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Abstract. In this paper, we first introduce the notion of (α, β)-intuitionistic fuzzy order relation for any two positive reals numbers α and β such that $(\alpha, \beta) \in]0, 1] \times [0, 1[$ with $\alpha + \beta \leq 1$. Secondly, we prove the existence of a maximal and a minimal (α, β)-intuitionistic fuzzy fixed points. Also, we establish the existence of the greatest and the least (α, β)-intuitionistic fuzzy fixed points. Furthermore, we give an (α, β)-intuitionistic fuzzy version of Tarski's fixed point Theorem.

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1. Introduction

In [6], Zadeh introduced the concept of fuzzy set. Let X be a nonempty set, with a generic element of X denoted by x. A fuzzy subset A of X is characterized by its membership function $\mu_A : X \to [0, 1]$ and $\mu_A(x)$ is interpreted as the degree of membership of element x in the fuzzy subset A for each $x \in X$. Let A and B be two fuzzy subsets of X. We say that A is included in B and we write $A \subseteq B$, if $\mu_A(x) \leq \mu_B(x)$, for all $x \in X$. A fuzzy multifunction is a map $T : X \to [0, 1]$ such that for every $x \in X$, $T(x)$ is a nonempty fuzzy set.

In [1], K. Antanassov introduced the definition of intuitionistic fuzzy set. An intuitionistic fuzzy set $A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\}$ in X is characterized by a membership function μ_A which associate with each element x in X a real number $\mu_A(x)$ in the interval $[0, 1]$ and a non-membership function ν_A which associate with each element x in X a real number $\nu_A(x)$ in the interval $[0, 1]$ with the condition

$0 \leq \mu_A(x) + \nu_A(x) \leq 1$, for all $x \in X$. In the sequel, we shall write $A = (\mu_A, \nu_A)$. We will call $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ intuitionistic index of the element x in A.

An intuitionistic fuzzy multifunction is a map $T : X \to [0,1]^X \times [0,1]^X$ such that for every $x \in X$, $T(x)$ is a nonempty intuitionistic fuzzy set. Let $(\alpha, \beta) \in]0,1] \times [0,1[$ such that $\alpha + \beta \leq 1$ and let $T : X \to [0,1]^X \times [0,1]^X$ be an intuitionistic fuzzy multifunction. An element x of X is said to be an (α, β)-intuitionistic fuzzy fixed point of T, if $\mu_{T(x)}(x) = \alpha$ and $\nu_{T(x)}(x) = \beta$. The set of all (α, β)-intuitionistic fuzzy fixed points of T is noted by $Fix(T)^{(\alpha,\beta)}$. For the case $\alpha = 1$ and $\beta = 0$, we shall simply say that x is a fixed point of T. The set of all fixed point of T is denoted by $Fix(T)$.

Let $\alpha \in]0,1]$ and let $T : X \to [0,1]^X$ be a fuzzy multifunction. In [4], the present author introduced the concept of α-fuzzy fixed point of the fuzzy multifunction T as an element x of X satisfying $\mu_{T(x)}(x) = \alpha$. Note that for every $\alpha \in]0,1]$ the concept of α-fuzzy fixed point for a fuzzy multifunction T (if its exists) coincides with that of $(\alpha, 1-\alpha)$-intuitionistic fuzzy fixed point of the intuitionistic fuzzy multifunction $T^{'}$ defined for every $y \in X$ as follows by setting: $\mu_{T'(x)}(y) = \mu_{T(x)}(y)$ and $\nu_{T'(x)}(y) = 1 - \mu_{T(x)}(y)$.

In [3], the present author introduced the notion of α-fuzzy order. In [4], he proved some α-fuzzy fixed points theorems for α-fuzzy monotone multifunctions.

In [2], P. Burillo and H. Bustince gave a definition of intuitionistic fuzzy ordered sets. In this paper, we first introduce the notion of (α, β)-intuitionistic fuzzy order relation for any two positive reals numbers α and β such that $(\alpha, \beta) \in]0,1] \times [0,1[$ with $\alpha + \beta \leq 1$. Notice that for every $\alpha \in]0,1]$, if $R = \mu_R$ is an α-fuzzy order relation defined on a nonempty set X, then $S = (\mu_R, 1 - \mu_R)$ is an $(\alpha, 1-\alpha)$-intuitionistic fuzzy order relation on X. Conversely, if $R = (\mu_R, \nu_R)$ is an (α, β)-intuitionistic fuzzy order relation defined on a nonempty set X, then it induce an α-fuzzy order relation R_α and a $(1-\beta)$-fuzzy order relation R_β defined on X, by setting for every $(x, y) \in X \times X$, $R_\alpha(x, y) = \mu_R(x, y)$ and $R_\beta(x, y) = 1 - \nu_R(x, y)$. Thus, the concept of (α, β)-intuitionistic fuzzy order generalizes that of the notion of α-fuzzy order relation.

The present paper is organized as follows. In the second section we recall and state some definitions and results for subsequence use. In the third section, we give an (α, β)-intuitionistic fuzzy Zorn's lemma (see Theorem 3.1). By using this result we prove in the fourth section the existence of a maximal and a minimal (α, β)-intuitionistic fuzzy fixed points (see Theorems 4.2 and 4.3). In the fifth section, we establish the existence of the greatest and the least (α, β)-intuitionistic fuzzy fixed points (see Propositions 5.3 and 5.4). Furthermore, we give an (α, β)-intuitionistic fuzzy version of Tarski's fixed point Theorem [5] (see Theorem 5.2). As consequences we obtain the existence of a maximal, a minimal, a least and a greatest fixed points of intuitionistic fuzzy monotone maps and multifunctions. Also, we reobtain some α-fixed points results given in [4].

2. Preliminaries

In this section, we recall some useful definitions and results which we shall need in the sequel.

An intuitionistic fuzzy relation R on $X \times X$ is an intuitionistic fuzzy set $R = (\mu_R, \nu_R)$ where $\mu_R : X \times X \to [0,1]$ and $\nu_R : X \times X \to [0,1]$ satisfy for all $(x,y) \in X \times X$ the condition

$$0 \leq \mu_R(x,y) + \nu_R(x,y) \leq 1.$$

The following expression is defined in [2], for every two intuitionistic fuzzy subsets A and B:

$$A \leq B \Longleftrightarrow \mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x) \quad \forall x \in X.$$

Let R be an intuitionistic fuzzy relation defined on X. As in [2] we define the following intuitionistic fuzzy relation $R \circ^{\vee;\wedge}_{\wedge,\vee} R$ by:

$$R \circ^{\vee;\wedge}_{\wedge,\vee} R = \{< (x,z), \mu_{R\circ^{\vee;\wedge}_{\wedge,\vee}R}(x,z), \nu_{R\circ^{\vee;\wedge}_{\wedge,\vee}R}(x,z) > | x,z \in X\},$$

where

$$\mu_{R\circ^{\vee;\wedge}_{\wedge,\vee}R}(x,z) = \vee_y\{\wedge[\mu_R(x,y), \mu_R(y,z)]\} = \sup_{y\in X}\{\inf[\mu_R(x,y), \mu_R(y,z)]\}$$

and

$$\nu_{R\circ^{\vee;\wedge}_{\wedge,\vee}R}(x,z) = \wedge_y\{\vee[\nu_R(x,y), \nu_R(y,z)]\} = \inf_{y\in X}\{\sup[\nu_R(x,y), \nu_R(y,z)]\},$$

whenever for every $(x,z) \in X \times X$, we have:

$$0 \leq \mu_{R\circ^{\vee;\wedge}_{\wedge,\vee}R}(x,z) + \nu_{R\circ^{\vee;\wedge}_{\wedge,\vee}R}(x,z) \leq 1.$$

Next, we introduce the definition of (α,β)-intuitionistic fuzzy order relation for every $(\alpha,\beta) \in]0,1] \times [0,1[$ such that $\alpha + \beta \leq 1$.

Definition 2.1. Let $(\alpha,\beta) \in]0,1] \times [0,1[$ such that $\alpha + \beta \leq 1$ and let X be a nonempty set. An (α,β)-intuitionistic fuzzy order relation on X is an intuitionistic fuzzy relation $R = (\mu_R, \nu_R)$ satisfying the following three properties:

(i) ((α,β)-if-reflexivity) for all $x \in X$, $\mu_R(x,x) = \alpha$ and $\nu_R(x,x) = \beta$,

(ii) ((α,β)-if-antisymmetry) for all $(x,y) \in X \times X$,

$$(\mu_R(x,y) + \mu_R(y,x) > \alpha \text{ and } \nu_R(x,y) + \nu_R(y,x) < \beta + 1) \text{ imply } (x = y),$$

(iii) ((α,β)-if-transitivity) $R \circ^{\vee;\wedge}_{\wedge,\vee} R \leq R$.

Let X be a nonempty set and let R be an (α,β)-intuitionistic fuzzy order relation defined on it. Then, X is called an (α,β)-intuitionistic fuzzy ordered set and we denote it by (X,R).

Let (X,R) be an (α,β)-intuitionistic fuzzy ordered set such that $R = (\mu_R, \nu_R)$. The (α,β)-intuitionistic fuzzy order R is said to be total, if for all $x, y \in X$, we have either

$$\mu_R(x,y) > \frac{\alpha}{2} \text{ and } \nu_R(x,y) < \frac{\beta+1}{2}$$

or

$$\mu_R(y,x) > \frac{\alpha}{2} \text{ and } \nu_R(y,x) < \frac{\beta+1}{2}.$$

An (α,β)-intuitionistic fuzzy ordered set (X,R) on which R is a total (α,β)-intuitionistic fuzzy order is called an (α,β)-intuitionistic R-fuzzy chain.

Let (X,R) be a nonempty (α,β)-intuitionistic fuzzy ordered set such that $R = (\mu_R, \nu_R)$ and let A be a nonempty subset of X.

(i) An element $a \in X$ is a R-upper bound of A, if

$$\mu_R(x,a) > \frac{\alpha}{2} \text{ and } \nu_R(x,a) < \frac{\beta+1}{2} \text{ for all } x \in A.$$

If a is a R-upper bound of A and $a \in A$, then a is called a R-greatest element of A and we denote it by $\max_R(A)$.

(ii) An element $b \in X$ is a R-lower bound of A, if

$$\mu_R(b,x) > \frac{\alpha}{2} \text{ and } \nu_R(b,x) < \frac{\beta+1}{2} \text{ for all } x \in A.$$

If b is a R-lower bound of A and $b \in A$, then b is called a R-least element of A and we denote it by $\min_R(A)$.

