Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 14, No. 3, (September 2017), pp. 265–278
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Dual hesitant fuzzy aggregation operators based on Bonferroni means and their applications to multiple attribute decision making

Huonian Tu, Chunyong Wang, Xiaoqiang Zhou, Shengda Tao

Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 14, No. 3, September 2017

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 14, No. 3, (September 2017), pp. 265–278
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Dual hesitant fuzzy aggregation operators based on Bonferroni means and their applications to multiple attribute decision making

Huonian Tu, Chunyong Wang, Xiaoqiang Zhou, Shengda Tao

Received 16 September 2016; Revised 19 December 2016; Accepted 25 January 2017

Abstract. In this paper, we investigate the multiple attribute decision making problems with dual hesitant fuzzy information. Motivated by the idea of Bonferroni mean and Choquet integral, we develop the aggregation techniques called the dual hesitant fuzzy choquet ordered Bonferroni mean operator for aggregating the dual hesitant fuzzy information. We research its properties and discuss its special cases. We also apply the newly defined operator to deal with multiple attribute decision making problems under dual hesitant fuzzy environment. Finally, an illustrative example is given to show the developed method and demonstrate its practicality and effectiveness.

2010 AMS Classification: 68T37, 94D05

Keywords: Multiple attribute decision making, Dual hesitant fuzzy set, Bonferroni mean, Choquet integral.

Corresponding Author: Huonian Tu (tuhuonian2015@163.com)

1. Introduction

Multiple attribute decision making (MADM) handles decision situations where a set of alternatives have to be assessed against multiple attributes before a final choice is selected. It can be used in many fields such as economics, management and engineering. There are two basic topics in MADM: one topic is that how the decision makers express their assessments, the other is how these assessments are aggregated.

In many decision making problems, it is difficult for a decision maker to give his assessments in crisp values due to ambiguity and incomplete information. Instead, it has become popular that these assessments are presented by fuzzy set or extensions of fuzzy set. Fuzzy set (FS), proposed by Zadeh, is a powerful tool to deal with vagueness and has received much attention [7, 11, 12]. Atanassov [2] generalized FS to intuitionistic fuzzy set (IFS), in which each element is expressed by an ordered pair denoting a membership degree and a non-membership degree. Then Torra and Narukawa [13, 14] generalized FS to hesitant fuzzy set which permits the membership having a set of possible values. Following this way, by giving different values to the membership degree or the non-membership degree of an element, several other famous extensions of FS have been developed, such as type-2 fuzzy sets (T2FSs) [34], type n fuzzy sets (TnFSs), fuzzy multisets (FMSs) [10, 26], interval-valued intuitionistic fuzzy sets (IVIFSs) [1].

Among the aggregation operators, the average mean (AM) and the geometric mean (GM) are two basic kinds. The AM and GM have been extended extensively. For example, in order to reorder the arguments before being aggregated, the ordered weighted averaging (OWA) operator[27] and the ordered weighted geometric (OWG) operator [6, 21] were proposed. For the continuous interval valued fuzzy information, Yager [29] developed a continuous ordered weighted averaging (C-OWA) operator. Then, Yager and Xu [33] further proposed the continuous ordered weighted geometric (C-OWG) operator. The AM and GM were also extended to linguistic fuzzy information, such as the linguistic weighted averaging (LWA) operator [19], the linguistic ordered weighted averaging (LOWA) operator [20], the linguistic weighted geometric averaging (LWGA) operator [18] and the linguistic ordered weighted geometric averaging (LOWGA) operator [18]. In order to deal with the aggregated arguments which were correlative, the power average (PA) operator [28], the power geometric (PG) operator [25] and the choquet ordered average operator [30] were introduced. As Bonferroni mean (BM) [3] can capture the interrelationship between input arguments, it has also been applied to construct aggregation operators. For example, Xu and Yager [24] investigated the BM under intuitionistic fuzzy environment. Xia et al. [23] proposed the geometric Bonferroni mean (GBM). Furthermore, Zhu et al. [36] developed the hesitant fuzzy geometric Bonferroni means.

Recently, Zhu et al. [35] introduced a dual hesitant fuzzy sets (DHFSs) which is another new extension of FSs. It is a comprehensive set containing FSs, IFSs, FMSs and HFSs as special cases under certain conditions. By several possible values for the membership and nonmembership degrees respectively, DHFSs can take much more information given by decision makers into account in multiple attribute decision making. In their work, some basic operations and properties for DHFSs were investigated. Then Wang et al. [17] investigated the multiple attribute decision making problem based on the aggregation operators with dual hesitant fuzzy information. They also developed some aggregation operators for aggregating dual hesitant fuzzy information including dual hesitant fuzzy weighted average (DHFWA) operator, dual hesitant fuzzy weighted geometric (DHFWG) operator, dual hesitant fuzzy ordered weighted average (DHFOWA) operator, dual hesitant fuzzy ordered weighted geometric (DHFOWG) operator, dual hesitant fuzzy hybrid average (DHFHA) operator and dual hesitant fuzzy hybrid geometric (DHFHG) operator. Ye proposed a correlation coefficient [31] and a cross-entropy measure [32] for DHFSs, then applied them to multiple attribute decision making under dual hesitant fuzzy environments. However, the existing dual hesitant fuzzy aggregation operators above only consider situations where all the attributes in the dual hesitant fuzzy set are independent. Nevertheless, attributes in DHFSs are usually correlative in real life. Fortunately, the Choquet integral [4] can characterize the correlations of the decision data. Motivated by BM and Choquet integral, we propose a dual hesitant fuzzy choquet ordered Bonferroni mean (DHFCOBM) operator, whose prominent characteristic is that it can consider both the interactions of the attributes and the correlations of the input arguments. It is worth mentioning that DHFCOBM can be regarded as an extension of DHFWA and DHFOWA.

