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Conjugate space of fuzzy soft normed linear space

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Abstract. In this paper the concept of conjugate space of the fuzzy soft normed linear space is introduced. Some properties of fuzzy soft conjugate space are emphasized and furthermore theorems related to these properties are established.

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1. Introduction

In 1965, Zadeh [10] defined a fuzzy set as a class of objects with a continuum of grades of membership. A new set theory called soft set dealing with uncertainties was initiated by Molodtsov [6] for the inadequacy of the parameterization tool of the fuzzy set theory. Fuzzy soft set is another mathematical tool developed by Maji et al [5] with the combination of fuzzy set theory and soft set theory. S.C. Cheng and J.N. Mordeson [2] introduced fuzzy norm and this led Bag and Samanta to establish fuzzy normed linear space [3]. Topological studies of fuzzy soft sets [4] were dealt by Tanay and Kandemir. In 2013, Azadeh et al. [1] coined fuzzy soft norm over a set and established the relationship between fuzzy soft norm and fuzzy norm over a set.

In this paper, the preliminary concepts like fuzzy soft sets, fuzzy soft linear operator and the definition of conjugate space have been recapitulated [7, 8, 9]. Later fuzzy soft linear transformation, continuity and conjugate space of fuzzy soft normed linear spaces are defined. Some related properties viz equivalent condition for continuity, natural imbedding theorem, linearity properties are proved.

2. Preliminaries

Definition 2.1 ([7]). Let N be a normed linear space. Then the set of all continuous linear transformations of N into $\mathbb{R}$ or $\mathbb{C}$ is denoted by $B(N,\mathbb{R})$ or $B(N,\mathbb{C})$.

N^* is called the conjugate space of N and its elements are said to be continuous linear functionals or briefly functional.

If the norm of a functional f, denoted by $\|f\|$, is given by

$$\begin{aligned}\|f\| &= \sup\{|f| : \|x\| \leq 1\}\\ &= inf\{k : k \geq 0 \text{ and } |f(x)| \leq k\|x\| \text{ for all } x\}\end{aligned}$$

then N^* is a Banach space.

Since the conjugate space N^* of a normed linear space N is itself a normed linear space, it is possible to form the conjugate space N^{**} of N and N^{**} is said to be the second conjugate space of N.

Theorem 2.2 ([7]). *Let N and N' be normed linear spaces and T a linear transformation of N into N'. Then the following conditions on T are all equivalent:*

(1) *T is continuous,*

(2) *T is continuous at the origin, in the sense that $x_n \to 0 \Rightarrow T(x_n) \to 0$,*

(3) *there exists a real number $k \geq 0$ with the property that $\|T(x)\| \leq k\|x\|$, for every $x \in N$,*

(4) *if $S = \{x : \|x\| \leq 1\}$ is the closed unit sphere in N then its image $T(S)$ is a bounded set in N'.*

Theorem 2.3 ([7]). *A non empty subset X of a normed linear space N is bounded if and only if $f(X)$ is a bounded set of numbers for each f in N^*.*

Definition 2.4 ([7]). Let L be the linear space of all scalar valued linear functions defined on N. The conjugate space N^* is clearly a linear space of L. Let T be a linear transformation of N into itself which is not necessarily continuous. Using T, define a linear transformation T' of L into itself. If f is in L, then $T'(f)$ is defined by $[T'(f)](x) = f(T(x))$.

Result 2.5 ([7]). *If T is a continuous operator on N, then it is possible to restrict T' to a mapping N^* into itself.*

Proof. Consider $f \in N^* \subseteq L$. Then $T'(f) \in T'(N^*)$. If S is a closed unit sphere in N, then T is continuous

$$\begin{aligned}&\Leftrightarrow T(S) \text{ is bounded}\\ &\Leftrightarrow f(T(S)) \text{ is bounded, for each } f \in N^*\\ &\Leftrightarrow [T'(f)](S) \text{ is bounded, for each } f \in N^*\\ &\Leftrightarrow T'(f) \in N^*, \text{ for each } f \in N^*.\end{aligned}$$

Thus $T'(N^*) \subseteq N^*$. So it is possible to restrict T' to a mapping N^* into itself.

Denote this restriction by T^* and is said to be the conjugate of T. □

Definition 2.6 ([1]). A pair (F, A) is called fuzzy soft set over X, if f is a mapping given by $f : A \to I^X$, $A \subseteq E$. Then for every $e \in A$, $f(e)$ is a fuzzy subset of X with membership function $f_e : X \to [0,1]$.

The fuzzy soft set is denoted by f_A and the set of all fuzzy soft sets is denoted by $\mathcal{F}S(X_E)$.

Definition 2.7 ([1])**.** Let X and Y be universe sets and E and E' be the corresponding parameter sets. The map $h_{up} : \mathcal{FS}(X_E) \to \mathcal{FS}(Y_{E'})$ is called a fuzzy soft map from X and Y which maps the fuzzy soft subset f_E of X to fuzzy soft subset $h_{up}(f_E)$ of Y and is defined as below

$$[h_{up}(f_E)]_{e'}(y) = \begin{cases} \sup\limits_{x=u^{-1}(y)} \left[\sup\limits_{e=p^{-1}(e')} f(e) \right](x) & \text{if } p^{-1}(e') \neq \phi \text{ and } u^{-1}(y) \neq \phi \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in X$ and $e \in E$, where $u : X \to Y$ and $p : E \to E'$ are ordinary functions.

Definition 2.8 ([1])**.** Let $f_E, g_E \in \mathcal{FS}(X_E)$. Then define $f_E \times g_E \in \mathcal{FS}(\tilde{X} \times \tilde{X})$ as the fuzzy soft multiplication by map $f \times g : E \to I^{X \times X}$, where $(f \times g)(e) : X \times X \to I$,

$$(f \times g)(e)(x_1, x_2) = (f_e \times g_e)(x_1, x_2) = \min\{f_e(x_1), g_e(x_2)\}.$$

Definition 2.9 ([1])**.** Let X be soft vector space over a scalar field $\mathbb{R}$ or $\mathbb{C}$. Let $h_1 : X \times X \to X$ the addition function on X, i.e., $h_1(x_1, x_2) = x_1 + x_2$, for all $x_1, x_2 \in X$. Let $f_E, g_E \in \mathcal{FS}(X_E)$. Then by using the fuzzy soft functional definition, the fuzzy soft addition, denoted by $h_1(f_E \times g_E) = f_E \oplus g_E$, is defined as

$$\begin{aligned} (f_E \oplus g_E)(e)(x) &= h_1(f_E \times g_E)(e)(x) \\ &= \sup_x [(f \times g)(e)](x) \\ &= \sup_{x_1, x_2} \min\{f_e(x_1), g_e(x_2)\}. \end{aligned}$$

