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Abstract. The notions of anti fuzzy ideals, bi-ideals and interior ideals of Partially Ordered Γ-Semigroups(POΓS) have been proposed in this paper. We characterize some properties of POΓS in terms of anti fuzzy ideals (AFI). We obtain equivalent statements on composition of AFI using the characteristic function and anti fuzzy bi-ideal(AFBI). Also we study the relationship between anti fuzzy product and union of AFI in a POΓS. Finally, we deliberate the necessary and sufficient condition of PO-Γ-semigroups.

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Keywords: PO-Γ-semigroup, Anti fuzzy sub Γ-semigroup, Anti fuzzy ideal, Anti fuzzy bi-ideal, Anti fuzzy interior ideal, Anti fuzzy product, Composition of anti fuzzy ideal, Normal anti fuzzy ideal, Fuzzy magnified translation.

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1. Introduction

The idea of fuzzy set was first considered by Zadeh [26]. The AFI of lie algebras was considered by Akram [1]. Fuzzy subgroups and anti fuzzy subgroups were considered by Biswas [2]. The notion of left regular PO-Γ-semigroups were studied by Lee et al.[11]. The notion of AFI in ternary semirings was considered by Nagaiah [14]. The notion of fuzzy ideal extension of ordered semigroups was considered by Xie [25]. AFI of Γ-rings was considered by Ozturk et al.[15]. Fuzzy groups were considered by Rosenfeld [17]. Γ-semigroups were considered by Sen [18]. Later on Sen et al.[19] considered only one sided Γ-semigroups. Moreover, POΓS (Partially Ordered Γ-semigroups) studied by Kwon et al.[10].

The notion of AFI in semigroups, characterizations of different classes in semigroups and the properties of their AFIs were considered by Khan et al.[8] and Shabir et al.[20]. Recently, Srinivas et al.[22] has been studied the concept of the notion of Γ-near-rings in terms of AFI and its properties. Further Nagaiah et al.[13] extended varies properties of PO-Γ-semigroups. Prime radicals of Γ-Semigroup were studied by Dutta et al.[3]. Bi-ideals in ordered Γ-Semigroups were considered by Thawhat [24] and fuzzy bi-ideals in semigroups were considered by Kuroki [9]. Moreover, properties of PO-Γ-semigroups in terms of fuzzy ideals were studied by Majumder et al.[12]. After that Pal et al.[16] studied the characterization of Γ-semigroup in terms of AFI. Kehayopulu et al.[7] introduced the regular ordered semigroups in terms of fuzzy sets. Many more researchers studied the different types of fuzzy ideals in ordered Γ-semigroups and its properties of PO-Γ-semigroups,for example see [4, 5, 6, 21, 23].

In this direction we study the Partially Ordered Γ-semigroups in terms of AFI.

2. Preliminaries

Definition 2.1 ([4])**.** Let S and Γ be two non-empty sets. Then S is called a Γ-semigroup, if there exists a mapping from $S \times \Gamma \times S \to S$, written as $(a, \alpha, b) \mapsto a\alpha b$ satisfying the identity $(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

Definition 2.2 ([12])**.** Let S be a Γ-semigroup. By subΓ-semigroup of S, we mean a non-empty subset A of S such that $A\Gamma A \subseteq A$.

Definition 2.3 ([4])**.** A Γ-semigroup S is called a PO-Γ-semigroup, if for any $a, b, c \in S$ and for $\alpha \in \Gamma, a \leq b$ implies $a\alpha c \leq b\alpha c$ and $c\alpha a \leq c\alpha b$.

Definition 2.4 ([21])**.** Let S be a PO-Γ-semigroup. A non-empty subset A of S is said to be right (resp. left) ideal of S, if

(i) $A\Gamma S \subseteq A$ (resp. $S\Gamma A \subseteq A$),
(ii) if $x \in A$ and $y \in S$ such that $y \leq x$, then $y \in A$.

Definition 2.5 ([21])**.** Let S be an PO-Γ-semigroup. A sub Γ-semigroup A of S is said to be bi-ideal of S, if

(i) $A\Gamma S\Gamma A \subseteq A$,
(ii) if $x \in A$ and $y \in S$ such that $y \leq x$, then $y \in A$.

Definition 2.6. A fuzzy subset μ of a non-empty set X is a function $\mu : X \to [0, 1]$.

Definition 2.7. A function $\Omega : S \to S^{'}$, where S and $S^{'}$ are POΓSs, is said to be homomorphism, if $\Omega(x\gamma y) = \Omega(x)\gamma\Omega(y)$, for all $x, y \in S$ and $\gamma \in \Gamma$.

