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Fuzzy soft hyperconnected spaces

A. Kandil, O. Tantawy, S. A. El-Sheikh, Sawsan S. S. El-Sayed

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Abstract. The purpose of this paper is to introduce concepts of fuzzy soft D-space, fuzzy soft extremally disconnected spaces and fuzzy soft hyperconnected spaces. The relationship between these concepts is investigated. The properties of fuzzy soft hyperconnected spaces are studied.

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Keywords: Fuzzy soft connected space, Fuzzy soft hyperconnected space, Fuzzy soft D-space, Fuzzy soft extremally disconnected space, Fuzzy soft regular open subset, Fuzzy soft almost S-continuous functions.

Corresponding Author: Sawsan S. S. El-Sayed (sawsan_809@yahoo.com; s.elsayed@mu.edu.sa)

1. Introduction

It is well known that the conception of fuzzy sets, firstly defined by Zadeh [29] in 1965. Chronologically in 1968 and 1976, Chang [9] and Lowen [16] redound the concept of fuzzy topological spaces to literature substantively by using this conception.

In 1999, Molodtsov [20] introduced the concept of soft set theory which is a completely new approach for modeling uncertainty. Molodtsov [20] established the fundamental results of this new theory and successfully applied the soft set theory into several directions, such as smoothness of functions, operations research, Riemann integration, game theory, theory of probability and so Maji et al. [18] defined and studied several basic notions of soft set theory in 2003. Shabir and Naz

[24] introduced the concept of soft topological space and studied neighborhoods and separation axioms.

Maji et al. [19] initiated the study involving both fuzzy sets and soft sets. In this paper, the notion of fuzzy soft sets was introduced as a fuzzy generalizations of soft sets and some basic properties of fuzzy soft sets are discussed in details. Maji et al. combined fuzzy sets and soft sets and introduced the concept of fuzzy soft sets. In 2011, Tanay et al. [26] gave the topological structure of fuzzy soft sets.

On the other hand, the notion of hyperconnected space is intoduced and studied by many authers. In 1992, Ajmal and Kohli [6] studied the the properties of hyperconnected spaces as their mappings into Hausdorff spaces. In 2012, E. Ekici et al. [10] introduced the notion of $*$-hyperconnected ideal topological spaces and investigated its charectedizations of properties. In 2014, Kandil et al. [15] introduced the notion of soft hyperconnected space and $*$-soft hyperconnected space and studied its properties. The fuzzy hyperconnected spece studied by Thangaraj et al. [27], Rashid et al. [22] and Alkhafaji et al. [8].

Our aim in this paper is to extend the idea of hyperconnected to fuzzy soft topological space and study its characterizations. In Section 3, we introduce the notions of fuzzy soft hyperconnected space, fuzzy soft D-space, and fuzzy soft extremally disconnected space. Also, we study the relation between them. In Section 4, we present the notion of fuzzy soft almost S-continuous function. The properties of fuzzy soft hyperconnected space in terms of the functions are investigated.

2. Preliminaries

Throughout this paper X denotes initial universe, E denotes the set of all possible parameters which are attributes, characteristic or properties of the objects in X, and the set of all subsets of X will be denoted by $P(X)$. In this section, we present the basic definitions and results of soft set theory which will be needed in the sequel.

Definition 2.1 ([9])**.** A fuzzy set A of a non-empty set X is characterized by a membership function $\mu_A : X \longrightarrow [0,1] = I$ whose value $\mu_A(x)$ represents the "degree of membership" of x in A, for $x \in X$.

Let I^X denotes the family of all fuzzy sets on X.

Definition 2.2 ([20])**.** Let A be a non-empty subset of E. A pair (F, A) denoted by F_A is called a soft set over X, where F is a mapping given by $F : A \to P(X)$. In other words, a soft set over X is a parametrized family of subsets of the universe X. For a particular, $e \in A$, $F(e)$ may be considered the set of e -approximate elements of the soft set (F, A) and if $e \notin A$, then $F(e) = \phi$, i.e, $F = \{F(e) : e \in A \subseteq E, F : A \to P(X)\}$.

Proposition 2.1 ([7])**.** *Every fuzzy set may be considered as a soft set.*

Definition 2.3 ([19]). Let $A \subseteq E$. A pair (f, A), denoted by f_A, is called a fuzzy soft set over X, where f is a mapping given by $f : A \longrightarrow I^X$ defined by $f_A(e) = \mu^e_{f_A}$; where $\mu^e_{f_A} = \overline{0}$, if $e \notin A$ and $\mu^e_{f_A} \neq \overline{0}$, if $e \in A$, where $\overline{0}(x) = 0$, $\forall x \in X$.

The family of all fuzzy soft sets over X denoted by $FSS(X)_E$.

Note that, a fuzzy soft set is a hybridizition of fuzzy sets and soft sets, in which a soft set is defined over a fuzzy set.

The family of all fuzzy soft sets over X with a fixed set of parameter E is denoted by $FSS(X)_E$.

Definition 2.4 ([19]). The complement of a fuzzy soft set (f, A), denoted by $(f, A)^c$, is defined by $(f, A)^c = (f^c, A)$, $f^c_A : E \longrightarrow I^X$ is a mapping given by $\mu^e_{f^c_A} = \overline{1} - \mu^e_{f_A}$ $\forall e \in E$, where $\overline{1}(x) = 1$ $\forall x \in X$.

Clearly, $(f^c_A)^c = f_A$.

Definition 2.5 ([26]). A fuzzy soft set f_E over X is said to be a null- fuzzy soft set, denoted by $\widetilde{0}_E$, if for all $e \in E$, $f_E(e) = \overline{0}$.