Definition 2.10 ([1])**.** Let $(X, E, \|.\|)$ be a fuzzy soft normed space. Let $\{(\tilde{x}_n)_E\}$ be a sequence of fuzzy soft points in X, say fuzzy soft sequence in X such that for all $e \in E$, $(\tilde{x}_n)_e(t) = \lambda_{e,n}$, if $t = x_n$ and $(\tilde{x}_n)_e(t) = 0$, if $t \neq x_n$. Suppose that $\tilde{x}_E$ be fuzzy soft point in X, such that for all $e \in E$, $\tilde{x}_e(t) = \gamma_e$, if $t = x$ and $\tilde{x}_e(t) = 0$, if $t \neq x$. Then the fuzzy soft sequence $\{(\tilde{x}_n)_E\}$ converges to fuzzy soft point $\tilde{x}_E$ if and only if $\lim\limits_{n\to\infty} |\lambda_{e,n} - \gamma_e| = 0$ and $\lim\limits_{n\to\infty} \|x_n - x\|^i_{e,\alpha} = 0$ for $i = 1, 2$. It is denoted by $\lim\limits_{n\to\infty} (\tilde{x}_n)_E = \tilde{x}_E$ or $\lim\limits_{n\to\infty} \|(\tilde{x}_n)_E - \tilde{x}_E\| = \tilde{0}_E$. The fuzzy soft sequence $\{(\tilde{x}_n)_E\}$ is called fuzzy soft convergent sequence in X.

Definition 2.11 ([1])**.** Let $(X, E, \|.\|)$ and $(Y, E', \|.\|)$ be fuzzy soft normed spaces, where E and E' are the parameter sets and X and Y are universal sets. Then $T : (X, E, \|.\|) \to (Y, E', \|.\|)$ is called fuzzy soft sequentially continuous, if for every fuzzy soft convergent sequence $\{(\tilde{x}_n)_E\}$ of $\tilde{X}$ such that $\lim\limits_{n\to\infty} (\tilde{x}_n)_E = \tilde{x}_E$, the fuzzy soft sequence $\{T(\tilde{x}_n)_E\}$ of $\tilde{Y}$ converges to fuzzy soft point $T\tilde{x}_E$.

Definition 2.12 ([1])**.** Let $\mathbb{R}$ be the set of all real numbers and E be the parameter set. Then the set of all fuzzy soft real numbers, say $\mathbb{R}_E$ is defined as set of all $\{f : E \to I^{\mathbb{R}}\}$, where for all $e \in E$, $f(e)$'s are fuzzy real numbers.

3. Conjugate space of fuzzy soft normed linear space

Definition 3.1 ([9])**.** Let $SSP(\tilde{X})$ or $S(\tilde{X})$ be the set of all soft points on fuzzy soft normed linear space $\tilde{X}$. Then the map $\tilde{X}_E$ from $SSP(\tilde{X})$ to I^X is called the fuzzy soft set on $SSP(\tilde{X})$ and the set of all fuzzy sets on $\tilde{X}$ is denoted as $\mathcal{F}(SSP(\tilde{X}))$ or $\mathcal{F}S(\tilde{X})$.

Definition 3.2 ([9])**.** Let $\tilde{X}_E : SSP(\tilde{X}) \to I^X$ be a fuzzy soft set. Then the norm of the fuzzy soft set $\tilde{X}_E$ is defined as $\|\tilde{X}_E\| = \sup\limits_{\|\tilde{x}_e\| \tilde{\leq} \tilde{1}} \|\tilde{X}_E(\tilde{x}_e)\|$ and the pair $(\tilde{X}_E, \|\|)$ is a fuzzy soft normed linear space.

Definition 3.3 ([9])**.** The fuzzy soft map T_{up} of $\mathcal{F}S(\tilde{X})$ into $\mathbb{R}_{E'}$ is defined as

$$[T_{up}(\tilde{X}_E)](e')(r) = \begin{cases} \sup\limits_{x=u^{-1}(r)} \left[\sup\limits_{e=p^{-1}(e')} \tilde{X}_E(e) \right](x), & \text{if } u^{-1}(r) \neq \phi, p^{-1}(e') \neq \phi \\ 0, & \text{otherwise,} \end{cases}$$

where $u : X \to \mathbb{R}$ and $p : E \to E'$. Then the norm of T_{up} is defined as

$$\|T_{up}\| = \sup\{\|T_{up}(\tilde{X}_E)\| : \|\tilde{X}_E\| \tilde{\leq} \tilde{1}\}$$

and

$$\|T_{up}\| = inf\{\tilde{k} : \|T_{up}(\tilde{X}_E)\| \tilde{\leq} \tilde{k} \|\tilde{X}_E\|\},$$

where $\|\tilde{X}_E\| = \sup\limits_{\|\tilde{x}_e\| \tilde{\leq} \tilde{1}} \|\tilde{X}_E(\tilde{x}_e)\|$.

Clearly, $\|T_{up}(\tilde{X}_E)\| \tilde{\leq} \|T_{up}\| \|\tilde{X}_E\|$.

Definition 3.4. Let $\tilde{X}_E, \tilde{Y}_E \tilde{\in} \mathcal{F}S(\tilde{X})$. Then define $\tilde{X}_E \times \tilde{Y}_E \tilde{\in} \mathcal{F}S(\tilde{X} \times \tilde{X})$ as the fuzzy soft multiplication by map $\tilde{X}_E \times \tilde{Y}_E : S(\tilde{X}) \times S(\tilde{X}) \to I$, $(\tilde{X}_E \times \tilde{Y}_E)(\tilde{x}_{e_1}, \tilde{x}_{e_2}) = \min\{\tilde{X}_E(\tilde{x}_{e_1}), \tilde{Y}_E(\tilde{x}_{e_2})\}$.