(iii) An element $s \in X$ is the R-supremum of A, if s is the least R-upper bound of A. When s exists, we shall write $s = \sup_R(A)$.

Similarly, $\ell \in X$ is the R-infimum of A, if ℓ is the greatest R-lower bound of A. When ℓ exists, we shall write $\ell = \inf_R(A)$.

(iv) An element $m \in A$ is called a R-maximal element of A, if $\mu_R(m,y) > \frac{\alpha}{2}$ and $\nu_R(m,y) < \frac{\beta+1}{2}$, for some $y \in A$, then $y = m$.

(v) An element $n \in A$ is called a R-minimal element of A, if $\mu_R(y,n) > \frac{\alpha}{2}$ and $\nu_R(y,n) < \frac{\beta+1}{2}$, for some $y \in A$, then $y = n$.

Next, we shall give some examples of (α,β)-intuitionistic fuzzy order relations.

Example 2.2. (1) Let $(\alpha,\beta) \in]0,1] \times [0,1[$ such that $\alpha+\beta \leq 1$, $a = \min\{\frac{\alpha}{4}, \frac{1-\beta}{4}\}$ and let $R = (\mu_R, \nu_R)$ be the following intuitionistic fuzzy relation defined on the real line $\mathbb{R}$ by setting:

$$\begin{cases} \mu_R(x,x) = \alpha \\ \mu_R(x,y) = a \text{ if } x < y \\ \mu_R(x,y) = \alpha - a \text{ if } x > y \end{cases}$$

and

$$\begin{cases} \nu_R(x,x) = \beta \\ \nu_R(x,y) = 1 - a \text{ if } x < y \\ \nu_R(x,y) = a + \beta \text{ if } x > y. \end{cases}$$

Then, R is an (α,β)-intuitionistic fuzzy order relation on $\mathbb{R}$.

(2) Let $(\alpha,\beta) \in]0,1] \times [0,1[$ such that $\alpha+\beta \leq 1$, $b = \min\{\frac{\alpha}{8}, \frac{1-\beta}{8}\}$ and let $R = (\mu_R, \nu_R)$ be the following intuitionistic fuzzy relation defined on the set of all naturel numbers $\mathbb{N}$ by setting:

$$\begin{cases} \mu_R(n,n) = \alpha \\ \mu_R(n,m) = b \text{ if } n < m \\ \mu_R(n,m) = \alpha - b \text{ if } n > m \end{cases}$$

and

$$\begin{cases} \nu_R(n,n) = \beta \\ \nu_R(n,m) = 1 - b \text{ if } n < m \\ \nu_R(n,m) = b + \beta \text{ if } n > m. \end{cases}$$

Then, R is an (α,β)-intuitionistic fuzzy order relation on $\mathbb{N}$.

(3) Let $\alpha \in]0,1]$, le A be a finite set defined by $A = \{a,b,c\}$ and let $R = (\mu_R, \nu_R)$ be the following intuitionistic fuzzy relation defined on A by setting:

$\mu_R(.,.)$	a	b	c
a	α	$0,55\alpha$	$0,53\alpha$
b	$0,23\alpha$	α	$0,52\alpha$
c	$0,15\alpha$	$0,15\alpha$	α

and

$\nu_R(.,.)$	a	b	c
a	$\frac{1-\alpha}{2}$	$0,08\alpha$	$0,06\alpha$
b	$0,05\alpha$	$\frac{1-\alpha}{2}$	$0,02\alpha$
c	$0,05\alpha$	$0,08\alpha$	$\frac{1-\alpha}{2}$

Then, R is an $(\alpha, \frac{1-\alpha}{2})$-intuitionistic fuzzy order relation on A.

In [3], the present author introduced the notion of α-fuzzy order as follows.

Definition 2.3 ([3])**.** Let $\alpha \in]0,1]$ and let X be a nonempty set. An α-fuzzy order relation is a fuzzy relation $R = \mu_R$ on X satisfying the following three properties:
(i) (α-f-reflexivity) for all $x \in X$, $\mu_R(x,x) = \alpha$,
(ii) (α-f-antisymmetry) for all $(x,y) \in X \times X$,

$$\mu_R(x,y) + \mu_R(y,x) > \alpha \text{ implies } x = y,$$

(iii) (α-fuzzy transitivity) for all $x, z \in X$,

$$r_\alpha(x,z) \geq \sup_{y \in X} \left[\min\{r_\alpha(x,y), r_\alpha(y,z)\}\right].$$

Notice that for every $\alpha \in]0,1]$, if $R = \mu_R$ is an α-fuzzy order relation defined on a nonempty set X, then $S = (\mu_R, 1 - \mu_R)$ is an $(\alpha, 1-\alpha)$-intuitionistic fuzzy order relation on X. Conversely, if $R = (\mu_R, \nu_R)$ is an (α, β)-intuitionistic fuzzy order relation defined on a nonempty set X, then it induce an α-fuzzy order relation R_α and a $(1-\beta)$-fuzzy order relation R_β defined on X, by setting for every $(x,y) \in X \times X$, $R_\alpha(x,y) = \mu_R(x,y)$ and $R_\beta(x,y) = 1 - \nu_R(x,y)$. Thus, the concept of (α,β)-intuitionistic fuzzy order generalizes that of the notion of α-fuzzy order relation.

Next, we shall give the definition of an intuitionistic fuzzy inverse relation.

Definition 2.4. Let R be an intuitionistic fuzzy relation defined on a nonempty set X such that $R = (\mu_R, \nu_R)$. The intuitionistic fuzzy inverse relation $S = (\mu_S, \nu_S)$ of R is defined by $\mu_S(x,y) = \mu_R(y,x)$ and $\nu_S(x,y) = \nu_R(y,x)$.

Let $(\alpha,\beta) \in]0,1] \times [0,1[$ such that $\alpha + \beta \leq 1$ and let $T : X \to [0,1]^X \times [0,1]^X$ be an intuitionistic fuzzy multifunction. Then, for every $x \in X$ we define the following subset $T_x^{(\alpha,\beta)}$ by:

$$T_x^{(\alpha,\beta)} = \{y \in X : \mu_{T(x)}(y) = \alpha \text{ and } \nu_{T(x)}(y) = \beta\}.$$

In this paper, we shall use the following definitions of (α,β)-intuitionistic fuzzy monotonicity.

Definition 2.5. Let (X,R) be a nonempty (α,β)-intuitionistic fuzzy ordered set such that $R = (\mu_R, \nu_R)$ and let $T : X \to [0,1]^X \times [0,1]^X$ be an intuitionistic fuzzy multifuction. We say that T is an (α,β)-intuitionistic R-fuzzy monotone multifunction, if the two following properties are satisfied:
(i) for all $x \in X$, $T_x^{(\alpha,\beta)} \neq \varnothing$;

(ii) for every $x, y \in X$, with $x \neq y$, if $\mu_R(x,y) > \frac{\alpha}{2}$ and $\nu_R(x,y) < \frac{\beta+1}{2}$, then for all $a \in T_x^{(\alpha,\beta)}$ and $b \in T_y^{(\alpha,\beta)}$, we have $\mu_R(a,b) > \frac{\alpha}{2}$ and $\nu_R(a,b) < \frac{\beta+1}{2}$.

Let X be a nonempty set and $(\alpha,\beta) \in]0,1] \times [0,1[$ such that $\alpha + \beta \leq 1$ and let $T : X \to [0,1]^X \times [0,1]^X$ be an intuitionistic fuzzy multifunction. We say that an element x of X is an (α,β)-intuitionistic fuzzy fixed point of T, if $\mu_{T(x)}(x) = \alpha$ and $\nu_{T(x)}(x) = \beta$. We denote by $Fix(T)^{(\alpha,\beta)}$ the set of all (α,β)-intuitionistic fuzzy fixed points of T. For $\alpha = 1$ and $\beta = 0$, we shall say simply that x is a fixed point of T.

For $\alpha = 1$ and $\beta = 0$, we have the following definition.

Definition 2.6. Let (X,R) be a nonempty (α,β)-intuitionistic fuzzy ordered set such that $R = (\mu_R, \nu_R)$ and let $T : X \to [0,1]^X \times [0,1]^X$ be an intuitionistic fuzzy multifuction. We say that T is an intuitionistic R-fuzzy monotone multifunction, if the two following properties are satisfied:

(i) for all $x \in X$, $T_x^{(1,0)} \neq \varnothing$,

(ii) for every $x, y \in X$, with $x \neq y$, if $\mu_R(x,y) > \frac{1}{2}$ and $\nu_R(x,y) < \frac{1}{2}$, then for all $a \in T_x^{(1,0)}$ and $b \in T_y^{(1,0)}$, we have $\mu_R(a,b) > \frac{1}{2}$ and $\nu_R(a,b) < \frac{1}{2}$.

Definition 2.7. Let (X,R) be a nonempty (α,β)-intuitionistic fuzzy ordered set such that $R = (\mu_R, \nu_R)$ and let $f : X \to X$ be a map. We say that f is an (α,β)-intuitionistic R-fuzzy monotone map, if for every $x, y \in X$, with $\mu_R(x,y) > \frac{\alpha}{2}$ and $\nu_R(x,y) < \frac{1+\beta}{2}$, then we have $\mu_R(f(x), f(y)) > \frac{\alpha}{2}$ and $\nu_R(f(x), f(y)) < \frac{1+\beta}{2}$.

Let X be a nonempty set and let $f : X \to X$ be a map. We say that an element x of X is a fixed point of f if $f(x) = x$. The set of all fixed points of f is denoted by $Fix(f)$.

Notice that if f is a map from X to X, then f is also an intuitionistic fuzzy multifunction by taking for every $x, y \in X$ $\mu_{f(x)}(y) = \chi_{\{f(x)\}}(y)$ and $\nu_{f(x)}(y) = 1 - \chi_{\{f(x)\}}(y)$ with $\chi_{\{f(x)\}}$ is the characteristic map of the crisp set $\{f(x)\}$.

Definition 2.8. Let (X,R) be a nonempty (α,β)-intuitionistic fuzzy ordered set. We say that (X,R) is an (α,β)-intuitionistic fuzzy ordered complete set, if every nonempty (α,β)-intuitionistic R-fuzzy chain has a R-supremum.

Next, we shall give some useful results concerning intuitionistic fuzzy inverse order relations.

