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Abstract. The extension of the uniformity on the family of fuzzy subsets L^Y to a uniformity on the family of fuzzy subsets L^X; $Y \subseteq X$ and the restriction of the uniformity on L^Y to a uniformity on L^Y are defined and studied. The induced (fuzzy) quasi-uniformity on $P^*(L)^X$ for each given (fuzzy) quasi-uniformity on L^X is defined. Moreover, the induced $(P^*(L), M)-$fuzzy quasi-uniformity on $P^*(L)^X$, for each $(L, M)-$fuzzy quasi-uniformity on L^X is studied. In each case, the relation between their interior operators is obtained. Finally, the relation between the category $\mathbf{Qunif(L, M)}$ of all $(L, M)-$fuzzy quasi uniformity spaces and all quasi-uniformly continuous functions, and the category $\mathbf{FQunif(P^*(L), M)}$ of all $(P^*(L), M)-$fuzzy quasi uniformity spaces and quasi-uniformly continuous fuzzy functions is outlined. It is remarked that all kinds of categories of quasi-uniform spaces and quasi-uniformly continuous functions can be derived from the category $\mathbf{FQunif(P^*(L), M)}$.

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1. Introduction

The notion of a uniform space was introduced by Andre Weil [23] in 1937. The first systematic exposition of the theory of uniform spaces was given by Bourbaki [2] in 1940. The quasi-uniformity is a very important concept and a convenient tool for investigating topology. The L-quasi-uniformity, introduced by Hutton [11], has been accepted by many authors and has attracted wide attention in the literature [9, 15, 16, 17, 18, 19, 27]. Rodabaugh in [20, 21] introduced a theory of fuzzy uniformities with applications to the fuzzy real lines. The extension of Hutton's quasi-uniformities and $[0, 1]-$fuzzy uniformity were considered in [8]. Later, in [22], fuzzy uniformities for lattices more general than $[0,1]$, namely $(L,M)-$fuzzy uniformities were considered. Further, in [21], there is a significant extension of Hutton's approach for quasi-uniformities without using filters explicitly, without any distributivity and with general tensor products generating the intersection axiom. In [24, 25, 26], the relationship between $(L,M)-$fuzzy topologies and $(L,M)-$fuzzy quasi-uniformities was investigated. The uniform operator approach of Rodabaugh [21] as generalization of Hutton [11] is based on powersets of the form $(L^X)^{L^X}$. In [5], the $(P^*(L),2)$-fuzzy topology on the fuzzy space $P^*(L)^X$ was studied, which is induced by an $(L,2)$-fuzzy topological space on L^X, where the lattice $P^*(L)$ is defined by $P^*(L)=\{M\subset L: 0_L\in M\}$. Interesting relations between the $(P^*(L),2)-$fuzzy topology on $P^*(L)^X$ and the $(L,2)$-fuzzy topology on L^X were obtained. These results have been a motivation to study the quasi-uniformity spaces on $P^*(L)^X$ to find out its relation with the quasi-uniformity spaces on L^X, where L is a complete lattice.

The outline of this paper is as follows: In section 2, the basic concepts and useful results which will be used in the sequel are given. In section 3, the extension of the uniformity on the fuzzy space L^Y to a uniformity on the fuzzy space L^X; $Y\subset X$ and the restriction of the fuzzy uniformity on L^X to a fuzzy uniformity on L^Y are defined and studied. In each case, the relation between their interior operators is obtained.In section 4, the induced quasi-uniformity on $P^*(L)^X$ for each given quasi- uniformity on L^X is defined and a fundamental relation between their interior operators is obtained. In section 5, the induced $(P^*(L),M)-$fuzzy quasi-uniformity on $P^*(L)^X$ for each given $(L,M)-$fuzzy quasi-uniformity on L^X is studied and the relation between their interior operators is obtained. Moreover, the relation between the category $\mathbf{Qunif(L,M)}$ of all $(L,M)-$fuzzy quasi uniformity spaces and all quasi-uniformly continuous functions, and the category $\mathbf{FQunif(P^*(L),M)}$ of all $(P^*(L),M)-$fuzzy quasi uniformity spaces and quasi-uniformly continuous fuzzy functions is outlined. Finally, it is remarked that all kinds of categories of quasi-uniform spaces and quasi-uniformly continuous functions can be derived from the category $\mathbf{FQunif(P^*(L),M)}$.

2. Preliminaries

Let X be a given universal set and L be a given lattice. Denote the smallest element of L by $0_L$and the greatest element of L by 1_L. Also denote the smallest fuzzy subset of L^X by $\mathbf{0}_X$ and the greatest fuzzy subset of L^X by $\mathbf{1}_X$. In [4, 5, 6, 7], the lattice of the form $P^*(L)=\{M\subset L: 0_L\in M\}$ was used. The algebraic structure $(P^*(L),\cup,\cap,')$ forms a complemented, completely distributive and complete lattice with $0_{P^*(L)}=\{0_L\}$ being the smallest element and $1_{P^*(L)}=L$ being the greatest element. The complementary operation is defined by $\prime: P^*(L)\to P^*(L)$, where $M'=(L-M)\bigcup\{0_L\}$

Definition 2.1 ([4, 5, 6, 7]). (Algebra of $P^*(L)-$fuzzy subsets) Let $V,U\in P^*(L)^X$, then the operations on $P^*(L)-$fuzzy subsets of X are defined as follows:

(i) $V\subset U$, if $V(x)\subseteq U(x)$, for all $x\in X$,
(ii) $(V\cap U)(x)=V(x)\cap U(x)$, for all $x\in X$,
(iii) $(V\cup U)(x)=V(x)\cup U(x)$, for all $x\in X$,

(iv) $(V - U)(x) = (V(x) - U(x)) \cup \{0_L\}$, for all $x \in X$,
(v) $Co(V)(x) = (L - V(x)) \cup \{0_L\}$, for all $x \in X$.

Remark 2.2. It is important to note that on the $P^*(L)-$fuzzy subsets the difference operation is defined. The difference operation does not depend on the existence of any complementary operation on L. Moreover, the operations on the $P^*(L)-$fuzzy subsets are defined through the corresponding operations on the set of membership values.

Definition 2.3 ([6, 7])**.** A $P^*(L)-$fuzzy subset $p(x_0, \lambda)$ is said to be a fuzzy point of X, if

$$p(x_0, \lambda)(x) = \begin{cases} \{0_L, \lambda\}, & x = x_0, \\ \{0_L\}, & x \neq x_0, \end{cases}$$

where $\lambda \in L - \{0_L\}$.

The fuzzy point $p(x, \lambda)$ of X belongs to $V \in P^*(L)^X$, if $x \in \{x \in X : V(x) \neq \{0_L\}\}$ and $\lambda \in V(x)$.

The concept of fuzzy topology on a set X was introduced by C. L. Chang in [3] as a collection of fuzzy subsets of I^X (where $I = [0, 1]$ is the closed unit interval of real numbers), satisfying the known axioms of the topology. This definition is extended to $L-$topology, where L is a complete lattice. Kubiak in [14] generalized the $L-$topology by introducing the $(L, M)-$fuzzy topology.

Definition 2.4 ([14])**.** Let L, M be complete lattices. A mapping $\tau : L^X \to M$ is called an $(L, M)-$ fuzzy topology on X, if it satisfies the following conditions:
(i) $\tau(\mathbf{0}_X) = \tau(\mathbf{1}_X) = 1_M$,
(ii) $\tau(A \wedge B) \geq \tau(A) \wedge \tau(B)$, for every $A, B \in L^X$,
(iii) $\tau(\bigvee_i A_i) \geq \bigwedge_i \tau(A_i)$, for every $\{A_i; i \in \alpha\} \subseteq L^X$.
In this case, τ is called $(L, M)-$fuzzy topology, (X, L, M, τ) is called fuzzy topological space and $\tau(A)$ is called the degree of openness of A, for each $A \in L^X$.

Uniformity is an important concept in topology, and the history of uniform spaces goes back to the late thirties.
An important class of functions $\varphi : L^X \to L^X$ satisfing the following properties:
I. $A \leq \varphi(A)$, for all $A \in L^X$,
II. $\varphi(\bigvee_{i\in\triangle} A_i) = \bigvee_{i\in\triangle} \varphi(A_i)$, for all families $\{A_i; i \in \triangle\} \subseteq L^X$.
In this article, the set of all mappings, satisfying conditions (I),(II) in L^X will be denoted by $H(L^X)$.

