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Soft ideal extension

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Abstract. Soft topological spaces based on soft set theory which is a new mathematical tool for dealing with uncertainties. In this paper, the notions of $\widetilde{I}_{\widetilde{c}}$ soft free ideal and soft $\widetilde{c}-$ ideals are introduced. Also, soft ideal extension of a given soft topological space is investigated via the concept of soft ideals.

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1. Introduction

The theory of soft sets was introduced by Molodtsov [21] as a new mathematical tool for dealing with uncertainties. After Molodtsov's work, some different applications of soft sets were studied. Maji et al. [18, 19] research deal with operations over soft set. Up to the present, research on soft sets has been very active and many important results have been achieved in the theoretical aspect.The algebraic structure of set theories is an important problem. Many researchers have contributed towards the algebraic structure of soft set theory. Aktaş and Çağman [2] defined soft groups and derived their basic properties. U. Acar et al. [1] introduced initial concepts of soft rings. F. Feng et al. [5] defined soft semirings and several related notions to establish a connection between soft sets and semirings. M. Shabir et al. [26] studied soft ideals over a semigroup. C. Gunduz and S. Bayramov [6, 7] introduced fuzzy soft modules and intuitionistic fuzzy soft modules and investigated some important properties of these modules. T. Y. Ozturk and S. Bayramov [23] defined chain complexes of soft modules and soft homology modules of them. T. Y. Ozturk et al. [24] introduced the concept of inverse and direct systems in the category of soft modules.

The idea of soft topological spaces was first given by Shabir and Naz [27] and theoretical studies of soft topological spaces were studied in [29, 25, 4, 8, 20]. In these studies, the concept of a soft point is expressed using different approaches. In the present study, we use the concept of a soft point as given in [3].

The concept of ideals in topological spaces was treated in the classic text by Vaidyanathaswamy [28], Kuratowski [17]. Jankovic and Hamlett [9] investigated further properties of ideal spaces. M. N. Mukherjee et al. [22] studied the concept of ideal extension and showed that under certain condition imposed on the ideals involved, the ideal extension space turns out to be the compactification of a given space. A.Kandil et al [12] introduced the notion of soft ideal in soft set theory and examined some properties of soft ideals. Later many studies of soft ideal spaces were investigated in [10, 11, 13, 14, 15, 16].

In the present study, the notions of $\widetilde{I_{\widetilde{c}}}$ soft free ideal and soft $\widetilde{c}-$ ideal are introduced. Also, soft ideal extension of a given soft topological spaces is investigated via the concept of soft ideals.

2. Preliminaries

Definition 2.1 ([21])**.** Let X be an initial universe set and E be a set of parameters. Let $P(X)$ denotes the power set of X and $A \subset E$. A pair (F, A) is called a soft set over X, where F is a mapping given by $F : A \to P(X)$.

The family of all these soft sets is denoted by $SS\left(X\right)_A$.

Definition 2.2 ([18])**.** For two soft sets (F, A) and (G, B) over X, (F, A) is called soft subset of (G, B), if

(i) $A \subset B$,

(ii) $\forall e \in A$, $F(e)$ and $G(e)$ are identical approximations.

This relationship is denoted by $(F, A)\widetilde{\subseteq}(G, B)$.

Definition 2.3 ([18])**.** The soft intersection of two soft sets (F, A) and (G, B) over X is the soft set (H, C), where $C = A \cap B$ and $\forall e \in C$, $H(e) = F(e) \cap G(e)$.

This is denoted by $(F, A)\widetilde{\cap}(G, B) = (H, C)$.

Definition 2.4 ([18])**.** The soft union of two soft sets (F, A) and (G, B) over X is the soft set, where $C = A \cup B$ and $\forall e \in C$,

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ F(e) \cup G(e) & \text{if } e \in A \cap B. \end{cases}$$

This relationship is denoted by $(F, A)\widetilde{\cup}(G, B) = (H, C)$.

Definition 2.5 ([18])**.** A soft set (F, A) over X is said to be a null soft set, denoted by Φ, if for all $e \in A$, $F(e) = \varnothing$(NULL set).