Proposition 2.9. *Let (X,R) be a nonempty (α,β)-intuitionistic fuzzy ordered set such that $R = (\mu_R, \nu_R)$ and let S be its intuitionistic fuzzy inverse relation such that $S = (\mu_S, \nu_S)$. Then, we have*

(1) *S is an (α,β)-intuitionistic fuzzy order relation on X,*

(2) *every (α,β)-intuitionistic R-fuzzy monotone multifunction $T : X \to [0,1]^X \times [0,1]^X$ is also (α,β)-intuitionistic S-fuzzy monotone,*

(3) *if a nonempty subset A of X has a R-infimum, then A has a S-supremum and $\inf_R(A) = \sup_S(A)$,*

(4) *if a nonempty subset A of X has a R-supremum, then A has a S-infimum and $\sup_R(A) = \inf_S(A)$,*

(5) *let (X,R) be a nonempty (α,β)-intuitionistic fuzzy ordered set. Then, every nonempty (α,β)-intuitionistic R-fuzzy chain is also an (α,β)-intuitionistic S-fuzzy chain,*

(6) *if* (X, R) *is an* (α, β)*-intuitionistic fuzzy ordered complete set, then* (X, S) *is also an* (α, β)*-intuitionistic fuzzy ordered complete set.*

Proof. Let (X, R) be a nonempty (α, β)-intuitionistic fuzzy ordered set such that $R = (\mu_R, \nu_R)$ and let S be its intuitionistic fuzzy inverse relation such that $S = (\mu_S, \nu_S)$.

(1) We shall prove that S is an (α, β)-intuitionistic fuzzy order relation on X.

(i) (α, β)-intuitionistic fuzzy reflexivity. Let $x \in X$. As R is (α, β)-intuitionistic fuzzy reflexive, we get $\mu_R(x, x) = \alpha$ and $\nu_R(x, x) = \beta$. Then, we have $\mu_S(x, x) = \alpha$ and $\nu_S(x, x) = \beta$. Thus, S is (α, β)-intuitionistic fuzzy reflexive relation.

(ii) (α, β)-intuitionistic fuzzy antisymmetry. Let $x, y \in X$ such that

$$\mu_S(x, y) + \mu_S(y, x) > \alpha \text{ and } \nu_S(x, y) + \nu_S(y, x) < \beta.$$

Then, we get

$$\mu_R(y, x) + \mu_R(x, y) > \alpha \text{ and } \nu_R(y, x) + \nu_S(x, y) < \beta.$$

As R is (α, β)-intuitionistic fuzzy antisymmetric, we obtain $x = y$. Then S is (α, β)-intuitionistic fuzzy antisymmetric relation.

(iii) Intuitionistic fuzzy transitivity. Let $x, y, z \in X$. Then

$$\begin{aligned}\mu_{S\circ^{\vee;\wedge}_{\wedge;\vee}S}(x, z) &= \vee_y\{\wedge[\mu_S(x, y), \mu_S(y, z)]\} = \vee_y\{\wedge[\mu_R(y, x), \mu_R(z, y)]\}\\ &= \vee_y\{\wedge[\mu_R(z, y), \mu_R(y, x)]\} = \mu_{R\circ^{\vee;\wedge}_{\wedge;\vee}R}(z, x) \leq \mu_R(z, x).\end{aligned}$$

Thus for every $x, z \in X$, we have $\mu_{S\circ^{\vee;\wedge}_{\wedge;\vee}S}(x, z) \leq \mu_S(x, z)$.

On the other hand, we have

$$\begin{aligned}\nu_{S\circ^{\vee;\wedge}_{\wedge;\vee}S}(x, z) &= \wedge_y\{\vee[\nu_S(x, y), \nu_S(y, z)]\} = \wedge_y\{\vee[\nu_R(y, x), \nu_R(z, y)]\}\\ &= \wedge_y\{\vee[\nu_R(z, y), \nu_R(y, x)]\} = \nu_{R\circ^{\vee;\wedge}_{\wedge;\vee}R}(z, x)\\ &\geq \nu_R(z, x).\end{aligned}$$

So for every $x, z \in X$, we get

$$\nu_{S\circ^{\vee;\wedge}_{\wedge;\vee}S}(x, z) \geq \nu_S(x, z).$$

Hence, S is (α, β)-intuitionistic fuzzy transitive.
Therefore, S is an (α, β)-intuitionistic fuzzy order relation on X.

(2) Let $T : X \to [0, 1]^X \times [0, 1]^X$ be an (α, β)-intuitionistic R-fuzzy monotone multifunction. Let $x \in X$. Then, $T_x^{(\alpha,\beta)} \neq \varnothing$. Let $x, y \in X$ such that $x \neq y$, $\mu_S(x, y) > \frac{\alpha}{2}$, $\nu_S(x, y) < \frac{\beta+1}{2}$, $a \in T_x^{(\alpha,\beta)}$ and $b \in T_y^{(\alpha,\beta)}$. Then, we have $\mu_R(y, x) > \frac{\alpha}{2}$ and $\nu_R(y, x) < \frac{\beta+1}{2}$. Since T is (α, β)-intuitionistic R-fuzzy monotone, we get $\mu_R(b, a) > \frac{\alpha}{2}$ and $\nu_R(b, a) < \frac{\beta+1}{2}$. Thus, we obtain $\mu_S(a, b) > \frac{\alpha}{2}$ and $\nu_S(a, b) < \frac{\beta+1}{2}$. So, T is (α, β)-intuitionistic S-fuzzy monotone multifunction.

(3) Let A be a nonempty subset of X such that $\ell = \inf_R(A)$. Then, for every $x \in A$, we have $\mu_R(\ell, x) > \frac{\alpha}{2}$ and $\nu_R(\ell, x) < \frac{\beta+1}{2}$. Thus, for every $x \in A$, we get $\mu_S(x, \ell) > \frac{\alpha}{2}$ and $\nu_S(x, \ell) < \frac{\beta+1}{2}$. So, ℓ is a S-upper bound of A.

Now, let k be another S-upper bound of A. Then, for every $x \in A$, we get $\mu_S(x, k) > \frac{\alpha}{2}$ and $\nu_S(x, k) < \frac{\beta+1}{2}$. Thus, for every $x \in A$, we get $\mu_R(k, x) > \frac{\alpha}{2}$ and $\nu_R(k, x) < \frac{\beta+1}{2}$. So, k is a R-lower bound of A. As $\ell = \inf_R(A)$, $\mu_R(k, \ell) > \frac{\alpha}{2}$ and $\nu_R(k, \ell) < \frac{\beta+1}{2}$. Hence, we get $\mu_S(\ell, k) > \frac{\alpha}{2}$ and $\nu_S(\ell, k) < \frac{\beta+1}{2}$. Therefore, ℓ is the least S-upper bound of A and thus, we deduce that we have $\ell = \sup_S(A)$.

(4) Let A be a nonempty subset A such that $m = \sup_R(A)$. Then, for every $x \in A$, we have $\mu_R(x, m) > \frac{\alpha}{2}$ and $\nu_R(x, m) < \frac{\beta+1}{2}$. Thus, for every $x \in A$, we get $\mu_S(m, x) > \frac{\alpha}{2}$ and $\nu_S(m, x) < \frac{\beta+1}{2}$. So, m is a S-lower bound of A.

Now, let n be an another S-lower bound of A. Then, for every $x \in A$, we get $\mu_S(n, x) > \frac{\alpha}{2}$ and $\nu_S(n, x) < \frac{\beta+1}{2}$. Thus, for every $x \in A$,, we get $\mu_R(x, n) > \frac{\alpha}{2}$ and $\nu_R(x, n) < \frac{\beta+1}{2}$. So, n is a R-upper bound of A. As $m = \sup_R(A)$, $\mu_R(m, n) > \frac{\alpha}{2}$ and $\nu_R(m, n) < \frac{\beta+1}{2}$. Hence, we get $\mu_S(n, m) > \frac{\alpha}{2}$ and $\nu_S(n, m) < \frac{\beta+1}{2}$. Therefore, m is the greatest S-lower bound of A and thus, we deduce that we have $m = \inf_S(A)$.

(5) Let C be a nonempty (α, β)-intuitionistic R-fuzzy chain. Then for every $x, y \in C$, we have

$$(\mu_R(x, y) > \frac{\alpha}{2} \text{ and } \nu_R(x, y) < \frac{\beta+1}{2}) \text{ or } (\mu_R(y, x) > \frac{\alpha}{2} \text{ and } \nu_R(y, x) < \frac{\beta+1}{2}).$$

Thus, we get

$$(\mu_S(y, x) > \frac{\alpha}{2} \text{ and } \nu_S(y, x) < \frac{\beta+1}{2}) \text{ or } (\mu_S(x, y) > \frac{\alpha}{2} \text{ and } \nu_S(x, y) < \frac{\beta+1}{2}).$$

So, C is a nonempty (α, β)-intuitionistic S-fuzzy chain.

(6) From (4) and (5), we easily deduce (6). □

In the sequel, we shall need the following result.

Lemma 2.10. *Let (X, R) be a nonempty (α, β)-intuitionistic fuzzy ordered set such that $R = (\mu_R, \nu_R)$. Let $x, y_0, z \in X$ such that*

$$\mu_R(x, y_0) > \frac{\alpha}{2}, \mu_R(y_0, z) > \frac{\alpha}{2}, \nu_R(x, y_0) < \frac{\beta+1}{2} \text{ and } \nu_R(y_0, z) < \frac{\beta+1}{2}.$$

Then, $\mu_R(x, z) > \frac{\alpha}{2}$ and $\nu_R(x, z) < \frac{\beta+1}{2}$.