Definition 2.5 ([12])**.** An $L-$quasi uniformity on X is a subset $\mathcal{D}$ of $H(L^X)$ such that
(LFU1) $\mathcal{D} \neq \varnothing$,
(LFU2) $\varphi \in \mathcal{D}, \varphi \leq \omega \in H(L^X) \Longrightarrow \omega \in \mathcal{D}$,
(LFU3) $\varphi, \omega \in \mathcal{D} \Longrightarrow \varphi \bigwedge \omega \in \mathcal{D}$, where $(\varphi \bigwedge \omega)(A) = \bigwedge_{A_1 \vee A_2 = A} (\varphi(A_1) \bigvee \omega(A_2))$,
(LFU4) $\varphi \in \mathcal{D} \Longrightarrow$ there exists $\omega \in \mathcal{D}$ such that $\omega o \omega \leq \varphi$.
An $L-$quasi uniformity $\mathcal{D}$ on X is called $L-$uniformity, if it satisfies the condition:
(LFU5) $\varphi \in \mathcal{D} \Longrightarrow \varphi^{-1} \in \mathcal{D}$, where $\varphi^{-1}(A) = \bigwedge\{B : \varphi(B') \leq A'\}$.
The pair $(X, \mathcal{D})$ is called an $L-$uniform space.

Proposition 2.6 ([12])**.** *$f \leq g$ if and only if $f^{-1} \leq g^{-1}$.*

Remark 2.7 ([9]). Let $(X, \mathcal{D})$ be an $L-$quasi-uniform space. The interior map is defined as follows:

(1) $Int\ :\ L^X \to L^X$, where

$$Int(V) = \bigvee\{U \in L^X\ : f(U) \leq V \ \text{ for some } f \in \mathcal{D}\}.$$

(2) If $Int\ :\ L^X \to L^X$ is an interior map, then $\tau = \{V \in L^X : Int(V) = V\}$ is a fuzzy topology.

Definition 2.8 ([17, 26]). An $L-$fuzzy quasi-uniformity is a mapping $\mathcal{U} : H(L^X) \to M$ such that

(FQU1) $\mathcal{U}(f_1) = 1_M$, where f_1 denotes the biggest element of $H(L^X)$, i.e.,

$$f_1(A) = \begin{cases} 0_X & : \ A = 0_X, \\ 1_X & : \ \text{otherwise}, \end{cases}$$

(FQU2) $\mathcal{U}(f \bigwedge g) = \mathcal{U}(f) \bigwedge \mathcal{U}(g)$, for all $f, g \in H(L^X)$,

(FQU3) $\mathcal{U}(f) = \bigvee_{g \circ g \leq f} \mathcal{U}(g)$, for all $f \in H(L^X)$.

The pair $(L^X, \mathcal{U})$ is called an $L-$fuzzy quasi-uniform space. Then any $L-$fuzzy quasi- uniformity is called an $L-$fuzzy uniformity if it also satisfies the following condition:

(FQU4) $\mathcal{U}(f) = \bigvee \mathcal{U}(f^{-1})$, for all $f \in H(L^X)$.

The interior operator is defined as follows.

Definition 2.9 ([13]). Let $(L^X, \mathcal{U})$ be an $L-$fuzzy quasi-uniform space. Define $\forall\ r \in L - \{1_L\}, A \in L^X$, the interior operator

$$Int_{\mathcal{U},r}(A) = \bigvee\{B \in L^X : f(B) \leq A, \mathcal{U}(f) > r\}.$$

This interior operator given by Kim [13] is different from Höle and Šostak $L-$fuzzy interior operator [10] in order to make it suitable to $L-$fuzzy uniformities.

Theorem 2.10 ([13]). *Let $(L^X, \mathcal{U})$ be an $L-$ fuzzy quasi-uniform space. The function $\tau_{\mathcal{U}} : L^X \to L$ is defined by: for all $A \in L^X$,*

$$\tau_{\mathcal{U}}(A) = \vee\{r \in L : Int_{\mathcal{U},r}(A) \geq A\}$$

is an $L-$ fuzzy topology on X.

3. Extended and restricted L-uniformities

Let X, Y be given ordinary sets, where $Y \subseteq X$ and L be a complete and completely distributive lattice. In this section, we discuss how to extend a given $L-$uniformity on L^Y to an $L-$uniformity on L^X. And conversely, we show that every $L-$uniformity on L^X induces an $L-$uniformity on L^Y. Through this article we shall consider only the families $D \subseteq (L^X)^{L^X}$ which satisfy conditions (I) and (II).

Notation 3.1. Let X, Y be given ordinary sets, where $Y \subseteq X$ and L be a given lattice. In this article, we shall use the following notations:

(1) For every $U \in L^X$, the restriction $U_{\downarrow Y}$ of U on L^Y is defined by:

$$U_{\downarrow Y}(x) = U(x); x \in Y.$$

(2) For every $A \in L^Y$ the extension $A_{\uparrow X}$ of A on L^X is defined by:

$$A_{\uparrow X}(x) = \begin{cases} A(x), & x \in Y, \\ 0_L, & x \in X - Y. \end{cases}$$

It is easy to notice that if $A_{\uparrow X} = U \vee V$ for some $U, V \in L^X$, then there exists $B, C \in L^Y$ such that $B_{\uparrow X} = U$ and $C_{\uparrow X} = V$.

(3) For every $f \in D \subseteq (L^X)^{L^X}$, the notation $f_{\downarrow Y} \in (L^Y)^{L^Y}$ denotes the restriction of f on L^Y which is defined by: $f_{\downarrow Y}(A) = (f(A_{\uparrow X}))_{\downarrow Y}$, $A \in L^Y$ or simply $(f(A_{\uparrow X}))_{\downarrow Y} \equiv f\,(A_{\uparrow X})_{\downarrow Y}$. The function $f_{\downarrow Y}$ is well defined, since $f \in H(L^X)$.

(4) For every $g \in G \subseteq (L^Y)^{L^Y}$, the notation $g_{\uparrow X} \in (L^X)^{L^X}$ denotes the extension of g on L^X which is defined by:

$$g_{\uparrow X}(U)(x) = \begin{cases} g(U_{\downarrow Y})(x), & x \in Y, \\ U(x), & x \in X - Y. \end{cases}$$

It is clear that $U \leq g_{\uparrow X}(U)$.

Lemma 3.2. *Let X, Y be given ordinary sets, where $Y \subseteq X$ and L be a given lattice. Then*

(1) $f \in H(L^X) \Rightarrow f_{\downarrow Y} \in H(L^Y)$,

(2) $g \in H(L^Y) \Rightarrow g_{\uparrow X} \in H(L^X)$.

Proof. (1) Let $f \in H(L^X)$. Then

(I) $A_{\uparrow X} \leq f(A_{\uparrow X})$, implies $A = (A_{\uparrow X})_{\downarrow Y} \leq (f(A_{\uparrow X}))_{\downarrow Y} = f_{\downarrow Y}(A)$.

(II) $f_{\downarrow Y}(\bigvee_i A_i) = (f((\bigvee_i A_i)_{\uparrow X}))_{\downarrow Y} = (f(\bigvee_i (A_i)_{\uparrow X}))_{\downarrow Y}$
$= (\bigvee_i f((A_i)_{\uparrow X}))_{\downarrow Y} = \bigvee_i f_{\downarrow Y}(A_i)$.

(2) $g \in H(L^Y)$. Then

(I) It is clear that $U \leq g_{\uparrow X}(U)$.

(II)

$$\begin{aligned} g_{\uparrow X}(\bigvee_i U_i)(x) &= \begin{cases} g((\bigvee_i U_i)_{\downarrow Y})(x), & x \in Y, \\ \bigvee_i U_i(x), & x \in X - Y \end{cases} \\ &= \begin{cases} \bigvee_i (g(U_i)_{\downarrow Y})(x), & x \in Y, \\ \bigvee_i U_i(x), & x \in X - Y \end{cases} \\ &= \bigvee_i \begin{cases} (g(U_i)_{\downarrow Y})(x), & x \in Y, \\ U_i(x), & x \in X - Y \end{cases} \\ &= \bigvee_i g_{\uparrow X}(U_i)(x). \end{aligned}$$

□

Lemma 3.3. *For every $f, h \in (L^X)^{L^X}$ and for every $g, k \in (L^Y)^{L^Y}$, the following relations are true:*

(1) $g_{\uparrow X} \bigwedge k_{\uparrow X} = (g \bigwedge k)_{\uparrow X}$, $g_{\uparrow X} \bigvee k_{\uparrow X} = (g \bigvee k)_{\uparrow X}$,

(2) $f_{\downarrow Y} \bigwedge h_{\downarrow Y} = (f \bigwedge h)_{\downarrow Y}$, $f_{\downarrow Y} \bigvee h_{\downarrow Y} = (f \bigvee h)_{\downarrow Y}$,

(3) $g, k \in (L^Y)^{L^Y}$, $g_{\uparrow X} = k_{\uparrow X} \Rightarrow g = k$,

(4) $(g_{\uparrow X})_{\downarrow Y} = g$,

(5) $g_{\uparrow X} \circ g_{\uparrow X} = (g \circ g)_{\uparrow X}$,

(6) $f_{\downarrow Y} \circ f_{\downarrow Y} = (f \circ f)_{\downarrow Y}$,
(7) $(g_{\uparrow X})^{-1} = (g^{-1})_{\uparrow X} = g^{-1}_{\uparrow X}$,
(8) $(f_{\downarrow Y})^{-1} = (f^{-1})_{\downarrow Y} = f^{-1}_{\downarrow Y}$.