Definition 2.8 ([16])**.** Let S be a POΓS and μ, λ be two fuzzy subsets of S. Then their anti product $\mu\Gamma\lambda$ of μ and λ is defined as

$$(\mu\Gamma\lambda)(x) = \begin{cases} \inf\{\max\{\mu(y), \lambda(z)\}\} & \text{if } x = y\alpha z \text{ for } y, z \in S \text{ and } \alpha \in \Gamma, \\ 1 & \text{otherwise.} \end{cases}$$

Definition 2.9. Let $f : X \to Y$ be a function. For a fuzzy set μ in Y, we define $f^{-1}(\mu)(x) = \mu(f(x))$ for every $x \in X$. For a fuzzy set λ in X, $f(\lambda)$ is defined by

$$(f(\lambda))(y) = \begin{cases} \sup \lambda(x) & \text{if } f(x) = y,\ x \in X \\ 0 & \text{if there is no such } x, \end{cases}$$

for each $y \in Y$.

Definition 2.10 ([16])**.** Let Ψ be a fuzzy subset of X and $\alpha \in [0, 1 - \sup\{\mu(x) : x \in X\}], \beta \in [0, 1]$. A mapping $\Psi^{c}_{\beta\alpha} : S \to [0, 1]$ is called a fuzzy magnified translation of

Ψ, if $\Psi^c_{\beta\alpha}(x) = \beta.\Psi(x) + \alpha$ for all $x \in X$. Ψ^T_α (obtained by putting β =1) and Ψ^T_β (obtained by putting $\alpha = 0$) are fuzzy translation and a fuzzy multiplication of Ψ.

Definition 2.11 ([16]). A PO-Γ-semigroup S is called left zero (right zero), if $x\gamma y = x$ (resp. $x\gamma y = y$), for all $x, y \in S, \gamma \in \Gamma$.

3. Anti fuzzy ideals

In this section we define an anti fuzzy sub Γ-semigroup, anti fuzzy left(right) ideal, anti fuzzy bi-ideal, anti fuzzy interior ideal of Partially Ordered Γ-semigroup and discuss an example of anti fuzzy ideal of Partially Ordered Γ-semigroup.

Definition 3.1. A fuzzy subset μ of a POΓS S is called an anti fuzzy sub Γ-semigroup of S, if

$$\mu(x\alpha y) \leq \max\{\mu(x), \mu(y)\}, \text{ for all } x, y \in S \text{ and } \alpha \in \Gamma.$$

Definition 3.2. A fuzzy subset μ of a POΓS S is called an anti fuzzy right(resp. left) ideal of S, if

(i) $x \leq y \Rightarrow \mu(x) \leq \mu(y)$ for all $x, y \in S$,

(ii) $\mu(x\alpha y) \leq \mu(x)$ (resp. $\mu(x\alpha y) \leq \mu(y)$), for all $x, y \in S$ and $\alpha \in \Gamma$.

A fuzzy subset μ of a POΓS S is called an AFI of S, if it is both an anti fuzzy left ideal and anti fuzzy right ideal.

Example 3.3. Let $S = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$ and $\Gamma = \{\{a, b, c\}\}$. If for all $A, C \in S$ and $B \in \Gamma, ABC = A \cap B \cap C$ and $A \leq C \Leftrightarrow A \subseteq C$, then in [23] Subrahmanyeswara Rao etc shown that S is partial order Γ-semigroup(shortly POΓS).

Now we find AFI of S. Let μ be fuzzy subset of S defined as follows:

$$\mu(A) = \begin{cases} 0.6 & \text{if order of A=3} \\ 0.2 & \text{if order of A=2} \\ 0.1 & \text{otherwise,} \end{cases}$$

for each $A \in S$.

It is easy to prove that μ is an AFI of the POΓS S.

Example 3.4. Let S be the set of all negative integers and Γ be the set of all negative even integers. Then S is a Γ-semigroup where $x\alpha y$ denote the usual multiplication of integers x, α, y with $x, y \in S$ and $\alpha \in \Gamma$. Then S is a POΓS. Let μ be fuzzy subset of S defined as follows;

$$\mu(x) = \begin{cases} 0.8 & \text{if } \ \text{x = 1} \\ 0.4 & \text{if } \ x = -2 \\ 0.2 & \text{if } \ x < -2, \end{cases}$$

for each $x \in S$.

It is easy to verify that μ is an AFI of a POΓS S.

Definition 3.5. A fuzzy sub Γ-semigroup μ of a POΓS S is called an anti fuzzy bi-ideal of S, if

(i) $x \leq y \Rightarrow \mu(x) \leq \mu(y)$, for all $x, y \in S$,

(ii) $\mu(x\alpha y\beta z) \leq \max\{\mu(x), \mu(z)\}$,for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.

Definition 3.6. A fuzzy sub Γ-semigroup μ of a POΓS S is called an anti fuzzy interior ideal of S, if
(i) $x \leq y \Rightarrow \mu(x) \leq \mu(y)$, for all $x, y \in S$,
(ii) $\mu(x\alpha a\beta y) \leq \mu(a)$, for all $x, y, a \in S$ and $\alpha, \beta \in \Gamma$.

Definition 3.7. An AFI μ of a POΓS S is said to be normal, if $\mu(0) = 1$.

