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Abstract. In this article the notions of α-n-convergent and α-n-Cauchy sequence in random n-normed space are introduced. Weak and strong n-bounded and n-compact linear operator in random n-normed space are discussed as well.

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Keywords: α-n-norms, α-n-convergent, α-n-Cauchy, n-bounded, n-compact, Random n-normed spaces.

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1. Introduction

The extensive treatment of general n-metrics was made by K. Menger in 1928. The notions of 2-norm and n-norm on a linear space were introduced by Gähler, see [1, 2, 4, 13, 15]. In 1962, A. N. Serstnev introduced the concept of random normed linear space [14]. In 2003, I. Jebril and R. Ali, studied bounded linear operators in probabilistic normed linear spaces [8]. In 2009, I. Jebril and R. Hatamleh introduced the concept of random n-normed linear space [9] as a generalization of n-normed space already introduced by Gunawan and Mashadi [10]. For more results in this subject, we refer the reader to [5, 11, 12, 16] for instance.

We now state some basic notions that will be needed later.

Definition 1.1 ([13]). A t-norm is a binary operation on unit interval $[0, 1]$, that is, a function $T : [0, 1] \times [0, 1] \to [0, 1]$ such that for all $x, y, z \in [0, 1]$, the four following axioms are satisfied:

($T1$) (Commutativity) $T(x, y) = T(y, x)$,
($T2$) (Associativity) $T(x, T(y, z)) = T(T(x, y), z)$,
($T3$) (Boundary condition) $T(x, 1) = x$,
($T4$) (Monotonicity) $T(x, y) \le T(x, z)$, whenever $y \le z$.

Definition 1.2 ([4]). Let $n \in \mathbb{N}$ and L be a real vector space of dimension $d \geq n$. If a real valued function $\|\bullet, \bullet, \ldots, \bullet\|$ on $L \times L \times \cdots \times L = L^n$, satisfies the following properties:

(i) $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,
(ii) $\|x_1, x_2, \ldots, x_n\|$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$,
(iii) $\|x_1, x_2, \ldots, \alpha x_n\| = |\alpha| \|x_1, x_2, \ldots, x_n\|$, where $\alpha \in \mathbb{R}$,
(iv) $\|x_1, x_2, \ldots, x_{n-1}, y + z\| \leq \|x_1, x_2, \ldots, x_{n-1}, y\| + \|x_1, x_2, \ldots, x_{n-1}, z\|$,

then $\|\bullet, \bullet, \ldots, \bullet\|$ is called an n-norm on L and the pair $(L, \|\bullet, \bullet, \ldots, \bullet\|)$ is called an n-normed linear space.

Definition 1.3 ([4]). Let $(L, \|\bullet, \bullet, \ldots, \bullet\|)$ be a n-normed space. The sequence (x_k) in L is said to be n-convergent to $l \in L$ (with respect to the n-norm), whenever

$$\lim_{k\to\infty} \|x_1, x_2, \ldots, x_{n-1}, x_k - l\| = 0,$$

for every $x_1, x_2, \ldots, x_{n-1} \in L$.

Definition 1.4 ([4]). Let $(L, \|\bullet, \bullet, \ldots, \bullet\|)$ be a n-normed space, the sequence (x_k) in L is said to be n-Cauchy (with respect to the n-norm), whenever

$$\lim_{k,l\to\infty} \|x_1, x_2, \ldots, x_{n-1}, x_k - x_l\| = 0,$$

for every $x_1, x_2, \ldots, x_{n-1} \in L$.

Definition 1.5 ([4]). Let $(L, \|\bullet, \bullet, \ldots, \bullet\|)$ be a n-normed space.

If every Cauchy sequence converges to an $l \in L$, then $(L, \|\bullet, \bullet, \ldots, \bullet\|)$ is said to be n-complete.

A n-complete n-normed space is called an n-Banach space.

Definition 1.6 ([13]). A function $f{:}\mathbb{R} \to \mathbb{R}_0^+$ is called a distribution function, if it is a non-decreasing and left continuous with $\sup_{t\in\mathbb{R}} f(t) = 1$.

By D^+, we denote the set of all distribution functions such that $f(0) = 0$.

If $a \in \mathbb{R}_0^+$, then $a \in D^+$, where

$$H_a(t) = \begin{cases} 0, & \text{if } t \leq a \\ 1, & \text{if } t > a. \end{cases}$$

It is obvious that $H_0 \geq f$ for all $f \in D^+$.

The notion of random normed space was introduced as follows:

Definition 1.7 ([10]). A random normed space is a triple (V, v, T), where V is a vector space, T is a continuous t-norm, and v is a mapping from V into D^+ such that the following conditions hold: for all p, q, r in V,

(i) $\varepsilon_0 = v_p$ if and only if $p = \theta, \theta$ being the null vector in V,
(ii) $v_{p+q} \geq T(v_p, v_q)$,
(iii) $v_{\lambda p} = v_p\left(\frac{x}{|\lambda|}\right)$, for all x and λ in $\mathbb{R}$.

