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Abstract. In this article, our main intention is to deal with the fuzzy matrices on the basis of reference function and thereafter to define arithmetic mean, geometric mean and harmonic mean of such matrices. Various properties of arithmetic, geometric and harmonic mean of these matrices are also studied. Further the concept discussed is illustrated with the help of numerical examples.

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1. Introduction

Fuzzy matrix has been proposed to represent fuzzy relation in a system based on fuzzy set theory, Ovehinnikov [11]. Fuzzy matrices were introduced first time by Thomason [15], who discussed the convergence of powers of fuzzy matrices. Several authors have presented a number of results on the convergence of power sequence of fuzzy matrices [10]. It plays an important role in fuzzy set theory and its applications. Uses of fuzzy matrices are most common in science and technology. It is well known that the matrix formulation of a mathematical formula gives extra advantage to handle/study the problem under consideration. When some problems are not solved by classical matrices, then the concepts of fuzzy matrices are used by Kim [10][3], represented some important result on the determinant of a square matrix. He defined the determinant of a square fuzzy matrix and contributed with very research works [7, 8, 9, 10] a lot to the study of determinant theory of square fuzzy matrices. The adjoint of square matrices was discussed in the works of Thomason and in Kim. Many works on fuzzy matrices and associated properties were carried out by different authors at different times, for example, the works of [6, 12, 13, 14] can be mentioned here. The need for a fundamentally different approach to the study of complementation of fuzzy sets motivated the development of a new approach to the study of fuzzy matrices on the basis of reference function. In the process, efforts have been made to represent fuzzy matrices on the basis of reference function and thereafter the operation of addition and multiplication of matrices are defined accordingly, Dhar[11]. Further some properties of addition and multiplication of matrices are put forward. Thereafter trace, transpose, determinant and adjoint of fuzzy matrices are discussed. The properties of trace, determinant and adjoint of square fuzzy matrices are studied, Dhar [2, 3, 4, 5]. It is observed that the properties of the trace, determinant as well as adjoint of square fuzzy matrices are analogous to the properties of determinant and adjoint of classical matrices except in some cases. Here special attention is given for arithmetic, geometric and harmonic mean of fuzzy matrices on the basis of reference function. It is important to mention here that the properties are studied taking into consideration of the complementation of fuzzy matrices because we in usual case it the results can be easily established but with complementation of matrices, our concept of representation would be more clear. Just as classical relation can be viewed as a set, fuzzy relation can be viewed as a fuzzy subset. It is again important to mention here the fact that since in case of fuzzy sets we prefer to represent it with the help of reference function and so the use of reference function in fuzzy matrices cannot be overlooked. But in case of usual matrices there would not be much difference because the membership value and membership functions are of course equal but in case of complementation it makes sense. It is for this reason we would like to deal with matrix complementation to show how the new representation also satisfy the properties which are seen in case of existing definition of fuzzy matrices.

2. Basic Definitions

2.1. **Fuzzy Sets.** If X is a universe of discourse and x is any particular element of X, then a fuzzy set F, defined on X may be written as a collection of ordered pairs

$$F = (x, \mu_F(x)), x \epsilon X, \tag{2.1}$$

where $\mu_{F(x):X \to [0,1]}$, is called the membership function or grade of membership of x in F and each pair $(x, \mu_{F(x)})$ is called a singleton.

2.2. **Fuzzy sets on the basis of reference function.** New definition of complementation of fuzzy sets is introduced by Baruah [1] in which the use of two functions in defining a fuzzy set is advocated. These are named as membership function and reference function. The new definition of complementation can be explained in the following way: Let $\mu_1(x)$ and $\mu_2(x)$ be two functions such that $0 \leq \mu_2(x) \leq \mu_1(x) \leq 1$. For a fuzzy number denoted by $x, \mu_1(x), \mu_2(x)$, we would call $\mu_1(x)$ as the fuzzy membership function and $\mu_2(x)$ as the reference function, so that $\mu_1(x) - \mu_2(x)$ is the fuzzy membership value for any x. In accordance with the process discussed above, a fuzzy set defined by

$$A = (x, \mu(x)), x \epsilon X \tag{2.2}$$

would be defined in this way as

$$A = \{x, \mu(x)), 0, x \epsilon X\} \tag{2.3}$$

so that the complement would become

$$A^c = \{x, 1, \mu(x)), x \epsilon X\}. \tag{2.4}$$

2.3. **Fuzzy Matrix.** A fuzzy matrix is a matrix which has its elements from [0, 1], called fuzzy unit interval.