Definition 2.6 ([26]). A fuzzy soft set f_E over$X$ is said to be an absolute fuzzy soft set, denoted by $\widetilde{1}_E$, if $f_E(e) = \overline{1}$ $\forall e \in E$.

Clearly we have $(\widetilde{0}_E)^c = \widetilde{1}_E$ and $(\widetilde{1}_E)^c = \widetilde{0}_E$.

Definition 2.7 ([26]). Let f_A and $g_B \in FSS(X)_E$. A fuzzy soft set f_A is called a fuzzy soft subset of g_B, denoted by $f_A \widetilde{\subseteq} g_B$, if $A \subseteq B$ and $\mu^e_{f_A}(x) \leq \mu^e_{g_B}(x)$ $\forall x \in X$, $\forall e \in E$.

Also, g_B is called a fuzzy soft superset of f_A denoted by $g_B \widetilde{\supseteq} f_A$.

If f_A is not a fuzzy soft subset of g_B, we write $f_A \widetilde{\nsubseteq} g_B$.

Definition 2.8 ([25]). Two fuzzy soft sets f_A and g_B on X are called equal, if f_A $\widetilde{\subseteq}$ g_B and $g_B \widetilde{\subseteq} f_A$.

Definition 2.9 ([23]). The union of two fuzzy soft sets f_A and g_B over the common universe X, denoted by $f_A \sqcup g_B$, is also a fuzzy soft set h_C, where $C = A \cup B$ and for all $e \in C$, $h_C(e) = \mu^e_{h_C} = \mu^e_{f_A} \vee \mu^e_{g_B}$ $\forall e \in C$.

Definition 2.10 ([26]). The intersection of two fuzzy soft sets f_A and g_B over the common universe X, denoted by $f_A \sqcap g_B$, is also a fuzzy soft set h_C, where $C = A \cap B$ and for all $e \in C$, $h_C(e) = \mu^e_{h_C} = \mu^e_{f_A} \wedge \mu^e_{g_B}$ $\forall e \in C$.

Definition 2.11 ([25, 26, 28]). Let τ be a collection of fuzzy soft sets over a universe X with a fixed set of parameters E. The collection τ is called a fuzzy soft topology on X, if

(i) $\widetilde{0}_E$ and $\widetilde{1}_E \in \tau$, where $\widetilde{0}_E(e) = \overline{0}$ and $\widetilde{1}_E(e) = \overline{1}$ $\forall e \in E$,

(ii) the union of any members of τ belongs to τ,

(iii) the intersection of any two members of τ belongs to τ.

The triplet (X, τ, E) is called a fuzzy soft topological space over X. Also, each member of τ is called an open fuzzy soft in (X, τ, E). The family of all fuzzy soft sets in X is denoted by $FSO(X)_E$.

Note that, the intersection of any family of fuzzy soft topologies on X is also a fuzzy soft topology on X.

Definition 2.12 ([25, 26])**.** Let (X, τ, E) be a fuzzy soft topological space. A fuzzy soft set f_A over X is said to be a closed fuzzy soft set in X, denoted by $f_A \in \tau^c$, if its complement f_A^c is an open fuzzy soft set.

The collection of all closed fuzzy soft sets in X is denoted by $FSC(X)_E$.

Definition 2.13 ([17])**.** Let (X, τ, E) be a fuzzy soft topological space and $g_B \in FSS(X)_E$.

If $\tau_{g_B} = \{g_B \sqcap f_A;\ f_A \in \tau\}$, then τ_{g_B} is called a fuzzy soft relative topology for g_B and $(g_B,\ \tau_{g_B},\ B)$ is called a fuzzy soft subspace of (X, τ, E).

If $g_B \in \tau$ (respectively, $g_B \in \tau^c$), then $(g_B,\ \tau_{g_B},\ B)$ is called a fuzzy soft open (respectively, closed) subspace of (X, τ, E).

Definition 2.14 ([23, 25, 26, 28])**.** Let (X, τ, E) be a fuzzy soft topological space and $f_A \in FSS(X)_E$. Then the closure of fuzzy soft set f_A, denoted by $Fcl(f_A)$, is the intersection of all closed fuzzy soft super sets of f_A, i.e.,

$$Fcl(f_A) = \sqcap\{h_C; h_C \text{ is closed fuzzy soft set and } f_A \mathrel{\widetilde{\subseteq}} h_C\}.$$

Clearly, $Fcl(f_A)$ is the smallest fuzzy soft closed set over X which contains f_A, and $Fcl(f_A)$ is a closed fuzzy soft set.

Definition 2.15 ([23, 25, 26, 28])**.** Let (X, τ, E) be a fuzzy soft topological space and $f_A \in FSS(X)_E$. Then the interior of fuzzy soft set g_B, denoted by $Fint(g_B)$, is the union of all fuzzy open soft subsets of g_B, i.e.,

$$Fint(g_B) = \sqcup\{h_C; h_C \text{ is fuzzy open soft set and } h_C \mathrel{\widetilde{\subseteq}} g_B\}.$$

Clearly, $Fint(g_B)$ is the largest open fuzzy soft set contained in g_B and $Fint(g_B)$ is fuzzy open soft set.

Definition 2.16 ([17])**.** The fuzzy soft set $f_A \in FSS(X)_E$ is called fuzzy soft point, if there exist $x \in X$ and $e \in E$ such that $\mu^e_{f_A}(x) = \alpha$ $(0 \leq \alpha \leq 1)$ and $\mu^e_{f_A}(y) = 0$, $\forall y \in X - \{x\}$, and this fuzzy soft point is denoted by x^e_α or f_e.

The set of all fuzzy soft points in X will be denoted by $FSP(X)_E$.