Definition 3.8. An AFI μ of a POΓS S is said to be complete, if it is normal and there exist $z \in S$ such that $\mu(z) = 0$.

4. Main results

In this section we study several properties of partially ordered Γ-semigroups in terms of AFIs.

Theorem 4.1. *Every AFI of a POΓS is an anti fuzzy bi-ideal of a POΓS.*

Proof. Let μ be AFI of a POΓS S. For any $x, y \in S$ with $x \leq y$, $\mu(x) \leq \mu(y)$.
case(i): Suppose μ is an anti fuzzy left ideal of a POΓS S. Then $\mu(x\alpha y) \leq \mu(y)$, for all $x, y \in S$ and $\alpha \in \Gamma$. Thus for any $x, y, z \in S$ and $\alpha, \beta \in \Gamma$, we have

$$\mu(x\alpha y\beta z) = \mu(x\alpha(y\beta z)) \leq \mu(y\beta z) \leq \mu(z).$$

Case(ii): Suppose μ is an anti fuzzy right ideal of a POΓS S. Then $\mu(x\alpha y) \leq \mu(x)$, for all $x, y \in S$ and $\alpha \in \Gamma$. Thus For any $x, y, z \in S$ and $\alpha, \beta \in \Gamma$, we have

$$\mu(x\alpha y\beta z) = \mu(x\alpha(y\beta z)) = \mu((x\alpha y)\beta z) \leq \mu(x\alpha y) \leq \mu(x).$$

From the both cases,

$$\mu(x\alpha y) \leq \mu(x) \vee \mu(y) = \max\{\mu(x), \mu(y)\}$$

and

$$\mu(x\alpha y\beta z) \leq \max\{\mu(x), \mu(z)\},$$

for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. So μ is an anti fuzzy bi-ideal of S. □

Example 4.2. The examples 3.3 and 3.4 are AFIs of POΓS S. We can easily verify that μ is an AFBI of a POΓS S.

Proposition 4.3 ([6]). *Let S be a POΓS and $\{\Omega_i\}_{i\in I}$ a non-empty family of fuzzy subsets of S. Then $\bigwedge_{i\in I} \Omega_i$ is a fuzzy subset of S.*

Proposition 4.4. *Let S be a POΓS and $\{\Omega_i\}_{i\in I}$ a non-empty family of fuzzy subsets of S. Then $\bigvee_{i\in I} \Omega_i$ is a fuzzy subset of S.*

Proof. Let $x \in S$. Then the set $\{\Omega_i(x)\}_{i\in I}$ is a non-empty bounded above subset of R. By the Completeness axiom, there exists the $\sup\{\Omega_i(x)\}_{i\in I}$ in R. Since $0 \leq \Omega_i(x) \leq 1$, for each $i \in I$, we have $0 \leq \sup\{\Omega_i(x)\}_{i\in I} \leq 1$. Thus $0 \leq (\bigvee_{i\in I} \Omega_i)(x) \leq 1$. If $x, y \in S$ is such that $x \leq y$, then $\{\Omega_i(x)\}_{i\in I} = \{\Omega_i(y)\}_{i\in I}$. Thus $\sup\{\Omega_i(x)\}_{i\in I} = \sup\{\Omega_i(y)\}_{i\in I}$. So $(\bigvee_{i\in I} \Omega_i)(x) = (\bigvee_{i\in I} \Omega_i)(y)$. Hence $\bigvee_{i\in I} \Omega_i$ is a fuzzy subset of S. □

Proposition 4.5. *Let S be a POΓS and $\{\Omega_i\}_{i\in I}$ a family of anti fuzzy Γ-semigroup of S. Then $\bigvee_{i\in I} \Omega_i$ is an anti fuzzy sub Γ-semigroup of S.*

Proof. By the proposition 4.4, we have $(\bigvee_{i\in I} \Omega_i)$ is fuzzy subset of S. Let $x, y \in S$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}(\bigvee_{i\in I} \Omega_i)(x\alpha y) &= \sup\{\Omega_i(x\alpha y)\}_{i\in I}\\ &\leq \sup\{\max\{\Omega_i(x), \Omega_i(y)\}\}_{i\in I}\\ &= \max\{\sup\{\Omega_i(x)\}_{i\in I}, \sup\{\Omega_i(y)\}_{i\in I}\}\\ &= \max\{(\bigvee_{i\in I} \Omega_i)(x), (\bigvee_{i\in I} \Omega_i)(y))\}.\end{aligned}$$

Thus $\bigvee_{i\in I} \Omega_i$ is anti fuzzy sub Γ-semigroup of S. □

Theorem 4.6. *Let S be a $PO\Gamma S$. Then the following statements are true.*

(1) *For any collection $\{\Omega_i\}_{i\in I}$ of an anti fuzzy left (resp. right) ideals of S, $\bigvee_{i\in I} \Omega_i$ is an anti fuzzy left (resp. right) ideal of S.*

(2) *For any collection $\{\Omega_i\}_{i\in I}$ of an AFBIs of S, $\bigvee_{i\in I} \Omega_i$ is an AFBI of S.*

(3) *For any collection $\{\Omega_i\}_{i\in I}$ of an anti fuzzy interior ideals of S, $\bigvee_{i\in I} \Omega_i$ is an anti fuzzy interior ideal of S.*

Proof. (1) By proposition 4.4, we have $\bigvee_{i\in I} \Omega_i$ is a fuzzy subset of S.

