(y))$. Then $\mathrm{h}(x \cdot y) \subset \mathrm{h}(x) \cap \mathrm{h}(y)$. Choose $\varepsilon = \mathrm{h}(x) \cap \mathrm{h}(y) \in \mathcal{P}([0, 1])$. Then $\mathrm{h}(x \cdot y) \subset \varepsilon$. Thus $x \cdot y \in L^-(\mathrm{h}; \varepsilon) \neq \varnothing$. By assumption, we have $L^-(\mathrm{h}; \varepsilon)$ is a prime subset of A and thus $x \in L^-(\mathrm{h}; \varepsilon)$ or $y \in L^-(\mathrm{h}; \varepsilon)$. So $\mathrm{h}(x) \subset \varepsilon = \mathrm{h}(x) \cap \mathrm{h}(y)$ or $\mathrm{h}(y) \subset \varepsilon = \mathrm{h}(x) \cap \mathrm{h}(y)$, a contradiction. Hence, $\overline{\mathrm{h}}(x \cdot y) \subseteq \overline{\mathrm{h}}(x) \cup \overline{\mathrm{h}}(y)$, for all $x, y \in A$. Therefore, $\overline{\mathrm{h}}$ is a prime hesitant fuzzy set on A. □

Theorem 2.40. *Let* h *be a hesitant fuzzy set on* A. *Then the following statements hold:*

(1) *if* $\overline{\mathrm{h}}$ *is a prime hesitant fuzzy UP-subalgebra (resp. prime hesitant fuzzy UP-filter, prime hesitant fuzzy UP-ideal, prime hesitant fuzzy strongly UP-ideal) of* A*, then for all* $\varepsilon \in \mathcal{P}([0,1])$*,* $L^{-}(\mathrm{h};\varepsilon)$ *is a prime UP-subalgebra (resp. prime UP-filter, prime UP-ideal, prime strongly UP-ideal) of* A *if* $L^{-}(\mathrm{h};\varepsilon)$ *is nonempty and* $E(\mathrm{h};\varepsilon)$ *is empty,*

(2) *if* $\mathrm{Im}(\mathrm{h})$ *is a chain and for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $L^{-}(\mathrm{h};\varepsilon)$ *of* A *is a prime UP-subalgebra (resp. prime UP-filter, prime UP-ideal, prime strongly UP-ideal) of* A*, then* $\overline{\mathrm{h}}$ *is a prime hesitant fuzzy UP-subalgebra (resp. prime hesitant fuzzy UP-filter, prime hesitant fuzzy UP-ideal, prime hesitant fuzzy strongly UP-ideal) of* A*.*

Proof. It is straightforward by Theorem 1.22, 2.37 and 1.11. □

Theorem 2.41. *Let* h *be a hesitant fuzzy set on* A*. Then the following statements hold:*

(1) *if* $\mathrm{Im}(\mathrm{h})$ *is a chain and* $\overline{\mathrm{h}}$ *is a weakly prime hesitant fuzzy set on* A*, then for all* $\varepsilon \in \mathcal{P}([0,1])$*,* $L^{-}(\mathrm{h};\varepsilon)$ *is a weakly prime subset of* A *if* $L^{-}(\mathrm{h};\varepsilon)$ *is nonempty,*

(2) *if* $\mathrm{Im}(\mathrm{h})$ *is a chain and for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $L^{-}(\mathrm{h};\varepsilon)$ *of* A *is a weakly prime subset of* A*, then* $\overline{\mathrm{h}}$ *is a weakly prime hesitant fuzzy set on* A*.*

Proof. (1) Assume that $\mathrm{Im}(\mathrm{h})$ is a chain and $\overline{\mathrm{h}}$ is a weakly prime hesitant fuzzy set on A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $L^{-}(\mathrm{h};\varepsilon) \neq \varnothing$. Let $x, y \in A$ be such that $x \neq y$ and $x \cdot y \in L^{-}(\mathrm{h};\varepsilon)$ and assume that $x \notin L^{-}(\mathrm{h};\varepsilon)$. Then $\mathrm{h}(x \cdot y) \subset \varepsilon$ and $\mathrm{h}(x) \not\subset \varepsilon$. Since $\overline{\mathrm{h}}$ is a weakly prime hesitant fuzzy set on A, we have $\overline{\mathrm{h}}(x \cdot y) \subseteq \overline{\mathrm{h}}(x) \cup \overline{\mathrm{h}}(y)$. By Lemma 2.22 (2), we have $[0,1] - \mathrm{h}(x \cdot y) \subseteq ([0,1] - \mathrm{h}(x)) \cup ([0,1] - \mathrm{h}(y)) = [0,1] - (\mathrm{h}(x) \cap \mathrm{h}(y))$. Thus $\varepsilon \supset \mathrm{h}(x \cdot y) \supseteq \mathrm{h}(x) \cap \mathrm{h}(y)$.