The notion of random n-normed space was introduced in [9] and [7] as itemized in the following:

Definition 1.8 ([9]). Let L be a linear space of dimension greater than one over a real field, T be continuous t-norm and let v be a mapping from $L \times L \times \cdots \times L = L^n$ into D^+. If the following conditions are satisfied for all $x_1, x_2, \ldots, x_n \in L$ and $t \in \mathbb{R}$,

(i) $v_{x_1,x_2,\ldots,x_n}(t) = H_0(t) \Leftrightarrow x_1, x_2, \ldots, x_n$ are linearly dependent, where $v_{x_1,x_2,\ldots,x_n}(t)$ denotes the value of $v_{x_1,x_2,\ldots,x_n}$,

(ii) $v_{x_1,x_2,\ldots,x_n}(t) \neq H_0(t)$, if $x_1, x_2, \ldots, x_n$ are linearly independent,

(iii) $v_{x_1,x_2,\ldots,x_n}$ is invariant under any permutation of $x_1, x_2, \ldots, x_n \in L^n$,

(iv) $v_{x_1,x_2,\ldots,\alpha x_n}(t) = v_{x_1,x_2,\ldots,x_n}\left(\frac{t}{|\alpha|}\right)$, for every $t > 0$, $\alpha \neq 0$, $\alpha \in \mathbb{R}$,

(v) $v_{x_1,x_2,\ldots,x_n+x'_n}(s+t) \geq T\left(v_{x_1,x_2,\ldots,x_n}(s), v_{x_1,x_2,\ldots,x'_n}(t)\right)$,
for all $x'_n \in L$ and $s \in \mathbb{R}$,
then (L^n, v, T) is called a random n-normed linear space (briefly R-n-NLS).

Definition 1.9. Let (L, v, T) be a R-n-NLS. Assume further that

$(R-n-N_6)$ For all $t \in (0, \infty)$, $v_{x_1,x_2,\ldots,x_n}(t) > 0$ implies that $x_1, x_2, \ldots, x_n$ are linearly dependent.

The following examples give us a R-n-NLS satisfying condition $(R-n-N_6)$.

Example 1.10. Let $(L, \|\bullet, \bullet, \ldots, \bullet\|)$ be a n-normed space. We can made a random n-normed space in a natural way, by setting

1. $v_{x_1,x_2,\ldots,x_n}(t) = H_0(t - \|x_1, x_2, \ldots, x_n\|)$, for every $x_1, x_2, \ldots, x_n \in L$, $t > 0$ and $T(a,b) = min(a,b)$, $a, b \in L$;

2. $v_{x_1,x_2,\ldots,x_n}(t) = \begin{cases} \frac{t}{t+\|x_1,x_2,\ldots,x_n\|}, & \text{when } t > 0, t \in \mathbb{R}, \\ 0, & \text{when } t \leq 0, \end{cases}$

for every $x_1, x_2, \ldots, x_n \in L, t > 0$ and $T(a,b) = ab$, for $a, b \in L$.

2. n-Banach random space

Let (L, v, T) be a R-n-NLS. Since T is continuous t-norm, then (L, v, T) becomes a Hausdorff linear topological space having as a fundamental system of neighborhood of the null vector θ the family

$$\{\mathcal{N}_\theta(\varepsilon, \lambda) : \varepsilon > 0, \lambda \in (0,1)\},$$

where

$$\mathcal{N}_\theta(\varepsilon, \lambda) = \left\{x_1, x_2, \ldots, x_{n-1} \in L : v_{x_1,x_2,\ldots,x_{n-1}}(\varepsilon) > 1 - \lambda\right\}.$$

The $\mathcal{N}_\theta(\varepsilon, \lambda)$ neighborhood determines a first countable Hausdorff topology. F-topology of sequences, i.e., $x_1, x_2, \ldots, x_{n-1} - y \in \mathcal{N}_\theta(\varepsilon, \lambda)$ means that $y \in \mathcal{N}_{x_1,x_2,\ldots,x_{n-1}}$ and vice versa.

A sequence $x = (x_k)_{k\in\mathbb{N}}$ in L is said to have F-convergence to $l \in L$, if for every $\varepsilon > 0, \lambda \in (0,1)$ and for each nonzero $x_1, x_2, \ldots, x_{n-1} \in L$ there exist a positive integer $k_0 \in \mathbb{N}$ such that

$$x_1, x_2, \ldots, x_{n-1}, x_k - l \in \mathcal{N}_\theta(\varepsilon, \lambda) \text{ for each } n \geq k_0.$$

Or equivalently,

$$x_1, x_2, \ldots, x_{n-1}, x_k \in \mathcal{N}_l(\varepsilon, \lambda) \text{ for each } n \geq k_0.$$

In this case, we write $F - lim x_1, x_2, \ldots, x_{n-1}, x_k = l$.

Definition 2.1 ([6])**.** Let (L, v, T) be R-n-NLS. A sequence (x_k) in L is said to be random n-convergent to $l \in L$, if

$$\lim_{k\to\infty} v_{x_1,x_2,\ldots,x_{n-1},x_k-l}\,(t) \;= 1,$$

for every $x_1, x_2, \ldots, x_{n-1} \in L$.
Or equivalently, for every $\varepsilon > 0$, $\lambda \in (0,1)$ and $x_1, x_2, \ldots, x_{n-1} \in L$, there exists $k_0 \in \mathbb{N}$ such that

$$v_{x_1,x_2,\ldots,x_{n-1},x_k-l}\,(\varepsilon) > 1-\lambda,$$

for all $k \geq k_0$.

Theorem 2.2. *Let (L, v, T) be a R-n-NLS. If a sequence (x_k) is random n-convergent to l with respect to the random n-norm v, then l is unique.*

Proof. Suppose that there exist elements l_1, l_2 $(l_1 \neq l_2)$ in L such that

$$\lim_{k\to\infty} v_{x_1,x_2,\ldots,x_{n-1},x_k-l_1}\,(t) = 1,$$

$$\lim{}_{k\to\infty} v_{x_1,x_2,\ldots,x_{n-1},x_k-l_2}\,(t) = 1.$$

Let $\varepsilon > 0$ choose $T\left((1-\lambda),(1-\lambda)\right) > 1-\varepsilon$, and for each nonzero $x_1, x_2, \ldots, x_{n-1} \in L$ such that $\mathcal{N}_{l_1}(\varepsilon,\lambda)$ and $\mathcal{N}_{l_2}(\varepsilon,\lambda)$ are disjoint neighborhoods of $l_1$and l_2.