Now, let $x, y \in S$ be such that $x \leq y$. Since Ω_i is a fuzzy left ideal of S, $\Omega_i(x) \leq \Omega_i(y)$, for all $i \in I$. Then $\sup\{\Omega_i(y)\}_{i\in I} \geq \Omega_i(y) \geq \Omega_i(x)$, for all $i \in I$. Thus $\sup\{\Omega_i(y)\}_{i\in I}$ is an upper bound of $\{\Omega_i(x)\}_{i\in I}$. So $\sup\{\Omega_i(y)\}_{i\in I} \geq \sup\{\Omega_i(x)\}_{i\in I}$. Hence $(\bigvee_{i\in I} \Omega_i)(x) \leq (\bigvee_{i\in I} \Omega_i)(y)$.

Let $x, y \in S$ and $\alpha \in \Gamma$. Since Ω_i is an anti fuzzy left ideal of S, we have $\Omega_i(x\alpha y) \leq \Omega_i(y)$, for all $i \in I$. Then

$$\begin{aligned}(\bigvee_{i\in I} \Omega_i)(x\alpha y) &= \sup\{\Omega_i(x\alpha y)\}_{i\in I}\\ &\leq \sup\{\Omega_i(y)\}_{i\in I}\\ &= (\bigvee_{i\in I} \Omega_i)(y).\end{aligned}$$

Thus $\bigvee_{i\in I} \Omega_i$ is an anti fuzzy left ideal of S.

(2) By proposition 4.5, we have $\bigvee_{i\in I} \Omega_i$ is an anti fuzzy sub Γ-semigroup of S. From (a), let $x, y \in S$ be such that $x \leq y$. Then $(\bigvee_{i\in I} \Omega_i)(x) \leq (\bigvee_{i\in I} \Omega_i)(y)$.

Let $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. Since Ω_i is an AFBI of S, we have $\Omega_i(x\alpha y\beta z) \leq \max\{\Omega_i(x), \Omega_i(z)\}$, for all $i \in I$. Thus

$$\begin{aligned}(\bigvee_{i\in I} \Omega_i)(x\alpha y\beta z) &= \sup\{\Omega_i(x\alpha y\beta z)\}_{i\in I}\\ &\leq \sup\{\max\{\Omega_i(x), \Omega_i(z)\}\}_{i\in I}\\ &= \max\{\sup\{\Omega_i(x)\}_{i\in I}, \sup\{\Omega_i(z)\}_{i\in I}\}\\ &= \max\{(\bigvee_{i\in I} \Omega_i)(x), (\bigvee_{i\in I} \Omega_i)(z))\}.\end{aligned}$$

So $\bigvee_{i\in I} \Omega_i$ is an AFBI of S.

(3) By proposition 4.5, we have $\bigvee_{i\in I}\Omega_i$ is an anti fuzzy sub Γ-semigroup of S. From (1), let $x, y \in S$ be such that $x \leq y$. Then $(\bigvee_{i\in I}\Omega_i)(x) \leq (\bigvee_{i\in I}\Omega_i)(y)$.

Let $x, y, a \in S$ and $\alpha, \beta \in \Gamma$. Since Ω_i is an anti fuzzy interior ideal of S, we have $\Omega_i(x\alpha a\beta y) \leq \Omega_i(a)$, for all $i \in I$. Thus

$$\begin{aligned}(\bigvee_{i\in I}\Omega_i)(x\alpha a\beta y) &= \sup\{\Omega_i(x\alpha a\beta y)\}_{i\in I}\\ &\leq \sup\{\Omega_i(a)\}_{i\in I}\\ &= (\bigvee_{i\in I}\Omega_i)(a).\end{aligned}$$

So $\bigvee\limits_{i\in I}\Omega_i$ is an anti fuzzy interior ideal of S. □

Theorem 4.7. *Let $\Omega : S \to S^{'}$ be an epimorphism on $PO\Gamma Ss$ S and $S^{'}$. If μ is an anti fuzzy sub Γ-semigroup of $S^{'}$, then $\Omega^{-1}(\mu)$ is an anti fuzzy sub Γ-semigroup of S, provided $\Omega^{-1}(\mu)$ is a non-empty.*

Proof. Let $x, y \in S$. Then $\Omega(x), \Omega(y) \in S^{'}$. Since μ is an anti fuzzy sub Γ-semigroup of $S^{'}$,

$$\mu((\Omega(x))\alpha(\Omega(y))) \leq \max\{\mu(\Omega(x)), \mu(\Omega(y))\} = \max\{\Omega^{-1}(\mu)(x), \Omega^{-1}(\mu)(y)\}.$$