Definition 2.6 ([18])**.** A soft set (F, A) over X is said to be an absolute soft set, denoted by $\widetilde{X}$, if for all $e \in A$, $F(e) = X$.

Definition 2.7 ([3])**.** Let (F, E) be a soft set over X. The soft set (F, E) is called a soft point, denoted by x_e, if for the element $e \in E$, $F(e) = \{x\}$ and $F(e') = \varnothing$ for all $e' \in E - \{e\}$.

Definition 2.8 ([3])**.** Let (F, E) be a soft set over X and $x \in X$. We say that x_e belongs to (F, E), denoted by $x_e\widetilde{\in}(F, E)$, if $x \in F\left(e\right)$, for the element $e \in E$.

Definition 2.9 ([3])**.** Two soft points x_e and $y_{e'}$ over a common universe X, we say that the soft points are different points, if $x \neq y$ or $e \neq e'$.

Definition 2.10 ([27])**.** Let τ be the collection of soft set over X. Then τ is said to be a soft topology on X, if it satisfies the following axioms:
(i) $\Phi, \widetilde{X}$ belong to τ,
(ii) the union of any number of soft sets in τ belongs to τ,
(iii)) the intersection of any two soft sets in τ belongs to τ.
The triplet (X, τ, E) is called a soft topological space over X.

Definition 2.11 ([27])**.** Let (X, τ, E) be a soft topological space over X. Then the members of τ are said to be soft open sets in X.

A soft set (F, E) over X is said to be soft closed set in X, if its relative complement belongs to τ.

Definition 2.12 ([27])**.** Let (X, τ, E) be a soft topological space over X and (F, E) be a soft set over X. Then the soft closure of (F, E), denoted by $\overline{(F, E)}$, is the intersection of all soft closed super sets of (F, E).

Clearly $\overline{(F, E)}$ is the smallest soft closed set over X which contains (F, E).

Definition 2.13. [3] Let (X, τ, E) be a soft topological space over X and $x_e \neq y_{e'}$. If there exists soft open set (F, E) such that $x_e \widetilde{\in} (F, E)$ and $y_{e'} \widetilde{\notin} (F, E)$, then (X, τ, E) is called a soft T_0-sapce.

In this paper, $\widetilde{c} : SS(X)_E \to SS(X)_E$ given by $\widetilde{c}((F, E)) = \overline{(F, E)}$ is a soft closure operator on $SS(X)_E$, for any $(F, E) \in SS(X)_E$. In general, we denote a soft topological space by $(X, \widetilde{c}, E)$, where $\widetilde{c}$ is the soft closure operator inducing the soft topology of the soft topological space.

Definition 2.14 ([12])**.** Let $\widetilde{I}$ be a non-null collection of soft sets over a universe X with the same set of parameters E. Then $\widetilde{I} \subset SS(X)_E$ is called a soft ideal on X if
(i) $(F, E) \in \widetilde{I}$ and $(G, E) \in \widetilde{I}$ implies $(F, E) \widetilde{\cup} (G, E) \in \widetilde{I}$,
(ii) $(F, E) \in \widetilde{I}$ and $(G, E) \widetilde{\subseteq} (F, E)$ implies $(G, E) \in \widetilde{I}$.

Example 2.15 ([12])**.** Let X be a universe set and E be a parameters set. The following families are soft ideal over X with the same set of parameters E:

(1) $\widetilde{I} = SS(X)_E = \{(F, E) : (F, E) \text{ is a soft set}\}$,
(2) $\widetilde{I} = \{\Phi\}$,
(3) $\widetilde{I} = \{(F, E) \in SS(X)_E : (F, E) \text{ is a finite sub soft set of } X\}$.

Theorem 2.16 ([12])**.** *Let $\widetilde{I}$ be a soft ideal over a universe X. Then the collection*

$$\widetilde{I}_e = \left\{F(e) : (F, E) \in \widetilde{I}\right\}$$

defines an ideal on X, for each $e \in E$.

Remark 2.17 ([12])**.** The converse of the Theorem 2.16 is not true in general.