Proof. (1)

$$\begin{aligned}
&(g_{\uparrow X} \bigwedge k_{\uparrow X})(U)(x)\\
&= \bigwedge_{U_1 \bigvee U_2 = U} (g_{\uparrow X}(U_1) \bigvee k_{\uparrow X}(U_2))(x)\\
&= \bigwedge_{U_1 \bigvee U_2 = U} \left(\left(\begin{cases} g\left(U_{1\downarrow Y}\right)(x), & x \in Y,\\ U_1(x), & x \in X - Y \end{cases} \right) \bigvee \left(\begin{cases} k\left(U_{2\downarrow Y}\right)(x), & x \in Y,\\ U_2(x), & x \in X - Y \end{cases} \right) \right)\\
&= \bigwedge\nolimits_{U_1 \bigvee U_2 = U} \begin{cases} (g\left(U_{1\downarrow Y}\right) \bigvee k\left(U_{2\downarrow Y}\right))(x), & x \in Y,\\ (U_1 \bigvee U_2)(x), & x \in X - Y \end{cases}\\
&= \begin{cases} \bigwedge_{U_{1\downarrow Y} \bigvee U_{2\downarrow Y} = U_{\downarrow Y}} (g\left(U_{1\downarrow Y}\right) \bigvee k\left(U_{2\downarrow Y}\right))(x), & x \in Y,\\ U(x), & x \in X - Y \end{cases}\\
&= \begin{cases} (g \bigwedge k)(U_{\downarrow Y})(x), & x \in Y,\\ U(x), & x \in X - Y \end{cases}\\
&= (g \bigwedge k)_{\uparrow X}(U)(x).
\end{aligned}$$

$$\begin{aligned}
(2)\quad (f_{\downarrow Y} \bigwedge h_{\downarrow Y})(A) &= \bigwedge\nolimits_{A_1 \bigvee A_2 = A} \left(f_{\downarrow Y}(A_1) \bigvee h_{\downarrow Y}(A_2) \right)\\
&= \bigwedge_{A_1 \bigvee A_2 = A} (f(A_{1\uparrow X})_{\downarrow Y} \bigvee h(A_{2\uparrow X})_{\downarrow Y})\\
&= \bigwedge_{A_{1\uparrow X} \bigvee A_{2\uparrow X} = A_{\uparrow X}} \left((f\left(A_{1\uparrow X}\right) \bigvee h\left(A_{2\uparrow X}\right))_{\downarrow Y} \right)\\
&= ((f \bigwedge h)(A_{\uparrow X}))_{\downarrow Y} = (f \bigwedge h)_{\downarrow Y}(A).
\end{aligned}$$

(3) The proof of (3) can be obtained directly.

$$\begin{aligned}
(4)\quad (g_{\uparrow X})_{\downarrow Y}(A_{\uparrow X})(x) &= [g_{\uparrow X}(A_{\uparrow X})(x)]_{\downarrow Y}\\
&= \left(\begin{cases} g\left((A_{\uparrow X})_{\downarrow Y}\right)(x), & x \in Y,\\ A_{\uparrow X}(x), & x \in X - Y \end{cases} \right)_{\downarrow Y}\\
&= \left(\begin{cases} g(A)(x), & x \in Y,\\ A_{\uparrow X}(x), & x \in X - Y \end{cases} \right)_{\downarrow Y}.
\end{aligned}$$

$$\begin{aligned}
(5)\quad (g_{\uparrow X} \circ g_{\uparrow X})(U)(x) &= g_{\uparrow X}(g_{\uparrow X}(U)(x))\\
&= g_{\uparrow X}\left(\begin{cases} g(U_{\downarrow Y})(x), & x \in Y,\\ U(x), & x \in X - Y \end{cases} \right)\\
&= \left(\begin{cases} g(g(U_{\downarrow Y}))(x), & x \in Y,\\ U(x), & x \in X - Y \end{cases} \right)\\
&= \left(\begin{cases} (g \circ g)(U_{\downarrow Y})(x), & x \in Y,\\ U(x), & x \in X - Y \end{cases} \right)
\end{aligned}$$

$$= (g \circ g)_{\uparrow X} (U) (x).$$

$$(6)\ \ (f_{\downarrow Y} \circ f_{\downarrow Y}) (A) = f_{\downarrow Y} (f_{\downarrow Y} (A)) = f_{\downarrow Y} \left(f (A_{\uparrow X})_{\downarrow Y} \right)$$
$$= [f((f(A_{\uparrow X})_{\downarrow Y})_{\uparrow X})]_{\downarrow Y}$$
$$\leq [f (f (A_{\uparrow X}))]_{\downarrow Y} = (f \circ f)_{\downarrow Y} (A).$$

(7) $\quad (g_{\uparrow X})^{-1} (U) (x)$

$$= \bigwedge \{V (x) : g_{\uparrow X} (V') \leq U'\}$$
$$= \bigwedge \left\{ V (x) : \left(\left\{ \begin{array}{ll} g (V'_{\downarrow Y}) (x), & x \in Y, \\ V'(x), & x \in X - Y \end{array} \right. \right) \leq U' (x) \right\}$$
$$= \left\{ \begin{array}{ll} \bigwedge \{V (x) \ : g (V'_{\downarrow Y}) (x) \leq U'_{\downarrow Y} (x) = (U_{\downarrow Y}(x))'\}, & x \in Y, \\ \bigwedge \{V (x) \ : V'(x) \leq U' (x)\}, & x \in X - Y \end{array} \right.$$
$$= \left\{ \begin{array}{ll} g^{-1} (U_{\downarrow Y}) (x), & x \in Y, \\ \bigwedge \{V (x) \ : U(x) \leq V (x)\}, & x \in X - Y \end{array} \right.$$
$$= \left\{ \begin{array}{ll} g^{-1} (U_{\downarrow Y}) (x), & x \in Y; \\ U (x), & x \in X - Y \end{array} \right.$$
$$= g^{-1}{}_{\uparrow X} (A) (x).$$

(8) $\quad f^{-1}{}_{\downarrow Y} (A) = f^{-1}(A_{\uparrow X})_{\downarrow Y}$

$$= \left(\bigwedge \{U \ : f (U') \leq (A_{\uparrow X})'\} \right)_{\downarrow Y}$$
$$= \bigwedge \{U_{\downarrow Y} \ : \ f (U') \leq (A_{\uparrow X})'\}$$
$$= \bigwedge \left\{ U_{\downarrow Y} \ : \ (A_{\uparrow X}) \leq (f (U'))' \right\}$$
$$= \bigwedge \left\{ U_{\downarrow Y} \ : \ A = (A_{\uparrow X})_{\downarrow Y} \leq ((f (U'))')_{\downarrow Y} = (f (U')_{\downarrow Y})' \right\}$$
$$= \bigwedge \left\{ U_{\downarrow Y} \ : f \left((U'_{\downarrow Y})_{\uparrow X} \right)_{\downarrow Y} \leq f (U')_{\downarrow Y} \leq A' \right\} = (f_{\downarrow Y})^{-1} (A).$$

□

Theorem 3.4. *Each $L-$uniformity $\mathcal{D}$ on L^Y; $Y \subset X$ can be extended to an $L-$uniformity $\mathcal{D}^*$ on L^X as follows: $\mathcal{D}^* = \{f : f \geq g_{\uparrow X}, g \in \mathcal{D}\}$.*

Proof. (i) Since $\mathcal{D} \neq \varnothing$, there exists $g \in \mathcal{D}$. Then $g_{\uparrow X} \in \mathcal{U}^* \neq \varnothing$.