Thus

$$\Omega^{-1}(\mu)(x\alpha y) = \mu(\Omega(x\alpha y)) = \mu(\Omega(x)\alpha\Omega(y)) \leq \max\{\Omega^{-1}(\mu)(x), \Omega^{-1}(\mu)(y)\}.$$

So $\Omega^{-1}(\mu)$ is anti fuzzy sub Γ-semigroup of S. □

Theorem 4.8. *Let $\Omega : S \to S^{'}$ be an epimorphism on $PO\Gamma Ss$ S and $S^{'}$. If μ is an anti fuzzy left (resp. right)ideal of $S^{'}$, then $\Omega^{-1}(\mu)$ is an anti fuzzy left (resp. right) of S, provided $\Omega^{-1}(\mu)$ is non-empty.*

Proof. By theorem 4.7, we have, $\Omega^{-1}(\mu)$ is an anti fuzzy sub Γ-semigroup of S. Let $x, y \in S$. Then $\Omega(x), \Omega(y) \in S^{'}$. For any $\alpha \in \Gamma$, we have

$$\begin{aligned}\Omega^{-1}(\mu)(x\alpha y) &= \mu(\Omega(x\alpha y))\\ &= \mu(\Omega(x)\alpha\Omega(y)))\\ &\leq \mu(\Omega(y))\\ &= \Omega^{-1}(\mu)(y).\end{aligned}$$

Let $x, y \in S$ be such that $x \leq y$. Then $\Omega(x), \Omega(y) \in S^{'}$ with $\Omega(x) \leq \Omega(y)$. Since μ is an anti fuzzy left ideal, $\mu(\Omega(x)) \leq \mu(\Omega(y))$. Thus $\Omega^{-1}(\mu)(x) \leq \Omega^{-1}(\mu)(y)$. So $\Omega^{-1}(\mu)$ is anti fuzzy left ideal of S. □

Theorem 4.9. *Let $\Omega : S \to S^{'}$ be an epimorphism on $PO\Gamma Ss$ S and $S^{'}$. If μ is an AFBI of $S^{'}$, then $\Omega^{-1}(\mu)$ is an AFBI of S, provided $\Omega^{-1}(\mu)$ is non-empty.*

Proof. By theorem 4.7, we have $\Omega^{-1}(\mu)$ is an anti fuzzy sub Γ-semigroup of S.

Let μ be an AFBI of S'. Let $x, y \in S$. Then $\Omega(x), \Omega(y) \in S'$. Thus for any $\alpha, \beta \in \Gamma$, we have

$$\begin{aligned}\Omega^{-1}(\mu)(x\alpha y\beta z) &= \mu(\Omega(x\alpha y\beta z)) \\ &= \mu(\Omega(x)\alpha\Omega(y)\beta\Omega(z)) \\ &\leq \max\{\mu(\Omega(x)), \mu(\Omega(y))\} \\ &= \max\{\Omega^{-1}(\mu)(x), \Omega^{-1}(\mu)(y)\}.\end{aligned}$$

Let $x, y \in S$ be such that $x \leq y$. Then $\Omega(x), \Omega(y) \in S'$ with $\Omega(x) \leq \Omega(y)$. Since μ is an AFBI, $\mu(\Omega(x)) \leq \mu(\Omega(y))$. So $\Omega^{-1}(\mu)(x) \leq \Omega^{-1}(\mu)(y)$. Hence $\Omega^{-1}(\mu)$ is an AFBI of S. □

Theorem 4.10. *Let $\Omega : S \to S'$ be an epimorphism on POΓSs S and S'. If μ is an anti fuzzy interior ideal of S', then $\Omega^{-1}(\mu)$ is anti fuzzy interior ideal in S, provided $\Omega^{-1}(\mu)$ is non-empty.*

Proof. Straight forward □

Theorem 4.11. *Let Ψ be an anti fuzzy left (resp. anti fuzzy right, anti fuzzy)ideal of a POΓS S. Then so is Ψ^α, for every real number $\alpha \geq 0$, where Ψ^α defined by $\Psi^\alpha(x) = (\Psi(x))^\alpha$, for all $x \in S$.*

Proof. Let Ψ be an anti fuzzy left ideal of a POΓS S. For any $x, y \in S$ and $\gamma \in \Gamma$, we have $\Psi(x\gamma y) \leq \Psi(y)$. Now $\Psi^\alpha(x\gamma y) = (\Psi(x\gamma y))^\alpha \leq (\Psi(y))^\alpha = \Psi^\alpha(y)$, for all $x, y \in S$ and $\gamma \in \Gamma$.