Definition 3.5. Let $S(\tilde{X})$ be soft vector space over a scalar field $\mathbb{R}$. Let $T : S(\tilde{X}) \times S(\tilde{X}) \to S(\tilde{X})$ be the addition function on $S(\tilde{X})$, i.e., $T(\tilde{x}_{e_1}, \tilde{x}_{e_2}) = \tilde{x}_{e_1} + \tilde{x}_{e_2}$, for all $\tilde{x}_{e_1}, \tilde{x}_{e_2} \tilde{\in} S(\tilde{X})$. Let $\tilde{X}_E, \tilde{Y}_E \tilde{\in} \mathcal{F}S(\tilde{X})$. Then by using the fuzzy soft functional definition, the fuzzy soft addition, denoted by $T(\tilde{X}_E \times \tilde{Y}_E) = \tilde{X}_E + \tilde{Y}_E$, is defined as

$$\begin{aligned} (\tilde{X}_E + \tilde{Y}_E)(e)(x) &= T(\tilde{X}_E \times \tilde{Y}_E)(e)(r) \\ &= \sup_x [(\tilde{X}_E \times \tilde{Y}_E)(e)](x) \\ &= \sup_{\tilde{x}_{e_1}, \tilde{x}_{e_2}} \min\{\tilde{X}_E(\tilde{x}_{e_1}), \tilde{Y}_E(\tilde{x}_{e_2})\}. \end{aligned}$$

Result 3.6. *T_{up} is linear.*

Proof. (i) $T_{up}(\tilde{X}_E + \tilde{Y}_E)(e')(r) = \sup_{x=u^{-1}(r)} \left[\sup_{e=p^{-1}(e')} (\tilde{X}_E + \tilde{Y}_E)(e) \right](x)$. Then

$$T_{up}(\tilde{X}_E + \tilde{Y}_E)(e')(r)$$

$$= \sup_{x=u^{-1}(r)} \left[\sup_{e=p^{-1}(e')} \left[\sup_{(\tilde{x}_{e_1}, \tilde{x}_{e_2}) = T^{-1}(\tilde{x}_{e_1} + \tilde{x}_{e_2})} \min\{\tilde{X}_E(\tilde{x}_{e_1}), \tilde{Y}_E(\tilde{x}_{e_1})\} \right](e) \right](x)$$

(3.1)

On one hand,

$$T_{up}(\tilde{X}_E)(e'_1)(r_1) + T_{up}(\tilde{Y}_E)(e'_2)(r_2)$$
$$= \sup_{x=u^{-1}(r)} \left[\sup_{e=p^{-1}(e')} \tilde{X}_E(e_1) \right](x_1) + \sup_{x=u^{-1}(r)} \left[\sup_{e=p^{-1}(e')} \tilde{Y}_E(e_2) \right](x_2)$$
$$= \sup_{x=x_1+x_2} \left[\sup_{e=e_1+e_2} (\tilde{X}_E + \tilde{Y}_E)(e) \right](x).$$

Thus

(3.2)

$$T_{up}(\tilde{X}_E)(e'_1)(r_1) + T_{up}(\tilde{Y}_E)(e'_2)(r_2)$$
$$= \sup_{x=u^{-1}(r)} \left[\sup_{e=p^{-1}(e')} \left[\sup_{(\tilde{x}_{e_1},\tilde{x}_{e_2})=T^{-1}(\tilde{x}_{e_1}+\tilde{x}_{e_2})} \min\{\tilde{X}_E(\tilde{x}_{e_1}), \tilde{Y}_E(\tilde{x}_{e_2})\} \right](e) \right](x).$$

From (3.1) and (3.2),

$$T_{up}(\tilde{X}_E + \tilde{Y}_E)(e')(r) = T_{up}(\tilde{X}_E)(e'_1)(r_1) + T_{up}(\tilde{Y}_E)(e'_2)(r_2).$$

So, $T_{up}(\tilde{X}_E + \tilde{Y}_E) = T_{up}(\tilde{X}_E) + T_{up}(\tilde{Y}_E)$.

(ii) Clearly, $T_{up}(\tilde{X}_E \times \tilde{Y}_E)(e')(r) = \sup_{x=u^{-1}(r)} \left[\sup_{e=p^{-1}(e')} (\tilde{X}_E \times \tilde{Y}_E)(e) \right](x)$. Then

$$T_{up}(\tilde{X}_E \times \tilde{Y}_E)(e')(r) = \sup_{x=u^{-1}(r)} \left[\sup_{e=p^{-1}(e')} \left[\min\{\tilde{X}_E(\tilde{x}_{e_1}), \tilde{Y}_E(\tilde{x}_{e_1})\} \right](e) \right](x). \tag{3.3}$$

On the other hand,

$$T_{up}(\tilde{X}_E)(e'_1)(r_1) \times T_{up}(\tilde{Y}_E)(e'_2)(r_2)$$
$$= \sup_{x=u^{-1}(r)} \left[\sup_{e=p^{-1}(e')} \tilde{X}_E(e_1) \right](x_1) \times \sup_{x=u^{-1}(r)} \left[\sup_{e=p^{-1}(e')} \tilde{Y}_E(e_2) \right](x_2)$$
$$= \sup_{x=x_1\times x_2} \left[\sup_{e=e_1\times e_2} (\tilde{X}_E \times \tilde{Y}_E)(e) \right](x).$$

Thus

(3.4)
$$T_{up}(\tilde{X}_E)(e'_1)(r_1) \times T_{up}(\tilde{Y}_E)(e'_2)(r_2)$$
$$= \sup_{x=u^{-1}(r)} \left[\sup_{e=p^{-1}(e')} \left[\min\{\tilde{X}_E(\tilde{x}_{e_1}), \tilde{Y}_E(\tilde{x}_{e_2})\} \right](e) \right](x).$$

From (3.3) and (3.4),

$$T_{up}(\tilde{X}_E \times \tilde{Y}_E)(e')(r) = T_{up}(\tilde{X}_E)(e'_1)(r_1) \times T_{up}(\tilde{Y}_E)(e'_2)(r_2).$$

So, $T_{up}(\tilde{X}_E \times \tilde{Y}_E) = T_{up}(\tilde{X}_E) \times T_{up}(\tilde{Y}_E)$. □

Definition 3.7. A fuzzy soft linear transformation T_{up} is said to be continuous, if $\lim_{n\to\infty} (\tilde{X}_E)_n = \tilde{X}_E$ implies that $\lim_{n\to\infty} T_{up}(\tilde{X}_E)_n = T_{up}(\tilde{X}_E)$.

Suppose that for all $\{(\tilde{X}_E)_n\}$, $(\tilde{X}_E)_n(\tilde{x}_e) = \lambda_{e,n}$ and $\tilde{X}_E(\tilde{x}_e) = \lambda'_e$, and for every sequence $\{T_{up}(\tilde{X}_E)_n\}$ in $\mathbb{R}_{E'}$, $(T_{up}(\tilde{X}_E)_n)_{e'}(r_n) = \alpha_{e',n}$ and $(T_{up}(\tilde{X}_E))_{e'}(r) = \alpha'_{e'}$. Then T_{up} is continuous if and only if $\lim_{n\to\infty} |\lambda_{e,n} - \lambda'_e| = 0$ and $\lim_{n\to\infty} |x_n - x| = 0$ implies that $\lim_{n\to\infty} |\alpha_{e',n} - \alpha'_{e'}| = 0$ and $\lim_{n\to\infty} |r_n - r| = 0$.