Since (x_k) is random n-convergent to $l_1$and l_2, for any $t > 0$ there exist $k_0 \in \mathbb{N}$ such that

$$v_{x_1,x_2,\ldots,x_{n-1},x_k-l_1}\left(\frac{t}{2}\right) > 1-\lambda,\ \forall k \geq k_0,$$

$$v_{x_1,x_2,\ldots,x_{n-1},x_k-l_2}\left(\frac{t}{2}\right) > 1-\lambda,\ \forall k \geq k_0.$$

Now let

$$\begin{aligned} v_{x_1,x_2,\ldots,x_{n-1},l_1-l_2}\,(t) &\geq v_{x_1,x_2,\ldots,x_{n-1},l_1-x_n+x_n-l_2}\left(\frac{t}{2}+\frac{t}{2}\right) \\ &\geq T\left(v_{x_1,x_2,\ldots,x_{n-1},x_n-l_1}\left(\frac{t}{2}\right), v_{x_1,x_2,\ldots,x_{n-1},x_n-l_2}\left(\frac{t}{2}\right)\right) \\ &> T\left((1-\lambda),(1-\lambda)\right). \end{aligned}$$

It follows that

$$v_{x_1,x_2,\ldots,x_{n-1},l_1-l_2}\,(t) > 1-\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we get

$$v_{x_1,x_2,\ldots,x_{n-1},l_1-l_2}\,(t) = 1,$$

for all $t > 0$, and non-zero element $x_1, x_2, \ldots, x_{n-1} \in L$. Hence $l_1 = l_2$. □

Lemma 2.3. *Let $(L, \|\bullet,\bullet,\ldots,\bullet\|)$ be a real n-normed space, and let (L, v, T) be a R-n-NLS induced by*

$$v_{x_1,x_2,\ldots,x_n}\,(t) = \frac{t}{t+\|x_1,x_2,\ldots,x_n\|},$$

where $x_1, x_2, \ldots, x_n \in L$ and $T > 0$. If the sequence (x_k) is n-convergent to $l \in L$ and nonzero $x_1, x_2, \ldots, x_{n-1} \in L$, then (x_k) is random n-convergent to $l \in L$.

Proof. Suppose that (x_k) *is* n-convergent to $l \in L^n$. Then

$$\lim_{k\to\infty} \|x_1, x_2, \ldots, , x_{n-1}, x_k - l\| = 0.$$

Then, for every $t > 0$ and for every $x_1, x_2, \ldots, , x_{n-1}, x_k \in L$, $\exists k_0 = k_0(t)$ such that

$$\|x_1, x_2, \ldots, , x_{n-1}, x_k - l\| < t, \ \forall k > k_0.$$

For every given $\varepsilon > 0$,

$$\begin{aligned} \frac{\varepsilon + \|x_1, x_2, \ldots, , x_{n-1}, x_k - l\|}{\varepsilon} &< \frac{\varepsilon + t}{\varepsilon}, \\ \frac{\varepsilon}{\varepsilon + \|x_1, x_2, \ldots, , x_{n-1}, x_k - l\|} &> \frac{\varepsilon}{\varepsilon + t}, \\ \frac{\varepsilon}{\varepsilon + \|x_1, x_2, \ldots, , x_{n-1}, x_k - l\|} &> 1 - \frac{t}{\varepsilon + 1}. \end{aligned}$$

Thus, by letting $\lambda = \frac{t}{\varepsilon + t} \in (0, 1)$, we have

$$v_{x_1, x_2, \ldots, , x_{n-1}, x_k - l}(t) \geq 1 - \lambda, \ \forall k > k_0.$$

So (x_k) is random n-convergent to $l \in L$. □

Definition 2.4 ([6])**.** Let (L, v, T) be R-n-NLS. A sequence (x_k) in L is said to be random n- Cauchy, if

$$\lim_{k,m\to\infty} v_{x_1, x_2, \ldots, x_{n-1}, x_k - x_m}(t) = 1,$$

for every $x_1, x_2, \ldots, x_{n-1} \in L$.
Or equivalent, for every $\varepsilon > 0$, $\lambda \in (0, 1)$ and $x_1, x_2, \ldots, x_{n-1} \in L$, there exists $k_0 \in \mathbb{N}$ such that

$$v_{x_1, x_2, \ldots, x_{n-1}, x_k - x_m}(\varepsilon) > 1 - \lambda,$$

for all $k, m \geq k_0$.

Proposition 2.5 ([6])**.** *In R-n-NLS* (L, v, T)*, every random* n*-convergent sequence is a random* n*-Cauchy sequence.*

If every random n-Cauchy sequence in L converges to an $l \in L$, then (L, v, T) is called a complete random n-normed space. A complete random n-normed space is then called a random n-Banach space.

3. Main results

In this section, we have discussed n-bounded linear operator, (strong boundedness and weak boundedness) in R-n-NLS. Then study α-n-norms, α-n-convergent, α-n-Cauchy in R-n-NLS. Also, study the notion of n-compact linear operator in R-n-NLS.

3.1. **Boundedness linear operator in R-n-NLS.** Definition of bounded linear operator in probabilistic normed space introduced by B. L. Guillén, J. A. R. Lallena and C. Sempi [3] then studied by Jebril and Ali in [8]. In this section, we introduce the definition of n-bounded linear operator in R-n-NLS. By virtue of this definition, we describe the boundedness linear operator in R-n-NLS and we prove some related results.