Proof. Let (X, R) be a nonempty (α, β)-intuitionistic fuzzy ordered set such that $R = (\mu_R, \nu_R)$. Let $x, y_0, z \in X$ such that

$$\mu_R(x, y_0) > \frac{\alpha}{2}, \mu_R(y_0, z) > \frac{\alpha}{2}, \nu_R(x, y_0) < \frac{\beta+1}{2} \text{ and } \nu_R(y_0, z) < \frac{\beta+1}{2}.$$

Then, we obtain

$$\inf[\mu_R(x, y_0), \mu_R(y_0, z)] > \frac{\alpha}{2} \text{ and } \sup[\mu_R(x, y_0), \mu_R(y_0, z)] < \frac{\beta+1}{2}.$$

Thus, we get

$$\sup_{y \in X}\{\inf[\mu_R(x, y), \mu_R(y, z)]\} > \frac{\alpha}{2} \text{ and } \inf_{y \in X}\{\sup[\mu_R(x, y), \mu_R(y, z)]\} < \frac{\beta+1}{2}.$$

On the oner hand, we know that

$$\mu_{R\circ^{\vee;\wedge}_{\wedge;\vee}R}(x, z) = \sup_{y \in X}\{\inf[\mu_R(x, y), \mu_R(y, z)]\}$$

and

$$\nu_{R\circ^{\vee;\wedge}_{\wedge;\vee}R}(x, z) = \inf_{y \in X}\{\sup[\nu_R(x, y), \nu_R(y, z)]\}.$$

Then, we obtain

$$\mu_{R\circ^{\vee;\wedge}_{\wedge;\vee}R}(x, z) > \frac{\alpha}{2} \text{ and } \nu_{R\circ^{\vee;\wedge}_{\wedge;\vee}R}(x, z) < \frac{\beta+1}{2}.$$

On the other hand, by our hypothesis, we know that R is (α,β)-intuitionistic fuzzy transitive. Thus, we have $R \circ^{\vee,\wedge}_{\wedge,\vee} R \leq R$. So, we get

$$\mu_{R\circ^{\vee,\wedge}_{\wedge,\vee}R}(x,z) \leq \mu_R(x,z) \text{ and } \nu_{R\circ^{\vee,\wedge}_{\wedge,\vee}R}(x,z) \geq \nu_R(x,z).$$

Hence, we obtain $\mu_R(x,z) > \frac{\alpha}{2}$ and $\nu_R(x,z) < \frac{\beta+1}{2}$. □

3. An (α,β)-intuitionistic fuzzy Zorn's lemma

In this section, we shall give an (α,β)-intuitionistic fuzzy Zorn's Lemma. More precisely we shall show the following result.

Theorem 3.1. *Let (X,R) be a nonempty (α,β)-intuitionistic fuzzy ordered set such that every nonempty (α,β)-intuitionistic R-fuzzy chain in X has an intuitionistic R-upper bound. Then, (X,R) has at least a R-maximal element.*

In order to prove Theorem 3.1, we shall show that every (α,β)-intuitionistic fuzzy order relation R defined on a nonempty set X induce a crisp order relation noted $\leq_R$ defined for every $(x,y) \in X^2$ by:

$$(x \leq_R y) \Leftrightarrow (\mu_R(x,y) > \frac{\alpha}{2} \text{ and } \nu_R(x,y) < \frac{\beta+1}{2}).$$

Lemma 3.2. *Let (X,R) be a nonempty (α,β)-intuitionistic fuzzy ordered set such that $R=(\mu_R,\nu_R)$ and let $\leq_R$ the crisp relation defined on X as above. Then $\leq_R$ is a crisp order relation.*

Proof. Let (X,R) be a nonempty (α,β)-intuitionistic fuzzy ordered set such that $R=(\mu_R,\nu_R)$ and let $\leq_R$ be the crisp relation defined on X, for every $(x,y)\in X^2$ by:

$$(x \leq_R y) \Leftrightarrow (\mu_R(x,y) > \frac{\alpha}{2} \text{ and } \nu_R(x,y) < \frac{\beta+1}{2}).$$

(i) Reflexivity: Let $x \in X$. As R is (α,β)-intuitionistic fuzzy reflexive and $0<\alpha$, $\mu_R(x,y)=\alpha > \frac{\alpha}{2}$. On the other hand, $0 \leq \beta < 1$. Then, we get $\nu_R(x,y)=\beta<\frac{\beta+1}{2}$. Thus, for every $x\in X$, we have $x \leq_R x$. So, $\leq_R$ is a crisp reflexive relation.

(ii) Antisymmetry: Let $x,y \in X$ such that $x \leq_R y$ and $y \leq_R x$. Then, we have

$$\mu_R(x,y) > \frac{\alpha}{2} \text{ and } \nu_R(x,y) < \frac{\beta+1}{2}$$

and

$$\mu_R(y,x) > \frac{\alpha}{2} \text{ and } \nu_R(y,x) < \frac{\beta+1}{2}.$$

Thus, we get

$$\mu_R(x,y)+\mu_R(y,x) > \alpha \text{ and } \nu_R(x,y)+\nu_R(y,x) < \beta+1.$$

On the other hand, we know that R is (α,β)-intuitionistic fuzzy antisymmetric. So, we deduce that we have $x=y$.

(iii) Transitivity: Let $x,y_0,z\in X$ such that $x \leq_R y_0$ and $y_0 \leq_R z$. Then, we get

$$\mu_R(x,y_0) > \frac{\alpha}{2} \text{ and } \nu_R(x,y_0) < \frac{\beta+1}{2}$$

and

$$\mu_R(y_0,z) > \frac{\alpha}{2} \text{ and } \nu_R(y_0,z) < \frac{\beta+1}{2}.$$

Thus, from Lemma 2.9, we get $\mu_R(x,z) > \frac{\alpha}{2}$ and $\nu_R(x,z) < \frac{\beta+1}{2}$. So, $\leq_R$ is a crisp transitive relation. Hence, we conclude that $\leq_R$ is a crisp order relation on X. □

Now, we are in position to prove Theorem 3.1.

Proof. Let (X,R) be a nonempty (α,β)-intuitionistic fuzzy ordered set such that every nonempty (α,β) intuitionistic R-fuzzy chain in X has an intuitionistic R-lower bound. Let $R = (\mu_R, \nu_R)$ and let $\leq_R$ the crisp order relation defined on X as Lemma 3.2. Let C be a nonempty (α,β)-intuitionistic R-fuzzy chain in X. Then, for every $x, y \in C$, we have

$$(\mu_R(x,y) > \frac{\alpha}{2} \text{ and } \nu_R(x,y) < \frac{\beta+1}{2}) \text{ or } (\mu_R(y,x) > \frac{\alpha}{2} \text{ and } \nu_R(y,x) < \frac{\beta+1}{2}).$$

Thus, C is a nonempty chain in $(X, \leq_R)$. By our hypothesis, C has a R-upper bound, s, say. So, for every $x \in C$, we have

$$\mu_R(x,s) > \frac{\alpha}{2} \text{ and } \nu_R(x,s) < \frac{\beta+1}{2}.$$

Hence, for every $x \in C$, we get $x \leq_R s$ and thus , s is a $\leq_R$-upper bound of C. Therefore, every nonempty chain in $(X, \leq_R)$ has a $\leq_R$-upper bound. From the crisp Zorn's Lemma, we deduce that $(X, \leq_R)$ has at least a maximal element, m, say.

Now, assume that there is an element x in X such that

$$\mu_R(m,x) > \frac{\alpha}{2} \text{ and } \nu_R(m,x) < \frac{\beta+1}{2}.$$

Then, $m \leq_R x$. As m is a maximal element in $(X, \leq_R)$, we deduce that we have $x = m$. Thus, m is a R-maximal element. □

As a consequence of Theorem 3.1, we reobtain the α-fuzzy Zorn's Lemma [[3], Lemma 3.6].

Corollary 3.3. *Let (X,R) be a nonempty α-fuzzy ordered set such that every nonempty α-fuzzy R-fuzzy chain has a R-upper bound. Then, (X,R) has at least a R-maximal element.*

For $\alpha = 1$ and $\beta = 0$, we have the following result.

Corollary 3.4. *Let (X,R) be a nonempty intuitionistic fuzzy ordered set such that every nonempty intuitionistic R-fuzzy chain in X has an intuitionistic R-upper bound. Then, (X,R) has at least a R-maximal element.*

4. Maximal and minimal (α,β)-intuitionistic fuzzy fixed points

In this section, we shall establish the existence of a maximal and a minimal (α,β)-intuitionistic fuzzy fixed points of (α,β)-intuitionistic fuzzy monotone multifunctions.

In this section we shall need the following key result.

Proposition 4.1. *Let (X,R) be a nonempty (α,β)-intuitionistic fuzzy complete ordered set such that $R=(\mu_R,\nu_R)$. Let $T: X\to[0,1]^X\times[0,1]^X$ be an (α,β)-intuitionistic R-fuzzy monotone multifunction. If there exist $a,b\in X$ such that $\mu_{T(a)}(b)=\alpha$, $\nu_{T(a)}(b)=\beta$, $\mu_R(a,b)>\frac{\alpha}{2}$ and $\nu_R(a,b)<\frac{\beta+1}{2}$, then the set*

$$P_{(\alpha,\beta)}=\left\{x\in X:\exists y\in T_x^{(\alpha,\beta)},\mu_R(x,y)>\frac{\alpha}{2}\ \textit{and}\ \nu_R(x,y)<\frac{\beta+1}{2}\right\}$$

is nonempty and has at least a R-maximal element.

Proof. Let (X,R) be a nonempty (α,β)-intuitionistic fuzzy complete ordered set such that $R=(\mu_R,\nu_R)$. Let $T: X\to[0,1]^X\times[0,1]^X$ be an (α,β)-intuitionistic R-fuzzy monotone multifunction. Assume that there exists $a,b\in X$ such that $\mu_{T(a)}(b)=\alpha$, $\nu_{T(a)}(b)=\beta$, $\mu_R(a,b)>\frac{\alpha}{2}$ and $\nu_R(a,b)<\frac{\beta+1}{2}$. Consider the following subset $P_{(\alpha,\beta)}$ of X defined by

$$P_{(\alpha,\beta)}=\left\{x\in X:\exists y\in T_x^{(\alpha,\beta)},\mu_R(x,y)>\frac{\alpha}{2}\ \text{and}\ \nu_R(x,y)<\frac{\beta+1}{2}\right\}.$$

By our hypothesis, $a\in P_{(\alpha,\beta)}$. Then, we get $P_{(\alpha,\beta)}\neq\varnothing$.

Claim 1: It is clear that the subset $(P_{(\alpha,\beta)},R)$ is a nonempty (α,β)-intuitionistic fuzzy complete ordered set. Indeed, if C is a nonempty (α,β)-intuitionistic R-fuzzy chain in $P_{(\alpha,\beta)}$, then we set $s=\sup_R(C)$. Thus, we claim that $s\in P_{(\alpha,\beta)}$.

On the contrary, assume that $s\notin P_{(\alpha,\beta)}$. Then, for every $x\in C$, we have $x\neq s$, $\mu_R(x,y)>\frac{\alpha}{2}$ and $\nu_R(x,y)<\frac{\beta+1}{2}$.