Example 2.18 ([12])**.** Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and let
$\widetilde{I} = \{\Phi, (F_1, E), (F_2, E), (F_3, E)\}$, where (F_1, E), (F_2, E) and (F_3, E) are soft sets over X defined as follows:

$$\begin{aligned} F_1(e_1) &= \{h_1\}, & F_1(e_2) = \{h_1\}, \\ F_2(e_1) &= \{h_1,\ h_3\}, & F_2(e_2) = \{h_2\}, \\ F_3(e_1) &= \{h_3\}, & F_3(e_2) = \{h_1,\ h_2\}. \end{aligned}$$

Then

$$\widetilde{I}_{e_1} = \{\varnothing,\ \{h_1\}, \{h_3\},\ \{h_1, h_3\}\}$$

and

$$\widetilde{I}_{e_2} = \{\varnothing,\ \{h_1\},\ \{h_2\},\ \{h_1, h_2\}\}$$

are ideals on X. But, $\widetilde{I}$ is not soft ideal on X. Indeed, $(F_1, E)\,\widetilde{\cup}\,(F_2, E) = (G, E) \notin \widetilde{I}$.

3. Soft ideal extension

Lemma 3.1. *Let $(X, \widetilde{c}, E)$ be a soft topological space and $x_e \widetilde{\in} SS(X)_E$. Then*

$$\widetilde{I}_{\widetilde{c}}(x_e) = \left\{(F, E) \in SS(X)_E : x_e \widetilde{\notin} \widetilde{c}((F, E))\right\}$$

is a soft ideal on $SS(X)_E$.

Proof. Let $(F, E), (G, E) \in SS(X)_E$ and $(G, E)\,\widetilde{\subseteq}\,(F, E)$. If $(F, E) \in \widetilde{I}_{\widetilde{c}}(x_e)$, then $x_e \widetilde{\notin} \widetilde{c}((F, E))$. Thus $x_e \widetilde{\notin} \widetilde{c}((G, E))$. So $(G, E) \in \widetilde{I}_{\widetilde{c}}(x_e)$.

Now, let $(F, E), (G, E) \in \widetilde{I}_{\widetilde{c}}(x_e)$. Then $x_e \widetilde{\notin} \widetilde{c}((F, E))$ and $x_e \widetilde{\notin} \widetilde{c}((G, E))$, respectively. Thus $x_e \widetilde{\notin} \widetilde{c}((F, E))\,\widetilde{\cup}\,\widetilde{c}((G, E))$. So $x_e \widetilde{\notin} \widetilde{c}\left((F, E)\,\widetilde{\cup}\,(G, E)\right)$. This means that $(F, E)\,\widetilde{\cup}\,(G, E) \in \widetilde{I}_{\widetilde{c}}(x_e)$. Hence $\widetilde{I}_{\widetilde{c}}(x_e)$ is a soft ideal on $SS(X)_E$. □

Definition 3.2. The soft ideal $\widetilde{I}_{\widetilde{c}}(x_e)$ on $(X, \widetilde{c}, E)$ is called a soft free ideal.

Lemma 3.3. *A soft topological space $(X, \widetilde{c}, E)$ is a soft T_0-space over X if and only if for any distinct soft points x_e, $y_{e\prime}$ of $SS(X)_E$, $\widetilde{c}(x_e) \neq \widetilde{c}(y_{e\prime})$.*

Proof. Let $x_e, y_{e\prime}$ be distinct two soft points of $SS(X)_E$. Then $x \neq y$ or $e \neq e\prime$. Since $(X, \widetilde{c}, E)$ is a soft T_0-space, there exists soft open set (F, E) such that $x_e \widetilde{\in} (F, E)$ and $y_{e\prime} \widetilde{\notin} (F, E)$. Thus (F, E) is a soft neighborhood of x_e. Since $y_{e\prime} \widetilde{\notin} (F, E)$, $(F, E)\,\widetilde{\cap}\,\{y_{e\prime}\} = \Phi$. So $x_e \widetilde{\notin} \widetilde{c}(y_{e\prime})$. It follows that $\widetilde{c}(x_e) \neq \widetilde{c}(y_{e\prime})$.