(ii) Let $f, h \in \mathcal{D}^*$. then there exists $g, k \in \mathcal{D}$ such that $f \geq g_{\uparrow X}$ and $h \geq k_{\uparrow X}$. Thus

$$(f \bigwedge h) (U) (x)$$
$$= (\bigwedge_{U_1 \bigvee U_2 = U} (f(U_1) \bigvee h(U_2)))(x)$$
$$\geq (\bigwedge_{U_1 \bigvee U_2 = U} (g_{\uparrow X}(U_1) \bigvee k_{\uparrow X}(U_2)))(x)$$
$$= \left\{ \begin{array}{l} (\bigwedge_{U_{1\downarrow Y} \bigvee U_{2\downarrow Y} = U_{\downarrow Y}} (g(U_{1\downarrow Y}) \bigvee k(U_{2\downarrow Y})))(x), x \in Y \\ (\bigwedge_{U_1 \bigvee U_2 = U} (U_1 \bigvee U_2)(x), x \in X - Y \end{array} \right.$$

$$= \begin{cases} (g \bigwedge k)(U_{\downarrow Y})(x), x \in Y, \\ U, x \in X - Y \end{cases}$$

$$= (g \bigwedge k)_{\uparrow X}(U)(x).$$

So $g \bigwedge k \in \mathcal{D}$. Hence $f \bigwedge h \in \mathcal{U}^*$.

(iii) Let $h \geq f \in \mathcal{D}^*$. Then there exists $g \in \mathcal{D}$ such that $h \geq f \geq g_{\uparrow X}$. Thus $h \in \mathcal{D}^*$.

(iv) If $f \in \mathcal{D}^*$, then there exists $g \in \mathcal{D}$ and $f \geq g_{\uparrow X}$. Since $g^{-1} \in \mathcal{D}$ and $f^{-1} \geq (g_{\uparrow X})^{-1} = g^{-1}{}_{\uparrow X}$, $f^{-1} \in \mathcal{D}^*$.

(v) Let $f \in \mathcal{D}^*$. Then there exists $g \in \mathcal{D}$ and $f \geq g_{\uparrow X}$. Thus there exists $k \in \mathcal{D}$ such that $k \circ k \leq g$. It follows that $k_{\uparrow X} \circ k_{\uparrow X} = (k \circ k))_{\uparrow X} \leq g_{\uparrow X} \leq f$. □

Theorem 3.5. *If $Int_{\mathcal{D}}$ and $Int_{\mathcal{D}^*}$ are the interior operators on the $L-$uniformity $\mathcal{D}$ on L^Y and the extended $L-$uniformity $\mathcal{D}^*$ on L^X respectively, then*

$$(Int_{\mathcal{D}^*}(U))_{\downarrow Y} = Int_{\mathcal{D}}(U_{\downarrow Y});\ U \in L^X.$$

Proof. Let $B \in \left\{C \in L^Y : g(C) \leq U_{\downarrow Y};\ \text{for some } g \in \mathcal{D}\right\}$. Then there exists $g \in \mathcal{D}$ such that $g(B) \leq U_{\downarrow Y}$. Using the definition of extension of functions, we get

$$g_{\uparrow X}(B_{\uparrow X})(x) = \begin{cases} g(B)(x), & x \in Y, \\ 0, & x \in X - Y \end{cases} \leq U(x).$$

Since $g_{\uparrow X} \in \mathcal{D}^*$, $B_{\uparrow X} \in \left\{V \in L^X : f(V) \leq U;\ \text{for some } f \in \mathcal{D}^*\right\}$. Thus,

$$Int_{\mathcal{D}}(U_{\downarrow Y}) \leq (Int_{\mathcal{D}^*}(U))_{\downarrow Y}.$$

Conversely, let $W \in \left\{V \in L^X\ : f(V) \leq U;\ \text{for some } f \in \mathcal{D}^*\right\}$. Then there exists $f \in \mathcal{D}^*$ such that $f(W) \leq U$. And for every $f \in \mathcal{D}^*$, there exists $g \in \mathcal{D}$ such that $f \geq g_{\uparrow X}$. Thus, if $g_{\uparrow X}(W) \leq f(W) \leq U$, then we have:

$$g_{\uparrow X}(W)(x) = \begin{cases} g(W_{\downarrow Y})(x), & x \in Y; \\ W(x), & x \in X - Y \end{cases} \leq U(x).$$

So, we get $g(W_{\downarrow Y}) \leq U_{\downarrow Y}$ which implies that

$$W_{\downarrow Y} \in \left\{C \in L^Y\ : g(C) \leq U_{\downarrow Y};\ \text{for some } g \in U\right\}.$$

Hence $(Int_{\mathcal{D}^*}(U))_{\downarrow Y} \leq Int_{\mathcal{D}}(U_{\downarrow Y})$. □

Theorem 3.6. *Let $\mathcal{D}$ be an $L-$uniformity on L^Y and $\mathcal{D}^*$ be the extended an $L-$uniformity on L^X. If $U \in L^X$ is $\mathcal{D}^*-$open fuzzy subset, then $U_{\downarrow Y}$ is $\mathcal{D}-$open fuzzy subset.*

Corollary 3.7. *Let $\mathcal{D}$ be the $L-$uniformity on L^Y, $\mathcal{D}^*$ be the extended $L-$uniformity on L^X and $Y \subset X$. Then $\tau_{\mathcal{D}} = \left\{U_{\downarrow Y} \in L^Y : U \in \tau_{\mathcal{D}^*}\right\}$, where $\tau_{\mathcal{D}}$ and $\tau_{\mathcal{D}^*}$ are the induced topologies by $L-$uniformity $\mathcal{D}$ on L^Y and $L-$uniformity $\mathcal{D}^*$ on L^X, respectively.*

Theorem 3.8. *Each $L-$uniformity $\mathcal{D}$ on L^X defines a restricted $L-$uniformity $\mathcal{D}^*$ on L^Y as follows: $\mathcal{D}^* = \{f_{\downarrow Y} : f \in \mathcal{D}\}$, where $Y \subset X$.*

Proof. (i) Since $\mathcal{D} \neq \varnothing$, there exists $f \in \mathcal{D}$ such that $g = f_{\downarrow Y} \in \mathcal{D}^* \neq \varnothing$.

(ii) Let $g, k \in \mathcal{D}^*$, *then there exists* $f, h \in \mathcal{D}$ such that

$$g = f_{\downarrow Y} \in \mathcal{D}^*, k = h_{\downarrow Y} \in \mathcal{D}^*.$$

Then, $f \bigwedge h \in \mathcal{D}$. Thus

$$g \bigwedge k = f_{\downarrow Y} \bigwedge E_{\downarrow Y} = (f \bigwedge E)_{\downarrow Y} \in \mathcal{D}^*.$$

(iii) Let $g \geq k \in \mathcal{D}^*$. Then, there exists $f \in \mathcal{D}$ such that $k = f_{\downarrow Y}$.
Define $h \in (L^X)^{L^X}$ as follows:

$$h(U)(x) = \begin{cases} g(U_{\downarrow Y})(x), & x \in Y, \\ f(U)(x), & x \in X - Y \end{cases}.$$

Then it is clear that $h \geq f \in \mathcal{D}$. Thus $h \in \mathcal{D}, g = (h)_{\downarrow Y} \in \mathcal{D}^*$.

(iv) If $g \in \mathcal{D}^*$, *then there exists* $f \in \mathcal{D}$ such that $g = f_{\downarrow Y} \in \mathcal{D}^*$. This implies that $f^{-1} \in \mathcal{D}$. Thus $g^{-1} = (f_{\downarrow Y})^{-1} = f^{-1}_{\downarrow Y} \in \mathcal{D}^*$.

(v) Let $g \in \mathcal{D}^*$ *then there exists* $f \in \mathcal{D}$ such that $g = f_{\downarrow Y} \in \mathcal{D}^*$. Then there exists $h \in \mathcal{D}$ *such that* $h \circ h \leq f$. Thus,

$$h_{\downarrow Y} \circ h_{\downarrow Y} = (h \circ h)_{\downarrow Y} \leq f_{\downarrow Y} = g.$$

□

Theorem 3.9. *If $Int_{\mathcal{D}}$ and $Int_{\mathcal{D}^*}$ are the interior operators on the $L-$uniformity $\mathcal{D}$ on L^X and the restricted $L-$uniformity $\mathcal{D}^*$ on L^Y, then*

$$(Int_{\mathcal{D}}(U))_{\downarrow Y} = Int_{\mathcal{D}^*}(U_{\downarrow Y}), U \in L^X.$$

Proof. Let $V \in \{W : f(W) \leq U \ for \ some \ f \in \mathcal{D}\}$. Then there exists $f \in \mathcal{D}$ such that $f(V) \leq U$. It follows that

$$f_{\downarrow Y}(V_{\downarrow Y}) \leq (f(V))_{\downarrow Y} \leq U_{\downarrow Y}; \ f_{\downarrow Y} \in \mathcal{D}^*.$$

Thus

$$V_{\downarrow Y} \in \{B : g(B) \leq U_{\downarrow Y} \ for \ some \ g \in \mathcal{D}^*\},$$

where $g = f_{\downarrow Y}$. So, $(Int_{\mathcal{D}}(U))_{\downarrow Y} \leq Int_{\mathcal{D}^*}(U_{\downarrow Y})$.