Definition 2.17 ([17])**.** The fuzzy soft point x^e_α is said to be belonging to the fuzzy soft set f_A, denoted by $x^e_\alpha \mathrel{\widetilde{\in}} f_A$, if for the element $e \in A$, $\alpha \leq \mu^e_{f_A}(x)$.

Definition 2.18 ([17])**.** A fuzzy soft topological space is said to be fuzzy soft T_2-space (FST_2-space, for short), if for every two fuzzy soft pionts x^e_α and y^t_β in X such that $x^e_\alpha \neq y^t_\beta$, there exist two open fuzzy soft sets f_A and g_B in X such that $x^e_\alpha \mathrel{\widetilde{\in}} f_A$, $y^t_\beta \mathrel{\widetilde{\in}} g_B$ and $f_A \sqcap g_B = \widetilde{0}_E$.

Definition 2.19 ([11, 28]). Let $FSS(X)_E$ and $FSS(Y)_K$ be families of fuzzy soft sets over X and$Y$, respectively. Let $u : X \longrightarrow Y$ and $p : E \longrightarrow K$ be mappings. Then the map f_{pu} is called a fuzzy soft mapping from $FSS(X)_E$ to $FSS(Y)_K$, denoted by $f_{pu} : FSS(X)_E \longrightarrow FSS(Y)_K$, such that:

(i) If $g_B \in FSS(X)_E$, then the image of g_B under the fuzzy soft mapping f_{pu}, denoted by $f_{pu}(g_B)$, is a fuzzy soft set over$Y$ defined by:

$$f_{pu}(g_B)(k)(y) = \begin{cases} \bigvee\limits_{u(x)=y} [\bigvee\limits_{p(e)=k} (g_B(e))](x) & \text{, if } x \in u^{-1}(y), \\ 0 & \text{, otherwise,} \end{cases} \quad \forall k \in p(E), \forall y \in Y,$$

(ii) if $h_C \in FSS(Y)_K$, then the inverse image of h_C under the fuzzy soft mapping f_{pu}, denoted by $f_{pu}^{-1}(h_C)$, is a fuzzy soft set over X defined by:

$$f_{pu}^{-1}(h_C)(e)(x) = \begin{cases} h_C(p(e))(u(x)) & \text{, for } p(e) \in C \\ 0 & \text{, otherwise,} \end{cases} \quad \forall e \in p^{-1}(K), x \in X.$$

Definition 2.20 ([13]). surjective (respectively, injective), if p and u are surjective (respectively, injective).

Also f_{pu} is said to be constant, if p and u are constant.

Definition 2.21 ([28]). Let (X, τ, E) and (Y, σ, K) be two fuzzy soft topological spaces. A function $f_{pu} : (X, \tau, E) \longrightarrow (Y, \sigma, K)$ is called:

(i) fuzzy soft continuous, if $f_{pu}^{-1}(h_C) \in \tau$ for every $h_C \in \sigma$.

(ii) fuzzy soft open if $f_{pu}(h_C) \in \sigma$ for every $h_C \in \tau$.

Definition 2.22. Let (X, τ, E) be a fuzzy soft topological space and $f_A \in FSS(X)_E$. Then f_A is called:

(i)[[12, 13]] a fuzzy soft semi-open set, if $f_A \tilde{\subseteq} Fcl(Fint(f_A))$,

(ii)[[1, 5]] a fuzzy soft pre-open set, if $f_A \tilde{\subseteq} Fint(Fcl(f_A))$,

(iii)[[2, 14]] a fuzzy soft β-open set, if $f_A \tilde{\subseteq} Fcl(Fint(Fcl(f_A))$,

(iv)[[3, 4]] a fuzzy soft α-open set, if $f_A \tilde{\subseteq} Fint(Fcl(Fint(f_A))$.

The collection of all fuzzy soft semi-open (respectively, pre-open, β-open, α-open) sets will be denoted by $FSSO(X)_E$ (respectively, $FSPO(X)_E$, $FS\beta O(X)_E$, $FS\alpha O(X)_E$).

Remark 2.1. The following diagram shows the relationship between the types of fuzzy soft sets in Definition 2.21.

$$\begin{array}{ccccc} \text{open fuzzy soft} & \Longrightarrow & \text{fuzzy soft } \alpha\text{-open} & \Longrightarrow & \text{fuzzy soft semi-open} \\ & & \Downarrow & & \Downarrow \\ & & \text{fuzzy soft pre-open} & \Longrightarrow & \text{fuzzy soft } \beta\text{-open} \end{array}$$

Definition 2.23. Let (X, τ, E) be a fuzzy soft topological space and $f_A \in FSS(X)_E$. f_A is called fuzzy soft semi-closed [12, 13] (respectively, pre-closed [1, 5], β-closed

[2, 14], α-closed [3, 4]) set in X, if its complement is fuzzy soft semi-open (respectively, pre-open, β-open, α-open) set.

The collection of all fuzzy soft semi-closed (respectively, pre-closed, β-closed, α-closed) sets will be denoted by $FSSC(X)_E$ (respectively, $FSPC(X)_E$, $FS\beta C(X)_E$, $FS\alpha C(X)_E$).

Definition 2.24 ([13])**.** The fuzzy soft semi-interior of a fuzzy soft set f_A in a fuzzy soft topological space (X, τ, E), denoted by $FSint(f_A)$, is the union of all fuzzy soft semi-open subset of f_A in X.

Clearly, $FSint(f_A)$ is a semi-open fuzzy soft set in X for every $f_A \in FSS(X)_E$.