Let $(L_1, \|\bullet, \bullet, \ldots, \bullet\|)$ be n-normed space and $(L_2, \|\bullet\|)$ be a normed space. Following is an extension of the notion of random n-bounded linear operator. In [15], S. M. Gozali et al. introduced the notion of n-bounded linear operator.

Definition 3.1 ([15])**.** An operator $T : (L_1, \|\bullet, \bullet, \ldots, \bullet\|) \to (L_2, \|\bullet\|)$ is an n-linear operator on L_1, if T is linear in each variable.

An n-linear operator is called n-bounded of type I, if there is a constant k such that for all $(x_1, x_2, \ldots, x_n) \in L_1^n$,

$$\|T(x_1, x_2, \ldots, x_n)\| \leq k \|x_1, x_2, \ldots, x_n\|.$$

Note that when $n = 1$, the above is reduced to the usual notion of bounded linear operator in normed space.

In the following, we will generalize the definition of n-bounded linear operator in n-normed space introduced in [15] by starting the definition of random n-bounded linear operator of type I in R-n-NLS.

Definition 3.2. Let $T : (L_1, v, T) \to (L_2, \mu, Q)$ be a linear operator, where (L_1, v, T) is R-n-NLS and (L_2, μ, Q) is R-NLS.

(i) The operator T is called weak random n-bounded of type I on L_1, if for every $\alpha \in (0, 1)$, there exist $k_\alpha > 0$ such that

$$v_{x_1,x_2,\ldots,x_n}\left(\frac{t}{k_\alpha}\right) \geq \alpha \Rightarrow \mu_{T(x_1,x_2,\ldots,x_n)}(t) \geq \alpha,$$

for all $(x_1, x_2, \ldots, x_n) \in L_1^n$ and $t \in \mathbb{R}$.

(ii) The operator T is called strong random n-bounded of type I on L_1 if there exist a positive real number k such that

$$\mu_{T(x_1,x_2,\ldots,x_n)}(t) \geq v_{x_1,x_2,\ldots,x_n}\left(\frac{t}{k}\right),$$

for all $(x_1, x_2, \ldots, x_n) \in L_1^n$ and $t \in \mathbb{R}$.

Theorem 3.3. *Let $T : (L_1, v, T) \to (L_2, \mu, Q)$ be a linear operator, where (L_1, v, T) is R-n-NLS and (L_2, μ, Q) is R-NLS. For all (L_2, μ, Q). If T is strong random n-bounded, then it is weak random n-bounded.*

Proof. Suppose that T is strong random n-bounded of type I. Then there exists k such that

$$\mu_{T(x_1,x_2,\ldots,x_n)}(t) \geq v_{x_1,x_2,\ldots,x_n}\left(\frac{t}{k}\right),$$

for all $(x_1, x_2, \ldots, x_n) \in L_1^n$ and $t \in \mathbb{R}$. Thus, for every $\alpha \in (0, 1)$, there exist $k_\alpha > 0$ such that

$$v_{x_1,x_2,\ldots,x_n}\left(\frac{t}{k_\alpha}\right) \geq \alpha \Rightarrow \mu_{T(x_1,x_2,\ldots,x_n)}(t) \geq \alpha,$$

for all $(x_1, x_2, \ldots, x_n) \in L_1^n$ and $t \in \mathbb{R}$. So T is weakly random n-bounded of type I. □

Let $(L_1, \|\bullet, \bullet, \ldots, \bullet\|)$ and $(L_2, \|\bullet, \bullet, \ldots, \bullet\|)$ be n-normed spaces. Following is an extension of the notion of random n-bounded linear operator. In [15], A. L. Soenjaya introduced the notion of n-bounded linear operator of type II. Motivated by this paper, we will generalize this concept in R-n-NLS.

Definition 3.4 ([15])**.** An operator $T : (L_1, \|\bullet, \bullet, \ldots, \bullet\|) \to (L_2, \|\bullet, \bullet, \ldots, \bullet\|)$ is called n-bounded of type II, if there is a constant k such that for all $x_1, x_2, \ldots, x_n \in L_1$,

$$\|T(x_1), T(x_2), \ldots, T(x_n)\| \leq k \|x_1, x_2, \ldots, x_n\|.$$

Definition 3.5. Let $T : (L_1, v, T) \to (L_2, \mu, Q)$ be a linear operator, where (L_1, v, T) and (L_2, μ, Q) are R-n-NLS.

(i) The operator T is called weakly random n-bounded of type II on L_1, if for every $\alpha \in (0, 1)$, there exist $k_\alpha > 0$ such that

$$v_{x_1,x_2,\ldots,x_n}\left(\frac{t}{k_\alpha}\right) \geq \alpha \Rightarrow \mu_{T(x_1),T(x_2),\ldots,T(x_n)}(t) \geq \alpha,$$

for all $x_1, x_2, \ldots, x_n \in L_1$ and $t \in \mathbb{R}$.

(ii) The operator T is called strong random n-bounded of type II on L_1, if there exist a positive real number k such that

$$\mu_{T(x_1),T(x_2),\ldots,T(x_n)}(t) \geq v_{x_1,x_2,\ldots,x_n}\left(\frac{t}{k}\right),$$

for all $x_1, x_2, \ldots, x_n \in L_1$ and $t \in \mathbb{R}$.