2.4. **Fuzzy matrix on the basis of reference function.** From the standpoint of the new definition of fuzzy sets on the basis of reference function, a fuzzy matrix $A = [a_{ij}]_{m\times n}]$ would be represented as $A = [a_{ij}, 0]_{m\times n}$ and similarly the complement of the matrix A would be represented by $A^c = [1, a_{ij}]_{m\times n}$

2.5. **Addition of fuzzy matrices on the basis of reference function.** Two fuzzy matrices are conformable for addition if the matrices are of same order. That is to say, when we wish to find addition of two matrices, the number of rows and columns of both the matrices should be same.If $A = [a_{ij}, r_{ij}]$ and $B = [b_{ij}, r_{ij}']$ be two matrices of same order then their addition can be defined as follows:

$$A + B = \{max(a_{ij}, b_{ij}), min(r_{ij}, r_{ij}'\}, \tag{2.5}$$

where a_{ij} stands for the membership function of the fuzzy matrix A for the ith row and jth column and r_{ij} is the corresponding reference function and b_{ij} stands for the membership function of the fuzzy matrix B for the ith row and jth column where r_{ij}' represents the corresponding reference function. The addition of two matrices can be expressed in the following way: $A + B = [C_{ij}]$,
where

$$C_{11} = \{max(a_{11}, b_{11}), min(r_{11}, r_{11}'\},$$
$$C_{12} = \{max(a_{12}, b_{12}), min(r_{12}, r_{12}'\},$$
$$C_{13} = \{max(a_{13}, b_{13}), min(r_{13}, r_{13}'\},$$

etc.

2.6. **Multiplication of fuzzy matrices on the basis of reference function.** The product of two fuzzy matrices under usual matrix multiplication is not a fuzzy matrix. It is due to this reason; a conformable operation analogous to the product which again happens to be a fuzzy matrix was introduced by many researchers which can be found in fuzzy literature. However, even for this operation if the product AB to be defined if the number of columns of the first fuzzy matrix is must be equal to the number of rows of the second fuzzy matrix. If this condition is satisfied then the multiplication of two fuzzy matrices A and B, will be defined in the following manner:

$$AB = \{maxmin(a_{ij}, b_{ji}), minmax(r_{ij}, r_{ji}')\}, \tag{2.6}$$

where $A = [a_{ij}, r_{ij}]$ and $B = [b_{ij}, r_{ij}']$ be two matrices of order conformable for multiplication, $1 \leq i \leq n$, $1 \leq j \leq n$ and a_{ij} stands for the membership function of the fuzzy matrix A for the ith row and jth column and r_{ij} is the corresponding reference function and b_{ij} stands for the membership function of the fuzzy matrix B for the ith row and jth column where r_{ij}' represents the corresponding reference function.

3. Mean of fuzzy matrices on the basis of reference function

3.1. **Arithmetic Mean.** Let $C = (\mu^A_{ij}, 0)$ and $D = (\mu^B_{ij}, 0)$ be two fuzzy matrices on the basis of reference function. Then the arithmetic mean of these two fuzzy matrices is defined as

$$C@D = (\frac{\mu^A_{ij} + \mu^B_{ij}}{2}, 0). \tag{3.1}$$

Let $A = (1, \mu^A_{ij})$ and $B = (1, \mu^B_{ij})$ be two complement fuzzy matrices on the basis of reference function. Then the arithmetic mean of these two fuzzy matrices is defined as

$$A@B = (1, \frac{\mu^A_{ij} + \mu^B_{ij}}{2}). \tag{3.2}$$

Example 3.1. (Numerical Example) Let us consider two fuzzy matrices A and B on the basis of reference function in the following way:

$$A = \begin{pmatrix} (1, 0.5) & (1, 0.3) \\ (1, 0.6) & (1, 0.4) \end{pmatrix}$$

and

$$B = \begin{pmatrix} (1, 0.3) & (1, 0.5) \\ (1, 0.6) & (1, 0.8) \end{pmatrix}.$$

Then

$$A@B = \begin{pmatrix} (1, 0.4) & (1, 0.4) \\ (1, 0.6) & (1, 0.6) \end{pmatrix}.$$

Proposition 3.2. (Properties of arithmetic mean of fuzzy matrices) *Let $A = (1, \mu^A_{ij})$, $B = (1, \mu^B_{ij})$ and $C = (1, \mu^B_{ij})$ be three complement fuzzy matrices on the basis of reference function. Then the arithmetic mean of these fuzzy matrices satisfy the following properties:*

(1) $A@A = A$,
(2) $A@B = B@A$,
(3) $A@(B + C) = (A@B) + (A@C)$,
(4) $(A + B)@C = (A@C) + (B@C)$.