If $\mathrm{h}(x) \subseteq \mathrm{h}(y)$, then $\varepsilon \supset \mathrm{h}(x) \cap \mathrm{h}(y) = \mathrm{h}(x)$. Thus $x \in L^{-}(\mathrm{h};\varepsilon)$, a contradiction.

If $\mathrm{h}(x) \supseteq \mathrm{h}(y)$, then $\varepsilon \supset \mathrm{h}(x) \cap \mathrm{h}(y) = \mathrm{h}(y)$. Thus $y \in L^{-}(\mathrm{h};\varepsilon)$.

So, $L^{-}(\mathrm{h};\varepsilon)$ is a weakly prime subset of A.

(2) Assume that $\mathrm{Im}(\mathrm{h})$ is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$. a nonempty subset $L^{-}(\mathrm{h};\varepsilon)$ of A is a weakly prime subset of A. Assume that there exist $x, y \in A$ such that $x \neq y$ and $\overline{\mathrm{h}}(x \cdot y) \nsubseteq \overline{\mathrm{h}}(x) \cup \overline{\mathrm{h}}(y)$. Since $\mathrm{Im}(\mathrm{h})$ is a chain, we have $\overline{\mathrm{h}}(x \cdot y) \supset \overline{\mathrm{h}}(x) \cup \overline{\mathrm{h}}(y)$. By Lemma 2.22 (2), we have $[0,1] - \mathrm{h}(x \cdot y) \supset ([0,1] - \mathrm{h}(x)) \cup ([0,1] - \mathrm{h}(y)) = [0,1] - (\mathrm{h}(x) \cap \mathrm{h}(y))$. Then $\mathrm{h}(x \cdot y) \subset \mathrm{h}(x) \cap \mathrm{h}(y)$. Choose $\varepsilon = \mathrm{h}(x) \cap \mathrm{h}(y) \in \mathcal{P}([0,1])$. Then $\mathrm{h}(x \cdot y) \subset \varepsilon$. Thus $x \cdot y \in L^{-}(\mathrm{h};\varepsilon) \neq \varnothing$. By assumption, we have $L^{-}(\mathrm{h};\varepsilon)$ is a weakly prime subset of A and thus $x \in L^{-}(\mathrm{h};\varepsilon)$ or $y \in L^{-}(\mathrm{h};\varepsilon)$. So $\mathrm{h}(x) \subset \varepsilon = \mathrm{h}(x) \cap \mathrm{h}(y)$ or $\mathrm{h}(y) \subset \varepsilon = \mathrm{h}(x) \cap \mathrm{h}(y)$, a contradiction. Hence, $\overline{\mathrm{h}}(x \cdot y) \subseteq \overline{\mathrm{h}}(x) \cup \overline{\mathrm{h}}(y)$, for all $x, y \in A$. Therefore, $\overline{\mathrm{h}}$ is a weakly prime hesitant fuzzy set on A. □

Theorem 2.42. *Let* h *be a hesitant fuzzy set on* A*. Then the following statements hold:*

(1) *if* $\mathrm{Im}(\mathrm{h})$ *is a chain and* $\overline{\mathrm{h}}$ *is a weakly prime hesitant fuzzy UP-subalgebra (resp. weakly prime hesitant fuzzy UP-filter, weakly prime hesitant fuzzy UP-ideal, weakly prime hesitant fuzzy strongly UP-ideal) of* A*, then for all* $\varepsilon \in \mathcal{P}([0,1])$*,* $L^{-}(\mathrm{h};\varepsilon)$ *is a weakly prime UP-subalgebra (resp. weakly prime UP-filter, weakly prime UP-ideal, weakly prime strongly UP-ideal) of* A *if* $L^{-}(\mathrm{h};\varepsilon)$ *is nonempty and* $E(\mathrm{h};\varepsilon)$ *is empty,*

(2) *if* $\mathrm{Im}(\mathrm{h})$ *is a chain and for all* $\varepsilon \in \mathcal{P}([0,1])$, *a nonempty subset* $L^{-}(\mathrm{h};\varepsilon)$ *of* A *is a weakly prime UP-subalgebra (resp. weakly prime UP-filter, weakly prime UP-ideal, weakly prime strongly UP-ideal) of* A, *then* $\overline{\mathrm{h}}$ *is a weakly prime hesitant fuzzy UP-subalgebra (resp. weakly prime hesitant fuzzy UP-filter, weakly prime hesitant fuzzy UP-ideal, weakly prime hesitant fuzzy strongly UP-ideal) of* A.

Proof. It is straightforward by Theorem 2.31 (resp. Theorem 2.33, Theorem 2.35, Theorem 2.37) and 2.41. □

Acknowledgements. The authors wish to express their sincere thanks to the referees for the valuable suggestions which lead to an improvement of this paper.

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