Theorem 3.6. *Let $T : (L_1, v, T) \to (L_2, \mu, Q)$ be a linear operator, where (L_1, v, T) and (L_2, μ, Q) are R-n-NLS. If T is strong random n-bounded of type II, then it is weakly random n-bounded of type II.*

Proof. Suppose that T is strong random n-bounded of type II. Then there exists k such that

$$\mu_{T(x_1),T(x_2),\ldots,T(x_n)}(t) \geq v_{x_1,x_2,\ldots,x_n}\left(\frac{t}{k}\right),$$

for all $x_1, x_2, \ldots, x_n \in L_1$ and $t \in \mathbb{R}$.
Thus, for every $\alpha \in (0, 1)$, there exist $k_\alpha > 0$ such that

$$v_{x_1,x_2,\ldots,x_n}\left(\frac{t}{k_\alpha}\right) \geq \alpha \Rightarrow \mu_{T(x_1),T(x_2),\ldots,T(x_n)}(t) \geq \alpha,$$

for all $x_1, x_2, \ldots, x_n \in L_1$ and $t \in \mathbb{R}$.
So T is weakly random n-bounded of type II. □

The converse of previous theorem is not true as confirmed by the following counter example.

Example 3.7. Let $(L, \|\bullet, \bullet, \ldots, \bullet\|)$ be n-normed space. Define $T(a, b) = min(a, b)$ and $Q(a, b) = max(a, b)$ for all $a, b \in [0, 1]$. Now we define

$$v_{x_1,x_2,\ldots,x_n}(t) = \frac{t^2 - \|x_1, x_2, \ldots, x_n\|^2}{t^2 + \|x_1, x_2, \ldots, x_n\|^2} \text{ and } \mu_{T(x_1,x_2,\ldots,x_n)}(t) = \frac{t}{t + \|x_1, x_2, \ldots, x_n\|},$$

where $t > \|x_1, x_2, \ldots, x_n\|$. Let $T : (L, v, T) \to (L, \mu, Q)$ defined by $T(x) = x$ for all $x \in L$. We choose $k_\alpha = \frac{1}{1-\alpha}$, $\forall \alpha \in (0, 1)$. Then for $t > \|x_1, x_2, \ldots, x_n\|$. Then

$$
\begin{aligned}
& v_{x_1,x_2,\ldots,x_n}\left(\frac{t}{k_\alpha}\right) \geq \alpha \\
\Rightarrow & \frac{\frac{t^2}{k_\alpha^2} - \|x_1,x_2,\ldots,x_n\|^2}{\frac{t^2}{k_\alpha^2} + \|x_1,x_2,\ldots,x_n\|^2} \geq \alpha \\
\Rightarrow & \frac{t^2(1-\alpha)^2 - \|x_1,x_2,\ldots,x_n\|^2}{t^2(1-\alpha)^2 + \|x_1,x_2,\ldots,x_n\|^2} \geq \alpha \\
\Rightarrow & t^2(1-\alpha)^2 - \|x_1, x_2, \ldots, x_n\|^2 \geq \alpha t^2(1-\alpha)^2 + \alpha\|x_1, x_2, \ldots, x_n\|^2 \\
\Rightarrow & \|x_1, x_2, \ldots, x_n\|^2 \leq \frac{t^2(1-\alpha)^3}{1+\alpha} \\
\Rightarrow & \|x_1, x_2, \ldots, x_n\| \leq \frac{t(1-\alpha)(1-\alpha)^{\frac{1}{2}}}{(1+\alpha)^{\frac{1}{2}}} \\
\Rightarrow & t + \|x_1, x_2, \ldots, x_n\| \leq \frac{t\left((1-\alpha)(1-\alpha)^{\frac{1}{2}} + (1+\alpha)^{\frac{1}{2}}\right)}{(1+\alpha)^{\frac{1}{2}}} \\
\Rightarrow & \frac{t}{t + \|x_1,x_2,\ldots,x_n\|} \leq \frac{(1+\alpha)^{\frac{1}{2}}}{(1-\alpha)(1-\alpha)^{\frac{1}{2}} + (1+\alpha)^{\frac{1}{2}}}.
\end{aligned}
$$

Now,

$$\frac{(1+\alpha)^{\frac{1}{2}}}{(1-\alpha)(1-\alpha)^{\frac{1}{2}} + (1+\alpha)^{\frac{1}{2}}} \geq \alpha \Rightarrow 1 + \alpha + \alpha^3 \geq \alpha^2.$$

Thus, we have

$$\frac{t}{t + \|x_1, x_2, \ldots, x_n\|} \geq \alpha, \ \forall t \in (0, 1).$$

So,

$$v_{x_1,x_2,\ldots,x_n}\left(\frac{t}{k_\alpha}\right) \geq \alpha \Rightarrow \mu_{T(x_1,x_2,\ldots,x_n)}(t) \geq \alpha \Rightarrow \mu_{x_1,x_2,\ldots,x_n}(t) \geq \alpha.$$

Hence T is weakly random n-bounded of type II.

Now, for $t > \|x_1, x_2, \ldots, x_n\|$,

$$
\begin{aligned}
& \mu_{T(x_1),T(x_2),\ldots,T(x_n)}(t) \geq v_{x_1,x_2,\ldots,x_n}\left(\frac{t}{k}\right) \\
\Leftrightarrow & \frac{t}{t + \|T(x_1),T(x_2),\ldots,T(x_n)\|} \geq \frac{\left(\frac{t}{k}\right)^2 - \|x_1,x_2,\ldots,x_n\|^2}{\left(\frac{t}{k}\right)^2 + \|x_1,x_2,\ldots,x_n\|^2} \\
\Leftrightarrow & \frac{t}{t + \|x_1,x_2,\ldots,x_n\|} \geq \frac{t^2 - k^2\|x_1,x_2,\ldots,x_n\|^2}{t^2 + k^2\|x_1,x_2,\ldots,x_n\|^2} \\
\Leftrightarrow & k^2 \geq \left(\frac{t^2}{2t\|(x_1,x_2,\ldots,x_n)\| + \|x_1,x_2,\ldots,x_n\|^2}\right)^{\frac{1}{2}} \\
\Leftrightarrow & k = \infty \text{ as } t \to \infty.
\end{aligned}
$$

Then T is not strongly random n-bounded of type II.