To facilitate our discussion, the remainder of this paper is organized as follows. Some basic concepts related to dual hesitant fuzzy sets are introduced in the next section. In Section 3, we propose a family of dual hesitant fuzzy aggregation operators based on Bonferroni means, and then develop new approaches to multiple attribute decision making problems based on these new operators. An illustrative example is also given to show the effectiveness of the developed approach. We conclude the paper and give some remarks in Section 4.

2. Preliminaries

2.1. **Dual hesitant fuzzy sets.**

Definition 2.1 ([13])**.** Let X be a reference set. Then we define hesitant fuzzy set on X in terms of a function that when applied to X returns a sunset of $[0,1]$.

To be easily understood, Xia and Xu [22] express the HFS by a mathematical symbol: $E = (\langle x, h_E(x)\rangle | x \in X)$, where $h_E(x)$ is a set of some values in $[0,1]$, denoting the possible membership degree of the element $x \in X$ to the set E. For convenience, Xia and Xu [22] call $h = h_E(x)$ a hesitant fuzzy element (HFE) and H the set of all HFEs when there is no confusion.

Nevertheless, HFSs only considers the membership degree of the element $x \in X$ to the set E and ignores the non-membership degree. In order to overcome this difficult, Zhu et al. [35] generalized HFSs to DHFS.

Definition 2.2 ([35])**.** Let X be a fixed set, then a dual hesitant fuzzy set D on X is defined as

$$D = \{\langle x, h(x), g(x)\rangle | x \in X\}$$

in which $h(x)$ and $g(x)$ are two sets of some values in $[0,1]$, denoting the possible membership degrees and nonmembership degrees of the element $x \in X$ to the set D, respectively, with conditions: $0 \leq \gamma, \eta \leq 1$ and $0 \leq \gamma^+ + \eta^+ \leq 1$, where $\gamma \in h(x)$, $\eta \in g(x)$, $\gamma^+ \in h^+(x) = \cup_{\gamma \in h(x)} max\{\gamma\}$, and $\eta^+ \in g^+(x) = \cup_{\eta \in g(x)} max\{\eta\}$ for $\forall x \in X$. For convenience, the pair $d(x) = \{h(x), g(x)\}$ is called a dual hesitant fuzzy element (DHFE) and denoted by $d = \{h, g\}$.

In order to compare two dual hesitant fuzzy elements, corresponding score function is defined as follows.

Definition 2.3 ([35])**.** Let $d_1 = \{h_1, g_1\}$ and $d_2 = \{h_2, g_2\}$ be any two DHFEs. Then the score function of $d_i\,(i = 1, 2)$ is

$$S(d_i) = \frac{1}{n(h_i)} \sum_{\gamma \in h_i} \gamma - \frac{1}{n(g_i)} \sum_{\eta \in g_i} \eta \,(i = 1, 2)$$

and the accuracy function of $d_i\,(i=1,2)$ is

$$P(d_i)=\frac{1}{n(h_i)}\sum_{\gamma\in h_i}\gamma+\frac{1}{n(g_i)}\sum_{\eta\in g_i}\eta\,(i=1,2)$$

where $n(h_i)$ and $n(g_i)$ are the numbers of the elements in h_i and g_i, respectively. Then

(i) if $S(d_1)>S(d_2)$, then d_1 is superior to d_2, denoted by $d_1\succ d_2$;
(ii) if $S(d_1)=S(d_2)$, then
 (1) if $P(d_1)=P(d_2)$, then d_1 is equivalent to d_2, denoted by $d_1\sim d_2$;
 (2) if $P(d_1)>P(d_2)$, then d_1 is superior to d_2, denoted by $d_1\succ d_2$.

Besides, some new operations on the DHFEs d, d_1 and d_2 are also introduced in [35]:

(i) $d^\lambda=\cup_{\gamma\in h,\eta\in g}\{\{\gamma^\lambda\},\{1-(1-\eta)^\lambda\}\},\lambda>0$,
(ii) $\lambda d=\cup_{\gamma\in h,\eta\in g}\{\{1-(1-\gamma)^\lambda\},\{\mu^\lambda\}\},\lambda>0$,
(iii) $d_1\oplus d_2=\cup_{\gamma_1\in h_1,\gamma_2\in h_2,\eta_1\in g_1,\eta_2\in g_2}\{\{\gamma_1+\gamma_2-\gamma_1\gamma_2\},\{\eta_1\eta_2\}\}$,
(iv) $d_1\otimes d_2=\cup_{\gamma_1\in h_1,\gamma_2\in h_2,\eta_1\in g_1,\eta_2\in g_2}\{\{\gamma_1\gamma_2\},\{\eta_1+\eta_2-\eta_1\eta_2\}\}$.

2.2. **Choquet integral and Bonferroni mean.**

Definition 2.4 ([9])**.** A fuzzy measure μ on the set X is a set function $\mu:\theta(X)\to[0,1]$ satisfying the following axioms and $\theta(X)$ is the set of all subsets of X:

(i) $\mu(\phi)=0,\mu(X)=1$,
(ii) $A\subseteq B$ implies $\mu(A)\le\mu(B)$, for all $A,B\subseteq X$,
(iii) $\mu(A\cup B)=\mu(A)+\mu(B)+\rho\mu(A)\mu(B)$, for all $A,B\subseteq X$ and $A\cap B=\phi$, where $\rho\in(-1,\infty)$.

Especially, if $\rho=0$, then the condition (iii) reduces to the axiom of additive measure: $\mu(A\cup B)=\mu(A)+\mu(B)$, for all $A,B\subseteq X$ and $A\cap B=\phi$. If all the elements in X are independent, then we have

$$\mu(A)=\sum_{x_i\in A}\mu(\{x_i\}),\forall A\subseteq X.$$

The discrete Choquet integral is a linear expression up to a reordering of the elements, which is defined as below.