Proof. Let us consider $\mu^C_{ij} < \mu^A_{ij} < \mu^B_{ij}$.

(1) $A@A = (1, \frac{\mu^A_{ij}+\mu^A_{ij}}{2})$
$= (1, 2\frac{\mu^A_{ij}}{2})$
$= (1, \mu^A_{ij})$
$= A$.

(2) From the definition of arithmetic mean of two complement fuzzy matrices, we have

$$A@B = (1, \frac{\mu^A_{ij} + \mu^B_{ij}}{2}).$$

Thus $B@A = (1, \frac{\mu^B_{ij}+\mu^A_{ij}}{2}) = (1, \frac{\mu^B_{ij}+\mu^A_{ij}}{2}) = (1, \frac{\mu^A_{ij}+\mu^B_{ij}}{2})$. So $A@B = B@A$.

(3) Since $\mu^C_{ij} < \mu^A_{ij} < \mu^B_{ij}$, it can be seen that

$$B + C = (1, \mu^C_{ij}).$$

Thus $A@(B+C) = (1, \mu_{ij}^A)@(1, \mu_{ij}^C) = (1, \frac{\mu_{ij}^A + \mu_{ij}^C}{2})$.

On the other hand, $A@B = (1, \frac{\mu_{ij}^A + \mu_{ij}^B}{2})$ and $A@C = (1, \frac{\mu_{ij}^A + \mu_{ij}^C}{2})$. So

$$(A@B) + (A@C) = \{1, min(\frac{\mu_{ij}^A + \mu_{ij}^B}{2}, \frac{\mu_{ij}^A + \mu_{ij}^C}{2}\}.$$

Since $\mu_{ij}^C < \mu_{ij}^B$, $\mu_{ij}^A + \mu_{ij}^C < \mu_{ij}^A + \mu_{ij}^B$. Hence $(A@B) + (A@C) = \{1, \frac{\mu_{ij}^A + \mu_{ij}^C}{2}\}$. Therefore $A@(B+C) = (A@B) + (A@C)$.

(4) The proof is similar to one of (3). □

3.2. **Geometric Mean of fuzzy matrices.** Let $C = (\mu_{ij}^A, 0)$ and $D = (\mu_{ij}^B, 0)$ be two fuzzy matrices on the basis of reference function. Then the arithmetic mean of these two fuzzy matrices is defined as

$$C \Upsilon D = (\sqrt{\mu_{ij}^A \mu_{ij}^A}, 0). \tag{3.3}$$

Let $A = (1, \mu_{ij}^A)$ and $B = (1, \mu_{ij}^B)$ be two complement fuzzy matrices on the basis of reference function. Then the geometric mean of these two fuzzy matrices is defined as

$$A \Upsilon A = (1, \sqrt{\mu_{ij}^A \mu_{ij}^A}). \tag{3.4}$$

Proposition 3.3. (Properties of geometric mean of fuzzy matrices) *Let* $A = (1, \mu_{ij}^A)$, $B = (1, \mu_{ij}^A)$ *and* $C = (1, \mu_{ij}^B)$ *be three complement fuzzy matrices on the basis of reference function. Then the geometric mean of these fuzzy matrices satisfy the following properties:*

(1) $A \Upsilon A = A$,
(2) $A \Upsilon B = B \Upsilon A$,
(3) $A \Upsilon (B+C) = (A \Upsilon B) + (A \Upsilon C)$,
(4) $(A+B) \Upsilon C = (A \Upsilon C) + (B \Upsilon C)$.

Proof. Let us consider $\mu_{ij}^C < \mu_{ij}^A < \mu_{ij}^B$.

(1) $A \Upsilon A = (1, \sqrt{\mu_{ij}^A \mu_{ij}^A}) = (1, \mu_{ij}^A) = A$.