Conversely, let

$$B \in \{C \ : g(C) \leq U_{\downarrow Y} \ for \ some \ g \in \mathcal{D}^*\}.$$

Then, there exists $g \in \mathcal{D}^*$ such that $g(B) \leq U_{\downarrow Y}$. Thus, there exists $f \in \mathcal{D}$ such that $g(B) = f_{\downarrow Y}(B) \leq U_{\downarrow Y}$. Using the definition of the restriction of functions, we get $(f(B_{\uparrow X}))_{\downarrow Y} \leq U_{\downarrow Y}$. So, $f(B_{\uparrow X}) \leq U$, $B_{\uparrow X} \leq Int_{\mathcal{D}}(U)$. Hence $B \leq (Int_{\mathcal{D}}(U))_{\downarrow Y}$. Therefore, $Int_{\mathcal{D}^*}(U_{\downarrow Y}) \leq (Int_{\mathcal{D}}(U))_{\downarrow Y}$ and the theorem is proved. □

Corollary 3.10. *If $Int_{\mathcal{D}}$ and $Int_{\mathcal{D}^*}$ are the interior operators on the $L-$uniformity $\mathcal{D}$ on L^X and the restricted $L-$uniformity $\mathcal{D}^*$ on L^Y, then*

$$(Int_{\mathcal{D}}(B_{\uparrow X}))_{\downarrow Y} = Int_{\mathcal{D}^*}(B), \ B \in L^Y.$$

Theorem 3.11. *Let $\mathcal{D}$ be an $L-$uniformity on L^X and $\mathcal{D}^*$ be the restricted $L-$uniformity on L^Y. If $U \in L^X$ is a $\mathcal{D}-$open fuzzy subset, then $U_{\downarrow Y}$ is a $\mathcal{D}^*-$open fuzzy subset.*

Corollary 3.12. *Let $\mathcal{D}$ be an $L-$uniformity on L^X and $\mathcal{D}^*$ be the restricted $L-$uniformity on L^Y, $Y \subset X$. Then $\tau_{\mathcal{D}^*} = \left\{ U_{\downarrow Y} \in L^Y : U \in \tau_{\mathcal{D}} \right\}$,where $\tau_{\mathcal{D}}$ and $\tau_{\mathcal{D}^*}$ are the induced topologies by the $L-$uniformity $\mathcal{D}$ on L^X and the $L-$uniformity $\mathcal{D}^*$ on L^Y .*

4. The induced $L-$uniformity on the family $P^*(L)^X$ due to a given $L-$uniformity on the family L^X

Let $U, V \in P^*(L)^X$ and $A, B \in L^X$. We shall say that $A \in U$, if $A(x) \in U(x)$, for every $x \in X$. It is clear that $U = V$, if $A \in U$ if and only if $A \in V$.

Definition 4.1. Every $A \in L^X$ defines $U_A, U_{A[}, U_{A]} \in P^*(L)^X$ as follows:
(i) $U_A(x) = \{0_L, A(x)\}$,
(ii) $U_{A]}(x) = [0_L, A(x)] = \{r \in L \ : 0_L \le r \le A(x)\}$ which is equivalent to

$$U_{A]} = \bigcup \{U_C : C \le A, C \in L^X\},$$

(iii) $U_{A[}(x) = [0_L, A(x)[= \{r \in L \ : 0_L \le r < A(x)\}$ which is equivalent to

$$U_{A[} = \bigcup \{U_C : C < A, C \in L^X\}.$$

One can show that the following lemma is valid.

Lemma 4.2. *If $A, \ A_i, B, B_i, C, C_i \in L^X$ and $U, V, U_i \in P^*(L)^X$, then the following properties are satisfied:*
(1) $A = \bigvee \{B : B \in U_A\}$,
(2) $A \vee B \in U_A \bigcup U_B, \ A \wedge B \in U_A \bigcup U_B$,
(3) *for every $A \in U$ and $B \in V$, $A \vee B \in U \cup V$,*
(4) *if $A \in \bigcup_{i\in\alpha} U_i$, then A can be written in the form*

$$A = \bigvee\nolimits_{i\in\alpha} A_i, \ A_i \in U_i, \ for\ all\ i \in \alpha,$$

where $A_i(x) = A(x)$, if $A(x) \in U_i(x)$ and $A_i(x) = 0_L$, if $A(x) \notin U_i(x)$,
(5) *if $B \le \bigvee_{i\in\alpha} C_i$, then B can be written in the form*

$$B = \bigvee_{i\in\alpha} B_i, \ B_i \le C_i, \ for\ all\ i \in \alpha,$$

where $B_i(x) = B(x)$, if $B(x) \le C_i(x)$ and $B_i(x) = 0_L$, if $B(x) > C_i(x)$.

Definition 4.3. Every function $f : L^X \to L^X$ induces the function

$$F_f : P^*(L)^X \to P^*(L)^X; \ F_f(U) = \bigcup \{U_B : B \le f(A), \ A \in U\}; \ U \in P^*(L)^X.$$

In this case, F_f is called the induced function by f in $P^*(L)^{\mathrm{X}}$.

Lemma 4.4. *If $f \in H(L^X)$, then*
(1) *The induced functions $F_f \in H(P^*(L)^X)$,*
(2) $F_f \circ F_g = F_{f\circ g}$,
(3) $F_f \cap G_g = N_{f\wedge g}$,
where $N_{f\wedge g}$ is the induced functions by $f \wedge g$ in $P^*(L)^X$.

Proof. (1) (I) F_f satisfies property (I). Since $A \leq f(A),\ A \in\ L^X$. Then

$$U = \bigcup\{U_A\ : A \in U\} \subset F_f(U) = \bigcup\{U_B : B \leq f(A), A \in U\}.$$

(II) F_f satisfies property (II). Since

$$\begin{aligned} F_f(\textstyle\bigcup_{i\in\alpha} U_i) &= \cup\{U_B\ : B \leq f(A), A \in \textstyle\bigcup_{i\in\alpha} U_i\} \\ &\supset \textstyle\bigcup\{U_B\ : B \leq f(A), A \in U_i\} \\ &= F^*_f(U_i),\ i \in \alpha. \end{aligned}$$

Then, $F_f(\bigcup_{i\in\alpha} U_i) \supset \bigcup_{i\in\alpha} F^*_f(U_i)$.

On one hand, for every $B \in F_f(\bigcup_{i\in\alpha} U_i)$, there exists $A \in L^X$ and $B \leq f(A), A \in \bigcup_{i\in\alpha} U_i$. The fuzzy subset A can be written in the form

$$A = \bigvee_{i\in\alpha} A_i,\ A_i \in U_i,\ i \in \alpha.$$

Then

$$f(A) \leq f(\bigvee_{i\in\alpha} A_i) = \bigvee_{i\in\alpha} f(A_i),\ A_i \in U_i,\ i \in \alpha.$$

Using Lemma 4.1, the fuzzy subset B can be written in the form

$$B = \bigvee_{i\in\alpha} B_i, B_i \leq f(A_i), i \in \alpha.$$

It is clear that $B_i \in F(U_i), i \in \alpha$. Thus,

$$B = \bigvee_{i\in\alpha} B_i \in \bigcup_{i\in\alpha} F^*_f(U_i),\ F_f(\textstyle\bigcup_{i\in\alpha} U_i) \subset \textstyle\bigcup_{i\in\alpha} F_f(U_i).$$

So the first requirement is proved.