Definition 2.5 ([15])**.** Let f be a positive real-valued function on X, and μ be a fuzzy measure on X. The discrete Choquet integral of f with respect to μ is defined by

$$C_\mu(f)=\Sigma_{i=1}^n f_{\sigma(i)}\left[\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)})\right].$$

where $(\sigma(1),\sigma(2),\cdots,\sigma(n))$ is a permutation of $(1,2,\cdots,n)$, such that $f_{\sigma(i-1)}\ge f_{\sigma(i)}$ for all $i=2,3,\cdots,n$, $A_{\sigma(k)}=\{x_{\sigma(j)}\mid j\le k\}$, for $k\ge1$, and $A_{\sigma(0)}=\phi$.

As an extension of the arithmetic average, the Bonferroni mean (BM) is a very practical aggregation operator, which considers the interrelationships among arguments.

Definition 2.6 ([3]). Let $p, q \geq 0$, and $a_i (i = 1, 2, \cdots, n)$ be a collection of non-negative numbers. If

$$B^{p,q}(a_1, a_2, \cdots, a_n) = \left(\frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^{n} a_i^p a_j^q \right)^{\frac{1}{p+q}},$$

then $B^{p,q}$ is called a Bonferroni mean (BM).

Particularly, if $q = 0$,the BM degenerates to the generalized mean operator [8] as the following:

$$B^{p,0}(a_1, a_2, \cdots, a_n) = \left(\frac{1}{n} \sum_{i,j=1}^{n} a_i^p \right)^{\frac{1}{p}}.$$

Further, if $p = 1$ and $q = 0$, then BM reduces to the well-known average mean:

$$B^{1,0}(a_1, a_2, \cdots, a_n) = \frac{1}{n} \sum_{i,j=1}^{n} a_i.$$

3. Major Section

Inspired by the Definition 2.6, we can define the dual hesitant fuzzy Bonferroni mean as follows:

Definition 3.1. Let d_j $(j = 1, 2, \cdots, n)$ be a collection of DHFEs and $p, q > 0$, then we define the the dual hesitant fuzzy Bonferroni mean (DHFBM) operator as

$$DHFBM^{p,q}(d_1, d_2, \cdots, d_n) = \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1; i \neq j}^{n} (d_i^p \otimes d_j^q) \right)^{\frac{1}{p+q}}.$$

According to the operational laws of DHFEs, we can drive the theorem below.

Theorem 3.2. *Let $d_j = \{h_j, g_j\}$ $(j = 1, 2, \cdots, n)$ be a collection of DHFEs. Then their aggregated value by using the DHFBM operator is also a DHFE, and*

$$DHFBM^{p,q}(d_1, d_2, \cdots, d_n) = \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1; i \neq j}^{n} (d_i^p \otimes d_j^q) \right)^{\frac{1}{p+q}}$$

$$= \bigcup_{\gamma_i \in h_i \eta_i \in g_i} \left\{ \left\{ \left(1 - \prod_{i \neq j}^{n} (1 - \gamma_i^p \gamma_j^q)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right\}, \left\{ 1 - \left(1 - \prod_{i \neq j}^{n} (1 - (1 - \eta_i)^p (1 - \eta_j)^q)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right\} \right\}$$

Proof. According to the operational laws of DHFEs, we can get

$$d_i^p = \bigcup_{\gamma_i \in h_i, \eta_i \in g_i} \{\{\gamma_i^p\}, \{1 - (1 - \eta_i)^p\}\}, d_j^q = \bigcup_{\gamma_j \in h_j, \eta_j \in g_j} \{\{\gamma_j^q\}, \{1 - (1 - \eta_j)^q\}\}$$

and

$$d_i^p \otimes d_j^q = \bigcup_{\gamma_i \in h_i, \gamma_j \in h_j, \eta_i \in g_i, \eta_j \in g_j} \{\{\gamma_i^p \gamma_j^q\}, \{1 - (1 - \eta_i)^p (1 - \eta_j)^q\}\}.$$

Then we acquire

$$\bigoplus_{i,j=1;i\neq j}^{n}(d_i^p\otimes d_j^q)$$
$$=\bigcup_{\gamma_i\in h_i,\gamma_j\in h_j,\eta_i\in g_i,\eta_j\in g_j}\left\{\left\{1-\prod_{i\neq j}^{n}\left(1-\gamma_i^p\gamma_j^q\right)\right\},\left\{\prod_{i\neq j}^{n}(1-(1-\eta_i)^p(1-\eta_j)^q)\right\}\right\}.$$

Thus

$$\frac{1}{n(n-1)}\bigoplus_{i,j=1;i\neq j}^{n}(d_i^p\otimes d_j^q)$$
$$=\bigcup_{\gamma_i\in h_i,\gamma_j\in h_j,\eta_i\in g_i,\eta_j\in g_j}\left\{\left\{1-\prod_{i\neq j}^{n}(1-\gamma_i^p\gamma_j^q)^{\frac{1}{n(n-1)}}\right\},\left\{\prod_{i\neq j}^{n}(1-(1-\eta_i)^p(1-\eta_j)^q)^{\frac{1}{n(n-1)}}\right\}\right\}.$$

Finally, we complete our proof by computing $(\frac{1}{n(n-1)}\bigoplus_{i,j=1;i\neq j}^{n}(d_i^p\otimes d_j^q))^{\frac{1}{p+q}}$. □

Obviously, the DHFBM operator has the following properties.

Theorem 3.3. *(Quasi-Idempotency). Let $d_j=\{h_j,g_j\}\,(j=1,2,\cdots,n)$ be a collection of DHFEs. If $d_1=d_2=\cdots=d_n=d=\{h,g\}=\{\{\gamma\},\{\eta\}\}$, then $DHFBM^{p,q}(d,d,\cdots,d)=d$.*

By Theorem 3.2, the proof is straightforward and we omit it here.