3.2. α-n-norms in R-n-NLS.

Definition 3.8. Let (L, v, T) be a R-n-NLS satisfying $(R - n - N_6)$. Define

$$\|x_1, x_2, \ldots, x_n\|_\alpha = \inf\{t : v_{x_1,x_2,\ldots,x_n}(t) \geq \alpha\}, \alpha \in (0, 1).$$

Then $\{\|x_1, x_2, \ldots, x_n\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of n-norms on L.

These n-norms are called α-n-norms on L corresponding to R-n-NLS on L.

Definition 3.9. Let (L, v, T) be a R-n-NLS and $\alpha \in (0, 1)$. A sequence (x_k) in L is said to be α-n-convergent to l, if

$$\lim_{k\to\infty} v_{x_1,x_2,\ldots,x_{n-1},x_k-l}(t) \ > \alpha, \ \forall t > 0,$$

for every $x_1, x_2, \ldots, x_{n-1} \in L$.

Theorem 3.10. *Let (L, v, T) be a R-n-NLS satisfying $(R - n - N_6)$. If (x_n) is α-n-convergent sequence in (L, v, T), then*

$$\lim_{n\to\infty} \|x_1, x_2, \ldots, x_{n-1}, x_k - l\|_\alpha \ = 0, \quad \forall \alpha \in (0, 1)\,.$$

Proof. Let (x_k) be an α-n-convergent sequence in (L, v, T) and suppose that it be α-n-convergent to l. For every $t > 0$ and $x_1, x_2, \ldots, x_{n-1} \in L$, choose $\alpha \in (0, 1)$. Then

$$\lim_{k\to\infty} v_{x_1,x_2,\ldots,x_{n-1},x_k-l}\,(t) > \alpha.$$

Thus, for all $t > 0$, there exist $k_0(t) \in \mathbb{N}$ such that

$$v_{x_1,x_2,\ldots,x_{n-1},x_k-l}\,(t) > \alpha, \ \forall t \geq 0.$$

So

$$\|x_1, x_2, \ldots, x_{n-1}, x_k - l\|_\alpha < t, \ \forall k \geq k_0\,(t)\,.$$

Hence, since $t > 0$ is arbitrary,

$$\lim_{k\to\infty} \|x_1, x_2, \ldots, x_{n-1}, x_k - l\|_\alpha = 0, \ \forall \alpha \in (0, 1)\,.$$

□

Theorem 3.11. *Let (L, v, T) be a R-n-NLS satisfying $(R - n - N_6)$ and (x_k) be a sequence in L. Then (x_k) is random n-convergent to l in (L, v, T) iff (x_k) is n-convergent to l in $(L, \|\bullet, \bullet, \ldots, \bullet\|_\alpha)$, for each $\alpha \in (0, 1)$.*

Proof. Suppose that (x_k) is a convergent sequence in (L, v, T) to l. For every $t > 0$ and $x_1, x_2, \ldots, x_{n-1} \in L$, choose $\alpha \in (0, 1)$. Then there exist $k_0 \in \mathbb{N}$ such that

$$v_{x_1,x_2,\ldots,x_{n-1},x_k-l}\,(t) > 1 - \alpha, \ \forall n \geq k_0.$$

Thus

$$\|x_1, x_2, \ldots, x_{n-1}, x_k - l\|_{1-\alpha} \leq t, \ \forall n \geq k_0.$$

So

$$\lim_{n\to\infty} \|x_1, x_2, \ldots, x_{n-1}, x_k - l\|_{1-\alpha} \ = 0, \ \forall \alpha \in (0, 1)\,.$$

Conversely, choose $x_1, x_2, \ldots, x_{n-1} \in L^n$. Let

$$\lim_{n\to\infty} \|x_1, x_2, \ldots, x_{n-1}, x_k - l\|_\alpha \ = 0, \quad \forall \alpha \in (0, 1)\,.$$

Fix $\alpha \in (0, 1)$ and $r > 0$. Then there exist $k_0 \in \mathbb{N}$ such that

$$\inf \left\{r > 0 : v_{x_1,x_2,\ldots,,x_{n-1},x_k-l}\,(r) \geq 1 - \alpha\right\} < t,$$

for all $k \geq k_0$. Thus, for all $k \geq k_0$, there exist $0 < t_n < t$ such that

$$v_{x_1,x_2,\ldots,,x_{n-1},x_k-l}\,(t_n) \geq 1 - \alpha.$$

This implies that

$$v_{x_1,x_2,\ldots,,x_{n-1},x_k-l}\,(t) \geq 1 - \alpha,$$

for all $k \geq k_0$. So the sequence (x_k) is convergent to l in (L, v, T). □

Theorem 3.12. *Let (L_1, v, T) and (L_2, μ, Q) are R-n-NLS satisfying $(R - n - N_6)$. If the linear operator $T : (L_1, \|\bullet, \bullet, \dots, \bullet\|_\alpha) \to (L_2, \|\bullet, \bullet, \dots, \bullet\|_\alpha)$ is bounded with respect to α-n-norms corresponding to v and μ, for each $\alpha \in (0,1)$, then $T : (L_1, v, T) \to (L_2, \mu, Q)$ is weakly random n-bounded on L_1 of type II.*