(2)

$$\begin{aligned} (F_f \circ F_g)(U) &= F_f(F_g(U)) \\ &= \textstyle\bigcup\{U_B\ : B \leq f(A),\ A \in F_g(U)\} \\ &= \textstyle\bigcup\{U_B\ : B \leq f(A) \textit{ and } A \leq g(D),\ D \in U\} \\ &= \textstyle\bigcup\{U_B\ : B \leq f(A) \textit{ and } f(A) \leq f(g(D)),\ D \in U\} \\ &= \textstyle\bigcup\{U_B\ : B \leq f(g(D)),\ D \in U\} \\ &= \textstyle\bigcup\{U_B\ : B \leq (f \circ g)(D),\ D \in U\} \\ &= F_{f\circ g}(U). \end{aligned}$$

(3) $(F_f \cap G_g)(U)$

$$\begin{aligned} &= \textstyle\bigcap_{U_1\cup U_2=U}(F_f(U_1) \bigcup G_g(U_2)) \\ &= \textstyle\bigcap_{U_1\cup U_2=U}[\bigcup\{U_B\ : B \leq f(A),\ A \in U_1\} \bigcup(\bigcup\{V_D\ : D \leq g(C),\ C \in U_2\})] \\ &= \textstyle\bigcap_{U_1\cup U_2=U} \bigcup\{U_B\ : B \leq f(A) \vee g(C),\ A \in U_1,\ C \in U_2\} \\ &= \textstyle\bigcup\{U_B : B \leq \bigwedge_{A\vee C=D}[f(A) \vee g(C)];\ A \in U_1,\ C \in U_2,\ D \in U\} \\ &= \textstyle\bigcup\{U_B\ : B \leq (f \wedge g)(D);\ D \in U\} \\ &= H_{f\wedge g}(U), \text{ where } N_{f\wedge g} : P^*(L)^X \to P^*(L)^X \text{ is the induced map by } f \wedge g. \end{aligned}$$

If $D \in \bigcap_{U_1\cup U_2=U} \bigcup\{U_B : B \leq f(A) \vee g(C);\ A \in U_1,\ C \in U_2\}$, then for every $A \in U_1,\ C \in U_2$ and $U_1 \cup U_2 = U$. It is valid that $B \leq f(A) \vee g(C)$. Thus,

$$D \leq \bigwedge_{A\vee C=D} [f(A) \vee g(C)]$$

and

$$D \in \bigcup \left\{ U_B : B \leq \bigwedge_{A \vee C = D} [f(A) \vee g(C)];\ A \in U_1,\ C \in U_2,\ D \in U \right\}.$$

Conversely, If

$$D \in \bigcup \left\{ U_B : B \leq \bigwedge_{A \vee C = D} [f(A) \vee g(C)];\ A \in U_1,\ C \in U_2,\ D \in U \right\},$$

then $D \leq f(A) \vee g(C)$, for every $A \in U_1,\ C \in U_2$ and $U_1 \cup U_2 = U$. It follows that

$$D \in \bigcap_{U_1 \cup U_2 = U} \bigcup \{U_B : B \leq f(A) \vee g(C);\ A \in U_1,\ C \in U_2\}.$$

□

Theorem 4.5. *Each $L-$ quasi-uniformity $\mathcal{D}$ on a set L^X induces $L-$ quasi-uniformity $\mathcal{D}^*$ on $P^*(L)^X$, for every lattice L, where*

$$\mathcal{D}^* = \{G :\ G \geq F_f \text{ where } F_f \text{ is the induced function by } f;\ f \in \mathcal{D}\}.$$

Proof. (Q1) Since $\mathcal{D} \neq \varnothing$, there exists $f \in \mathcal{D}$ and $F_f \in \mathcal{D}^* \neq \varnothing$.

(Q2) Let $\Phi, \Psi \in \mathcal{D}^*$. Then there exists $f, g \in \mathcal{D}$ such that

$$\Phi(U) \geq F_f(U) = \bigcup \{U_B : B \leq f(A),\ A \in U\},$$

$$\Psi(V) \geq G_g(V) = \bigcup \{V_D : D \leq g(A),\ A \in V\}.$$

Thus, $(\Phi \bigcap \Psi)(U) = \bigcap_{U_1 \cup U_2 = U} (\Phi(U_1) \bigcup \Psi(U_2)$
$\geq \bigcap_{U_1 \cup U_2 = U} (F_f(U_1) \bigcup G_g(U_2))$
$= (F_f \cap G_g)(U) = H_{f \wedge g}(U)$.

Since $(f \wedge g) \in \mathcal{D}$, $H_{f \wedge g}$ and $(\Phi \bigcap \Psi) \in \mathcal{D}^*$.

(Q3) Let $F \geq G \in \mathcal{D}^*$. Then there exists $k \in \mathcal{D}$ such that $F \geq G \geq K_k$, where $K_k \in \mathcal{D}^*$ is the function, which is induced by the function $k \in \mathcal{D}$. It follows that $F \in \mathcal{D}^*$.

(Q4) Let $\Psi \in \mathcal{D}^*$. Then there exists $F_f \in \mathcal{D}^*$ and $\Psi \geq F_f$. Thus, there exists $g \in \mathcal{U}$ such that $g \circ g \leq f$. It follows that

$F_f(U) = \bigcup \{U_B : B \leq f(A),\ A \in U\}$
$\supseteq \bigcup \{U_B : B \leq (g \circ g)(A),\ A \in U\}$
$= \bigcup \{U_B : B \leq g(g(A)),\ A \in U\}$
$= \bigcup \{U_B : B \leq g(D),\ D = g(A) \in G(U)\} = (G_g \circ G_g)(U)$. □

Theorem 4.6. *If $Int : L^X \to L^X$ given by*

$$Int(A) = \bigvee \left\{ B \in L^X : f(B) \leq A, \quad \text{for some } f \in \mathcal{D} \right\}$$

is the interior map on the L-uniformity $\mathcal{D}$ on L^X
and
$Int^* : P^*(L)^X \to P^*(L)^X$ given by

$$Int^*(U) = \bigcup \{V \in P^*(L)^X : F(V) \subseteq U;\ \text{for some } F \in \mathcal{D}^*\}$$

is the interior map on the induced $P^*(L)-$uniformity $\mathcal{D}^*$ on $P^*(L)^X$, then

$$Int^*(U) = \bigcup \{U_A : A \leq Int(B),\ B \in U\}.$$

Proof. $Int^*(U) = \bigcup\{V : F_f(V) \subseteq U;\ F_f \in \mathcal{D}^*\}$
$= \bigcup\{V : A \in V \text{ then } B \in U \text{ for } B \leq f(A),\ f \in \mathcal{D}\}$
$= \bigcup\{U_A : B \in U \text{ for } B \leq f(A),\ f \in \mathcal{D}\}$
$= \bigcup\{U_A : A \leq \bigvee\{C : f(C) \leq B\};\ f \in \mathcal{D};\ B \in U\}$
$= \bigcup\{U_A : A \leq Int(B);\ B \in U\}.$ □

Corollary 4.7. *If $Int\ : L^X \to L^X$ is the interior map on the $L-$uniformity $\mathcal{D}$ on L^X and $Int^* : P^*(L)^X \to P^*(L)^X$ is the interior map on the induced $P^*(L)-$uniformity $\mathcal{D}^*$ on $P^*(L)^X$, then each $\mathcal{D}-$open fuzzy subset C defines $\mathcal{D}^*-$open fuzzy subsets $U_{C]}$ and $U_{C[}$, since*

$$Int^*(U_{C]}) = \bigcup \{U_A : A \leq Int(B);\ B \in U_{C]}\} = \bigcup \{U_A :\ A \leq C\} = U_{C]}$$

$$Int^*(U_{C[}) = \bigcup \{U_A : A \leq Int(B);\ B \in U_{C[}\} = \bigcup \{U_A : A < C\} = U_{C[}.$$

Corollary 4.8. *Each $\mathcal{D}-$fuzzy topology $\tau_{\mathcal{D}}$ on L^X defines $\mathcal{U}^*-$fuzzy topology $\tau^*_{\mathcal{D}}$ on $P^*(L)^X$ and $\tau^*_{\mathcal{D}} \supset \{U_{C]}, U_{C[} :\ C \in \tau_{\mathcal{D}}\}$.*

5. The induced $(P^*(L), M)-$fuzzy uniformity on the family $P^*(L)^X$ due to a given $(L, M)-$fuzzy uniformity on the family L^X

In this section, we define the induced $(P^*(L), M)-$fuzzy uniformity on $P^*(L)^X$ for every $(L, M)-$fuzzy uniformity on L^X.

Lemma 5.1. *Let $G \in H(P^*(L)^X)$. If $G \geq F_f$ for a function $f \in H(L^X)$, then there exists a unique function $h_G \in H(L^X)$ and $G \geq F_{h_G} \geq F_f$,*
(h_G is called the greatest associated function with G).