Let $x, y \in S$ be such that $x \leq y$. Since Ψ is anti fuzzy left ideal, $\Psi(x) \leq \Psi(y)$, for all $x, y \in S$. Then $(\Psi(x))^\alpha \leq (\Psi(y))^\alpha$, for all $\alpha \geq 0$. Thus $\Psi^\alpha(x) \leq \Psi^\alpha(y)$. So Ψ^α is an anti fuzzy left ideal of S. □

Theorem 4.12. *Let Ψ be an anti fuzzy interior ideal of a POΓS S. Then so is Ψ^α, for every real number $\alpha \geq 0$, where Ψ^α defined by $\Psi^\alpha(x) = (\Psi(x))^\alpha$, for all $x \in S$.*

Proof. Let Ψ be an anti fuzzy interior ideal of a POΓS S. Let $x, y \in S$ and $\gamma, \beta \in \Gamma$. Then we have $\Psi(x\gamma a\beta y) \leq \Psi(a)$. Thus

$$\Psi^\alpha(x\gamma a\beta y) = (\Psi(x\gamma a\beta y))^\alpha \leq (\Psi(a))^\alpha = \Psi^\alpha(a),$$

for all $x, y \in S$ and $\gamma, \beta \in \Gamma$.

Let $x, y \in S$ be such that $x \leq y$. Since Ψ is anti fuzzy interior ideal, $\Psi(x) \leq \Psi(y)$, for all $x, y \in S$. So $(\Psi(x))^\alpha \leq (\Psi(y))^\alpha$ for all $\alpha \geq 0$. So $\Psi^\alpha(x) \leq \Psi^\alpha(y)$. Hence Ψ^α is an anti fuzzy interior ideal of S. □

5. Composition of anti fuzzy ideals

In this section we prove equivalent statements on composition of AFIs using the characteristic function and AFBI. Also we study the relationship between anti fuzzy product and union of AFIs in a POΓSs.

Theorem 5.1. *A fuzzy subset μ of a POΓS S is an anti fuzzy sub Γ-semigroup of S if and only if $\mu\Gamma\mu \supseteq \mu$.*

Proof. Let μ be an anti fuzzy sub Γ-semigroup of S. Then for any $x \in S$, we have

$$\begin{aligned}(\mu\Gamma\mu)(x) &= \begin{cases} \inf\{\max\{\mu(y), \mu(z)\}\} & \text{if } x = y\alpha z \text{ for } y, z \in S \text{ and } \alpha \in \Gamma, \\ 1 & \text{otherwise.}\end{cases} \\ &\geq \begin{cases} \inf\{\mu(y\alpha z)\} & \text{if } x = y\alpha z \text{ for } y, z \in S \text{ and } \alpha \in \Gamma, \\ 1 & \text{otherwise.}\end{cases} \\ &= \begin{cases} \mu(x) \\ 1 \end{cases} \\ &\geq \mu(x).\end{aligned}$$

Thus $\mu\Gamma\mu \supseteq \mu$.

Conversely, suppose that $\mu \subseteq \mu\Gamma\mu$. Then for any $x \in S$, $\alpha \in \Gamma$, we have

$$\begin{aligned}\mu(x\alpha y) &\leq \mu\Gamma\mu(x\alpha y) \\ &\leq \max\{\mu(x), \mu(y)\}.\end{aligned}$$

Thus μ is an anti fuzzy sub Γ-semigroup of S. □

Theorem 5.2. *In a POΓS S, the following statements are equivalent:*

(1) *μ is an AFBI of S,*

(1) *$\mu\Gamma\mu \supseteq \mu$, $\mu\Gamma\lambda\Gamma\mu \supseteq \mu$, and for any $x \in S$, $x \leq y$ implies $\mu(x) \leq \mu(y)$, where λ is the characteristic function of S.*

Proof. Assume that μ is an AFBI of S. Then μ is an anti fuzzy sub Γ-semigroup of S. So by a theorem $\mu \subseteq \mu\Gamma\mu$. Let $a \in S$. Suppose there exists $x, y, p, q \in S, \alpha, \beta \in \Gamma$ such that $a = x\alpha y$ and $x = p\beta q$. Since μ is an AFBI of S, we obtain $\mu(p\beta q\alpha y) \leq \max\{\mu(x), \mu(y)\}$. Then

$$\begin{aligned}(\mu\Gamma\lambda\Gamma\mu)(a) &= \inf_{a=x\alpha y}\{\max\{(\mu\Gamma\lambda)(x), \mu(y)\}\} \\ &= \inf_{a=x\alpha y}\{\max[\inf_{x=p\beta q}\{\max\{\mu(p), \lambda(q)\}\}], \mu(y)\} \\ &= \inf_{a=x\alpha y}\{\max[\inf_{x=p\beta q}\{\max\{\mu(p), 0\}\}], \mu(y)\} \\ &= \inf_{a=x\alpha y}\{\max\{\mu(p), \mu(y)\}\} \\ &\geq \mu(p\beta q\alpha y) = \mu(x\alpha y) = \mu(a).\end{aligned}$$

Thus we have $\mu\Gamma\lambda\Gamma\mu \supseteq \mu$. Otherwise $(\mu\Gamma\lambda\Gamma\mu)(a) = 1$. So $\mu\Gamma\lambda\Gamma\mu \supseteq \mu$