Conversely, for the soft points x_e, $y_{e\prime}$ of $SS(X)_E$, $x_e \neq y_{e\prime}$ and $\widetilde{c}(x_e) \neq \widetilde{c}(y_{e\prime})$. Let $z_{e\prime\prime} \widetilde{\notin} \widetilde{c}(y_{e\prime})$, for soft point $z_{e\prime\prime} \widetilde{\in} SS(X)_E$. Then, there exists a soft neighborhood (F, E) of the soft point $z_{e\prime\prime}$ such that $(F, E)\,\widetilde{\cap}\,\{y_{e\prime}\} = \Phi$. Thus $y_{e\prime} \widetilde{\notin} (F, E)$. Since $z_{e\prime\prime} \widetilde{\in} \widetilde{c}(x_e)$, $(F, E)\,\widetilde{\cap}\,\{x_e\} \neq \Phi$. So $x_e \widetilde{\in} (F, E)$. Hence $(X, \widetilde{c}, E)$ is a soft T_0-space. □

Example 3.4. Let $X = \left\{x^1, x^2\right\}$, $E = \{e_1, e_2\}$ and let $\tau = \left\{\Phi, \tilde{X}, (F_1, E), (F_2, E), (F_3, E)\right\}$, where

$$\begin{aligned} F_1(e_1) &= \left\{x^1\right\}, & F_1(e_2) = \left\{x^2\right\}, \\ F_2(e_1) &= \varnothing, & F_2(e_2) = \left\{x^1\right\}, \\ F_3(e_1) &= \left\{x^1\right\}, & F_3(e_2) = X. \end{aligned}$$

Then X is a soft topological space and

$$\widetilde{I}_{\widetilde{c}}\left(x^1_{e_1}\right)=\left\{(F,E)\in SS\left(X\right)_E : x^1_{e_1}\widetilde{\notin}\widetilde{c}\left((F,E)\right)\right\}=\left\{\left\{x^1_{e_2}\right\},\ \left\{x^2_{e_1}\right\}\right\}$$

is a soft free ideal.

Proposition 3.5. *A soft space $(X,\widetilde{c},E)$ is a soft T_0-space if and only if for any two different soft points x_e, $y_{e'}$ of $SS\left(X\right)_E$, $\widetilde{I}_{\widetilde{c}}\left(x_e\right)\neq\widetilde{I}_{\widetilde{c}}\left(y_{e'}\right)$.*

Proof. Let x_e, $y_{e'}$ be two different soft points in a soft T_0-space $(X,\widetilde{c},E)$. Suppose that $x_e\widetilde{\notin}\widetilde{c}\left(\{y_{e'}\}\right)$.Then $\{y_{e'}\}\widetilde{\in}\ \widetilde{I}_{\widetilde{c}}\left(x_e\right)$ but $\{y_{e'}\}\widetilde{\notin}\widetilde{I}_{\widetilde{c}}\left(y_{e'}\right)$. Thus $\widetilde{I}_{\widetilde{c}}\left(x_e\right)\neq\widetilde{I}_{\widetilde{c}}\left(y_{e'}\right)$.

Conversely, let x_e, $y_{e'}$ be two different soft points in X. Then $\widetilde{I}_{\widetilde{c}}\left(x_e\right)\neq\widetilde{I}_{\widetilde{c}}\left(y_{e'}\right)$. Now, let $(F,E)\in SS\left(X\right)_E$ such that $(F,E)\in\widetilde{I}_{\widetilde{c}}\left(x_e\right)$. Then $(F,E)\notin\widetilde{I}_{\widetilde{c}}\left(y_{e'}\right)$. It follows that $x_e\widetilde{\notin}\widetilde{c}\left((F,E)\right)$ and $y_{e'}\widetilde{\in}\widetilde{c}\left((F,E)\right)$. Thus $x_e\widetilde{\in}\ SS\left(X\right)_E\setminus\widetilde{c}\left((F,E)\right)$ but $y_{e'}\widetilde{\notin}\ SS\left(X\right)_E\setminus\widetilde{c}\left((F,E)\right)$. So $(X,\widetilde{c},E)$ is a soft T_0-space. □

Definition 3.6. A soft ideal $\widetilde{I}$ on a soft topological space $(X,\widetilde{c},E)$ is called a soft $\widetilde{c}-$ ideal, if for all $(F,E)\in SS\left(X\right)_E$, $(F,E)\in\widetilde{I}\Rightarrow\widetilde{c}\left((F,E)\right)\in\widetilde{I}$.