Definition 2.25 ([13])**.** Let (X, τ, E) be a fuzzy soft topological space and $f_A \in FSS(X)_E$. Then, $f_A \in FSSO(X)_E$, if and only if there exists $g_B \in \tau$ such that $g_B \mathrel{\widetilde{\subseteq}} f_A \mathrel{\widetilde{\subseteq}} Fcl(g_B)$.

Definition 2.26 ([13])**.** Let (X, τ, E) and (Y, σ, K) be fuzzy soft topological spaces. A function $f_{pu} : (X, \tau, E) \longrightarrow (Y, \sigma, K)$ is called:

(i) fuzzy soft semi-continuous, if $f_{pu}^{-1}\ (g_B)$ is a fuzzy soft semi-open set in X, for every $g_B \in \sigma$,

(ii) fuzzy soft semi-open, if $f_{pu}\ (g_B)$ is a fuzzy soft semi-open set in Y, for every $g_B \in \tau$.

Definition 2.27 ([21])**.** A subset f_A of a fuzzy soft topological space (X, τ, E) is called:

(i) fuzzy soft regular open if $f_A = Fint(Fcl(f_A))$,

(ii) fuzzy soft regular closed if its complement is fuzzy soft regular open.

Clearly, if f_A is a fuzzy soft regular closed, then $f_A = Fcl(Fint(f_A))$.

The collection of all fuzzy soft regular open (respectively, regular closed) will be denoted by $FSRO(X)_E$ (respectively, $FSRC(X)_E$).

Remark 2.2. The following diagram shows the relationship between the subsets of fuzzy soft topological space (X, τ, E):

$$\begin{array}{ccccc}
 & & FSSO(X)_E & \Longrightarrow & FS\beta O(X)_E \\
 & & \Uparrow & & \Uparrow \\
 & & FS\alpha O(X)_E & \Longrightarrow & FSPO(X)_E \\
 & & \Uparrow & & \\
FSRO(X)_E & \Longrightarrow & FSO(X)_E & & \\
\Downarrow & & & & \\
FSSC(X)_E & & & &
\end{array}$$

Definition 2.28 ([17])**.** A fuzzy soft topological space (X, τ, E) is said to be fuzzy soft connected, if there does not exist two non-unll disjoint fuzzy soft open sets f_A and g_B in X such that $f_A \sqcup g_B = \widetilde{1}_E$.

Otherwise, (X, τ, E) is called fuzzy soft disconnected space.

3. Fuzzy soft hyperconnected space

In this section, we introduce the notions of fuzzy soft hyperconnected space, fuzzy soft D-space, and fuzzy soft extremally disconnected space. Also, we study the relationship between them.

Definition 3.1. A fuzzy soft topological space (X, τ, E) is said to be fuzzy soft hyperconnected space, if $f_A \sqcap g_B \neq \widetilde{0}_E$, for every non-null open fuzzy soft sets f_A and g_B in X.

Definition 3.2. A subset f_E of a fuzzy soft topological space (X, τ, E) is called:
(i) fuzzy soft dense set, if $Fcl(f_E) = \widetilde{1}_E$,
(ii) fuzzy soft nowhere dense, if $Fint(Fcl(f_E)) = \widetilde{0}_E$.

Definition 3.3. A fuzzy soft topological space (X, τ, E) is said to be fuzzy soft D-space, if every non-null open fuzzy soft set in X is a fuzzy soft dense.

Lemma 3.1. *Every fuzzy soft D-space is fuzzy soft hyperconnected space.*

Proof. Let (X, τ, E) be a fuzzy soft D-space. Suppose that f_A and g_B are non-null open fuzzy soft sets in X such that $f_A \sqcap g_B = \widetilde{0}_E$. Then, $Fcl(f_A) \widetilde{\subseteq} g^c_B$. Thus, f_A is not fuzzy soft dense. This is a contradiction. So, (X, τ, E) is fuzzy soft hyperconnected space. □

Remark 3.1. The converse of Lemma 3.1 is not true in general as shown by the following example.

Example 3.4. Let $X = \{a, b\}$, $E = \{e_1\}$ and $\tau = \{\widetilde{1}_E, \widetilde{0}_E, \{(e_1, \{a_{0.1}, b_{0.1}\})\}, \{(e_1, \{a_{0.5}, b_{0.5}\})\}\}$. Then, (X, τ, E) is fuzzy soft hyperconnected but it is not fuzzy soft D-space.

Definition 3.5. A fuzzy soft topological space (X, τ, E) is said to be fuzzy soft extremally disconnected space, if the closure of every fuzzy soft open set in X is fuzzy soft open.

Lemma 3.2. *Every fuzzy soft D-space is fuzzy soft extremally disconnected space.*

Proof. Let (X, τ, E) be a fuzzy soft D-space. Then, $Fcl(f_A) = \widetilde{1}_E$ for every fuzzy soft open subset f_A of X. Thus, $Fcl(f_A) \in \tau$ for every fuzzy soft open subset f_A of X. So (X, τ, E) is fuzzy soft extremally disconnected space. □

Remark 3.2. The converse of Lemma 3.2 is not true in general as shown by the following example.

Example 3.6. The fuzzy soft discrete topological space is fuzzy soft extremally disconnected space but it is not fuzzy soft D-space.

Lemma 3.3. (1) *Every fuzzy soft hyperconnected space is fuzzy soft connected.*

(2) *Every soft hyperconnected space is a fuzzy soft hyperconnected space.*

(3) *Every hyperconnected space is a fuzzy hyperconnected space.*

Proof. (1) Let (X, τ, E) be fuzzy soft hyperconnected space. Then, $f_A \sqcap g_B \neq \widetilde{0}_E$ for every non-null open fuzzy soft subsets f_A and g_B of X. Thus, $\widetilde{1}_E$ can not be written as a union of non-null disjoint open fuzzy soft sets in X. So, (X, τ, E) is fuzzy soft connected space.