Conversely, let us assume that (2) holds. Since $\mu \subseteq \mu\Gamma\mu$, μ is an anti fuzzy sub Γ-semigroup of S. Let $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. Then we have

$$\begin{aligned}\mu(x\alpha y\beta z) = \mu(a) &\leq (\mu\Gamma\lambda\Gamma\mu)(a) \\ &= \inf_{a=x\alpha y\beta z}\{\max\{(\mu\Gamma\lambda)(x\alpha y), \mu(z)\}\} \\ &\leq \max\{(\mu\Gamma\lambda)(p), \mu(z)\}(let\ p = x\alpha y) \\ &= \max\{\inf_{p=x\alpha y}\{\max\{\mu(x), \lambda(y)\}\}, \mu(z)\} \\ &\leq \max\{\max\{\mu(x), 0\}, \mu(z)\} \\ &= \max\{\mu(x), \mu(z)\}.\end{aligned}$$

Since any $x, y \in S, x \leq y$ implies $\mu(x) \leq \mu(y)$. Thus μ is an AFBI of S. □

Theorem 5.3. *Let Ψ_1 be an anti fuzzy right ideal and Ψ_2 be an anti fuzzy left ideal of a POΓSs of S. Then $\Psi_1\Gamma\Psi_2 \supseteq \Psi_1 \cup \Psi_2$.*

Proof. Let Ψ_1 be an anti fuzzy right ideal and Ψ_2 be an anti fuzzy left ideal of S. Then for any $x \in S$, we have

$$\begin{aligned}
(\Psi_1\Gamma\Psi_2)(x) &= \begin{cases} \inf\{\max\{\Psi_1(y), \Psi_2(z)\}\} & \text{if } x = y\alpha z \text{ for } y, z \in S \text{ and } \alpha \in \Gamma \\ 1 & \text{otherwise} \end{cases} \\
&\geq \begin{cases} \inf\{\max\{\Psi_1(y\alpha z), \Psi_2(y\alpha z)\}\} & \text{if } x = y\alpha z \text{ for } y, z \in S \text{ and } \alpha \in \Gamma \\ 1 & \text{otherwise} \end{cases} \\
&= \begin{cases} \max\{\Psi_1(x), \Psi_2(x)\} \\ 1 \end{cases} \\
&= \begin{cases} (\Psi_1 \cup \Psi_2)(x) \\ 1. \end{cases}
\end{aligned}$$

Thus $\Psi_1\Gamma\Psi_2 \supseteq \Psi_1 \cup \Psi_2$. □

6. Normal anti fuzzy ideal

In this section we study the normal, complete AFIs of partially ordered Γ-semigroups. Also we characterize fuzzy magnified translation.

Theorem 6.1. *Let Ψ be an anti fuzzy left ideal (resp. anti fuzzy right ideal, AFI) of a $PO\Gamma S$ of S and t be a fixed element of S such that $\Psi(0) \neq \Psi(t)$. Define a fuzzy set $\Psi^\star$ in S by $\Psi^\star(x) = \frac{\Psi(x)-\Psi(t)}{\Psi(0)-\Psi(t)}$ for all $x \in S$. Then $\Psi^\star$ is a complete anti fuzzy left ideal (resp. anti fuzzy right ideal, AFI) of S.*

Proof. Let Ψ be an anti fuzzy left ideal of S and $x, y \in$S, $\gamma \in \Gamma$. Then

$$\begin{aligned}
\Psi^\star(x\gamma y) &= \frac{\Psi(x\gamma y)-\Psi(t)}{\Psi(0)-\Psi(t)} \\
&\leq \frac{\Psi(y)-\Psi(t)}{\Psi(0)-\Psi(t)} \\
&= \Psi^\star(y).
\end{aligned}$$

Let $x, y \in S$ be such that $x \leq y$. Then $\Psi(x) \leq \Psi(y)$ implies $\Psi(x) - \Psi(t) \leq \Psi(y) - \Psi(t)$. Thus $\frac{\Psi(x)-\Psi(t)}{\Psi(0)-\Psi(t)} \leq \frac{\Psi(y)-\Psi(t)}{\Psi(0)-\Psi(t)}$. So $\Psi^\star(x) \leq \Psi^\star(y)$. Hence $\Psi^\star$ is an anti fuzzy left ideal of S. Since $\Psi^\star(0) = \frac{\Psi(0)-\Psi(t)}{\Psi(0)-\Psi(t)} = 1$, $\Psi^\star$ is normal anti fuzzy left ideal of S. Since $t \in S$, $\Psi^\star(t) = \frac{\Psi(t)-\Psi(t)}{\Psi(0)-\Psi(t)} = 0$. Therefore $\Psi^\star$ is a complete anti fuzzy left ideal of S. □