Proposition 3.7. *Every soft free ideal is also a soft $\widetilde{c}-$ ideal.*

Proof. Let $\widetilde{I}_{\widetilde{c}}\left(x_e\right)$ be any soft free ideal on $(X,\widetilde{c},E)$. Suppose that $(F,E)\in SS\left(X\right)_E$ is such that $(F,E)\in\widetilde{I}_{\widetilde{c}}\left(x_e\right)$. Then $x_e\widetilde{\notin}\widetilde{c}\left((F,E)\right)$. Thus $x_e\widetilde{\notin}\widetilde{c}\left(\widetilde{c}\left((F,E)\right)\right)=\widetilde{c}\left((F,E)\right)$. So $\widetilde{c}\left((F,E)\right)\in\widetilde{I}_{\widetilde{c}}\left(x_e\right)$. Hence $\widetilde{I}_{\widetilde{c}}\left(x_e\right)$ is a soft $\widetilde{c}-$ ideal. □

Let $(X,\widetilde{c},E)$ and $\left(Y,\widetilde{k},E\right)$ be two soft topological spaces, where $\widetilde{k}$ is a soft closure operator on $SS\left(Y\right)_E$, $(\psi,1_E)$ is a soft homeomorphism of $(X,\widetilde{c},E)$ onto a soft subspace of $\left(Y,\widetilde{k},E\right)$ such that

$$\widetilde{k}\left((\psi,1_E)\left((X,E)\right)\right)=(Y,E).$$

Then $\left(Y,\widetilde{k},E\right)$ is called a soft extension of $(X,\widetilde{c},E)$ and denoted by

$$\varepsilon=\left((\psi,1_E),\left(Y,\widetilde{k},E\right)\right).$$

Let

$$\varepsilon_1=\left((\psi_1,1_E),\left(Y_1,\widetilde{k}_1,E\right)\right)$$

and

$$\varepsilon_2=\left((\psi_2,1_E),\left(Y_2,\widetilde{k}_2,E\right)\right)$$

be two soft extensions of a soft topological spaces $(X,\widetilde{c},E)$.

Definition 3.8. The soft extensions ε_1 and ε_2 are said to be soft topologically equivalent, if there exists a soft homeomorphism $(h,1_E)$ of $\left(Y_1,\widetilde{k}_1,E\right)$ onto $\left(Y_2,\widetilde{k}_2,E\right)$ with $(h,1_E)\circ(\psi_1,1_E)=(\psi_2,1_E)$.

Definition 3.9. Let $\varepsilon = \left((\psi, 1_E), \left(Y, \widetilde{k}, E\right)\right)$ be a soft extension of a soft space $(X, \widetilde{c}, E)$. Then the soft strength $\widetilde{S}(y_e, \varepsilon)$ of an arbitrary soft point y_e of $SS(Y)_E$ on $(X, \widetilde{c}, E)$ is defined by

$$\begin{aligned}\widetilde{S}(y_e, \varepsilon) &= \left\{(F,E) \in SS(X)_E : y_e \widetilde{\notin} \widetilde{k}((\psi, 1_E)((F,E)))\right\} \\ &= \left\{(F,E) \in SS(X)_E : (\psi, 1_E)((F,E)) \in \widetilde{I}_{\widetilde{k}}(y_e)\right\}.\end{aligned}$$

Remark 3.10. $\widetilde{S}(y_e, \varepsilon)$ is a soft $\widetilde{c}-$ideal on $(X, \widetilde{c}, E)$.