Definition 3.8. If $(\mathcal{F}S(\tilde{X}), \|.\|)$ and $(\mathbb{R}_{E'}, \|.\|)$ are the fuzzy soft normed linear spaces, then the set of all continuous fuzzy soft linear transformations of $\mathcal{F}S(\tilde{X})$ into $\mathbb{R}_{E'}$ is denoted by $B(\mathcal{F}S(\tilde{X}), \mathbb{R}_{E'})$ or $[\mathcal{F}S(\tilde{X})]^*$. $[\mathcal{F}S(\tilde{X})]^*$ is called the conjugate space of $\mathcal{F}S(\tilde{X})$ and its elements are continuous fuzzy soft linear functional or briefly functional.

Clearly $([\mathcal{F}S(\tilde{X})]^*, \|.\|)$ is a Banach space.

Since $[\mathcal{F}S(\tilde{X})]^*$ is itself a fuzzy soft normed linear space it is possible to form conjugate space $[\mathcal{F}S(\tilde{X})]^{**}$ of $[\mathcal{F}S(\tilde{X})]^*$ and is said to be the second conjugate space of $\mathcal{F}S(\tilde{X})$.

The functional $S_{up} : [\mathcal{F}S(\tilde{X})]^* \to \mathbb{R}_{E'}$, of $[\mathcal{F}S(\tilde{X})]^{**}$ is defined as

$$S_{up}(T_{up})(e')(r) =$$
$$\begin{cases} \sup\limits_{x=u^{-1}(r)} \left[\sup\limits_{e=p^{-1}(e')} \left[\sup\limits_{\tilde{X}_E = T_{up}^{-1}(\tilde{r}_{e'})} T_{up}(\tilde{X}_E) \right] (e) \right] (x), & \text{if } u^{-1}(r) \neq \phi, p^{-1}(e') \neq \phi \\ 0, & \text{otherwise.} \end{cases}$$

Its norm is defined as $\|S_{up}\| = \sup\{\|S_{up}(T_{up})\| : \|T_{up}\| \tilde{\leq} \tilde{1}\}$.
Clearly, $\|S_{up}(T_{up})\| \tilde{\leq} \|T_{up}\| \|S_{up}\|$.

Theorem 3.9. *Let T_{up} be a fuzzy soft linear functional of $\mathcal{F}S(\tilde{X})$ into $\mathbb{R}_{E'}$ and $([\mathcal{F}S(\tilde{X})]^*, \|.\|)$ be a fuzzy soft normed linear space. Then the following conditions are equivalent:*

(1) *T_{up} is continuous,*

(2) *$S = \{\tilde{X}_E : |\lambda'_e| \tilde{\leq} \tilde{1}, \lambda'_e = \tilde{X}_E(\tilde{x}_e)\}$ is a closed fuzzy soft unit sphere in $\mathcal{F}S(\tilde{X})$, then its image $T_{up}(S)$ is a bounded set in $\mathbb{R}_{E'}$.*

Proof. Assume that T_{up} is continuous. For all $\{(\tilde{X}_E)_n\}$, let $(\tilde{X}_E)_n(\tilde{x}_e) = \lambda_{e,n}$ and $\tilde{X}_E(\tilde{x}_e) = \lambda'_e$, and for every sequence $\{T_{up}(\tilde{X}_E)_n\}$ in $\mathbb{R}_{E'}$ such that $(T_{up}(\tilde{X}_E)_n)_{e'}(r_n) = \alpha_{e',n}$ and $(T_{up}(\tilde{X}_E))_{e'}(r) = \alpha'_{e'}$, $\lim_{n\to\infty} |\lambda_{e,n} - \lambda'_e| = 0$ and $\lim_{n\to\infty} |x_n - x| = 0$ implies that $\lim_{n\to\infty} |\alpha_{e',n} - \alpha'_{e'}| = 0$ and $\lim_{n\to\infty} |r_n - r| = 0$. Suppose $S = \{\tilde{X}_E : |\lambda'_e| \tilde{\leq} \tilde{1}, \lambda'_e = \tilde{X}_E(\tilde{x}_e)\}$ is a closed fuzzy soft unit sphere in $\mathcal{F}S(\tilde{X})$. If $(T_{up}(\tilde{X}_E)_n)_{e'}(r_n) = \alpha_{e',n}$, then

$$|(T_{up}(\tilde{X}_E)_n)_{e'}(r_n)| = |\alpha_{e',n}|. \tag{3.5}$$

Claim: $T_{up}(S)$ is bounded.
Let $\{(\tilde{X}_E)_n\}$ be a sequence in $\mathcal{F}S(\tilde{X})$ such that

$$\|T_{up}(\tilde{X}_E)_n\| \tilde{\leq} \|T_{up}\| \|(\tilde{X}_E)_n\|. \tag{3.6}$$

But

$$\begin{aligned}\|T_{up}\| &= sup\{|T_{up}(\tilde{X}_E)_n| : \|(\tilde{X}_E)_n\| \tilde{\leq} \tilde{1}\}\\ &= sup\{|(T_{up}(\tilde{X}_E)_n)_{e'}(r_n)| : \|(\tilde{X}_E)_n(\tilde{x}_e)\| \tilde{\leq} \tilde{1}\}.\end{aligned}$$

Since $\alpha_{e',n} \to \alpha'_{e'}$, $\|T_{up}\| = sup\{|\alpha_{e',n}| : \|\lambda_{e,n}\| \tilde{\leq} \tilde{1}\} \tilde{\leq} \alpha_{e'}$, by (3.5).
From (3.6),

$$\begin{aligned}&\|T_{up}(\tilde{X}_E)_n\| \tilde{\leq} \|T_{up}\| \|(\tilde{X}_E)_n(\tilde{x}_e)\|,\\ &\|T_{up}(\tilde{X}_E)_n\| \tilde{\leq} \alpha_{e'} |\lambda_{e,n}| \tilde{\leq} \alpha_{e'},\ \forall\ (\tilde{X}_E)_n \in S,\\ &\|T_{up}(\tilde{X}_E)_n\| \tilde{\leq} \alpha_{e'},\ \forall\ (\tilde{X}_E)_n \in S.\end{aligned}$$

Then $T_{up}(S)$ is a bounded set in $\mathbb{R}_{E'}$.