Proof. Choose $x_1, x_2, \dots, x_n \in L_1$. For any $\alpha \in (0,1)$, there exist k_α such that for all $x_1, x_2, \dots, x_n \in L_1$,

$$\|T(x_1), T(x_2), \dots, T(x_n)\|_\alpha \leq k_\alpha \|x_1, x_2, \dots, x_n\|_\alpha .$$

Then for non zero $x_1, x_2, \dots, x_n$ and $t > 0$,

$$\begin{aligned} \|x_1, x_2, \dots, k_\alpha x_n\|_\alpha &\leq t \Rightarrow \|T(x_1), T(x_2), \dots, T(x_n)\|_\alpha \leq t. \\ inf\{r : v_{x_1,x_2,\dots,k_\alpha x_n}(r) \geq \alpha\} &\leq t \Rightarrow inf\left\{r : \mu_{T(x_1),T(x_2),\dots,T(x_n)}(r) \geq \alpha\right\} \leq t. \end{aligned}$$

i.e.,

$$\begin{aligned} inf\{r : v_{x_1,x_2,\dots,k_\alpha x_n}(r) \geq \alpha\} &\leq t \Leftrightarrow v_{x_1,x_2,\dots,k_\alpha x_n}(t) \geq \alpha, \\ inf\left\{r : \mu_{T(x_1),T(x_2),\dots,T(x_n)}(r) \geq \alpha\right\} &\leq t \Leftrightarrow \mu_{T(x_1),T(x_2),\dots,T(x_n)}(t) \geq \alpha. \end{aligned}$$

Thus, for any $\alpha \in (0,1)$, there exist $k_\alpha > 0$ such that for all $t \in \mathbb{R}$, $x_1, x_2, \dots, x_n \in L_1$,

$$v_{x_1,x_2,\dots,k_\alpha x_n}\left(\frac{t}{k_\alpha}\right) \geq \alpha \Rightarrow \mu_{T(x_1),T(x_2),\dots,T(x_n)}(t) \geq \alpha,$$

that is, T is weakly random n-bounded on L_1 of type II. □

Definition 3.13. Let (L, v, T) be a R-n-NLS and $\alpha \in (0,1)$. A sequence (x_k) in L is said to be $\alpha-n$-Cauchy, if

$$\lim_{k\to\infty} v_{x_1,x_2,\dots,x_{n-1},x_k - x_{k+p}}(t) \geq \alpha,$$

for all $t > 0, p = 1, 2, \dots$.

Theorem 3.14. *Let (L, v, T) be a R-n-NLS satisfying $(R - n - N_6)$. Then every n-Cauchy sequence in $(L,)$ is an α-n-Cauchy sequence in (L, v, T).*

Proof. Let $\alpha_0 \in (0,1)$ and (x_k) be a n-Cauchy sequence in $\left(L, \|\bullet, \bullet, \dots, \bullet\|_{\alpha_0}\right)$. Then

$$\lim_{k\to\infty} \|x_1, x_2, \dots, x_{n-1}, x_k - x_{k+p}\|_{\alpha_0} = 0, \; p = 1, 2, 3, \dots.$$

Thus for a given $\varepsilon > 0$, there exist a positive integer $N(\varepsilon)$ such that

$$\|x_1, x_2, \dots, x_{n-1}, x_k - x_{k+p}\|_{\alpha_0} < \alpha, \; \forall n \geq N(\varepsilon), \; p = 1, 2, 3, \dots.$$

It follows that

$$inf\left\{t > 0 : v_{x_1,x_2,\dots,x_{n-1},x_k - x_{k+p}}(t) \geq \alpha_0\right\} < \varepsilon, \; \forall n \geq N(\varepsilon), \; p = 1, 2, 3, \dots.$$

So, for all $n \geq N(\varepsilon), p = 1, 2, 3, \dots$ there exists $t(n, p, \varepsilon) < \varepsilon$ such that

$$v_{x_1,x_2,\dots,x_{n-1},x_k - x_{k+p}}(t(n, p, \varepsilon)) \geq \alpha_0.$$

Hence,

$$v_{x_1,x_2,\dots,x_{n-1},x_k - x_{k+p}}(\varepsilon) \geq \alpha_0, \text{ for all } n \geq N(\varepsilon), p = 1, 2, 3, \dots.$$

Therefore, since $\varepsilon > 0$ is arbitrary,

$$\lim_{k\to\infty} v_{x_1,x_2,\dots,x_{n-1},x_k - x_{k+p}}(\varepsilon) \geq \alpha_0 \text{ for all } t > 0.$$

This means that (x_k) is an α_0-n-Cauchy sequence in (L, v, T). Since $\alpha_0 \in (0,1)$ is arbitrary, every n-Cauchy sequence in $(L, \|\bullet, \bullet, \ldots, \bullet\|_\alpha)$ is an α-n-Cauchy sequence in (L, v, T), for each $\alpha \in (0,1)$. □

Theorem 3.15. *In R-n-NLS (L, v, T). Every α-n-convergent sequence is an α-n-Cauchy sequence.*

Proof. Assume that (x_k) is $\alpha-n$-convergent to l and $\alpha \in (0,1)$. Then we have

$$\lim_{k\to\infty} v_{x_1,x_2,\ldots,x_{n-1},x_k-x_{k-l}}(t) \geq \alpha \text{ for all } t > 0.$$

Now, for all $p = 1, 2, 3, \ldots,$

$$\begin{aligned} &v_{x_1,x_2,\ldots,x_{n-1},x_k-x_{k+p}}(t) \\ &= v_{x_1,x_2,\ldots,x_{n-1},x_k-x+x-x_{k+p}}\left(\tfrac{t}{2}+\tfrac{t}{2}\right) \\ &\geq T\left(v_{x_1,x_2,\ldots,x_{n-1},x_k-x}\left(\tfrac{t}{2}\right), v_{x_1,x_2,\ldots,x_{n-1},x-x_{k+p}}\left(\tfrac{t}{2}\right)\right). \end{aligned}$$