Proposition 3.11. *Let $\varepsilon = \left((\psi, 1_E), \left(Y, \widetilde{k}, E\right)\right)$ be a soft extension of a soft space $(X, \widetilde{c}, E)$. Then for all $x_e \widetilde{\in} SS(X)_E$, $\widetilde{S}((\psi, 1_E)(x_e), \varepsilon) = \widetilde{I}_{\widetilde{c}}(x_e)$.*

Proof.

$$\begin{aligned}\widetilde{S}((\psi, 1_E)(x_e), \varepsilon) &= \left\{(F,E) \in SS(X)_E : (\psi, 1_E)(x_e) \widetilde{\notin} \widetilde{k}((\psi, 1_E)((F,E)))\right\} \\ &= \{(F,E) \in SS(X)_E : (\psi, 1_E)(x_e) \\ &\quad \widetilde{\notin} \widetilde{k}(((\psi, 1_E)((F,E))) \widetilde{\cap} ((\psi, 1_E)((X,E))))\right\} \\ &= \left\{(F,E) \in SS(X)_E : (\psi, 1_E)(x_e) \widetilde{\notin} (\psi, 1_E)(\widetilde{c}((F,E)))\right\} \\ &= \left\{(F,E) \in SS(X)_E : x_e \widetilde{\notin} \widetilde{c}((F,E))\right\} = \widetilde{I}_{\widetilde{c}}(x_e)\end{aligned}$$

is obtained. □

Theorem 3.12. *Two soft equivalent extensions of a soft topological space have identical soft strength systems.*

Proof. Let $\varepsilon_1 = \left((\psi_1, 1_E), \left(Y_1, \widetilde{k_1}, E\right)\right)$ and $\varepsilon_2 = \left((\psi_2, 1_E), \left(Y_2, \widetilde{k_2}, E\right)\right)$ be two soft equivalent extensions of a soft topological space $(X, \widetilde{c}, E)$. From Definition 3.8, we have a soft homeomorphism $(h, 1_E)$ of $\left(Y_1, \widetilde{k_1}, E\right)$ onto $\left(Y_2, \widetilde{k_2}, E\right)$ such that $(h, 1_E) \circ (\psi_1, 1_E) = (\psi_2, 1_E)$. Now, we want to show that for all soft point $y_e \widetilde{\in} \left(Y_1, \widetilde{k_1}, E\right)$,

$$\widetilde{S}(y_e, \varepsilon_1) = \widetilde{S}((h, 1_E)(y_e), \varepsilon_2)$$

is satisfied. Indeed,

$$\begin{aligned}(F,E) &\in \widetilde{S}(y_e, \varepsilon_1) \Leftrightarrow y_e \widetilde{\notin} \widetilde{k_1}((\psi_1, 1_E)((F,E))) \\ &\Leftrightarrow (h, 1_E)(y_e) \widetilde{\notin} (h, 1_E)\left(\widetilde{k_1}((\psi_1, 1_E)((F,E)))\right).\end{aligned}$$

Since

$$\begin{aligned}(h, 1_E)\left(\widetilde{k_1}((\psi_1, 1_E)((F,E)))\right) &= \widetilde{k_2}((h, 1_E)(\psi_1, 1_E)((F,E))) \\ &= \widetilde{k_2}((\psi_2, 1_E)((F,E))),\end{aligned}$$

$(F,E) \in \widetilde{S}((h, 1_E)(y_e), \varepsilon_2)$. . □

Now, by a compactification $(\alpha X, E)$ of a soft local compact Hausdorff space (X, E), we mean a soft extension $\left((\alpha, 1_E), \left(Y, \widetilde{k}, E\right)\right)$. Here, $(Y, E) = (\alpha X, E)$ is a soft compact space

$$(\alpha, 1_E) : (X, E) \to (\alpha(X), E)$$

is a soft homeomorphism and $\widetilde{k}(\alpha(X), E) = (Y, E)$.

Particularly, we denote $\widetilde{k}((G, E)) = \overline{(G, E)}$, where $(G, E) \in SS(Y)_E$.

Theorem 3.13. *The soft strengths of different soft points of any soft compactification $(\alpha X, E)$ of a soft local Hausdorff space $(X, \widetilde{c}, E)$ are different.*