Theorem 3.4. *(Quasi-Monotonicity). Let $d_j=\{h_j,g_j\}(j=1,2,\cdots,n)$ be a collection of DHEEs, where $h_j=\{\gamma_{j1},\gamma_{j2},\cdots,\gamma_{jk}\},g_j=\{\eta_{j1},\eta_{j2},\cdots,\eta_{js}\}(j=1,2,\cdots,n)$. Let $d'_j=\{h'_j,g'_j\}(j=1,2,\cdots,n)$ be another collection of DHEEs, where $h'_j=\{\gamma'_{j1},\gamma'_{j2},\cdots,\gamma'_{jk}\},g'_j=\{\eta'_{j1},\eta'_{j2},\cdots,\eta'_{js}\}(j=1,2,\cdots,n)$. If $\gamma_{jp}\leq\gamma'_{jp},\eta_{jq}\geq\eta'_{jq},\forall p=1,2,\cdots,k,\forall q=1,2,\cdots,s$, then*

$$DHFBM^{p,q}(d_1,d_2,\cdots,d_n)\leq DHFBM^{p,q}(d'_1,d'_2,\cdots,d'_n).$$

Proof. Since $\gamma_{jp}\leq\gamma'_{jp},\eta_{jq}\geq\eta'_{jq},\forall p=1,2,\cdots,k,\forall q=1,2,\cdots,s$, we have

$$\left(1-\prod_{i,j=1;i\neq j}^{n}(1-\gamma_i^p\gamma_j^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\leq\left(1-\prod_{i,j=1;i\neq j}^{n}(1-{\gamma_i'}^{p}{\gamma_j'}^{q})^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},$$

$$1-\left(1-\prod_{i,j=1;i\neq j}^{n}(1-(1-\eta_i)^p(1-\eta_j)^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$
$$\geq 1-\left(1-\prod_{i,j=1;i\neq j}^{n}\left(1-(1-\eta_i)'^{p}(1-\eta_j)'^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}.$$

Then, by Theorem 3.2 and Definition 2.3, we complete the proof. □

Theorem 3.5. *(Commutativity). Let $d_j (j = 1, 2, \cdots, n)$ be a collection of DHFEs, and $(\widetilde{d}_1, \widetilde{d}_2, \cdots, \widetilde{d}_n)$ be any permutation of $(d_1, d_2, \cdots, d_n)$. Then*

$$DHFBM^{p,q}(d_1, d_2, \cdots, d_n) = \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1;i\neq j}^{n} (d_i^p \otimes d_j^q) \right)^{\frac{1}{p+q}}$$

$$= \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1;i\neq j}^{n} (\widetilde{d}_i^p \otimes \widetilde{d}_j^q) \right)^{\frac{1}{p+q}} = DHFBM^{p,q}(\widetilde{d}_1, \widetilde{d}_2, \cdots, \widetilde{d}_n).$$

When we change the parameters p and q of the DHFBM, we can get some special cases as below.

Case 1. If $q \to 0$, then by Theorem 3.2, we have

$$\lim_{q\to 0} DHFBM^{p,q}(d_1, d_2, \cdots, d_n)$$

$$= \bigcup_{\gamma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ \left(1 - \prod_{i=1}^{n} (1-\gamma_i^p)^{\frac{1}{n}} \right)^{\frac{1}{p}} \right\}, \left\{ 1 - \left(1 - \prod_{i=1;}^{n} (1-(1-\eta_i)^p)^{\frac{1}{n}} \right)^{\frac{1}{p}} \right\} \right\}$$

$$= (\frac{1}{n} \bigoplus_{i=1}^{n} d_i^p)^{\frac{1}{p}} = DHFBM^{p,0}(d_1, d_2, \cdots, d_n)$$

which we call the dual hesitant fuzzy generalized mean (DHFGM) operator.

Case 2. If $p = 2, q \to 0$, then the DHFGM reduces to

$$DHFBM^{2,0}(d_1, d_2, \cdots, d_n) = (\frac{1}{n} \bigoplus_{i=1}^{n} d_i^2)^{\frac{1}{2}}$$

which we call the dual hesitant fuzzy square mean (DHFSM) operator.

Case 3. If $p = 1, q \to 0$, then the DHFGM degenerates to

$$DHFBM^{1,0}(d_1, d_2, \cdots, d_n) = \frac{1}{n} \bigoplus_{i=1}^{n} d_i$$

which we call the dual hesitant fuzzy mean (DHFM) operator.

Case 4. If $p = q = 1$, then by Theorem 3.2, we obtain

$$DHFBM^{1,1}(d_1, d_2, \cdots, d_n) = \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1;i\neq j}^{n} (d_i \otimes d_j) \right)^{\frac{1}{2}}$$

which we call the dual hesitant fuzzy interrelated square mean (DHFISM) operator.

In practical decision, we have to deal with complicated situations considering both the relations among individual arguments and the importance of them. In what follows, we develop some weighted dual hesitant fuzzy aggregations based on BM.

In some practical applications, we have to consider not only the importance of individual arguments but also the relations among attributes. Then, by giving weights to each attribute, we can develop the dual hesitant fuzzy weighted Bonferroni mean as below.

Definition 3.6. Let $d_i(i = 1, 2, \cdots, n)$ be a collection of DHFEs, $p, q \geq 0$, and $w = (w_1, w_2, \cdots, w_n)^T$ be the weight of d_i, where w_i denotes the importance degree of d_i, satisfying $w_i > 0$ and $\sum_{i=1}^{n} = 1$. Then

$$DHFWBM^{p,q}(d_1, d_2, \cdots, d_n) = \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1; i \neq j}^{n} (w_i d_i^p \otimes w_j d_j^q) \right)^{\frac{1}{p+q}}$$

is called the dual hesitant fuzzy weighted Bonferroni mean (DHFWBM).

Remark 3.7. Let $d_i = \{h_i, g_i\}$, suppose there is only one fuzzy value in each h_i and g_i $(i = 1, 2, \cdots, n)$, then $DHFWBM^{1,0}(d_1, d_2, \cdots, d_n) = \frac{1}{n} \bigoplus_{i=1}^{n} (w_i d_i) = \frac{1}{n} DHFWA(d_1, d_2, \cdots, d_n)$. That is to say, the DHFWBM can reduce to DHFWA [17].

Sometimes, we may need to weight the ordered positions of the dual hesitant fuzzy arguments instead of weighting the arguments themselves. In this case, we can develop the ordered weighted operators as follows.