(2) It follows from the fact that every soft set can be consided as a fuzzy soft set.

(3) It follows from the fact that every set can be consided as a fuzzy set. □

Remark 3.3. The converse of Lemma 3.3 is not true in general as shown by the following example.

Example 3.7. Let $X = \{a,\ b\}$, $E = \{e_1,\ e_2\}$ and $\tau = \{\widetilde{1}_E,\ \widetilde{0}_E,\ \{(e_1, \{b_1\})\}, \{(e_1, \{a_1\})\}\}$. Then, (X, τ, E) is fuzzy soft connected but it is not fuzzy soft hyperconnected space.

Remark 3.4. By Lemmas 3.1, 3.2 and 3.3, we have the following diagram:

fuzzy soft D-space	$\Longrightarrow$	fuzzy soft hyperconnected	$\Longrightarrow$	fuzzy soft connected
$\Downarrow$		$\Uparrow$		
fuzzy soft extremally disconnected		soft hypercommscted spaces		

Theorem 3.1. *If (X, τ, E) is fuzzy soft D-space, then every fuzzy soft subset f_A of X is fuzzy soft dense or fuzzy soft nowhere dense.*

Proof. Let (X, τ, E) be a fuzzy soft D-space and f_A be a fuzzy soft subset of X. Suppose that f_A is not fuzzy soft nowhere dense. Then, $Fint(Fcl(f_A)) \neq \widetilde{0}_E$. Since $Fint(Fcl(f_A))$ is non-null open fuzzy soft subset of X, then $Fcl(Fint(Fcl(f_A))) = \widetilde{1}_E$. Since $\widetilde{1}_E = Fcl(Fint(Fcl(f_A))) \widetilde{\subseteq} Fcl(Fcl(f_A)) = Fcl(f_A)$, $Fcl(f_A) = \widetilde{1}_E$. Thus, f_A is fuzzy soft dense set in X. □

Theorem 3.2. *A fuzzy soft topological space (X, τ, E) is fuzzy soft hyperconnected space if and only if $f_A \sqcap g_B \neq \widetilde{0}_E$ for every non-null fuzzy soft semi-open subsets f_A and g_B of X.*

Proof. Let (X, τ, E) be a fuzzy soft hyperconnected space. Suppose that $f_A \sqcap g_B = \widetilde{0}_E$, for some non-null fuzzy soft semi-open sets f_A and g_B of X. Then by Theorem 2.1, there exist open fuzzy soft sets h_C and s_D in X such that $h_C \widetilde{\subseteq} f_A \widetilde{\subseteq} Fcl(h_C)$ and $s_D \widetilde{\subseteq} g_B \widetilde{\subseteq} Fcl(s_D)$. Since f_A and g_B are non-null, h_C and s_D are non-null. Moreover, we have $h_C \sqcap s_D \widetilde{\subseteq} f_A \sqcap g_B = \widetilde{0}_E$. Thus, $h_C \sqcap s_D = \widetilde{0}_E$. This is a contradiction. So, $f_A \sqcap g_B \neq \widetilde{0}_E$.

Conversely, suppose that $f_A \sqcap g_B \neq \widetilde{0}_E$, for every non-null fuzzy soft semi-open subsets f_A and g_B of X. Since every open fuzzy soft set is a fuzzy soft semi-open, $f_A \sqcap g_B \neq \widetilde{0}_E$ for every non-null open fuzzy soft subsets f_A and g_B of X. Then, (X, τ, E) is fuzzy soft hyperconnected space. □

Theorem 3.3. *If (X, τ, E) is fuzzy soft D-space, then every non-null fuzzy soft β-open subset f_A of X is fuzzy soft dense.*

Proof. Let (X, τ, E) be fuzzy soft D-space and f_A be any non-null fuzzy soft β-open subset of X. Then, $Fint(Fcl(f_A)) \neq \widetilde{0}_E$ this implies f_A is not fuzzy soft nowhere dense. By Theorem 3.1, we have f_A is a fuzzy soft dense set. □

Corollary 3.1. *For a fuzzy soft topological space (X, τ, E), the following properties are equivalent:*

(1) *(X, τ, E) is a fuzzy soft D-space,*

(2) *$f_A \sqcap g_B \neq \widetilde{0}_E$ for every non-null fuzzy soft semi-open subset f_A and fuzzy soft β-open subset g_B of X,*

(3) *$f_A \sqcap g_B \neq \widetilde{0}_E$ for every non-null fuzzy soft semi-open subset f_A and fuzzy soft pre-open subset g_B of X.*

Proof. The proof is obvious from Theorem 3.3. □

Theorem 3.4. *Every fuzzy soft open subspace of fuzzy soft D-space is fuzzy soft D-space.*

Proof. Let (X, τ, E) be a fuzzy soft D-space and f_A be a fuzzy soft open set in X. Let g_B be a fuzzy soft open subset in the open subspace (f_A, τ_{f_A}, A). Then, $g_B = f_A \sqcap h_C$ for some $h_C \in \tau$. Thus, g_B is an open fuzzy soft set in X as an intersection of two open fuzzy soft sets. So $Fcl(g_B) = \widetilde{1}_E$. Since

$$Fcl_{f_A}(g_B) = f_A \sqcap Fcl(g_B) = f_A \sqcap \widetilde{1}_E = f_A,$$

(f_A, τ_{f_A}, A) is fuzzy soft D-space. □

Theorem 3.5. *Every fuzzy soft open subspace of fuzzy soft hyperconnected space is fuzzy soft hyperconnected.*