On the one hand, we know that $C\subset P_{(\alpha,\beta)}$. Then, for every $x\in C$, there exists $y\in X$, such that $\mu_{T(x)}(y)=\alpha$, $\nu_{T(x)}(y)=\beta,\mu_R(x,y)>\frac{\alpha}{2}$ and $\nu_R(x,y)<\frac{\beta+1}{2}$. By our hypothesis, T is a (α,β)-intuitionistic R-fuzzy monotone multifunction. Thus, $T_s^{(\alpha,\beta)}\neq\varnothing$. So, there is $t\in X$ such that $\mu_{T(s)}(t)=\alpha$, $\nu_{T(t)}(s)=\beta$. As T is (α,β)-intuitionistic R-fuzzy monotone, $x\neq s$, $\mu_R(x,y)>\frac{\alpha}{2}$ and $\nu_R(x,y)<\frac{\beta+1}{2}$, we get

$$\mu_R(y,t)>\frac{\alpha}{2}\ \text{and}\ \nu_R(y,t)<\frac{\beta+1}{2}.$$

On the other hand, we know that we have

$$\mu_R(x,y)>\frac{\alpha}{2}\ \text{and}\ \nu_R(x,y)<\frac{\beta+1}{2}.$$

From Lemma 2.9, we deduce that for every $x\in C$, we have

$$\mu_R(x,t)>\frac{\alpha}{2}\ \text{and}\ \nu_R(x,t)<\frac{\beta+1}{2}.$$

Then, t is a R-upper bound of C. As $s=\sup_R(C)$, we get

$$\mu_R(s,t)>\frac{\alpha}{2}\ \text{and}\ \nu_R(s,t)<\frac{\beta+1}{2}.$$

Thus, $s\in P_{(\alpha,\beta)}$. That is a contradiction. So, $s\in P_{(\alpha,\beta)}$. Hence, $(P_{(\alpha,\beta)},R)$ is a nonempty (α,β)-intuitionistic fuzzy complete ordered set.

Claim 2: The subset $(P_{(\alpha,\beta)},R)$ has at least a R-maximal element. Indeed, by Claim 1, $(P_{(\alpha,\beta)},R)$ is a nonempty (α,β)-intuitionistic fuzzy complete ordered set.

Then, by (α, β)-intuitionistic fuzzy Zorn's Lemma (Theorem 3.1), we deduce that $(P_{(\alpha,\beta)}, R)$ has at least a R-maximal element, m, say. □

Next, we shall prove the existence of an (α, β)-intuitionistic R-maximal fuzzy fixed point.

Theorem 4.2. *Let (X, R) be a nonempty (α, β)-intuitionistic fuzzy complete ordered set such that $R = (\mu_R, \nu_R)$. Let $T : X \to [0,1]^X \times [0,1]^X$ be an (α, β)-intuitionistic R-fuzzy monotone multifunction. If there exist $a, b \in X$ such that $\mu_{T(a)}(b) = \alpha$, $\nu_{T(a)}(b) = \beta$, $\mu_R(a, b) > \frac{\alpha}{2}$ and $\nu_R(a, b) < \frac{\beta+1}{2}$, then the set $Fix(T)^{(\alpha,\beta)}$ of all (α, β)-intuitionistic fuzzy fixed points of T is nonempty and has at least a R-maximal element which is an element of the set $P_{(\alpha,\beta)}$ defined as above.*

Proof. Let (X, R) be a nonempty (α, β)-intuitionistic fuzzy ordered complete set such that $R = (\mu_R, \nu_R)$. Let $T : X \to [0,1]^X \times [0,1]^X$ be an (α, β)-intuitionistic R-fuzzy monotone multifunction. Assume that there exists $a, b \in X$ such that $\mu_{T(a)}(b) = \alpha$, $\nu_{T(a)}(b) = \beta$, $\mu_R(a, b) > \frac{\alpha}{2}$ and $\nu_R(a, b) < \frac{\beta+1}{2}$. Consider the following subset $P_{(\alpha,\beta)}$ of X defined by

$$P_{(\alpha,\beta)} = \left\{ x \in X : \exists y \in T_x^{(\alpha,\beta)}, \mu_R(x, y) > \frac{\alpha}{2} \text{ and } \nu_R(x, y) < \frac{\beta + 1}{2} \right\}.$$

From Proposition 4.1, the subset $P_{(\alpha,\beta)}$ is nonempty and has at least a maximal element, m, say.

Claim 1: It is obvious that $m \in Fix(T)^{(\alpha,\beta)}$. Indeed, assume on the contrary that $m \notin Fix(T)^{(\alpha,\beta)}$. As $m \in P_{(\alpha,\beta)}$, there is an element $n \in X$ such that

$$n \neq m, \mu_{T(m)}(n) = \alpha, \nu_{T(t)}(s) = \beta, \mu_R(m, n) > \frac{\alpha}{2} \text{ and } \nu_R(m, n) < \frac{\beta + 1}{2}.$$

Let $p \in T_n^{(\alpha,\beta)}$. As $n \neq m$ and T is (α, β) intuitionistic R-fuzzy monotone, we get

$$\mu_R(n, p) > \frac{\alpha}{2} \text{ and } \nu_R(n, p) < \frac{\beta + 1}{2}.$$

Then, $n \in P_{(\alpha,\beta)}$. That is a contradiction with the fact that m is a R-maximal element of $P_{(\alpha,\beta)}$. Thus, $m \in Fix(T)^{(\alpha,\beta)}$.

Claim 2: The element m is a R-maximal element of $Fix(T)^{(\alpha,\beta)}$. Indeed, as $Fix(T)^{(\alpha,\beta)} \subset P_{(\alpha,\beta)}$ and m is a maximal element of $P_{(\alpha,\beta)}$, m is a maximal element of $Fix(T)^{(\alpha,\beta)}$. □

For (α, β)-intuitionistic fuzzy monotone maps, we have the following result.

Corollary 4.3. *Let (X, R) be a nonempty (α, β)-intuitionistic fuzzy ordered complete set such that $R = (\mu_R, \nu_R)$ and let $f : X \to X$ be an (α, β)-intuitionistic fuzzy monotone map. If there is an element $a \in X$ such that $\mu_R(a, f(a)) > \frac{\alpha}{2}$ and $\nu_R(a, f(a)) < \frac{1+\beta}{2}$, then the set $Fix(f)$ of all fixed points of f is nonempty and has at least a R-maximal element.*

As a consequence of Theorem 4.2, we obtain the following result.

Corollary 4.4. *Let (X, R) be a nonempty intuitionistic fuzzy complete ordered set such that $R = (\mu_R, \nu_R)$. Let $T : X \to [0,1]^X \times [0,1]^X$ be a intuitionistic R-fuzzy monotone multifunction. If there exist $a, b \in X$ such that $\mu_{T(a)}(b) = 1$, $\nu_{T(a)}(b) = 0$, $\mu_R(a, b) > \frac{1}{2}$ and $\nu_R(a, b) < \frac{1}{2}$, then the set $Fix(T)$ of all fixed points of T is nonempty and has at least a R-maximal element.*

From Theorem 4.2, we reobtain the existence of a maximal α-fuzzy fixed for α-fuzzy monotone multifunction [[3], Theorem 3.1].

Corollary 4.5. *Let (X, R) be a nonempty α-fuzzy complete ordered set such that $R = \mu_R$. Let $T : X \to [0,1]^X$ be an α-fuzzy monotone multifunction. If there exist $a, b \in X$ such that $\mu_{T(a)}(b) = \alpha$ and $\mu_R(a, b) > \frac{\alpha}{2}$, then the set $Fix(T)^\alpha$ of all α-fuzzy fixed points of T is nonempty and has at least a R-maximal element.*

By using Propositions 2.8 and 4.1 and Theorem 4.2, we obtain the following dual result.

Theorem 4.6. *Let (X, R) be a nonempty (α, β)-intuitionistic fuzzy ordered set such that $R = (\mu_R, \nu_R)$. Assume that every nonempty (α, β)-intuitionistic R-fuzzy chain has a R-infimum. Let $T : X \to [0,1]^X \times [0,1]^X$ be an (α, β)-intuitionistic R-fuzzy monotone multifunction. If there exist $a, b \in X$ such that $\mu_{T(a)}(b) = \alpha$, $\nu_{T(a)}(b) = \beta$, $\mu_R(b, a) > \frac{\alpha}{2}$ and $\nu_R(b, a) < \frac{\beta+1}{2}$, then the set $Fix(T)^{(\alpha,\beta)}$ of all (α, β)-intuitionistic fuzzy fixed points of T is nonempty and has at least a R-minimal element.*

Proof. Let (X, R) be a nonempty (α, β)-intuitionistic fuzzy ordered set such that $R = (\mu_R, \nu_R)$. Assume that every nonempty (α, β)-intuitionistic fuzzy R-chain has a R-infimum. Let $T : X \to [0,1]^X \times [0,1]^X$ be an (α, β)-intuitionistic R-fuzzy monotone multifunction. Let S be the (α, β)-intuitionistic fuzzy inverse relation of R. From Proposition 2.8, S is an (α, β)-intuitionistic fuzzy order relation. By our hypothesis, since every nonempty (α, β)-intuitionistic R-fuzzy chain has a R-infimum, by Proposition 2.8, (X, S) is a nonempty (α, β)-intuitionistic fuzzy ordered complete set. As $\mu_R(b, a) > \frac{\alpha}{2}$ and $\nu_R(b, a) < \frac{\beta+1}{2}$, we get $\mu_S(a, b) > \frac{\alpha}{2}$ and $\nu_R(a, b) < \frac{\beta+1}{2}$.

On the other hand by Proposition 2.8, $T : X \to [0,1]^X \times [0,1]^X$ is an (α, β) intuitionistic S-fuzzy monotone multifunction. Then, by using Theorem 4.1, $Fix(T)^{(\alpha,\beta)}$ has a S-maximal element m, say. Now, let $x \in Fix(T)^{(\alpha,\beta)}$ such that $\mu_R(m, x) > \frac{\alpha}{2}$ and $\nu_R(m, x) < \frac{\beta+1}{2}$. Then, we get $\mu_S(x, m) > \frac{\alpha}{2}$ and $\nu_S(x, m) < \frac{\beta+1}{2}$. As m is a R-maximal element of $Fix(T)^{(\alpha,\beta)}$ and $x \in Fix(T)^{(\alpha,\beta)}$, we deduce that we have $x = m$. Thus, m is a R-minimal element of $Fix(T)^{(\alpha,\beta)}$. □

For (α, β)-intuitionistic fuzzy monotone maps, we obtain the following consequence.

Corollary 4.7. *Let (X, R) be a nonempty (α, β)-intuitionistic fuzzy ordered set such that $R = (\mu_R, \nu_R)$. Assume that every nonempty (α, β)-intuitionistic R-fuzzy chain has a R-infimum. Let $f : X \to X$ be an (α, β)-intuitionistic fuzzy monotone map. If there is an element $a \in X$ such that $\mu_R(f(a), a) > \frac{\alpha}{2}$ and $\nu_R(f(a), a) < \frac{1+\beta}{2}$, then the set of all fixed points $Fix(f)$ of f is nonempty and has at least a R-minimal element.*

As a consequence of Theorem 4.6, we obtain the following result.