Definition 3.8. Let $d_i(i = 1, 2, \cdots, n)$ be a collection of DHFEs, $p, q \geq 0$, and $w = (w_1, w_2, \cdots, w_n)^T$ be the associated weight vector such that $w_i > 0$ and $\sum_{i=1}^{n} = 1$. $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $1, 2, \cdots, n$,such that $d_{\sigma(j-1)} \geq d_{\sigma(j)}$ for all $j = 2, 3, \cdots, n$. Then

$$DHFOWBM^{p,q}(h_1, h_2, \cdots, h_n) = \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1, i \neq j}^{n} (w_i d_{\sigma(i)}^p \otimes w_j d_{\sigma(j)}^q) \right)^{\frac{1}{p+q}}$$

are called the dual hesitant fuzzy ordered weighted Bonferroni mean (DHFOWBM).

Remark 3.9. Let $d_i = \{h_i, g_i\}$, suppose there is only one fuzzy value in each h_i and g_i $(i = 1, 2, \cdots, n)$, then $DHFOWBM^{1,0}(d_1, d_2, \cdots, d_n) = \frac{1}{n} \bigoplus_{i=1}^{n} (w_i h_{\sigma(i)}) = \frac{1}{n} DHFOWA(d_1, d_2, \cdots, d_n)$. That is to say, the DHFOWBM can reduce to DHFOWA [17].

If we want to not only weight the dual hesitant fuzzy arguments but also weight the ordered positions of the dual hesitant fuzzy arguments, we can propose the following hybrid average operators.

Definition 3.10. Let $d_i(i = 1, 2, \cdots, n)$ be a collection of DHFEs, $p, q \geq 0$, and $w = (w_1, w_2, \cdots, w_n)^T$ be the associated weight vector such that $w_i > 0$ and $\sum_{i=1}^{n} = 1$. $\widetilde{d}_{\sigma(j)}$ is the j-th largest element of the dual hesitant fuzzy arguments ($\widetilde{d}_j = (nw_j)d_j, j = 1, 2, \cdots, n$). Then, we call

$$DHFHBM^{p,q}(d_1, d_2, \cdots, d_n) = \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1, i \neq j}^{n} (w_i \widetilde{d}_{\sigma(i)}^p \otimes w_j \widetilde{d}_{\sigma(j)}^q) \right)^{\frac{1}{p+q}}$$

the dual hesitant fuzzy hybrid Bonferroni mean (DHFHBM).

Remark 3.11. Let $d_i = \{h_i, g_i\}$, suppose there is only one fuzzy value in each h_i and g_i $(i = 1, 2, \cdots, n)$, then $DHFHBM^{1,0}(d_1, d_2, \cdots, d_n) = \frac{1}{n}\left(\bigoplus_{i=1}^{n}(w_i\widetilde{d}_{\sigma(i)})\right) = \frac{1}{n}DHFHA(d_1, d_2, \cdots, d_n)$. That is to say, DHFHBM can reduce to DHFHA [17].

However, the above aggregation operators are based on the assumption that the attributes are independent. In real decision making problems, these are usually interactions among attributes. As we all know, the Choquet integral [4] can depict the correlations of attributes. In what follows, we shall develop the dual hesitant fuzzy choquet ordered averaging operator based on the famous Choquet integral.

Definition 3.12. Let $d_i (i = 1, 2, \cdots, n)$ be a collection of DHFEs on X, μ be a fuzzy measure on X and $p, q \geq 0$. Then, we call

$$DHFCOBM_{\mu}^{p,q}(d_1, d_2, \cdots, d_n)$$
$$= \left(\frac{1}{n(n-1)}\bigoplus_{i,j=1,i\neq j}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)})d_{\sigma(i)}^{p} \otimes (\mu(A_{\sigma(j)})-\mu(A_{\sigma(j-1)})d_{\sigma(j)}^{q})\right)^{\frac{1}{p+q}}$$

the dual hesitant fuzzy choquet ordered Bonferroni mean (DHFCOBM), where $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $(1, 2, \cdots, n)$,such that $d_{\sigma(j-1)} \geq d_{\sigma(j)}$ for all $j = 2, 3, \cdots, n$, $A_{\sigma(k)} = \{x_{\sigma(j)} \mid j \leq k\}$, for $k \geq 1$, and $A_{\sigma(0)} = \phi$.

Remark 3.13. If $\mu(\{x_{\sigma(j)}\}) = \mu(\{A_{\sigma(j)}\}) - \mu(\{A_{\sigma(j-1)}\}), j = 1, 2, \cdots, n$, then DHFCOBM degenerates into DHFWBM. Let $w_j = \mu(\{A_{\sigma(j)}\}) - \mu(\{A_{\sigma(j-1)}\}), j = 1, 2, \cdots, n$, then DHFCOBM degenerates into DHFOWBM. In addition, suppose there is only onefuzzy value in each $h_i, g_i (i = 1, 2, \cdots, n)$ and let $p = 1, q = 0$, then $DHFCOBM_{\mu}^{1,0}(d_1, d_2, \cdots, d_n) = \frac{1}{n}\left(\bigoplus_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)})d_{\sigma(i)}\right) = \frac{1}{n}DHFCOA_{\mu}(d_1, d_2, \cdots, d_n)$. This is the so-called dual hesitant fuzzy choquet ordered averaging operator proposed by Wang et al. [16].

Next, we shall utilize the DHFCOBM operator to multiple attribute decision making under dual hesitant fuzzy environment. The following assumptions or notations are used to represent the MADM problems for evaluation of theses with dual hesitant fuzzy information. Let $A = \{A_1, A_2, \cdots, A_m\}$ be a discrete set of alternatives, and $G = \{G_1, G_2, \cdots, G_n\}$ be the state of nature. If the decision makers provide several values for the alternative A_i under the attribute G_j with anonymity, these values can be considered as a dual hesitant fuzzy element $d_{ij} = \{h_{ij}, g_{ij}\}$. Suppose that the decision matrix $D = (d_{ij})_{m\times n}$ is the dual hesitant fuzzy decision matrix, where $d_{ij} = \{h_{ij}, g_{ij}\}, (i = 1, 2, \cdots, m, j = 1, 2, \cdots, n)$ are in the form of DHFEs. In the following, we apply the DHFCOBM operator to the multiple attribute decision making problems for evaluation of theses with dual hesitant fuzzy information.