Proof. Let $G \geq F_f$, for some f. Consider the family of functions

$$F = \{k : G \geq F_k \text{ and } k \in H(L^X)\}.$$

It is clear that F is not empty, since $f \in F$. Define the function $h_G : L^X \to L^X$, where $h_G(A) = \bigvee_{k\in F} k(A)$. It is also clear that the function $h_G \in H(L^X)$. Since $A \leq f(A)$, $A \leq \bigvee_{k\in F} k(A) = h_G(A)$. Moreover,

$$h_G(\bigvee_{i\in\alpha} A_i) = \bigvee_{k\in F} k(\bigvee_{i\in\alpha} A_i) = \bigvee_{k\in F}(\bigvee_{i\in\alpha} k(A_i))$$
$$= \bigvee_{i\in\alpha}(\bigvee_{k\in F} k(A_i)) = \bigvee_{i\in\alpha} h_G(A_i).$$

The uniqueness of h_G follows from its definition. □

Notation 5.2. In this section, we use the notation $1_{L,X}$ for the greatest fuzzy subset in L^X, $1_{P^*(L),X}$ for the greatest fuzzy subset in $P^*(L)^X$, $0_{L,X}$ for the smallest subset in L^X and $0_{P^*(L),X}$ for the smallest subset in $P^*(L)^X$.

Lemma 5.3. *The associated function $h_F \in H(L^X)$ with the function $F \in H(P^*(L)^X)$ satisfies the following properties:*

(1) *if f_1 is the greatest element in $H(L^X)$, then F_{f_1} is the greatest element in $H(P^*(L)^X)$,*

(2) $h_{F_f} = f$,

(3) $h_{G\cap K} = h_G \wedge h_K$,
(4) $h_{F_f} \circ h_{F_g} = h_{F_{f\circ g}}$.

Proof. (1) Since f_1 is the greatest element in $H(L^X)$, $f_1(A) = 1_{L,X}$, for all $A \in L^X$ and $A \neq 0_{L,X}$. Then for $U \neq 0_{P^*(L),X}$, we have that
$F_{f_1}(U) = \bigcup\{U_B : B \leq f_1(A);\ A \in U\}$
$= \bigcup\left\{U_B : B \leq 1_{L,X};\ A \in U \neq 0_{P^*(L),X}\right\}$
$= 1_{P^*(L),X}$.

(2) Since $F_f(U) = \bigcup\{U_B\ : B \leq f(A),\ A \in U\},\ U \in P^*(L)^X$ and $F_f \geq F_f$, $h_{F_f} \geq f$. But, if $g > f$, then there exists $A \in L^X$ such that $g(A) > f(A)$ from which it follows that $F_g(U_A) > F_f(U_A)$. Thus $h_{F_f} = f$.

(3) Let $G, K \in H(P^*(L)^X)$. Then
$(G \cap K)(U) = \bigcap_{U_1\cup U_2=U}(G(U_1) \cap K(U_2))$
$\geq \bigcap_{U_1\cup U_2=U}(F_{h_G}(U_1) \cap F_{h_K}(U_2))$
$= (F_{h_G} \cap F_{h_K})(U) = F_{h_G\wedge h_K}(U)$.
It follows that $h_{G\cap K} \geq h_G \wedge h_K$.

On the other hand, since $G \supseteq G\cap K$ and $K \supseteq G\cap K$, $h_G \geq h_{G\cap K}$ and $h_K \geq h_{G\cap K}$. Thus, it follows that $h_G \wedge h_K \geq h_{G\cap K}$. So $h_{G\cap K} = h_G \wedge h_K$.

(4) Since $F_f \circ F_g = F_{f\circ g}$, from $h_{F_f} = f$ and $h_{F_g} = g$,

$$h_{F_f} \circ h_{F_g} = f \circ g = h_{F_{f\circ g}}.$$

□

Theorem 5.4. *Each $(L, M)-$fuzzy quasi uniformity $\mathcal{U} : H(L^X) \to M$ on L^X induces $(P^*(L), M)-$fuzzy quasi uniformity $\mathcal{U}^* : H(P^*(L)^X) \to M$ on $P^*(L)^X$, which is defined as follows:*
For every $G \in H(P^*(L)^X)$; $\mathcal{U}^*(G) = \mathcal{U}(h_G)$,
if there exists the greatest function $h_G : L^X \to L^X$ in $H(L^X)$ associated with G and if such greatest function h_G does not exist we put $\mathcal{U}^*(G) = 0_M$.

Proof. (FQU1) $\mathcal{U}^*(F_{f_1}) = \mathcal{U}(f_1 = 1_M$, where f_1 is the greatest element in $H(L^X)$.

(FQU2) Let $G, K \in P^*(L)^X$. Then we have
$\mathcal{U}^*(G \cap K) = \mathcal{U}(h_{G\cap K} = \mathcal{U}(h_G \wedge h_K$
$= \mathcal{U}(h_G) \bigwedge \mathcal{U}(h_K) = \mathcal{U}^*(G) \bigwedge \mathcal{U}^*(K)$.

(FQU3) Clearly, $\mathcal{U}^*(K) = \mathcal{U}(h_K) = \bigvee_{g\circ g\leq h_K} \mathcal{U}(g) = \bigvee_{F_g\circ F_g\leq F_{h_K}} \mathcal{U}(h_{F_g})$. Let $G \in \mathcal{U}^*$ satisfying that $G \circ G \leq F_{h_K}$. Then $(F_g \circ F_g)(U) \leq (G \circ G)(U) \leq F_{h_K}(U)$. But $h_G = h_{F_g}$. Thus we have

$$\mathcal{U}^*(K) = \bigvee_{G\circ G\leq F_{h_K}} \mathcal{U}(h_G) = \bigvee_{G\circ G\leq K} \mathcal{U}^*(G).$$

If $G \circ G \leq F_{h_K} \leq H \leq K$, then $h_H = h_K$. Otherwise, h_K is not the greatest element for K, which mean that if $h_H > h_K$, then $K \geq F_{h_H} > F_{h_K}$ which is a contradiction with the definition of h_K. It is known that if $\mathcal{U}^*$: $H(L^X) \to M$ is an $(L, M)-$fuzzy uniformity, then the family $\mathcal{U}^*_r = \{f : \mathcal{U}^*(f) \geq r\}$ is a uniformity on L^X for every $r \in M, r > 0_M$. □

Theorem 5.5. *If $\mathcal{U} : H(L^X) \to M$ is the $(L, M)-$fuzzy quasi uniformity on the family L^X and $\mathcal{U}^* : H(P^*(L)^X) \to M$ is the induced $(P^*(L), M)-$fuzzy quasi uniformity on the family $P^*(L)^X$, then*

$$Int_{\mathcal{U}^*,r}(U) = \bigcup \{U_A : A \leq Int_{\mathcal{U},r}(C),\ C \in U\}.$$

Corollary 5.6. *Let $\mathcal{U} : H(L^X) \to M$ be $(L, M)-$fuzzy quasi uniformity on the family L^X and $\mathcal{U}^* : H(P^*(L)^X) \to M$ be the induced $(P^*(L), M)-$fuzzy quasi uniformity on the family $P^*(L)^X$. Then $\mathcal{U}^*$ defines the $(P^*(L), M)-$fuzzy topology on $P^*(L)^X$ by the relation:*

$$\begin{aligned}\tau_{\mathcal{U}^*}(V) &= \bigvee \{r \in M : Int_{\mathcal{U}^*,r}(V) \supseteq V\} \\ &= \bigvee \{r \in M : \bigcup \{V_A : A \leq Int_{\mathcal{U},r}(C), C \in V\} \supseteq V\}.\end{aligned}$$

Corollary 5.7. *If $\mathcal{U} : H(L^X) \to M$ is the $(L, M)-$fuzzy quasi uniformity on the family L^X and $\mathcal{U}^* : H(P^*(L)^X) \to M$ is the induced $(P^*(L), M)-$fuzzy quasi uniformity on the family $P^*(L)^X$, then $\tau_{\mathcal{U}^*}(U_C]) = \tau_{\mathcal{U}}(C)$, if $\tau_{\mathcal{U}}(C) = r > 0$.*

Proof.

$$\begin{aligned}\tau_{\mathcal{U}^*}(U_{C]}) &= \bigvee \left\{r \in M : Int_{\mathcal{U}^*,r}(U_{C]}) \supseteq U_{C]}\right\} \\ &= \bigvee \left\{r \in M : \bigcup \left\{U_A : A \leq Int_{\mathcal{U},r}(B),\ B \in U_{C]}\right\} \supseteq U_{C]}\right\} \\ &= \bigvee \{r \in M : \bigcup \{U_A : A \leq Int_{\mathcal{U},r}(B), B \leq C\} \supseteq \{U_A : A \leq C\}\}] \\ &\geq \bigvee \{r \in M : Int_{\mathcal{U},r}(C) \geq C\} \\ &= \tau_{\mathcal{U}}(C)\end{aligned}$$ □

Definition 5.8 ([25, 26]). Let $(X,\mathcal{U}_1), (Y,\mathcal{U}_2)$ be two $(L, M)-$fuzzy quasi-uniform spaces. A function $F : (X,\mathcal{U}_1) \to (Y,\mathcal{U}_2)$ is said to be quasi-uniformly continuous function, if for every $g \in H(L^Y)$, $\mathcal{U}_2(g) \leq F_L{}^{\Leftarrow}(g)$, where, for all $A \in L^X$,

$$F_L{}^{\Leftarrow}(g)(A) = F_L{}^{\leftarrow}(g(F_L{}^{\rightarrow}(A))).$$

It is clear that the identity function $Id : (X,\mathcal{U}) \to (X,\mathcal{U})$ and the composition of the quasi-uniformly continuous functions are quasi- uniformly continuous function.