(2) From the definition of arithmetic mean of two complement fuzzy matrices, we have

$$A \Upsilon B = (1, \sqrt{\mu_{ij}^A \mu_{ij}^B}).$$

Thus $B \Upsilon A = (1, \sqrt{\mu_{ij}^B \mu_{ij}^A}) = (1, \sqrt{\mu_{ij}^A \mu_{ij}^B})$. So $A \Upsilon B = B \Upsilon A$.

(3) Since $\mu_{ij}^C < \mu_{ij}^A < \mu_{ij}^B$, it can be seen that

$$B + C = (1, \mu_{ij}^C).$$

Then $A \Upsilon (B+C) = (1, \sqrt{\mu_{ij}^A \mu_{ij}^C})$.

On the other hand, $A \Upsilon B = (1, \sqrt{\mu_{ij}^A \mu_{ij}^B})$ and $A \Upsilon C = (1, \sqrt{\mu_{ij}^A \mu_{ij}^C})$. Thus

$$(A \Upsilon B) + (A \Upsilon C) = (1, min(\sqrt{\mu_{ij}^A \mu_{ij}^B}, \sqrt{\mu_{ij}^A \mu_{ij}^C}).$$

Since $\mu_{ij}^C < \mu_{ij}^B$, $\mu_{ij}^A\mu_{ij}^C < \mu_{ij}^A\mu_{ij}^B$. So $\sqrt{\mu_{ij}^A\mu_{ij}^C} < \sqrt{\mu_{ij}^A\mu_{ij}^B}$. Hence $(A\Upsilon B)+(A\Upsilon C) = (1, \sqrt{\mu_{ij}^A\mu_{ij}^C})$. Therefore $A\Upsilon(B+C) = (A\Upsilon B)+(A\Upsilon C)$.

(4) The proof is similar to one of (3). □

3.3. **Relation between Arithmetic mean and geometric mean of fuzzy matrices on the basis of reference function.** Let $C = (\mu_{ij}^A, 0)$ and $D = (\mu_{ij}^B, 0)$ be two fuzzy matrices on the basis of reference function. Then the arithmetic mean of these two fuzzy matrices is defined as

$$A@B = (\frac{\mu_{ij}^A + \mu_{ij}^B}{2}, 0).$$

Again the geometric mean of these two fuzzy matrices is defined as

$$A\Upsilon B = (\sqrt{\mu_{ij}^A\mu_{ij}^B}, 0).$$

The membership value of $A@B$ is $(\frac{\mu_{ij}^A+\mu_{ij}^B}{2} - 0)$ which is equal to $\frac{\mu_{ij}^A+\mu_{ij}^B}{2}$
and
the membership value of $A\Upsilon B$ is $(\sqrt{\mu_{ij}^A\mu_{ij}^B} - 0)$ which is equal to $\sqrt{\mu_{ij}^A\mu_{ij}^B}$
From the above results, it is clear that

$$\frac{\mu_{ij}^A + \mu_{ij}^B}{2} \geq \sqrt{\mu_{ij}^A\mu_{ij}^B}.$$

Then it can be said that A. M of fuzzy matrices on the basis of reference function is greater than or equal to the geometric mean of fuzzy matrices on the basis of reference function.

On the other hand, if the complement of the above mentioned matrices $A = (1, \mu_{ij}^A)$, $B = (1, \mu_{ij}^A)$ are considered, then it can be observe that

$$A@B = (1, \frac{\mu_{ij}^A + \mu_{ij}^B}{2}).$$

Again the geometric mean of these two fuzzy matrices is defined as

$$A\Upsilon B = (1, \sqrt{\mu_{ij}^A\mu_{ij}^B}).$$

The membership value of $A@B$ is $(1 - \frac{\mu_{ij}^A+\mu_{ij}^B}{2})$
and
the membership value of $A\Upsilon B$ is $(1 - \sqrt{\mu_{ij}^A\mu_{ij}^B})$.
From the above results, it is clear that

$$(1 - \frac{\mu_{ij}^A + \mu_{ij}^B}{2}) \leq (1 - \frac{\mu_{ij}^A + \mu_{ij}^B}{2}).$$

Thus it can be said that when the complement fuzzy matrices on the basis of reference function is considered, A. M of fuzzy matrices on the basis of reference function is less than or equal to the geometric mean of fuzzy matrices on the basis of reference function.