Conversely, consider $\lim\limits_{n\to\infty} |\lambda_{e,n} - \lambda'_e| = 0$ and $\lim\limits_{n\to\infty} |x_n - x| = 0$.
Then

$$\begin{aligned}&\|T_{up}(\tilde{X}_E)_n - T_{up}(\tilde{X}_E)\| = \|T_{up}((\tilde{X}_E)_n - (\tilde{X}_E))\|\\ \Rightarrow\ &\|T_{up}(\tilde{X}_E)_n - T_{up}(\tilde{X}_E)\| \tilde{\leq} \|T_{up}(\tilde{X}_E)_n - (\tilde{X}_E)\|\\ \Rightarrow\ &\|(T_{up}(\tilde{X}_E)_n)_{e'}(r_n) - (T_{up}(\tilde{X}_E))_{e'}(r)\| \tilde{\leq} \|T_{up}\| \|(\tilde{X}_E)_n(\tilde{x}_e) - (\tilde{X}_E)(\tilde{x}_e)\|\\ \Rightarrow\ &\|\alpha_{e',n} - \alpha'_{e'}\| \tilde{\leq} \|T_{up}\| \|\lambda_{e,n} - \lambda'_e\| \to 0 \text{ as } n \to \infty,\end{aligned}$$

i.e., $\lim\limits_{n\to\infty} \|\alpha_{e',n} - \alpha'_{e'}\| = 0$ and $\lim\limits_{n\to\infty} |r_n - r| = 0$.
Thus T_{up} is continuous. □

Theorem 3.10. *Let $\mathcal{FS}(\tilde{X})$ be a fuzzy soft normed linear space and $(\tilde{x}_E)_0$ be a non zero function in $\mathcal{FS}(\tilde{X})$. Then there exists a functional $(T_{up})_0$ in $[\mathcal{FS}(\tilde{X})]^*$ such that $\alpha_{e',0} = \|\lambda_{e,0}\|$, where $(T_{up}(\tilde{X}_E)_0)_{e'}(r) = \alpha_{e',0}$ and $(\tilde{X}_E)_0(\tilde{x}_e) = \lambda_{e,0}$.*

Proof. Define $\mathcal{FS}(\tilde{Y}) = \{\tilde{\alpha}_r(\tilde{X}_E)_0\}$ a subspace of $\mathcal{FS}(\tilde{X})$ spanned by $(\tilde{X}_E)_0$. Then T_{up} on $\mathcal{FS}(\tilde{Y})$ is defined as

$$T_{up}(\tilde{\alpha}_r(\tilde{X}_E)_0) = \tilde{\alpha}_r \|(\tilde{X}_E)_0\|. \tag{3.7}$$

If $\tilde{X}_E \in \mathcal{FS}(\tilde{Y})$, then

$$\tilde{X}_E = \tilde{\alpha}_r(\tilde{X}_E)_0. \tag{3.8}$$

This implies that

$$\begin{aligned}T_{up}(\tilde{X}_E) &= T_{up}(\tilde{\alpha}_r(\tilde{X}_E)_0)\\ &= \tilde{\alpha}_r \|(\tilde{X}_E)_0\|\\ &= \tilde{\alpha}_r \sup\{\|(\tilde{X}_E)_0(\tilde{x}_e)\| : \|\tilde{x}_e\| \tilde{\leq} \tilde{1}\}\\ &= sup\{\|\tilde{\alpha}_r(\tilde{X}_E)_0(\tilde{x}_e)\| : \|\tilde{x}_e\| \tilde{\leq} \tilde{1}\}\\ &= \|\tilde{\alpha}_r(\tilde{X}_E)_0\|.\end{aligned}$$

Using (3.8),

$$T_{up}(\tilde{X}_E) = \|\tilde{X}_E\|. \tag{3.9}$$

If $\tilde{\alpha}_r = \tilde{1}$, then from (3.8), $\tilde{X}_E = (\tilde{X}_E)_0$ and

$$T_{up}(\tilde{X}_E)_0 = \|(\tilde{X}_E)_0\|. \tag{3.10}$$

It follows that

$$\begin{aligned}
\|T_{up}\| &= sup\{\|T_{up}(\tilde{X}_E)\| : \|\tilde{X}_E\|\tilde{\leq}\tilde{1}\} \\
\|T_{up}\| &= sup\{\|\tilde{X}_E\| : \|\tilde{X}_E\|\tilde{\leq}\tilde{1}\} \\
\|T_{up}\| &= \tilde{1}. \qquad (3.11)
\end{aligned}$$

By Hahn Banach theorem, T_{up} can be extended to a functional $(T_{up})_0$ in $[\mathcal{F}S(\tilde{X})]^*$ such that $(T_{up})_0(\tilde{X}_E)_0 = \|(\tilde{X}_E)_0\|$ with $\|(T_{up})_0\| = \tilde{1}$. Thus $((T_{up})_0(\tilde{X}_E)_0)_{e'}(r) = \|(\tilde{X}_E)_0(\tilde{x}_e)\|$. So $\alpha_{e',0} = \|\lambda_{e,0}\|$. □

Theorem 3.11. (**Natural Imbedding Theorem**) *Let $\mathcal{F}S(\tilde{X})$ be a fuzzy soft normed linear space. Then each $\tilde{X}_E$ in $\mathcal{F}S(\tilde{X})$ induces a functional S_{up} defined by $S_{up}(T_{up}) = T_{up}(\tilde{X}_E)$ such that $\|S_{up}\| = \|\tilde{X}_E\|$.*

Proof. For any $\tilde{X}_E\tilde{\in}\mathcal{F}S(\tilde{X})$, $S_{up}(T_{up}) = T_{up}(\tilde{X}_E)$.
To check S_{up} is linear.