Step 1. Confirm the fuzzy measures μ of attributes of G and attributes sets of G.

Step 2. Utilize the decision information given in matrix D, and the DHFCOBM operator

$$\widetilde{d}_i = DHFCOBM_{\mu}^{p,q}(d_1, d_2, \cdots, d_n)$$
$$=\left(\frac{1}{n(n-1)}\bigoplus_{i,j=1,i\neq j}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)})d_{\sigma(i)}^{p} \otimes (\mu(A_{\sigma(j)})-\mu(A_{\sigma(j-1)})d_{\sigma(j)}^{q})\right)^{\frac{1}{p+q}}$$

to derive the overall preference values $\widetilde{d}_i(i=1,2,\cdots,m)$ of the alternative A_i.

Step 3. Calculate the scores $s(\widetilde{d}_i)(i=1,2,\cdots,m)$ of the overall dual hesitant fuzzy values $\widetilde{d}_i(i=1,2,\cdots,m)$ by Definition 2.3.

Step 4. Rank all the alternatives $A_i(i=1,2,\cdots,m)$ in accordance with the scores $s(\widetilde{d}_i)(i=1,2,\cdots,m)$ and select the best one(s).

Step 5. End.

Example 3.14. Here, we will present a numerical example (adapted from [5, 17]) to show evaluation of theses with dual hesitant fuzzy information in order to illustrate the proposed method. There are five theses $A_i\,(i=1,2,3,4,5)$, and we want to select the best one. Four attributes are selected by experts to evaluate the theses: (1) G_1 is the language of a thesis; (2) G_2 is the innovation; (3) G_3 is the rigor; (4) G_4 is the structure of the thesis. In order to avoid influence each other, the experts are required to evaluate the five theses $A_i\,(i=1,2,3,4,5)$ under the above four attributes in anonymity and the decision matrix $D=(d_{ij})_{5\times 4}$ is presented in Table 1, where $d_{ij}=\{h_{ij},g_{ij}\},(i=1,2,3,4,5,j=1,2,3,4)$ are in the form of DHFEs. The fuzzy measure of attribute $G_j\,(j=1,2,\cdots,4)$ and attribute sets of G are as follows: $\mu(G_1)=0.30$, $\mu(G_2)=0.35$, $\mu(G_3)=0.30$, $\mu(G_4)=0.22$, $\mu(G_1,G_2)=0.70$, $\mu(G_1,G_3)=0.60$, $\mu(G_1,G_4)=0.55$, $\mu(G_2,G_3)=0.50$, $\mu(G_2,G_4)=0.45$, $\mu(G_3,G_4)=0.40$, $\mu(G_1,G_2,G_3)=0.82$, $\mu(G_1,G_2,G_4)=0.87$, $\mu(G_1,G_3,G_4)=0.75$, $\mu(G_2,G_3,G_4)=0.60$, $\mu(G_1,G_2,G_3,G_4)=1.00$.

Table 1. Dual hesitant fuzzy decision matrix $\widetilde{D}$

	G_1	G_2	G_3	G_4
A_1	{{0.1, 0.2},{0.5}}	{{0.3},{0.2,0.5}}	{{0.3},{0.5}}	{{0.4},{0.2,0.3}}
A_2	{{0.4,0.7},{0.3}}	{{0.3, 0.5},{0.4}}	{{0.1},{0.6,0.7}}	{{0.2},{0.3, 0.4}}
A_3	{{0.6, 0.8},{0.2}}	{{0.4},{0.3, 0.5}}	{{0.6},{0.2}}	{{0.4, 0.7},{0.3}}
A_4	{{0.4},{0.2}}	{{0.6, 0.8},{0.2}}	{{0.5},{0.1}}	{{0.6},{0.2,0.3}}
A_5	{{0.4, 0.7},{0.1}}	{{0.3},{0.6}}	{{0.4},{0.2}}	{{0.3},{0.6}}

Next, we apply the developed approach to evaluate these theses with dual hesitant fuzzy information.

Step 1. We use the decision information given in matrix D, and the DHFCOBM operator to obtain the overall preference values $\widetilde{d}_i$ of the thesis $A_i\,(i=1,2,3,4,5)$. Take thesis A_1 for example, (take $p=q=1$), there are 4160 numbers in $\widetilde{d}_1$ and we omit them here. When assigning different values to the parameter p,q, we can obtain different dual hesitant fuzzy values.

Step 2. Calculate the scores $s(\widetilde{d}_i)\,(i=1,2,3,4,5)$ of the overall dual hesitant fuzzy values $\widetilde{d}_i\,(i=1,2,3,4,5)$ of the thesis A_i: $s(\widetilde{d}_1)=-0.720127$, $s(\widetilde{d}_2)=-0.698057$, $s(\widetilde{d}_3)=-0.526878$, $s(\widetilde{d}_4)=-0.503364$, $s(\widetilde{d}_5)=-0.627032$.

Step 3. Rank all the alternatives $A_i(i=1,2,\cdots,m)$ in accordance with the values of $s(\widetilde{d}_i)$: $A_4\succ A_3\succ A_5\succ A_2\succ A_1$. Note that $\succ$ means "preferred to". Thus, the best thesis is A_4.

Remark 3.15. In order to show the merit of the proposed method, we utilized some existing methods proposed by Wang [17] and Wang [16] to solve this illustrate example. For simplicity, we omit the calculation process and only list the results in Table 2 and Table 3.