Corollary 4.8. *Let (X, R) be a nonempty intuitionistic fuzzy ordered set such that $R = (\mu_R, \nu_R)$. Assume that every nonempty intuitionistic R-fuzzy chain has a R-infimum. Let $T : X \to [0,1]^X \times [0,1]^X$ be an intuitionistic R-fuzzy monotone multifunction. If there exist $a, b \in X$ such that $\mu_{T(a)}(b) = 1$, $\nu_{T(a)}(b) = 0$, $\mu_R(b,a) > \frac{1}{2}$ and $\nu_R(b,a) < \frac{1}{2}$, then the set $Fix(T)$ of all fixed points of T is nonempty and has at least a R-minimal element.*

By using Theorem 4.6, we reobtain the existence of a minimal α-fuzzy fixed for α-fuzzy monotone multifunction [[4], Theorem 3.3].

Corollary 4.9. *Let (X, R) be a nonempty α-fuzzy ordered set such that $R = \mu_R$. Assume that every nonempty α-fuzzy chain in (X, R) has a R-infimum. Let $T : X \to [0,1]^X$ be an α-fuzzy monotone multifunction. If there exist $a, b \in X$ such that $\mu_{T(a)}(b) = \alpha$, $\mu_R(b,a) > \frac{\alpha}{2}$, then the set $Fix(T)^\alpha$ of all α-fuzzy fixed points of T is nonempty and has at least a R-minimal element.*

5. An (α, β)-intuitionistic fuzzy version of Tarski's fixpoint theorem for intuitionistic fuzzy monotone multifunctions

In this section we shall give an (α, β)-intuitionistic fuzzy version of Tarski's fixpoint Theorem [[5], Theorem 1] for (α, β)-intuitionistic fuzzy monotone multifunctions. First, we give the following definition.

Definition 5.1. Let (X, R) be a nonempty (α, β)-intuitionistic fuzzy ordered set. We say that (X, R) is an (α, β)-intuitionistic fuzzy complete lattice, if every nonempty subset of X has a R-infimum and a R-supremum.

The main result in this section is the following result.

Theorem 5.2. *Let (X, R) be a nonempty (α, β)-intuitionistic fuzzy complete lattice and let $T : X \to [0,1]^X \times [0,1]^X$ be an (α, β)-intuitionistic R-fuzzy monotone multifuction. Then, the set of all (α, β)-intuitionistic fuzzy fixed points $Fix(T)^{(\alpha,\beta)}$ of T is a nonempty (α, β)-intuitionistic R-fuzzy complete lattice.*

In order to prove Theorem 5.2, we shall need the following result.

Proposition 5.3. *Let (X, R) be a nonempty (α, β)-intuitionistic fuzzy ordered set such that $R = (\mu_R, \nu_R)$ and every nonempty subset of X has a R-supremum. Let $T : X \to [0,1]^X \times [0,1]^X$ be an (α, β) intuitionistic R-fuzzy monotone multifunction. If there exist $a, b \in X$ such that $\mu_{T(a)}(b) = \alpha$, $\nu_{T(a)}(b) = \beta$, $\mu_R(a,b) > \frac{\alpha}{2}$ and $\nu_R(a,b) < \frac{\beta+1}{2}$, then the set $Fix(T)^{(\alpha,\beta)}$ of all (α, β)-intuitionistic fuzzy fixed points of T is nonempty and has a R-greatest element. Furthermore, we have*

$$\max_R(Fix(T)^{(\alpha,\beta)}) = \sup_R \left\{ x \in X : \exists y \in T_x^{(\alpha,\beta)}, \mu_R(x,y) > \frac{\alpha}{2} \text{ and } \nu_R(x,y) < \frac{\beta+1}{2} \right\}.$$

Proof. Let (X, R) be a nonempty (α, β)-intuitionistic fuzzy ordered set such that $R = (\mu_R, \nu_R)$ and every nonempty subset of X has a R-supremum. Let $T : X \to [0,1]^X \times [0,1]^X$ be an (α, β)-intuitionistic R-fuzzy monotone multifunction. Let

$P_{(\alpha,\beta)}$ the subset of X defined as above. By our hypothesis, $a \in P_{(\alpha,\beta)}$. Then, $P_{(\alpha,\beta)} \neq \varnothing$. Then, set $k = \sup_R(P_{(\alpha,\beta)})$.

Claim 1: It is clear that $k \in P_{(\alpha,\beta)}$. Indeed, assume on the contrary that $k \notin P_{(\alpha,\beta)}$. Let $x \in P_{(\alpha,\beta)}$. Then there is $y \in X$, such that $\mu_{T(x)}(y) = \alpha$, $\nu_{T(x)}(y) = \beta, \mu_R(x,y) > \frac{\alpha}{2}$ and $\nu_R(x,y) < \frac{\beta+1}{2}$. By our hypothesis, T is an (α,β)-intuitionistic R-fuzzy monotone multifunction. Thus, $T_k^{(\alpha,\beta)} \neq \varnothing$. So, there is $z \in X$ such that $\mu_{T(k)}(z) = \alpha$ and $\nu_{T(k)}(z) = \beta$. As T is (α,β)-intuitionistic fuzzy monotone, we get

$$\mu_R(y,z) > \frac{\alpha}{2} \text{ and } \nu_R(y,z) < \frac{\beta+1}{2}.$$

On the other hand, we know that we have

$$\mu_R(x,y) > \frac{\alpha}{2} \text{ and } \nu_R(x,y) < \frac{\beta+1}{2}.$$

Then from Lemma 2.9, we obtain

$$\mu_R(x,z) > \frac{\alpha}{2} \text{ and } \nu_R(x,z) < \frac{\beta+1}{2} \text{ for every } x \in P_{(\alpha,\beta)}.$$

Thus, z is a R-upper bound of $P_{(\alpha,\beta)}$. As $k = \sup_R(P_{(\alpha,\beta)})$, we get

$$\mu_R(k,z) > \frac{\alpha}{2} \text{ and } \nu_R(k,z) < \frac{\beta+1}{2}.$$

So, $k \in P_{(\alpha,\beta)}$. That is a contradiction. Hence, $k \in P_{(\alpha,\beta)}$.

Claim 2: It is obvious that $k \in Fix(T)^{(\alpha,\beta)}$. Indeed, from Proposition 4.1, we know that the subset $Fix(T)^{(\alpha,\beta)}$ has at least a R-maximal element m, which an element of $P_{(\alpha,\beta)}$. Since $k = \sup(P_{(\alpha,\beta)})$, we get $\mu_R(m,k) > \frac{\alpha}{2}$ and $\nu_R(m,k) < \frac{\beta+1}{2}$. As from Claim 1, $k \in P_{(\alpha,\beta)}$, so we deduce that we have $m = k$. Thus, $k = \max((P_{(\alpha,\beta)})$ and $k \in Fix(T)^{(\alpha,\beta)}$.

Claim 3: It is clear that $k = \max_R(Fix(T)^{(\alpha,\beta)})$. Indeed, as $k = \sup_R(P_{(\alpha,\beta)})$, and $Fix(T)^{(\alpha,\beta)} \subset P_{(\alpha,\beta)}$, k is a R-upper bound of $Fix(T)^{(\alpha,\beta)}$. From claim 2, $k \in Fix(T)^{(\alpha,\beta)}$. Then, we obtain $k = \max_R(Fix(T)^{(\alpha,\beta)})$. □

For (α,β)-intuitionistic fuzzy monotone maps, we have the following result.

Corollary 5.4. *Let (X,R) be a nonempty (α,β)-intuitionistic fuzzy ordered set such that $R = (\mu_R, \nu_R)$ and every nonempty subset of X has a R-supremum. Let $f : X \to X$ be an (α,β)-intuitionistic fuzzy monotone map. If there is an element $a \in X$ such that $\mu_R(a, f(a)) > \frac{\alpha}{2}$ and $\nu_R(a, f(a)) < \frac{1+\beta}{2}$, then the set $Fix(f)$ of all fixed points of f is nonempty and has a R-greatest element. Furthermore, we have*

$$\max_R(Fix(f)) = \sup_R \left\{ x \in X : \mu_R(x, f(x)) > \frac{\alpha}{2} \text{ and } \nu_R(x, f(x)) < \frac{\beta+1}{2} \right\}.$$

As a consequence of Proposition 5.3, we obtain the following corollary.

Corollary 5.5. *Let (X,R) be a nonempty intuitionistic fuzzy ordered set such that $R = (\mu_R, \nu_R)$ and every nonempty subset of X has a R-supremum. Let $T : X \to [0,1]^X \times [0,1]^X$ be an intuitionistic R-fuzzy monotone multifunction. If there exist $a, b \in X$ such that $\mu_{T(a)}(b) = 1$, $\nu_{T(a)}(b) = 0$, $\mu_R(a,b) > \frac{1}{2}$ and $\nu_R(a,b) < \frac{1}{2}$, then* *the set $Fix(T)$ of all fixed points of T is nonempty and has a R-greatest element. Furthermore, we have*

$$\max_R(Fix(T)) = \sup_R \left\{ x \in X : \exists y \in T_x^{(1,0)}, \mu_R(x,y) > \frac{1}{2} \text{ and } \nu_R(x,y) < \frac{1}{2} \right\}.$$

From Proposition 5.3, we reobtain the existence of the greatest α-fuzzy fixed for α-fuzzy monotone multifunction [[4], Theorem 4.1].

Corollary 5.6. *Let (X,R) be a nonempty α-fuzzy ordered set such that $R = \mu_R$ and every nonempty subset of X has a R-supremum. Let $T : X \to [0,1]^X$ be an α-fuzzy monotone multifunction. If there exist $a, b \in X$ such that $\mu_{T(a)}(b) = \alpha$ and $\mu_R(a,b) > \frac{\alpha}{2}$, then the set $Fix(T)^\alpha$ of all α-fuzzy fixed points of T is nonempty and has a R-greatest element. Furthermore, we have*

$$\max_R(Fix(T)^\alpha) = \sup_R \left\{ x \in X : \exists y \in X \text{ such that } \mu_{T(x)}(y) = \alpha \text{ and } \mu_R(x,y) > \frac{\alpha}{2} \right\}.$$

Combining Propositions 2.8 and 5.3, we get the following dual result.