(i) $S_{up}(T_{up} + T'_{up})(e')(r)$

$$\begin{aligned}
&= \sup_{x=u^{-1}(r)} \left[\sup_{e=p^{-1}(e')} \left[\sup \left[T_{up}(\tilde{X}_E)_1 + T_{up}(\tilde{X}_E)_2\right]\right](e)\right](x) \\
&= \sup_{x=u^{-1}(r)} \left[\sup_{e=p^{-1}(e')} \left[\sup T_{up}(\tilde{X}_E)_1\right](e)\right](x) \\
&\qquad + \sup_{x=u^{-1}(r)} \left[\sup_{e=p^{-1}(e')} \left[\sup T_{up}(\tilde{X}_E)_2\right](e)\right](x) \\
&= S_{up}(T_{up})(e')(r) + S_{up}(T'_{up})(e')(r).
\end{aligned}$$

(ii) $S_{up}(\alpha T_{up})(e')(r)$

$$\begin{aligned}
&= \sup_{x=u^{-1}(r)} \left[\sup_{e=p^{-1}(e')} \left[\sup \alpha T_{up}(\tilde{X}_E)\right](e)\right](x) \\
&= \alpha \left[\sup_{x=u^{-1}(r)} \left[\sup_{e=p^{-1}(e')} \left[\sup T_{up}(\tilde{X}_E)\right](e)\right](x)\right] \\
&= \alpha S_{up}(T_{up})(e')(r).
\end{aligned}$$

Then S_{up} is linear.

Claim: S_{up} is continuous.
By definition,

$$\begin{aligned}
\|S_{up}\| &= sup\{\|S_{up}(T_{up})\| : \|T_{up}\|\tilde{\leq}\tilde{1}\} \\
&= sup\{\|T_{up}(\tilde{X}_E)\| : \|T_{up}\|\tilde{\leq}\tilde{1}\} \\
&\tilde{\leq} sup\{\|T_{up}\|\|\tilde{X}_E\| : \|T_{up}\|\tilde{\leq}\tilde{1}\} \\
&= \|\tilde{X}_E\|.
\end{aligned}$$

Then

$$\|S_{up}\|\tilde{\leq}\|\tilde{X}_E\|. \tag{3.12}$$

Thus S_{up} is bounded which implies $S_{up}\tilde{\in}[\mathcal{F}S(\tilde{X})]^{**}$. By the above theorem, $\alpha_{e',0} = \|\lambda_{e,0}\|$, where $(T_{up}(\tilde{X}_E)_0)_{e'}(r) = \alpha_{e',0}$ and $(\tilde{X}_E)_0(\tilde{x}_e) = \lambda_{e,0}$. This implies that

$((T_{up})_0(\tilde{X}_E)_0)_{e'}(r) = \|(\tilde{X}_E)_0(\tilde{x}_e)\|$ and $(T_{up})_0(\tilde{X}_E)_0 = (\tilde{X}_E)_0$.
It follows that

$$\begin{aligned}
\tilde{X}_E &= T_{up}(\tilde{X}_E),\\
\tilde{X}_E &= S_{up}(T_{up})\\
&\tilde{\leq}\|S_{up}(T_{up})\|\\
&\tilde{\leq} sup\{\|S_{up}(T_{up})\| : \|T_{up}\|\tilde{\leq}\tilde{1}\}\\
&= \|S_{up}\|.
\end{aligned}$$

So

$$\tilde{X}_E\tilde{\leq}\|S_{up}\|. \tag{3.13}$$

So from (3.12) and (3.13), $\tilde{X}_E = \|S_{up}\|$. □

Theorem 3.12. *A norm preserving mapping $\tilde{X}_E \to S_{up}$ is isometric isomorphism of $\mathcal{F}S(\tilde{X})$ into $[\mathcal{F}S(\tilde{X})]^{**}$.*

Proof. From Theorem 3.11, $\tilde{X}_E = \|S_{up}\|$, the mapping $\tilde{X}_E \to S_{up}$ is isometric and bijective.
Claim: The mapping $\tilde{X}_E \to S_{up}$ is linear.
(i) For any $T_{up}\tilde{\in}[\mathcal{F}S(\tilde{X})]^*$, $S_{up}(T_{up}) = T_{up}(\tilde{X}_E)$. Then

$$\begin{aligned}
&(S_{up})_{\tilde{X}_E+\tilde{Y}_E}(T_{up})(e')(r)\\
&= T_{up}(\tilde{X}_E + \tilde{Y}_E)(e')(r)\\
&= \sup_{x=u^{-1}(r)}\left[\sup_{e=p^{-1}(e')}(\tilde{X}_E + \tilde{Y}_E)(e)\right](x),\\
&(S_{up})_{\tilde{X}_E+\tilde{Y}_E}(T_{up})(e')(r)\\
&= \sup_{x=u^{-1}(r)}\left[\sup_{e=p^{-1}(e')}(\tilde{X}_E)(e)\right](x)+\sup_{x=u^{-1}(r)}\left[\sup_{e=p^{-1}(e')}(\tilde{Y}_E)(e)\right](x),\\
&(S_{up})_{\tilde{X}_E+\tilde{Y}_E}(T_{up})(e')(r) = (T_{up})(\tilde{X}_E)(e')(r) + (T_{up})(\tilde{Y}_E)(e')(r),\\
&(S_{up})_{\tilde{X}_E+\tilde{Y}_E}(T_{up})(e')(r) = (S_{up})_{\tilde{X}_E}(T_{up})(e')(r) + (S_{up})_{\tilde{Y}_E}(T_{up})(e')(r),\\
&(S_{up})_{\tilde{X}_E+\tilde{Y}_E}(T_{up}) = (S_{up})_{\tilde{X}_E}(T_{up}) + (S_{up})_{\tilde{Y}_E}(T_{up}).
\end{aligned}$$

Thus, $(S_{up})_{\tilde{X}_E+\tilde{Y}_E} = (S_{up})_{\tilde{X}_E} + (S_{up})_{\tilde{Y}_E}$.
(ii) For any scalar α,

$$\begin{aligned}
(S_{up})_{\alpha\tilde{X}_E}(T_{up})(e')(r) &= T_{up}(\alpha\tilde{X}_E)(e')(r)\\
&= \sup_{x=u^{-1}(r)}\left[\sup_{e=p^{-1}(e')}(\alpha\tilde{X}_E)(e)\right](x)\\
&= \alpha\sup_{x=u^{-1}(r)}\left[\sup_{e=p^{-1}(e')}(\tilde{X}_E)(e)\right](x),
\end{aligned}$$

$$\begin{aligned}
&(S_{up})_{\alpha\tilde{X}_E}(T_{up})(e')(r) = \alpha(T_{up})(\tilde{X}_E)(e')(r),\\
&(S_{up})_{\alpha\tilde{X}_E}(T_{up})(e')(r) = \alpha(S_{up})_{\tilde{X}_E}(T_{up})(e')(r),\\
&(S_{up})_{\alpha\tilde{X}_E}(T_{up}) = \alpha(S_{up})_{\tilde{X}_E}(T_{up}).
\end{aligned}$$