Definition 5.9 ([4, 5, 6, 7]). Let X, Y be given nonempty sets and L, K be given lattices. The fuzzy function $\mathbf{F} = (F, \{f_x\}_{x\in X})$ from L^X into K^Y or simply the fuzzy function $\mathbf{F} = (F, f_x) : X \to Y$ is defined as an ordered pair (F, f_x), where $F : X \to Y$, is a function from the set X to the set Y, and for all $x \in X$, $f_x : L \to K$ is a function from the lattice L to the lattice K, satisfying the following conditions:

(i) $f_x(0_L) = 0_K$ and $f_x(1_L) = 1_K$,
(ii) f_x is a non decreasing function, for all $x \in X$.

The action of the fuzzy function $\mathbf{F} = (F, f_x)$ on the $L-$fuzzy subsets A of X and the inverse image of the $K-$fuzzy subset B of Y are defined as follows:

$$\mathbf{F}_L^{\rightarrow}(A)(y) = \begin{cases} \bigvee_{y=F(x)} f_x(A(x)), & F^{-1}(y) \neq \varnothing, \\ 0_K, & F^{-1}(y) = \varnothing \end{cases},\ y \in Y,\ \textit{and}\ A \in L^X,$$

$$\mathbf{F}_L^{\leftarrow}(B)(x) = \bigvee f_x^{-1}(B(F(x))),\ x \in X\ \textit{and}\ B \in K^Y,$$

where the supremum is taken over the set of values $f_x^{-1}(B(F(x)))$.

The fuzzy function $\mathbf{F} = (F, f_x)$ from L^X to L^Y is called a uniform fuzzy function, if $f_x = f$; for all $x \in X$. The ordinary functions are embedded in the family of fuzzy functions as uniform fuzzy functions in which $f_x = id_L$ is the identity function.

Definition 5.10. Let $(X, \mathcal{U}_1), (Y, \mathcal{U}_2)$ be two $(L, M)-$fuzzy quasi-uniform spaces. A fuzzy function $\mathbf{F} = (F, f_x) : (X, \mathcal{U}_1) \to (Y, \mathcal{U}_2)$ is said to be quasi-uniformly continuous fuzzy function if for every $g \in H(L^Y)$, $\mathcal{U}_2(g) \leq \mathcal{U}_1(\mathbf{F}^{\Leftarrow}(g))$, where $\mathbf{F}^{\Leftarrow}(g) \in H(L^X)$ is defined as: for all $A \in L^X$, $\mathbf{F}^{\Leftarrow}(g)(A) = \mathbf{F}_L^{\leftarrow}(g(\mathbf{F}_L^{\rightarrow}(A)))$.

The definitions of category and related topics can be found in [1].

The family of all $(L, M)-$ fuzzy quasi-uniform spaces and quasi-uniformly continuous functions form a category that will be denoted by $\mathbf{Qunif(L, M)}$. While the family of all $(L, M)-$ fuzzy quasi-uniform spaces and quasi-uniformly continuous fuzzy functions form a category that will be denoted by $\mathbf{FQunif(L, M)}$.
Then, the following functor $\mathbf{R}$ is well defined as follows:

$$\mathbf{R} : \mathbf{Qunif(L,M)} \to \mathbf{FQunif}(\mathbf{P}^*(\mathbf{L}), \mathbf{M}); \quad \mathbf{R}(\mathcal{U}) = \mathcal{U}^*, \quad \mathbf{R}(\mathbf{F}) = (\mathbf{F}, \mathbf{id_L}),$$

where $\mathcal{U}^*$ is $(P^*(L), M)-$fuzzy quasi uniformity on the fuzzy family $P^*(L)^X$ which is induced by $(L, M)-$fuzzy quasi uniformity $\mathcal{U}$ on the fuzzy family L^X.

Theorem 5.11. *The functor* $\boldsymbol{R}$ *embedded the category* $\mathbf{Qunif(L, M)}$ *in the category* $\mathbf{FQunif}(\mathbf{P}^*(\mathbf{L}), \mathbf{M})$.

Proof. Any lattice L can be embedded in the lattice $P^*(L)$ by the embedding function $e : L \to P^*(L), e(r) = \{0_L, r\}$, which implies that the family L^X can be embedded in the family $P^*(L)^X$ by the embedding function $i : L^X \to P^*(L)^X, i(A)(x) = \{0_L, A(x)\}$. Then, the family $|\mathbf{Qunif(L, M)}|$ of all $(L, M)-$ fuzzy quasi-uniform spaces can be embedded in the family $|\mathbf{FQunif}(\mathbf{P}^*(\mathbf{L}), \mathbf{M})|$ of all $(P^*(L), M)-$ fuzzy quasi-uniform spaces by the one to one correspondence between $|\mathbf{Qunif(L, M)}|$ and $|\mathbf{FQunif}(\mathbf{L}^*, \mathbf{M})|$, where $L^* = \{\{o_L, r\} : r \in L\}$. Moreover, the family of Zadeh$'$s functions $\{F_L^{\rightarrow} : L^X \to L^Y\}$ can be embedded in the family of all fuzzy functions $\{\mathbf{F}_{P^*(L)}^{\rightarrow} : P^*(L)^X \to P^*(L)^Y\}$ by embedding $(F \to (F, id_L))$. This shows that the functor $\mathbf{R}$ is embedded and the proof is obtained. □

Remark 5.12. All kinds of categories of quasi-uniform spaces and quasi-uniformly continuous functions can be derived from the category $\mathbf{FQunif}(\mathbf{P}^*(\mathbf{L}), \mathbf{M})$ as follows:

(1) $\mathbf{FQunif}(\mathbf{P}^*(\mathbf{L}), \mathbf{M})$ derives the category **Qunif** of all ordinary quasi-uniform spaces and ordinary quasi-uniformly continuous functions, whenever

$$P^*(L) = P^*(\{0, 1\}), M = \{o, 1\}, \ \mathbf{F} = (F, id_{\{0,1\}})$$

(2) $\mathbf{FQunif}(\mathbf{P}^*(\mathbf{L}), \mathbf{M})$ derives the category **fQunif** of all fuzzifying quasi-uniform spaces and quasi-uniformly continuous functions, whenever

$$P^*(L) = P^*(\{0, 1\}), \ \mathbf{F} = (F, id_M),$$

(3) $\mathbf{FQunif}(\mathbf{P}^*(\mathbf{L}), \mathbf{M})$ derives the category **LQunif** of all $L-$quasi-uniform spaces and quasi-uniformly continuous functions, whenever

$$P^*(L) = \{\{0, r\} : r \in L\}, \ M = L, \ \mathbf{F} = (F, id_L)$$

(4) $\mathbf{FQunif}(\mathbf{P}^*(\mathbf{L}), \mathbf{M})$ derives the category $\mathbf{Qunif}(\mathbf{L}, \mathbf{M})$ of all $(L, M)-$quasi-uniform spaces and quasi-uniformly continuous fuzzy functions, whenever

$$P^*(L) = \{\{0, r\} : r \in L\},\ \mathbf{F} = (F, id_L).$$

6. Conclusion

From the study of $(L, M)-$quasi uniformity spaces and the $(P^*(L), M)-$quasi uniformity spaces we can advocate that every quasi uniformity in the category $\mathbf{Qunif}(\mathbf{L}, \mathbf{M})$ is isomorphic to at least one quasi uniformity in the category $\mathbf{FQunif}$ $(\mathbf{P}^*(\mathbf{L}), \mathbf{M})$. Moreover, all kinds of categories of quasi-uniform spaces and quasi-uniformly continuous functions can be derived from the category $\mathbf{FQunif}(\mathbf{P}^*(\mathbf{L}), \mathbf{M})$.

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