Proposition 5.7. *Let (X,R) be a nonempty (α,β)-intuitionistic fuzzy ordered set such that $R = (\mu_R, \nu_R)$ and every nonempty subset of X has a R-infumum. Let $T : X \to [0,1]^X \times [0,1]^X$ be an (α,β)-intuitionistic R-fuzzy monotone multifunction. If there exist $a, b \in X$ such that $\mu_{T(a)}(b) = \alpha$, $\nu_{T(a)}(b) = \beta$, $\mu_R(b,a) > \frac{\alpha}{2}$ and $\nu_R(b,a) < \frac{\beta+1}{2}$, then the set $Fix(T)^{(\alpha,\beta)}$ of all (α,β)-intuitionistic fuzzy fixed points of T is nonempty and has a R-least element. Furthermore, we have*

$$\min_R(Fix(T)^{(\alpha,\beta)}) = \inf_R \left\{ x \in X : \exists y \in T_x^{(\alpha,\beta)}, \mu_R(x,y) > \frac{\alpha}{2} \text{ and } \nu_R(x,y) < \frac{\beta+1}{2} \right\}.$$

Proof. Let (X,R) be a nonempty (α,β)-intuitionistic fuzzy ordered set such that $R = (\mu_R, \nu_R)$. Assume that every nonempty subset of X has a R-infimum. Let $T : X \to [0,1]^X \times [0,1]^X$ be an (α,β)-intuitionistic R-fuzzy monotone multifunction. Let S be the (α,β)-intuitionistic fuzzy inverse relation of R. From Proposition 2.8, S is an (α,β)-intuitionistic fuzzy order relation. Since by our hypothesis every nonempty subset of X has a R-infimum, by Proposition 2.8, every nonempty subset of X has a S-infimum. As $\mu_R(b,a) > \frac{\alpha}{2}$ and $\nu_R(b,a) < \frac{\beta+1}{2}$, we get $\mu_S(a,b) > \frac{\alpha}{2}$ and $\nu_R(a,b) < \frac{\beta+1}{2}$.

On the other hand, by Proposition 2.8, $T : X \to [0,1]^X \times [0,1]^X$ is an (α,β)-intuitionistic S-fuzzy monotone multifunction. Then, by using Proposition 5.3, $Fix(T)^{(\alpha,\beta)}$ has a S-greatest element ℓ, say. Thus, for every $x \in Fix(T)^{(\alpha,\beta)}$, we have

$$\mu_S(x,\ell) > \frac{\alpha}{2} \text{ and } \nu_S(x,\ell) < \frac{\beta+1}{2}.$$

So, for every $x \in Fix(T)^{(\alpha,\beta)}$, we get

$$\mu_R(\ell,x) > \frac{\alpha}{2} \text{ and } \nu_R(\ell,x) < \frac{\beta+1}{2}.$$

Hence, ℓ is the R-least element of $Fix(T)^{(\alpha,\beta)}$. □

For (α,β)-intuitionistic fuzzy monotone maps, we have the following result.

Corollary 5.8. *Let (X,R) be a nonempty (α,β)-intuitionistic fuzzy ordered set such that $R=(\mu_R,\nu_R)$ and every nonempty subset of X has a R-infimum. Let $f:X\to X$ be an (α,β)-intuitionistic fuzzy monotone map. If there is an element $a\in X$ such that $\mu_R(f(a),a)>\frac{\alpha}{2}$ and $\nu_R(f(a),a)<\frac{1+\beta}{2}$, then the set $Fix(f)$ of all fixed points of f is nonempty and has and has a R-least element. Furthermore, we have*

$$\min_R(Fix(f))=\inf_R\left\{x\in X:\mu_R(f(x),x)>\frac{\alpha}{2}\text{ and }\nu_R(f(x),x)<\frac{\beta+1}{2}\right\}.$$

As a consequence of Proposition 5.7, we obtain the following corollary.

Corollary 5.9. *Let (X,R) be a nonempty intuitionistic fuzzy ordered set such that $R=(\mu_R,\nu_R)$ and every nonempty subset of X has a R-infimum. Let $T:X\to[0,1]^X\times[0,1]^X$ be an intuitionistic R-fuzzy monotone multifunction. If there exist $a,b\in X$ such that $\mu_{T(a)}(b)=1$, $\nu_{T(a)}(b)=0$, $\mu_R(a,b)>\frac{1}{2}$ and $\nu_R(a,b)<\frac{1}{2}$, then the set $Fix(T)$ of all fixed points of T is nonempty and has a R-least element. Furthermore, we have*

$$\min_R(Fix(T))=\inf_R\left\{x\in X:\exists y\in T_x^{(1,0)},\mu_R(y,x)>\frac{1}{2}\text{ and }\nu_R(y,x)<\frac{1}{2}\right\}.$$

From Proposition 5.7, we reobtain the existence of the least α-fuzzy fixed for α-fuzzy monotone multifunction [[4], Theorem 4.2].

Corollary 5.10. *Let (X,R) be a nonempty α-fuzzy ordered set such that $R=\mu_R$ and every nonempty subset of X has a R-infimum. Let $T:X\to[0,1]^X$ be an α-fuzzy monotone multifunction. If there exist $a,b\in X$ such that $\mu_{T(a)}(b)=1$ and $\mu_R(a,b)>\frac{\alpha}{2}$, then the set $Fix(T)^\alpha$ of all α-fuzzy fixed points of T is nonempty and has a R-least element. Furthermore, we have*

$$\min_R(Fix(T)^\alpha)=\inf_R\left\{x\in X:\exists y\in X\text{ such that }\mu_{T(x)}(y)=\alpha\text{ and }\mu_R(x,y)>\frac{\alpha}{2}\right\}.$$

To prove Theorem 5.2, we shall need what follows.

Let (X,R) be a nonempty (α,β)-intuitionistic fuzzy complete lattice and let $T:X\to[0,1]^X\times[0,1]^X$ be (α,β)-intuitionistic R-fuzzy monotone multifunction. By Proposition 5.7, the set of all (α,β)-intuitionistic fuzzy fixed points $Fix(T)^{(\alpha,\beta)}$ of T is nonempty and has a R-least element, ℓ, say. For every nonempty subset A of $Fix(T)^{(\alpha,\beta)}$ we associate the following subset E of X defined by $x\in E$ if and only if there exists $y\in X$ such that $y\in T_x^{(\alpha,\beta)}$,

$$\mu_R(x,y)>\frac{\alpha}{2},\nu_R(x,y)<\frac{\beta+1}{2},$$

$$\mu_R(y,z)>\frac{\alpha}{2}\text{ and }\nu_R(y,z)<\frac{\beta+1}{2}\text{ for every }z\in A.$$

Since $\ell\in E$, $E\neq\varnothing$. Then, $t=\sup_R(E)$ exists in (X,R).

To proof Theorem 5.2, we shall need the following technical lemma.

Lemma 5.11. *Let (X,R) be a nonempty (α,β)-intuitionistic fuzzy complete lattice and let $T:X\to[0,1]^X\times[0,1]^X$ be an (α,β)-intuitionistic R-fuzzy monotone multifunction. Let us suppose that E is defined as above and $t=\sup_R(E)$. Then, we have*

(1) $t \in E$,

(2) $t \in Fix(T)^{(\alpha,\beta)}$.

Proof. Let (X, R) be a nonempty (α, β)-intuitionistic fuzzy complete lattice and let $T : X \to [0,1]^X \times [0,1]^X$ be an (α, β)-intuitionistic R-fuzzy monotone multifunction. Let A be a nonempty subset of $Fix(T)^{(\alpha,\beta)}$ and E be the subset of X defined as above.

(1) It is clear that The element t belongs to E. On the contrary, suppose that $t \notin E$. Let $a \in T_t^{(\alpha,\beta)}$ be given. By our hypothesis, for all $x \in E$, there exists $y_x \in T_x^{(\alpha,\beta)}$ with

$$\mu_R(x, y_x) > \frac{\alpha}{2} \text{ and } \nu_R(x, y_x) < \frac{\beta+1}{2}. \tag{5.8}$$

Since $\mu_R(x,t) > \frac{\alpha}{2}$ and $\nu_R(x,t) < \frac{\beta+1}{2}$, for all $x \in E$ and T is (α, β)-intuitionistic R-fuzzy monotone, we get

$$\mu_R(y_x, a) > \frac{\alpha}{2} \text{ and } \nu_R(y_x, a) < \frac{\beta+1}{2}. \tag{5.9}$$

From (5.8), (5.9) and by using Lemma 2.9, we deduce that we have

$$\mu_R(x, a) > \frac{\alpha}{2} \text{ and } \nu_R(x, a) < \frac{\beta+1}{2} \text{ for all } x \in E. \tag{5.10}$$

Then, the element a is a R-upper bound of E. Since $t = \sup_R(E)$, we obtain

$$\mu_R(t, a) > \frac{\alpha}{2} \text{ and } \nu_R(t, a) < \frac{\beta+1}{2}. \tag{5.11}$$

Now let $z \in A$. By our hypothesis, we know that for all $x \in E$, there exists $y_x \in T_x^{(\alpha,\beta)}$ such that

$$\mu_R(y_x, z) > \frac{\alpha}{2} \text{ and } \nu_R(y_x, z) < \frac{\beta+1}{2}, \text{ for all } z \in A. \tag{5.12}$$

By using (5.8), (5.12) and Lemma 2.9, we obtain

$$\mu_R(x, z) > \frac{\alpha}{2} \text{ and } \nu_R(x, z) < \frac{\beta+1}{2}, \text{ for all } x \in E. \tag{5.13}$$

Then, it follows from (5.13) that each element z of A is a R-upper bound of E. From this and as $t = \sup_R(E)$, we get

$$\mu_R(t, z) > \frac{\alpha}{2} \text{ and } \nu_R(t, z) < \frac{\beta+1}{2}, \text{ for all } z \in A. \tag{5.14}$$

Combining (5.14) and our assumption that $t \notin E$, we get $t \notin Fix(T)^{(\alpha,\beta)}$. Thus, $t \neq z$ for all $z \in A$. As T is an (α, β)-intuitionistic R-fuzzy monotone multifunction, $t \neq z$, $z \in T(z)^{(\alpha,\beta)}$, $a \in T(t)^{(\alpha,\beta)}$, and by (5.14), we get

$$\mu_R(a, z) > \frac{\alpha}{2} \text{ and } \nu_R(a, z) < \frac{\beta+1}{2}, \text{ for all } z \in A. \tag{5.15}$$

So, from (5.11) and (5.15), we deduce that we have $t \in E$. That is a contradiction. Hence $t \in E$.