Theorem 6.2. *Let Ψ be an anti fuzzy left (resp. right)ideal of a $PO\Gamma S$ of S and Ψ^+ be a fuzzy set in S given by $\Psi^+(x) = \Psi(x) + 1 - \Psi(0)$, for all $x \in S$. Then Ψ^+ is a normal anti fuzzy left (resp. right) ideal of S.*

Proof. Let $x, y \in S$, $\alpha \in \Gamma$ and Ψ be anti fuzzy left ideal of S. Then $\Psi^+(x\alpha y) = \Psi(x\alpha y) + 1 - \Psi(0) \leq \Psi(y) + 1 - \Psi(0) = \Psi(y)$. Thus Ψ^+ is anti fuzzy left ideal of S. Also $\Psi^+(0) = \Psi(0) + 1 - \Psi(0) = 1$. So Ψ^+ is normal anti fuzzy left ideal of S. □

Theorem 6.3. *Let Ψ be an AFBI of a $PO\Gamma S$ S and $\Psi^\star$ be fuzzy subset of S, defined by $\Psi^\star(x) = \frac{\Psi(x)}{\Psi(1)}$, for all $x \in S$. Then $\Psi^\star$ an AFBI of S.*

Proof. Let Ψ be an AFBI of a POΓS. For any $x, y \in S$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{array}{rcl}\Psi^{\star}(x\alpha y\beta z) & = & \frac{\Psi(x\alpha y\beta z)}{\Psi(1)} \\ & \leq & \frac{\max\{\Psi(x),\Psi(z)\}}{\Psi(1)} \\ & = & \max\{\frac{\Psi(x)}{\Psi(1)}, \frac{\Psi(y)}{\Psi(1)}\} \\ & = & \max\{\Psi^{\star}(x), \Psi^{\star}(y)\}.\end{array}$$

Let $x, y \in S$ be such that $x \leq y$. Since Ψ is an AFBI, $\Psi(x) \leq \Psi(y)$, for all $x, y \in S$. Thus $\frac{\Psi(x)}{\Psi(1)} \leq \frac{\Psi(y)}{\Psi(1)}$. So $\Psi^{\star}(x) \leq \Psi^{\star}(y)$. Hence $\Psi^{\star}$ is an AFBI of S. □

Proposition 6.4. *Let Ψ be an anti fuzzy left (resp. anti fuzzy right, anti fuzzy)ideal of a POΓS S. Then the fuzzy magnified translation $\Psi^c_{\beta\alpha}$ of Ψ is an anti fuzzy left (resp. anti fuzzy right, anti fuzzy) ideal of S.*

Proof. Let Ψ be an anti fuzzy left ideal of POΓS S. Let $x, y \in S$ be such that $x \leq y$. Then $\Psi(x) \leq \Psi(y)$, for all $x, y \in S$. Since $\beta \in [0, 1]$ $\beta.\Psi(x) \leq \beta.\Psi(y)$. Thus $\beta.\Psi(x)+\alpha \leq \beta.\Psi(y)+\alpha$, for all $\alpha \in [0, 1-\sup\{\mu(x) : x \in S\}]$. So $\Psi^c_{\beta\alpha}(x) \leq \Psi^c_{\beta\alpha}(y)$.
Again let $x, y \in S, \gamma \in \Gamma$, we have

$$\begin{array}{rcl}\Psi^c_{\beta\alpha}(x\gamma y) & = & \beta.\Psi(x\gamma y) + \alpha \\ & \leq & \beta.\ \mu(y) + \alpha \\ & = & \Psi^c_{\beta\alpha}(y).\end{array}$$

Hence $\Psi^c_{\beta\alpha}$ is an anti fuzzy left ideal of S. □

Proposition 6.5. *Let Ψ be an anti fuzzy left ideal (anti fuzzy right ideal) of a left zero (right zero) POΓS S. Then the fuzzy magnified translation $\Psi^c_{\beta\alpha}$ of Ψ is constant function.*

Proof. Let S be a left zero PO–Γ-semigroup S. Then $x\gamma y = x$(resp. $x\gamma y = y$), for all $x, y \in S, \gamma \in \Gamma$. Thus for any $x, y \in S, \gamma \in \Gamma$, we have

$$\begin{array}{rcl}\Psi^c_{\beta\alpha}(x) & = & \beta.\ \mu(x) + \alpha \\ & = & \beta.\ \mu(x\gamma y) + \alpha \\ & \leq & \beta.\ \mu(y) + \alpha \\ & = & \Psi^c_{\beta\alpha}(y), \\ \Psi^c_{\beta\alpha}(y) & = & \beta.\ \mu(y) + \alpha \\ & = & \beta.\ \mu(y\gamma x) + \alpha \\ & \leq & \beta.\ \mu(x) + \alpha \\ & = & \Psi^c_{\beta\alpha}(x).\end{array}$$

Thus $\Psi^c_{\beta\alpha}(x) = \Psi^c_{\beta\alpha}(y)$. Hence $\Psi^c_{\beta\alpha}$ is a constant function. □

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