So, $(S_{up})_{\alpha\tilde{X}_E} = \alpha(S_{up})_{\tilde{X}_E}$. □

Theorem 3.13. *A non empty subset $\mathcal{F}S(\tilde{Y})$ of a fuzzy soft normed linear space $\mathcal{F}S(\tilde{X})$ is bounded if and only if $T_{up}(\mathcal{F}S(\tilde{Y}))$ is a bounded set of fuzzy soft real number for each T_{up} in $[\mathcal{F}S(\tilde{X})]^*$.*

Proof. Assume that for each $\tilde{Y}_E\tilde{\in}\mathcal{F}S(\tilde{Y})$, $\|\tilde{Y}_E(\tilde{y}_e)\|\tilde{\leq}\tilde{M}_1$ and for every $T_{up}\tilde{\in}[\mathcal{F}S(\tilde{X})]^*$,

$$\|T_{up}\| = sup\{\|T_{up}(\tilde{Y}_E)\| : \|\tilde{Y}_E\|\tilde{\leq}\tilde{1}\}.$$

Then

$$\begin{aligned}\|T_{up}(\tilde{Y}_E)\|&\tilde{\leq}\|T_{up}\|\|\tilde{Y}_E\|\\ &= \|T_{up}\|\|\tilde{Y}_E(\tilde{y}_e)\|\\ &\tilde{\leq}\tilde{k}\tilde{M}_1 = \tilde{M}.\end{aligned}$$

Thus $\|T_{up}(\tilde{Y}_E)\|\tilde{\leq}\tilde{M}$, for each $\tilde{Y}_E\tilde{\in}\mathcal{F}S(\tilde{Y})$. So $T_{up}(\mathcal{F}S(\tilde{Y})$ is bounded.

Conversely, assume that for each $\tilde{Y}_E\tilde{\in}\mathcal{F}S(\tilde{Y})$,

$$\|T_{up}(\tilde{Y}_E)\|\tilde{\leq}\tilde{M}. \tag{3.14}$$

Then for each $(S_{up})_{\tilde{Y}_E}\tilde{\in}[\mathcal{F}S(\tilde{X})]^{**}$, $(S_{up})_{\tilde{Y}_E}(T_{up}) = T_{up}(\tilde{Y}_E)$. Thus $\|(S_{up})_{\tilde{Y}_E}(T_{up})\| = \|T_{up}(\tilde{Y}_E)\|$. Since the mapping $\tilde{X}_E \to S_{up}$ is isometric isomorphism, it follows that $\|\tilde{Y}_E\| = \|(S_{up})_{\tilde{Y}_E}\|$. Also,

$$\|(S_{up})_{\tilde{Y}_E}\| = sup\{\|(S_{up})_{\tilde{Y}_E}(T_{up})\| : \|T_{up}\|\tilde{\leq}\tilde{1}\},$$
$$\|(S_{up})_{\tilde{Y}_E}\| = sup\{\|T_{up}(\tilde{Y}_E)\| : \|T_{up}\|\tilde{\leq}\tilde{1}\}.$$

Using (3.14), $\|(S_{up})_{\tilde{Y}_E}\|\tilde{\leq}\tilde{M}$, i.e., $\|\tilde{Y}_E\|\tilde{\leq}\tilde{M}$. So, $\|\tilde{Y}_E(\tilde{y}_e)\|\tilde{\leq}\tilde{M}$, $\forall\, \tilde{Y}_E\tilde{\in}\mathcal{F}S(\tilde{Y})$. Hence $\mathcal{F}S(\tilde{Y})$ is bounded. □

Definition 3.14. Let L be the fuzzy soft linear space of all scalar valued fuzzy soft linear functional defined on $\mathcal{F}S(\tilde{X})$. The conjugate space $[\mathcal{F}S(\tilde{X})]^*$ is clearly a fuzzy soft normed linear subspace of L. Let h_{up} be a fuzzy soft linear transformation of $\mathcal{F}S(\tilde{X})$ into itself which is not necessarily continuous.

Using h_{up} define a fuzzy soft linear transformation h'_{up} of L into itself as follows: if T_{up} is in L, then $h'_{up}(T_{up})$ is defined by $[h'_{up}(T_{up})](\tilde{X}_E) = T_{up}(h_{up}(\tilde{X}_E))$.

Theorem 3.15. *If h_{up} is a continuous fuzzy soft operator on $\mathcal{F}S(\tilde{X})$ then it is possible to restrict an operator h'_{up} on L to a mapping $[\mathcal{F}S(\tilde{X})]^*$ into itself.*

Proof. Consider $T_{up}\tilde{\in}[\mathcal{F}S(\tilde{X})]^*\tilde{\subseteq}L$. Then

$$h'_{up}(T_{up})\tilde{\in}h'_{up}([\mathcal{F}S(\tilde{X})]^*). \tag{3.15}$$

Suppose h_{up} is continuous fuzzy soft linear operator. If S is a closed fuzzy soft unit sphere in $\mathcal{F}S(\tilde{X})$, then

$$\begin{aligned}&h_{up}(S) \text{ is bounded fuzzy soft linear space in } \mathcal{F}S(\tilde{X})\\ \Leftrightarrow\ &T_{up}(h_{up}(S)) \text{ is bounded, for each } T_{up}\tilde{\in}[\mathcal{F}S(\tilde{X})]^*\\ \Leftrightarrow\ &[h'_{up}(T_{up})](S) \text{ is bounded, for each } T_{up}\tilde{\in}[\mathcal{F}S(\tilde{X})]^*\\ \Leftrightarrow\ &h'_{up}(T_{up}) \text{ is continuous on } \mathcal{F}S(\tilde{X})\\ \Leftrightarrow\ &\end{aligned}$$

$$h'_{up}(T_{up})\tilde{\in}[\mathcal{F}S(\tilde{X})]^*. \tag{3.16}$$

Thus from (3.15) and (3.16), $h'_{up}([\mathcal{F}S(\tilde{X})]^*)\tilde{\subseteq}[\mathcal{F}S(\tilde{X})]^*$. So it is possible to restrict h'_{up} to a mapping $[\mathcal{F}S(\tilde{X})]^*$ into itself.