(2) It is obvious that $t \in Fix(T)^{(\alpha,\beta)}$. Assume on the contrary, that $t \notin Fix(T)^{(\alpha,\beta)}$. By (1) above, $t \in E$. Then there exists $a \in X$ such that $a \in T_t^{(\alpha,\beta)}$,

$$\mu_R(t,a) > \frac{\alpha}{2} \text{ and } \nu_R(t,a) < \frac{\beta+1}{2}, \tag{5.16}$$

$$\mu_R(a,z) > \frac{\alpha}{2}, \nu_R(a,z) < \frac{\beta+1}{2} \text{ for every } z \in A. \tag{5.17}$$

Since by our assumption, $t \notin Fix(T)^{(\alpha,\beta)}$, then $a \neq t$. Next, we shall distinguish the following two cases.

Case (i): Suppose that $a \in A$. Since $a \in T_a^{(\alpha,\beta)}$, from (5.17), we get $a \in E$. Since $t = \sup_R(E)$,

$$\mu_R(a,t) > \frac{\alpha}{2} \text{ and } \nu_R(a,t) < \frac{\beta+1}{2}. \tag{5.18}$$

Then From (5.16) and (5.18), we get

$$\mu_R(t,a) + \mu_R(a,t) > \alpha \text{ and } \nu_R(t,a) + \nu_R(a,t) < \beta + 1. \tag{5.19}$$

Since R is (α,β)-intuitionistic fuzzy antisymmetric, by using (5.19), we deduce that we have $a = t$. That is a contradiction. Thus, $t \in Fix(T)^{(\alpha,\beta)}$.

Case (ii): Suppose that $a \notin A$. Let $b \in T_a^{(\alpha,\beta)}$. Since $a \neq t$ and T is (α,β)-intuitionistic R-fuzzy monotone, from (5.16), we get

$$\mu_R(a,b) > \frac{\alpha}{2} \text{ and } \nu_R(a,b) < \frac{\beta+1}{2}. \tag{5.20}$$

Now, let $z \in A$. Since $z \neq a$, T is (α,β)-intuitionistic R-fuzzy monotone. Then form (5.17), we obtain

$$\mu_R(b,z) > \frac{\alpha}{2} \text{ and } \nu_R(b,z) < \frac{\beta+1}{2} \text{ for every } z \in A. \tag{5.21}$$

Thus it follows from (5.20) and (5.21) that $a \in E$.

On the other hand, we know that $t = \sup_R(E)$, Thus

$$\mu_R(a,t) > \frac{\alpha}{2} \text{ and } \nu_R(a,t) < \frac{\beta+1}{2}. \tag{5.22}$$

From (5.18) and (5.22), we obtain

$$\mu_R(t,a) + \mu_R(a,t) > \alpha \text{ and } \nu_R(t,a) + \nu_R(a,t) < \beta + 1. \tag{5.23}$$

Since R is (α,β)-intuitionistic fuzzy antisymmetric, by using (5.23), we obtain $a = t$. That is a contradiction. So, $t \in Fix(T)^{(\alpha,\beta)}$. □

Now we are able to give the proof of Theorem 5.2.

Proof. Theorem 5.2: Let (X,R) be a nonempty (α,β)-intuitionistic fuzzy complete lattice and let $T : X \to [0,1]^X \times [0,1]^X$ be an (α,β)-intuitionistic R-fuzzy monotone multifunction. Let A be a nonempty subset of $Fix(T)^{(\alpha,\beta)}$ and let E be the subset of X defined as above.

Case (i): We shall prove that the greatest element of all R-lower bounds of A which are elements of $Fix(T)^{(\alpha,\beta)}$ belongs to $Fix(T)^{(\alpha,\beta)}$. Let E be the subset of X defined above by: $x \in E$ if and only if there exists $y \in X$ such that $y \in T_x^{(\alpha,\beta)}$,

$$\mu_R(x,y) > \frac{\alpha}{2}, \nu_R(x,y) < \frac{\beta+1}{2},$$

$$\mu_R(y,z) > \frac{\alpha}{2} \text{ and } \nu_R(y,z) < \frac{\beta+1}{2} \text{ for every } z \in A.$$

Consider the following subset F of X defined by

$$F = \left\{ x \in Fix(T)^{(\alpha,\beta)} : \mu_R(x,z) > \frac{\alpha}{2} \text{ and } \nu_R(x,z) < \frac{\beta+1}{2} \text{ for every } z \in A \right\}.$$

Then by Proposition 5.7, the R-least fixed point ℓ of T exists in (X,R). Since $\ell \in F$, $F \neq \varnothing$ and $m = \sup_R(F)$ exists in (X,R). Also, as $\ell \in E$, $E \neq \varnothing$ and $t = \sup_R(E)$ exists in (X,R). By (1) of Lemma 5.11, we know that $t \in E$. Thus, there exists $a \in T_t^{(\alpha,\beta)}$ such that

$$\mu_R(t,a) > \frac{\alpha}{2} \text{ and } \nu_R(t,a) < \frac{\beta+1}{2}. \tag{5.24}$$

$$\mu_R(a,z) > \frac{\alpha}{2}, \nu_R(a,z) < \frac{\beta+1}{2} \text{ for every } z \in A. \tag{5.25}$$

By (5.24), (5.25) and Lemma 2.9, we get

$$\mu_R(t,z) > \frac{\alpha}{2} \text{ and } \nu_R(t,z) < \frac{\beta+1}{2} \text{ for every } z \in A. \tag{5.26}$$

From Lemma 5.11, we know that t is an (α,β)-fuzzy fixed point of T. Using this and (5.26), we get $t \in F$. As $m = \sup_R(F)$, we obtain

$$\mu_R(t,m) > \frac{\alpha}{2} \text{ and } \nu_R(t,m) < \frac{\beta+1}{2}. \tag{5.27}$$

Since $F \subseteq E$, we get

$$\mu_R(m,t) > \frac{\alpha}{2} \text{ and } \nu_R(m,t) < \frac{\beta+1}{2}. \tag{5.28}$$

Combining (5.27) and (5.28), we obtain

$$\mu_R(m,t) + \mu_R(t,m) > \alpha \text{ and } \nu_R(m,t) + \nu_R(t,m) < \beta + 1. \tag{5.29}$$

Since R is (α,β)-intuitionistic fuzzy antisymmetric, from (5.29), we get $t = m$. So, $m \in Fix(T)^{(\alpha,\beta)}$. Hence, the greatest element of all R-lower bounds of A which are elements of $Fix(T)^{(\alpha,\beta)}$ is the element m.

Case (ii): We shall prove that the least element of all R-upper bounds of A which are elements of $Fix(T)^{(\alpha,\beta)}$ belongs to $Fix(T)^{(\alpha,\beta)}$. Let G and H be the following two subsets of X defined by: $x \in G$ if and only if there exists $y \in X$ such that $y \in T_x^{(\alpha,\beta)}$ and

$$\mu_R(y,x) > \frac{\alpha}{2}, \nu_R(y,x) < \frac{\beta+1}{2},$$

$$\mu_R(z,y) > \frac{\alpha}{2} \text{ and } \nu_R(z,y) < \frac{\beta+1}{2} \text{ for every } z \in A$$

and

$$H = \left\{ x \in Fix(T)^{(\alpha,\beta)} : \mu_R(z,x) > \frac{\alpha}{2} \text{ and } \nu_R(z,x) < \frac{\beta+1}{2} \text{ for every } z \in A \right\}.$$

From Proposition 5.3, the R-greatest (α,β)-fuzzy fixed point, g of T exists in (X,R). As $g \in H$, $H \neq \varnothing$ and $n = \inf_R(H)$ exists in (X,R). Since $g \in G$, $G \neq \varnothing$ and $p = \inf_R(G)$ exists in (X,R).

Let S be the intuitionistic fuzzy inverse order relation of R such that $S = (\mu_S, \nu_S)$. Then, we get $x \in G$ if and only if there exists $y \in X$ such that $y \in T_x^{(\alpha,\beta)}$,

$$\mu_S(x,y) > \frac{\alpha}{2}, \nu_S(x,y) < \frac{\beta+1}{2},$$

$$\mu_S(y,z) > \frac{\alpha}{2} \text{ and } \nu_S(y,z) < \frac{\beta+1}{2} \text{ for every } z \in A$$

and

$$H = \left\{ x \in Fix(T)^{(\alpha,\beta)} : \mu_S(x,z) > \frac{\alpha}{2} \text{ and } \nu_S(x,z) < \frac{\beta+1}{2} \text{ for every } z \in A \right\}.$$

Now, as T is an (α,β)-intuitionistic R-fuzzy monotone multifunction, from Proposition 2.8, T is also an (α,β)-intuitionistic S-fuzzy monotone multifunction. Also, by Proposition 2.8, we deduce that (X,S) is an (α,β)-intuitionistic fuzzy complete lattice. Since $g \in G$, we set $\lambda = \sup_S(G)$. Then by Lemma 5.11, we deduce that $\lambda \in G$ and λ is an (α,β)-fuzzy fixed point of T. From the first step above, we deduce that $\lambda = \sup_S(H)$.

On the other hand, from Proposition 2.8, we get

$$p = \inf_R(G) = \sup_S(G) = \lambda$$

and

$$\lambda = \sup_S(H) = \inf_R(H) = n.$$

That we have $\lambda = n = p$. Then from Lemma 5.11, we get $n \in Fix(T)^{(\alpha,\beta)}$. Thus, the R-least element of all R-upper bounds of A which are elements of $Fix(T)^{(\alpha,\beta)}$ is the element n. □

As consequences of Theorem 5.2, we obtain the following results.

Corollary 5.12. *Let (X,R) be a nonempty (α,β)-intuitionistic fuzzy complete lattice and let $f : X \to X$ be an (α,β)-intuitionistic fuzzy monotone map. Then, the set $Fix(f)$ of all fixed points of f is a nonempty (α,β)-intuitionistic fuzzy complete lattice.*

Corollary 5.13. *Let (X,R) be a nonempty intuitionistic fuzzy complete lattice and let $T : X \to [0,1]^X \times [0,1]^X$ be an intuitionistic R-fuzzy monotone multifunction. Then, the set $Fix(T)$ of all fixed points of T is a nonempty intuitionistic fuzzy complete lattice.*

For the case of α-fuzzy complete lattice, we obtain the following result.

Corollary 5.14. *Let (X,R) be a nonempty α-fuzzy complete lattice and let $T : X \to [0,1]^X$ be an α-fuzzy monotone multifunction. Then, the set $Fix(T)^\alpha$ of all α-fuzzy fixed points of T is a nonempty α-fuzzy complete lattice.*

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