3.4. **Harmonic mean of fuzzy matrices.** Let $C = (\mu_{ij}^A, 0)$ and $B = (\mu_{ij}^B, 0)$ be two fuzzy matrices on the basis of reference function. Then the harmonic mean of these two fuzzy matrices is defined as

$$A \bigodot B = (2\frac{\mu_{ij}^A \mu_{ij}^B}{\mu_{ij}^A + \mu_{ij}^B}, 0). \tag{3.5}$$

Let $A = (1, \mu_{ij}^A)$ and $B = (1, \mu_{ij}^B)$ be two complement fuzzy matrices on the basis of reference function. Then the harmonic mean of these two fuzzy matrices is defined as

$$A \bigodot B = (1, 2\frac{\mu_{ij}^A \mu_{ij}^B}{\mu_{ij}^A + \mu_{ij}^B}). \tag{3.6}$$

Proposition 3.4. (Properties of harmonic mean of fuzzy matrices) *Let* $A = (1, \mu_{ij}^A)$ *,* $B = (1, \mu_{ij}^A)$ *and* $C = (1, \mu_{ij}^B)$ *be three complement fuzzy matrices on the basis of reference function. Then the harmonic mean of these fuzzy matrices satisfy the following properties:*

(1) $A \bigodot A = A$,
(2) $A \bigodot B = B \bigodot A$,
(3) $A \bigodot (B + C) = (A \bigodot B) + (A \bigodot C)$,
(4) $(A + B) \bigodot C = (A \bigodot C) + (B \bigodot C)$.

Proof. Let us consider $\mu_{ij}^C < \mu_{ij}^A < \mu_{ij}^B$.

(1) $A \bigodot A = (1, 2\frac{\mu_{ij}^A \mu_{ij}^A}{\mu_{ij}^A + \mu_{ij}^A}) = (1, 2\frac{\mu_{ij}^A}{2}) = (1, \mu_{ij}^A) = A$.

(2) From the definition of arithmetic mean of two complement fuzzy matrices, we have

$$A \bigodot B = (1, 2\frac{\mu_{ij}^A \mu_{ij}^B}{\mu_{ij}^A + \mu_{ij}^B}).$$

Thus $B \bigodot A = (1, 2\frac{\mu_{ij}^B \mu_{ij}^A}{\mu_{ij}^B + \mu_{ij}^A}) = (1, 2\frac{\mu_{ij}^B \mu_{ij}^A}{\mu_{ij}^B + \mu_{ij}^A}) = (1, 2\frac{\mu_{ij}^A + \mu_{ij}^B}{\mu_{ij}^A + \mu_{ij}^B})$. So $A \bigodot B = B \bigodot A$.

(3) Since $\mu_{ij}^C < \mu_{ij}^A < \mu_{ij}^B$, it can be seen that

$$B + C = (1, \mu_{ij}^C).$$

Then $A \bigodot (B + C) = (1, \mu_{ij}^A) \bigodot (1, \mu_{ij}^C) = (1, 2\frac{\mu_{ij}^A \mu_{ij}^C}{\mu_{ij}^A + \mu_{ij}^C})$.

On the other hand, $A \bigodot B = (1, 2\frac{\mu_{ij}^A \mu_{ij}^B}{\mu_{ij}^A + \mu_{ij}^B})$ and $A \bigodot C = (1, 2\frac{\mu_{ij}^A \mu_{ij}^C}{\mu_{ij}^A + \mu_{ij}^C})$. Thus

$$(A \bigodot B) + (A \bigodot C) = \{1, min(2\frac{\mu_{ij}^A \mu_{ij}^B}{\mu_{ij}^A + \mu_{ij}^B}, 2\frac{\mu_{ij}^A \mu_{ij}^C}{\mu_{ij}^A + \mu_{ij}^C}\}).$$

Since $\mu_{ij}^C < \mu_{ij}^B$, $\mu_{ij}^A + \mu_{ij}^C < \mu_{ij}^A + \mu_{ij}^B$. So $(A \bigodot B) + (A \bigodot C) = \{1, 2\frac{\mu_{ij}^A + \mu_{ij}^C}{\mu_{ij}^A + \mu_{ij}^C}\}$.
Hence $A \bigodot (B + C) = (A \bigodot B) + (A \bigodot C)$.

(4) The proof is similar to one of (3). □

4. Conclusions

In this article, the arithmetic, geometric and harmonic mean of fuzzy matrices on the basis of reference function are discussed.**The relation between arithmetic mean and geometric mean of fuzzy matrices on the basis of reference function is studied.** Further, the various means of fuzzy matrices on the basis of reference function and various other properties associated with such matrices are to be studied and discussed in future works.

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