Denote this restriction $h'_{up} : [\mathcal{F}S(\tilde{X})]^* \to [\mathcal{F}S(\tilde{X})]^*$ by h^*_{up} and is said to be the conjugate of h_{up}. □

Definition 3.16. The conjugate of h_{up} on $[\mathcal{F}S(\tilde{X})]^*$ is defined as $[h^*_{up}(T_{up})](\tilde{X}_E) = T_{up}(h_{up}(\tilde{X}_E))$, where

$$[h^*_{up}(T_{up})](\tilde{X}_{E_2})(e'_2)(r_2)$$
$$= \sup_{(r_1)=u^{-1}(r_2)} \left[\sup_{e'_1=p^{-1}(e'_2)} \left[\sup_{\tilde{X}_{E_1}\tilde{\in}[\mathcal{F}S(\tilde{X})]^*} T_{up}(\tilde{X}_{E_1})\right](e'_1)\right](r_1)$$

and

$$T_{up}(h_{up}(\tilde{X}_E))(e')(r) = \sup_{(r_1)=u^{-1}(r_2)} \left[\sup_{e'_1=p^{-1}(e'_2)} h_{up}(\tilde{X}_E)(e)\right](x).$$

Theorem 3.17. *If h_{up} is a continuous fuzzy soft normed linear operator on $\mathcal{F}S(\tilde{X})$ and a conjugate h^*_{up} is a continuous operator on $[\mathcal{F}S(\tilde{X})]^*$, then mapping $h_{up} \to h^*_{up}$ is an isometric isomorphism of $\mathcal{F}S(\tilde{X})$ into $[\mathcal{F}S(\tilde{X})]^*$ which reverses products and preserves the identity transformation.*

Proof. Define the norm of $\|h^*_{up}\|$ by

$$|h^*_{up}\| = sup\{\|h^*_{up}(T_{up})\| : \|T_{up}\|\tilde{\leq}\tilde{1}\}.$$

Then

$$\begin{aligned}\|h^*_{up}\| &= sup\{\|h^*_{up}(T_{up})(\tilde{X}_E)\| : \|T_{up}\|\tilde{\leq}\tilde{1}, \|\tilde{X}_E\|\tilde{\leq}\tilde{1}\}\\ &= sup\{\|T_{up}(h_{up}(\tilde{X}_E))\| : \|T_{up}\|\tilde{\leq}\tilde{1}, \|\tilde{X}_E\|\tilde{\leq}\tilde{1}\}\\ &\tilde{\leq} sup\{\|T_{up}\|\|h_{up}\|\|\tilde{X}_E(\tilde{x}_e)\| : \|T_{up}\|\tilde{\leq}\tilde{1}, \|\tilde{X}_E(\tilde{x}_e)\|\tilde{\leq}\tilde{1}\}\\ &\tilde{\leq}\|T_{up}\|.\end{aligned}$$

Thus

$$\|h^*_{up}\|\tilde{\leq}\|T_{up}\|. \tag{3.17}$$

Conversely, define the norm of h_{up} by $\|h_{up}\| = sup\{\|h_{up}(\tilde{X}_E)\| : \|\tilde{X}_E\|\tilde{\leq}\tilde{1}\}$. Then by Theorem 3.9, $\alpha_{e',0} = \|\lambda_{e,0}\|$, where $(T_{up}(\tilde{X}_E)_0)_{e'}(r) = \alpha_{e',0}$ and $(\tilde{X}_E)_0(\tilde{x}_e) = \lambda_{e,0}$. This implies $(T_{up}(\tilde{X}_E)_0)_{e'}(r) = \|(\tilde{X}_E)_0(\tilde{x}_e)\|$. Thus $T_{up}(\tilde{X}_E)_0 = \|(\tilde{X}_E)_0\|$. It follows that, $\|h_{up}(\tilde{X}_E)\| = T_{up}(h_{up}(\tilde{X}_E))\tilde{\leq}\|T_{up}(h_{up}(\tilde{X}_E))\|$. So

$$\begin{aligned}\|h_{up}\|&\tilde{\leq} sup\{\|T_{up}(h_{up}(\tilde{X}_E))\| : \|T_{up}\|\tilde{\leq}\tilde{1}, \|\tilde{X}_E\|\tilde{\leq}\tilde{1}\}\\ &= sup\{\|h^*_{up}(T_{up})(\tilde{X}_E)\| : \|T_{up}\|\tilde{\leq}\tilde{1}, \|\tilde{X}_E\|\tilde{\leq}\tilde{1}\}\\ &\tilde{\leq} h^*_{up}.\end{aligned}$$

Hence

$$h_{up}\tilde{\leq}h^*_{up}. \tag{3.18}$$

Therefore from (3.17) and (3.18), $h_{up} = h^*_{up}$.

Claim 1: The mapping $h_{up} \to h^*_{up}$ is linear.

By the definition, $[(\alpha h'_{up} + \beta h''_{up})^*(T_{up})](\tilde{X}_E) = T_{up}[(\alpha h'_{up} + \beta h''_{up})(\tilde{X}_E)]$.
Since T_{up} is linear,

$$\begin{aligned}[(\alpha h'_{up} + \beta h''_{up})^*(T_{up})](\tilde{X}_E) &= T_{up}[\alpha h'_{up}](\tilde{X}_E) + T_{up}[\beta h''_{up}](\tilde{X}_E)\\ &= \alpha T_{up}(h'_{up}(\tilde{X}_E)) + \beta T_{up}(h''_{up}(\tilde{X}_E))\\ &= \alpha[h'^*_{up}(T_{up})](\tilde{X}_E) + \beta[h''^*_{up}(T_{up})](\tilde{X}_E).\end{aligned}$$

Then, $(\alpha h'_{up} + \beta h''_{up})^* = \alpha h'^*_{up} + \beta h''^*_{up}$.
Claim 2: $(h'_{up}h''_{up})^* = h''^*_{up}h'^*_{up}$.

$$\begin{aligned}[(h'_{up}h''_{up})^*(T_{up})](\tilde{X}_E) &= T_{up}[h'_{up}h''_{up}(\tilde{X}_E)]\\ &= T_{up}[h'_{up}(h''_{up}(\tilde{X}_E))]\\ &= [h'^*_{up}(T_{up})](h''_{up}(\tilde{X}_E))\\ &= h''^*_{up}[h'^*_{up}(T_{up})](\tilde{X}_E)\\ &= [(h''^*_{up}h'^*_{up})(T_{up})](\tilde{X}_E).\end{aligned}$$

Then $(h'_{up}h''_{up})^* = h''^*_{up}h'^*_{up}$.
Claim 3: $I^* = I$.

$$\begin{aligned}[I^*(T_{up})](\tilde{X}_E) &= T_{up}(I(\tilde{X}_E))\\ &= I(T_{up}(\tilde{X}_E)).\end{aligned}$$

Then $I^* = I$.

Thus the mapping $h_{up} \to h^*_{up}$ reverses the products and preserves identity. □

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