Thus

$$\begin{aligned} &\lim_{k\to\infty} v_{x_1,x_2,\ldots,x_{n-1},x_k-x_{k+p}}(t) \\ &\geq T\left(\lim_{k\to\infty} v_{x_1,x_2,\ldots,x_{n-1},x_k-x}\left(\tfrac{t}{2}\right), \lim_{k\to\infty} v_{x_1,x_2,\ldots,x_{n-1},x-x_{k+p}}\left(\tfrac{t}{2}\right)\right). \end{aligned}$$

So (x_k) is an α-n-cauchy sequence in (L, v, T). □

The converse of the previous theorem is in general not true as explained by the following example.

Example 3.16. Let $(L, \|\bullet, \bullet, \ldots, \bullet\|)$ be an n-normed space and defined $T(a,b) = min\{a,b\}$, for all $a, b \in [0,1]$. Define

$$v_{x_1,x_2,\ldots,x_n}(t) = \begin{cases} \frac{t}{t+r\|x_1,x_2,\ldots,x_n\|}, & t > 0,\ t \in \mathbb{R} \\ 0, & t \leq 0, \end{cases}$$

where $r > 0$. Then (L, v, T) is an R-n-NLS. Also,

(1) (x_k) is a n-Cauchy sequence in $(L, \|\bullet, \bullet, \ldots, \bullet\|)$ if and only if (x_k) is an $\alpha-n$-Cauchy sequence in (L, v, T).

(2) (x_k) is a n-convergent sequence in $(L, \|\bullet, \bullet, \ldots, \bullet\|)$ if and only if (x_k) is an $\alpha-n$-convergent sequence in (L, v, T).

3.3. **n-compact linear operator in R-n-NLS.**

Definition 3.17. A subset D of a R-n-NLS (L, v, T) is said to be n-random bounded, if there exist $t > 0$ and $0 < r < 1$

$$v_{x_1,x_2,\ldots,x_n}(t) \geq 1 - r,$$

for all $x_1, x_2, \ldots, x_n \in D$.

Definition 3.18. A subset D of a R-n-NLS (L, v, T) is said to be n-compact, if every sequence (x_k) in D has a subsequence converging to an element of D.

Definition 3.19. Let (L_1, v, T) and (L_2, μ, Q) are R-n-NLS. A linear operator $T : (L_1, v, T) \to (L_2, \mu, Q)$ is called a compact, if it maps every n-bounded random sequence (x_k) in L_1 onto a sequence $(T(x_k))$ in L_2 which has a convergent subsequence.

Example 3.20. Let $(L_1, \|\bullet, \bullet, \ldots, \bullet\|_1)$ and $(L_2, \|\bullet, \bullet, \ldots, \bullet\|_2)$ be two n-normed linear spaces and $T : (L_1, \|\bullet, \bullet, \ldots, \bullet\|_1) \to (L_2, \|\bullet, \bullet, \ldots, \bullet\|_2)$ be a compact operator, where v and μ are n-random norms induced by norms $(L_1, \|\bullet, \bullet, \ldots, \bullet\|_1)$ and $(L_2, \|\bullet, \bullet, \ldots, \bullet\|_2)$, respectively defined by

$$v_{x_1,x_2,\ldots,x_n}(t) = \begin{cases} \frac{t}{t+\|x_1,x_2,\ldots,x_n\|}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

$$\mu_{T(x_1),T(x_2),\ldots,T(x_n)}(t) = \begin{cases} \frac{t}{t+\left\|T(x_1),T(x_2),\ldots,T(x_n)\right\|}, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Lemma 3.21. *Let (L_1, v, T) and (L_2, μ, Q) be R-n-NLS. Let a linear operator $T : (L_1, v, T) \to (L_2, \mu, Q)$ be a n-compact operator satisfying $(R - n - N_6)$. If the n-norms $\|\bullet, \bullet, \ldots, \bullet\|_\alpha^1$ and $\|\bullet, \bullet, \ldots, \bullet\|_\alpha^2$ are α-n-norms induced v and μ respectively, then $T : \left(L_1, \|\bullet, \bullet, \ldots, \bullet\|_\alpha^1\right) \to \left(L_2, \|\bullet, \bullet, \ldots, \bullet\|_\alpha^2\right)$ is a n-compact operator for $\alpha \in (0, 1)$.*

Proof. We will show that for each bounded sequence (x_k) in $\left(L_1, \|\bullet, \bullet, \ldots, \bullet\|_\alpha^1\right)$, the sequence $(T(x_{n_k}))$ has a convergent subsequence in $\left(L_2, \|\bullet, \bullet, \ldots, \bullet\|_\alpha^2\right)$.

It is clear that there exists $t > 0$ such that

$$\|x_1, x_2, \ldots, x_{n-1}, x_k\|_\alpha^1 < t, \text{ for all } k \in \mathbb{N}.$$

Then

$$v_{x_1,x_2,\ldots,x_{n-1},x_k}(t) \geq \alpha, \text{ for all } k.$$

Thus (x_k) is random n-bounded in (L_1, v, T). So $(T(x_k))$ has a convergent subsequence $(T(x_{n_k}))$ in (L_2, μ, Q). Hence by Theorem 3.3, $(T(x_{n_k}))$ is convergent under $\|\bullet, \bullet, \ldots, \bullet\|_\alpha^2$. The proof is completed. □

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