Proof. Let (X, τ, E) be a fuzzy soft hyperconnected space and f_A be an open fuzzy soft set in X. Let g_B and h_C be non-null open fuzzy soft subset in the fuzzy soft open subspace (f_A, τ_{f_A}, A). Then, $g_B = f_A \sqcap s_D$ and $h_C = f_A \sqcap u_N$, for some $s_D, u_N \in \tau$. Thus, g_B and h_C are non-null open fuzzy soft sets in X as an intersection of open fuzzy soft sets. So, $g_B \sqcap h_C \neq \widetilde{0}_E$. Hence, (f_A, τ_{f_A}, A) is fuzzy soft hyperconnected space. □

Definition 3.8. A function $f_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ is said to be fuzzy soft almost S-continuous, if for every non-null fuzzy soft regular open set g_B of Y with $f_{pu}^{-1}(g_B) \neq \widetilde{0}_E$, $FSint(f_{pu}^{-1}(g_B)) \neq \widetilde{0}_E$.

Theorem 3.6. *Every fuzzy soft semi-continuous function $f_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ is fuzzy soft almost S-continuous.*

Proof. Let $f_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ be a fuzzy soft semi-continuous function and g_B be any non-null fuzzy soft regular open subset of Y with $f_{pu}^{-1}(g_B) \neq \widetilde{0}_E$. Then, $f_{pu}^{-1}(g_B)$ is non-null fuzzy soft semi-open. Thus $FSint(f_{pu}^{-1}(g_B)) = f_{pu}^{-1}(g_B) \neq \widetilde{0}_E$. So, f_{pu} is fuzzy soft almost S-continuous function. □

Corollary 3.2. *For a fuzzy soft topological spaces, we have the following diagram:*

$$\begin{array}{ccccc} \textit{fuzzy soft} & \Longrightarrow & \textit{fuzzy soft} & \Longrightarrow & \textit{fuzzy soft} \\ \textit{continuous} & & \textit{semi-continuous} & & \textit{almost } S\textit{-continuous} \end{array}$$

Remark 3.5. The converse of Theorem 3.6 is not true in general. The following example shows a fuzzy soft almost S-continuous function may be not fuzzy soft semi-continuous.

Example 3.9. Let $X = Y = \{a,\ b\}$, $E = K = \{e_1,\ e_2\}$ and $f_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ be the constant fuzzy soft mapping where $\tau = \{\widetilde{1}_E,\ \widetilde{0}_E,\ \{(e_1,\ \{a_{0.5}\})\}\}$ and $\sigma = \{\widetilde{1}_E,\ \widetilde{0}_E,\ h_E = \{(e_1,\ \{a_{0.4},\ b_{0.4}\})\},\ \{(e_2,\ \{a_{0.4},\ b_{0.4}\})\},\ \{(e_1,\ \{a_{0.4},\ b_{0.4}\}),\ (e_2,\ \{a_{0.4},\ b_{0.4}\})\},\ g_E = \{(e_1,\ \{a_{0.5},\ b_{0.5}\}),\ (e_2,\ \{a_{0.5},\ b_{0.5}\})\}\}$ be fuzzy soft topological spaces on X and Y, respectively such that $u(x) = a$, for every $x \in X$ and $p(e) = e_1$, for every $e \in E$. Then, g_E is the only fuzzy soft regular open set in Y with $f_{pu}^{-1}(g_E) = g_E \neq \widetilde{0}_E$. Moreover, $f_{pu}^{-1}(g_E)$ is a fuzzy soft semi-open set in X. Thus $FSint(f_{pu}^{-1}(g_E)) = f_{pu}^{-1}(g_E) = g_E \neq \widetilde{0}_E$. So, f_{pu} is fuzzy soft almost S-continuous function. But, h_E is an open fuzzy soft set in Y, $f_{pu}^{-1}(h_E) = \{(e_1,\ \{a_{0.4},\ b_{0.4}\}),\ (e_2,\ \{a_{0.4},\ b_{0.4}\})\}$ which is not fuzzy soft semi open set in X. Hence, f_{pu} is not fuzzy soft semi-continuous function.

Theorem 3.7. *The following properties hold for a fuzzy soft hyperconnected space (X, τ, E):*

(1) *Every fuzzy soft almost S-continuous function $f_{pu} : (X, \tau, E) \to (Y, \sigma, K)$, where (Y, σ, K) is a FST_2-space, is a constant,*

(2) *Every fuzzy soft semi-continuous function $f_{pu} : (X, \tau, E) \to (Y, \sigma, K)$, where (Y, σ, K) is a FST_2-space, is a constant.*

Proof. (1) Let (X, τ, E) be a fuzzy soft hyperconnected space and (Y, σ, K) be a FST_2-space. Suppose that f_{pu} is a fuzzy soft almost S-continuous function such that f_{pu} is not constant. Then, there exist two fuzzy soft points x_α^e and y_β^t of X such that $f_{pu}(x_\alpha^e) \neq f_{pu}(y_\beta^t)$. Since (Y, σ, K) is a FST_2-space, there exist two open fuzzy soft sets f_A and g_B in Y such that $f_{pu}(x_\alpha^e)\ \widetilde{\in}\ f_A$, $f_{pu}(y_\beta^t)\ \widetilde{\in}\ g_B$ and $f_A \sqcap g_B = \widetilde{0}_K$. Take $h_C = Fint(Fcl(f_A))$ and $s_D = Fint(Fcl(g_B))$. This implies that h_C and s_D are non-null fuzzy soft regular open sets in Y with $h_C \sqcap s_D = \widetilde{0}_K$. Since f_{pu} is a fuzzy soft almost S-continuous function, $FSint(f_{pu}^{-1}(h_C)) \neq \widetilde{0}_E$ and