Table 2. Scores for theses obtained by the existing operators. (Let $w = (0.22, 0.23, 0.15, 0.4)^T$)

operators	$s(\tilde{d}_1)$	$s(\tilde{d}_2)$	$s(\tilde{d}_3)$	$s(\tilde{d}_4)$	$s(\tilde{d}_5)$	*Ranking* $\succ$
DHFWA	-0.027916	-0.044882	0.303905	0.384994	0.0424899	A_4, A_3, A_5, A_1, A_2
DHFWG	-0.092636	-0.149213	0.248196	0.345339	-0.113935	A_4, A_3, A_1, A_5, A_2
DHFOWA	-0.118141	-0.12896	0.280061	0.36394	0.0788862	A_4, A_3, A_5, A_1, A_2
DHFOWG	-0.183517	-0.275715	0.214003	0.319019	-0.0754086	A_4, A_3, A_5, A_1, A_2
DHFCOA	-0.118141	0.0188229	0.34354	0.38197	0.169648	A_4, A_3, A_5, A_2, A_1

Table 3. Scores for theses obtained by new operator DHFCOBM.

new operators	$s(\tilde{d}_1)$	$s(\tilde{d}_2)$	$s(\tilde{d}_3)$	$s(\tilde{d}_4)$	$s(\tilde{d}_5)$	*Ranking* $\succ$
$DHFCOBM^{1,1}$	-0.720127	-0.698057	-0.526878	-0.503364	-0.627032	A_4, A_3, A_5, A_2, A_1
$DHFCOBM^{1,0}$	-0.712216	-0.664281	-0.504271	-0.465212	-0.579535	A_4, A_3, A_5, A_2, A_1
$DHFCOBM^{0,1}$	-0.712216	-0.664281	-0.504271	-0.465212	-0.579535	A_4, A_3, A_5, A_2, A_1
$DHFCOBM^{2,0}$	-0.514869	-0.434002	-0.236848	-0.196797	-0.328211	A_4, A_3, A_5, A_2, A_1

From Table 2 and Table 3, we can compare these methods as follows.

First, we find that the rankings in Table 2 are usually different from Table 3. The reason may be that there are interdependent phenomena among attributes or input arguments in this numerical example. For example, $\mu(G_1)+\mu(G_2)+\mu(G_3)+\mu(G_4) = 0.30 + 0.35 + 0.30 + 0.22 > 1 = \mu(G_1, G_2, G_3, G_4)$ also tells us that the attributes are correlative. The DHFCOBM operator can perform aggregation of attributes when they are correlative and it allows argument values to support each other in the aggregation process. However, the existing operators, such as DHFWA and DHFOWG, always suppose that the attributes are independent, and each attribute is given a fixed weight subjectively. So the DHFCOBM operator is a better choice here.

Second, with the aid of fuzzy measure μ in the DHFCOBM operator, we can define a weight on not only each attribute but also each combination of attributes. During the calculation, the weight vectors can be obtained by the source decision information automatically. However, for other operators such as DHFOWA and DHFWG, the weight vectors must be given by experts in advance. Thus, our method is more reasonable and objective. Compare DHFCOBM with DHFCOA in Table 2 and Table 3, the rankings are the same which indicates that both DHFCOBM and DHFCOA are equipped with Choquet integral.

Third, the DHFCOBM operator can accommodate situations in which the input arguments are dual hesitant fuzzy information. As dual hesitant fuzzy set is a comprehensive set containing FSs, IFSs, FMSs and HFSs as special cases, our method can be widely used.

Fourth, the DHFCOBM operator has additional parameters p and q which control the power. In Table 3, the scores vary with parameters p and q, which make decision making more flexible and can meet the different needs of different decision makers. That is to say, the decision makers can choose the value of the parameters according to their preferences.

4. Conclusions

In this paper, we have investigated the multiple attribute decision making problem based on the DHFCOBM operator with dual hesitant fuzzy information. Firstly, some operational laws of dual hesitant fuzzy elements and score function of dual hesitant fuzzy elements have been introduced. Then, motivated by the ideal of Bonferroni mean and Choquet integral, the dual hesitant fuzzy choquet ordered Bonferroni mean (DHFCOBM) operator has been developed. Its advantage is that it can consider not only the importance of the attributes but also the correlation among the input arguments, which makes it more feasible and practical. At the same time we have introduced several aggregation operators for DHFSs based on BM, and discussed their basic relationships. As different parameters can be chosen in these aggregation operators, the decision becomes more flexible. Next, we have applied the DHFCOBM operator to multiple attribute decision making problems with dual hesitant fuzzy information. Finally, an illustrative example for evaluation of theses has been given to demonstrate its practicality and effectiveness. In the future, we will apply the dual hesitant fuzzy multiple attribute decision making to other domains.

Acknowledgements. We would like to thank the editor and the anonymous referees for their valuable comments, which have been very helpful in improving the paper. This work was supported by the doctor's scientific research foundation of Hezhou University (No.HZUBS201505), Guangxi University of Finance and Economics (No.2015XK30), China Postdoctoral Science Foundation(No.2015M572482), Young project of Hunan Provincial Education Department(No.15B104) and National Social Science Fund(No. 16XTJ002).

References

[1] K. T. Atanassov and G. Gargov, Interval valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 31 (1989) 343–349.
[2] K. T. Atanassov and P. Rangasamy, Intuitionistic Fuzzy Sets, Fuzzy Sets and Systems 20 (1986) 87–96.
[3] C. Bonferroni, Sulle medie multiple di potenze, Boll.un.mat.ital. 5 (3) (1950) 267–270.
[4] G. Choquet, Theory of capacities, Annales-Institut Fourier 5 (1954) 131–295.
[5] T. Y. Chen, Optimistic and pessimistic decision making with dissonance reduction using interval-valued fuzzy sets, Information Sciences An International Journal 181 (3) (201) 479–502.
[6] F. Chiclana, The ordered weighted geometric operator: Properties and application (2000) 985–991.
[7] V. Derhami, Similarity of learned helplessness in human being and fuzzy reinforcement learning algorithms, Journal of Intelligent and Fuzzy Systems 24 (24) (2013) 347–354.
[8] H. Dyckhoff and W. Pedrycz, Generalized means as model of compensative connectives, Fuzzy Sets and Systems 14 (1984) 143–154.
[9] M. Grabisch, M. Sugeno and T. Murofushi, Fuzzy Measures and Integrals: Theory and Applications, Sringer 2000.
[10] S. Miyamoto, Multisets and Fuzzy Multisets, Springer 2000.
[11] A. F. Meyabadi and M. Ehsan, A heuristic fuzzy decision-based solving of redispatching problem for congestion management in restructured power systems, Journal of Intelligent and Fuzzy Systems 24 (3) (2013) 563–574.