$FSint(f_{pu}^{-1}(s_D)) \neq \widetilde{0}_E$. Thus, $FSint(f_{pu}^{-1}(h_C))$ and $FSint(f_{pu}^{-1}(s_D))$ are non-null fuzzy soft semi-open sets in X. So,

$$\begin{aligned} & FSint(f_{pu}^{-1}(h_C)) \sqcap FSint(f_{pu}^{-1}(s_D)) \\ &= FSint[f_{pu}^{-1}(h_C) \sqcap (f_{pu}^{-1}(s_D)] \\ &\widetilde{\subseteq} f_{pu}^{-1}(h_C) \sqcap f_{pu}^{-1}(s_D) \\ &= f_{pu}^{-1}(h_C \sqcap s_D) = \widetilde{0}_E. \end{aligned}$$

This contradicts Theorem 3.2. Hence, f_{pu} is a constant.

(2) Let (X, τ, E) be a fuzzy soft hyperconnected space and (Y, σ, K) be a FST_2-space. Let f_{pu} be a fuzzy soft semi-continuous function. Then by Theorem 3.6, f_{pu} is a fuzzy soft almost S-continuous. Thus by (1), t f_{pu} is a constant. □

Theorem 3.8. *Let (X, τ, E) be a fuzzy soft hyperconnected space and $f_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ be a fuzzy soft almost S-continuous surjection. Then, (Y, σ, K) is a fuzzy soft hyperconnected space.*

Proof. Suppose that (Y, σ, K) is not a fuzzy soft hyperconnected space. Then, there exist two non-null disjoint open fuzzy soft sets f_A and g_B in Y. Take $h_C = Fint(Fcl(f_A))$ and $s_D = Fint(Fcl(g_B))$. Thus, h_C and s_D are non-null fuzzy soft regular open sets in Y such that $h_C \sqcap s_D = \widetilde{0}_K$. So we have

$$FSint(f_{pu}^{-1}(h_C)) \sqcap FSint(f_{pu}^{-1}(s_D)) \widetilde{\subseteq} f_{pu}^{-1}(h_C) \sqcap f_{pu}^{-1}(s_D) = f_{pu}^{-1}(h_C \sqcap s_D) = \widetilde{0}_E.$$

Since f_{pu} is a fuzzy soft almost S-continuous surjection, $FSint(f_{pu}^{-1}(h_C))$ and $FSint(f_{pu}^{-1}(s_D))$ are non-null fuzzy soft semi-open sets in X. Hence by Theorem 3.2, (X, τ, E) is not a fuzzy soft hyperconnected space. This is a contradiction. □

Corollary 3.3. *If (X, τ, E) is a fuzzy soft hyperconnected space and $f_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ is a fuzzy soft continuous surjection, then (Y, σ, K) is a fuzzy soft hyperconnected space.*

Proof. It follows from Theorem 3.8 and Corollary 3.2. □

Definition 3.10. A function $f_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ is said to be a fuzzy soft almost S-open, if $FSint(f_{pu}(f_A)) \neq \widetilde{0}_E$, for every non-null fuzzy soft regular open set f_A of X.

Remark 3.6. For a mapping $f_{pu} : (X, \tau, E) \to (Y, \sigma, K)$, we have the following diagram:

fuzzy soft open function	$\Longrightarrow$	fuzzy soft semi-open function	$\Longrightarrow$	fuzzy soft almost S-open function

Theorem 3.9. *If (Y, σ, K) is a fuzzy soft hyperconnected space and $f_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ is a fuzzy soft almost S-open injection, then (X, τ, E) is a fuzzy soft hyperconnected.*

Proof. Let f_A and g_B be any non-null open fuzzy soft sets of X. Take $h_C = Fint(Fcl(f_A))$ and $s_D = Fint(Fcl(g_B))$. Then, h_C and s_D are non-null fuzzy soft regular open sets in X. Since f_{pu} is a fuzzy soft almost S-open function, $FSint(f_{pu}(h_C)) \neq \widetilde{0}_E$ and $FSint(f_{pu}(s_D)) \neq \widetilde{0}_E$. Since (Y, σ, K) is a fuzzy soft hyperconnected space, $\widetilde{0}_E \neq FSint(f_{pu}(h_C)) \sqcap FSint(f_{pu}(s_D)) \widetilde{\subseteq} f_{pu}(h_C) \sqcap f_{pu}(s_D)$. Since f_{pu} is a fuzzy soft injection, $h_C \sqcap s_D \neq \widetilde{0}_E$. Thus, $f_A \sqcap g_B \neq \widetilde{0}_E$. So (X, τ, E) is a fuzzy soft hyperconnected. $\square$

Corollary 3.4. *If (Y, σ, K) is a fuzzy soft hyperconnected space and $f_{pu} : (X, \tau, E) \to (Y, \sigma, K)$ is a fuzzy soft open injection, then (X, τ, E) is a fuzzy soft hyperconnected space.*

Proof. It follows from Theorem 3.9 and Remark 3.6. $\square$

4. Conclusions

In This paper, we introduce the concept of fuzzy soft hyperconnected space and study its properties. We show that every fuzzy soft hyperconnected space is a fuzzy soft connected and every fuzzy soft D-space is both fuzzy soft hyperconnected space and fuzzy soft extremally disconnected space. Also, we present that every subset of a fuzzy soft D-space is either fuzzy soft dense set or fuzzy soft nowhere dense. Furthermore, every fuzzy soft open subspace of a fuzzy soft hyperconnected space is fuzzy soft hyperconnected.

References

[1] A. M. Abd El-Latif, Characterizations of Fuzzy Soft Pre Separation Axioms, Journal of New Theory 7 (2015) 47–63.

[2] A. M. Abd El-Latif, Fuzzy soft separation axioms based on fuzzy β -open soft sets, Ann. Fuzzy Math. Inform. 11 (2) (2016) 223–239.

[3] A. M. Abd El-Latif, Fuzzy Soft α-Connectedness in Fuzzy Soft Topological Spaces, Math. Sci. Lett. 1 (2016) 85–91.

[4] A. M. Abd El-Latif and R. Hosny, On Soft Separation Axioms via Fuzzy α-Open Soft Sets, Inf. Sci. Lett. 1 (2016) 1–9.

[5] A. M. Abd El-Latif and R. Hosny, Fuzzy soft pre-connected properties in fuzzy soft topological spaces, South Asian Journal of Mathematics 5 (2015) 202–213.

[6] N. Ajmal and J. K. Kohli, Properties of Hyperconnected Spaces, Their Mappings into Hausdorff Spaces and Embedding into Hyperconnected Spaces, Acta Math. Hung. 60 (1-2) (1992) 41–49.

[7] H. Aktaş and N. Çağman, Soft sets and soft groups, Inform. Sci. 177 (2007) 2726–2735.

[8] M. A. Alkhafaji and S. A. Alkanee, On Fuzzy Semi Extremely Disconnected in Fuzzy Topological Space, IOSR Journal of Mathematics 5 (2013) 42–45.

[9] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–193.

[10] E. Ekici and T. Noiri, $*$-Hyperconnected Ideal Topological Spaces, Annals of the Alexandru Ioan Cuza University - Mathematics 58 (2012) 121–129.

[11] P. K. Gain, P. Mukherjee and R. P. Chakraborty, On Some Decompositions of Fuzzy Soft Continuity, Journal of New Theory 4 (2015) 39–52.

[12] A. Kandil, O. A. El Tantawy, S. A. El-Sheikh, and A. M. Abd El-latif, Fuzzy Soft Semi Connected Properties in Fuzzy Soft Topological Spaces, Mathematical Sciences Letters 2 (2015) 171–179.

[13] A. Kandil, O. A. El Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Some Fuzzy Soft Topological Properties Based on Fuzzy Semi-Open Soft Sets, South Asian Journal of Mathematics 4 (2014) 154–169.

[14] A. Kandil, O. A. El Tantawy, S. A. El-Sheikh, A. M. Abd El-latif and S. El-Sayed, Fuzzy Soft Connectedness Based on Fuzzy β -Open Soft Sets, Journal of Mathematics and Computer Applications Research 2 (1) (2015) 37–46 .

[15] A. Kandil, O. A. El Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Soft Connectedness Via Soft Ideals, Journal of New Results in Science 4 (2014) 90–108.

[16] R. Lowen, Fuzzy Topological Spaces and Fuzzy Compactness. J. Math. Anal. Appl. 56 (1976) 621–33.

[17] J. Mahanta and P. K. Das, Results on Fuzzy Soft Topological Spaces, arXiv:1203.0634v1, (2012).

[18] P. K. Maji, R. Biswas and A. R. Roy, Soft Set Theory, Computers Math. Appl. 45 (2003) 555–562.

[19] P. K. Maji, R. Biswas and A. R. Roy, Fuzzy Soft Sets, J. Fuzzy Math. 9 (3) (2001) 589–602.

[20] D. Molodtsov, Soft Set Theory-First Results, Computers Math. Appl. 37(4/5) (1999) 19–31.

[21] P. Mukherjee, R. P. Chakraborty and C. Park, On Fuzzy Soft δ-Open Sets and Fuzzy Soft δ-Continuity, Ann. Fuzzy Math. Inform. 11 (2) (2016) 327–340.

[22] M. H. Rashid and D. M. Ali, Certain Fetures of Fuzzy Contra-Continuous Functions, Journal of Bangladesh Academy of Sciences 32 (2008) 71–77.

[23] S. Roy and T. K. Samanta, An Introduction to Open and Closed Sets on Fuzzy Soft Topological Spaces, Ann. Fuzzy Math. Inform. 6 (2) (2013) 425–431.

[24] M. Shabir and M. Naz, On Soft Topological Spaces, Computers and Mathematics with Applications 61 (2011) 1786–1799.

[25] T. Simsekler and S. Yuksel, Fuzzy Soft Topological spaces, Ann.Fuzzy Math. Inform. 5 (1) (2013) 87–96.

[26] B. Tanay and M.B. Kandemir, Topological Structures of Fuzzy Soft Sets, Computers and Mathematics with Applications 61 (2011) 412–418.

[27] G. Thangaraj and C. Anbazhagan, Some Remarks on Fuzzy P-Spaces , Gen. Math. Notes 26 (2015) 8–16 .

[28] B. P. Varol and H. Aygün, Fuzzy Sot Topology, Hacettepe Journal of Mathematics and Statistics 41 (3) (2012) 40-419.

[29] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

A. Kandil (dr.ali kandil@yahoo.com)
Department of Mathematics, Faculty of Science, Helwan University, Cairo, Egypt
O. Tantaway (drosamat@yahoo.com)
Department of Mathematics, Faculty of Science, Zagazig University, Cairo, Egypt
S. A. EL-SHEIKH (sobhyelsheikh@yahoo.com)
Department of Mathematics, Faculty of Education, Ain Shams University, Roxy 11757, Cairo, Egypt
Sawsan El Sayed (sawsan_809@yahoo.com)

Department of Mathematics, Faculty of Education, Ain Shams University, Roxy 11757, Cairo, Egypt