[12] S. Mathew and M. S. Sunitha, Cycle connectivity in fuzzy graphs, Journal of Intelligent and Fuzzy Systems 24 (3) (2013) 549–554.
[13] V. Torra, Hesitant fuzzy sets, International Journal of Intelligent Systems 25 (6) (2010) 529–539.
[14] V. Torra and Y. Narukawa, On hesitant fuzzy sets and decision, IEEE International Conference on Fuzzy Systems (2009) 1378–1382.
[15] Z. Wang and G.J. Klir, Fuzzy Measure Theory, Springer Berlin 35 (1-2)(1992) 3–10.
[16] C. Y. Wang, Q. G. Li and X. Q. Zhou, Multiple Attribute Decision Making Based on Generalized Aggregation Operators under Dual Hesitant Fuzzy Environment, Journal of Applied Mathematics 2014 (2)(2014) 701–708.
[17] H. J. Wang, X. F. Zhao and G. W. Wei, Dual hesitant fuzzy aggregation operators in multiple attribute decision making, Journal of Intelligent and Fuzzy Systems Applications in Engineering and Technology 26 (5) (2014) 2281–2290.
[18] Z. S. Xu, A method based on linguistic aggregation operators for group decision making with linguistic preference relations, Inform. Sci. 166 (2004) 19–30.
[19] Z. S. Xu, On generalized induced linguistic aggregation operators, International Journal of General Systems 35 (1) (2006) 17–28.
[20] Z. S. Xu, EOWA AND EOWG OPERATORS FOR AGGREGATING LINGUISTIC LABELS BASED ON LINGUISTIC PREFERENCE RELATIONS, International Journal of Uncertainty Fuzziness and Knowledge-Based Systems 12 (6) (2011) 791–810.
[21] Z. S. Xu and Q. L. Da, The ordered weighted geometric averaging operators, International Journal of Intelligent Systems 17 (7) (2002) 709–716.
[22] M. M. Xia and Z. S. Xu, Hesitant fuzzy information aggregation in decision making, International Journal of Approximate Reasoning 52 (3) (2011) 395–407.
[23] M. M. Xia, Z. S. Xu and B. Zhu, Geometric Bonferroni means with their application in multicriteria decision making, Knowledge-Based Systems 40 (1) (2013) 88–100.
[24] Z. S. Xu and R. R. Yager, Intuitionistic fuzzy Bonferroni means, IEEE Transactions on Systems Man and Cybernetics Part B Cybernetics A Publication of the IEEE Systems Man and Cybernetics Society 41 (2) (2011) 568–578.
[25] Z. S. Xu and R. R. Yager, Power-Geometric Operators and Their Use in Group Decision Making, IEEE Transactions on Fuzzy Systems 18 (1) (2010) 94–105.
[26] R. R. Yager, ON THE THEORY OF BAGS, International Journal of General Systems 13 (1) (1986) 23–37.
[27] R. R. Yager, On ordered weighted averaging aggregation operators in multicriteria decision-making, IEEE Transactions on Systems Man and Cybernetics 18 (1) (1988) 183–190.
[28] R. R. Yager, The power average operator, Systems Man and Cybernetics Part A Systems and Humans IEEE Transactions on 31 (6) (2001) 724–731.
[29] R. R. Yager, OWA aggregation over a continuous interval argument with applications to decision making, IEEE Transactions on Systems Man and Cybernetics Part B Cybernetics A Publication of the IEEE Systems Man and Cybernetics Society 34 (5) (2004) 1952–63.
[30] R. R. Yager, CHOQUET AGGREGATION USING ORDER INDUCING VARIABLES, International Journal of Uncertainty Fuzziness and Knowledge-Based Systems 12 (1) (2011) 69–88.
[31] J. Ye, Correlation coefficient of dual hesitant fuzzy sets and its application to multiple attribute decision making, Applied Mathematical Modelling 38 (2) (2014) 659–666.
[32] J. Ye, Cross-Entropy of Dual Hesitant Fuzzy Sets for Multiple Attribute Decision-Making 8 (3) (2016) 20–30.
[33] R. R. Yager and Z. S. Xu, The continuous ordered weighted geometric operator and its application to decision making, Fuzzy Sets and Systems 157 (2006) 1393–1402.
[34] L. A. Zadeh, The Concept of a Linguistic Variable and its Application to Approximate Reasoning, Springer 1974.
[35] B. Zhu, Z.S. Xu and M.M. Xia, Dual Hesitant Fuzzy Sets, Journal of Applied Mathematics 2012(11)(2012)2607–2645.
[36] B. Zhu, Z. S. Xu and M. M. Xia, Hesitant fuzzy geometric Bonferroni means, Information Sciences An International Journal 205 (1) (2012) 72–85.

HUONIAN TU (tuhuonian2015@163.com)
Guangxi University of Finance and Economics, Nanning, Guangxi 530003, P.R. China

CHUNYONG WANG (wangchunyong2006@sina.com)
College of Science, Hezhou University, Hezhou, Guangxi 542800, P.R. China

XIAOQIANG ZHOU (zxq0923@163.com)
School of Computer, Hunan Institute of Science and Technology, Yueyang, Hunan 414006, P.R. China

SHENGDA TAO (sdtao3@163.com)
College of Science, Hezhou University, Hezhou, Guangxi 542800, P.R. China