Proof. Let $y^1_{e_1}$, $y^2_{e_2}$ be two different soft points of $(\alpha X, E)$. Then there are disjoint soft open sets (G, E) and (H, E) in $(\alpha X, E)$ and $y^1_{e_1} \widetilde{\in} (G, E)$, $y^2_{e_2} \widetilde{\in} (H, E)$. It is clear that $y^2_{e_2} \widetilde{\notin} \overline{(G, E) \cap (\alpha(X), E)}$. We take $(F, E) = (\alpha, 1_E)^{-1}((G, E))$. Thus

$$(\alpha, 1_E)((F, E)) = (G, E) \widetilde{\cap} (\alpha(X), E).$$

So $y^2_{e_2} \widetilde{\notin} \overline{(\alpha, 1_E)((F, E))}$. Hence $(F, E) \in \widetilde{S}\left(y^2_{e_2}, (\alpha X, E)\right)$. Since $y^1_{e_1} \widetilde{\in} \overline{(\alpha, 1_E)((F, E))}$, $(F, E) \notin \widetilde{S}\left(y^1_{e_1}, (\alpha X, E)\right)$. This means that $\widetilde{S}\left(y^1_{e_1}, (\alpha X, E)\right) \neq \widetilde{S}\left(y^2_{e_2}, (\alpha X, E)\right)$. □

Theorem 3.14. *Let $(\alpha X, E)$ and $(\gamma X, E)$ be two soft compactifications of a soft local compact Hausdorff space $(X, \widetilde{c}, E)$ with identical soft strength systems. Then $(\alpha X, E)$ and $(\gamma X, E)$ are soft topologically equivalent.*

Proof. We have

$$\left\{\widetilde{S}(y_e, (\alpha X, E)) : y_e \widetilde{\in} SS(Y)_E\right\} = \left\{\widetilde{S}(z_e, (\gamma X, E)) : z_e \widetilde{\in} SS(\gamma X)_E\right\}.$$

Then Theorem 3.13 shows that none of the soft elements in either soft set is repeated. Thus, for all soft point $y_e \widetilde{\in} (\alpha X, E)$, we can associate a unique soft point $z_e \widetilde{\in} (\gamma X, E)$ such that $\widetilde{S}(y_e, (\alpha X, E)) = \widetilde{S}(z_e, (\gamma X, E))$. So we can define a soft mapping

$$(f, 1_E) : (\alpha X, E) \to (\gamma X, E)$$

by $(f, 1_E)(y_e) = z_e$. It is clear that $(f, 1_E)$ is a soft bijective mapping of $(\alpha X, E)$ onto $(\gamma X, E)$ and for each soft point $y_e \widetilde{\in} (\alpha X, E)$,

$$\widetilde{S}(y_e, (\alpha X, E)) = \widetilde{S}((f, 1_E)(y_e), (\gamma X, E)). \tag{3.1}$$

Moreover, for each soft point $x_e \widetilde{\in} SS(X)_E$,

$$\widetilde{S}((\alpha, 1_E)(x_e), (\alpha X, E)) = \widetilde{S}((f, 1_E) \circ (\alpha, 1_E)(x_e), (\gamma X, E)).$$

This means that

$$\widetilde{S}((f, 1_E) \circ (\alpha, 1_E)(x_e), (\gamma X, E)) = \widetilde{I_{\widetilde{c}}}(x_e).$$

Since $\widetilde{S}((\gamma, 1_E)(x_e), (\gamma X, E)) = \widetilde{I_{\widetilde{c}}}(x_e)$,

$$\widetilde{S}((f, 1_E)((\alpha, 1_E)(x_e)), (\gamma X, E)) = \widetilde{S}((\gamma, 1_E)(x_e), (\gamma X, E)).$$

It follows that $(f, 1_E)((\alpha, 1_E)(x_e)) = (\gamma, 1_E)(x_e)$, for each $x_e \widetilde{\in} SS(X)_E$. Hence $(f, 1_E) \circ (\alpha, 1_E) = (\gamma, 1_E)$.

Now we need only to show that $(f, 1_E)$ is a soft homeomorphism of $(\alpha X, E)$ onto $(\gamma X, E)$. It is seen that from (3.1), we have for any $(F,E) \in SS(X)_E$, $y_e \widetilde{\notin} \overline{(\alpha, 1_E)((F,E))}$. Then $(f, 1_E)(y_e) \widetilde{\notin} \overline{(\gamma, 1_E)((F,E))}$. Thus

$$(f, 1_E)\left(\overline{(\alpha, 1_E)((F,E))}\right) = \overline{(\gamma, 1_E)((F,E))}.$$

So

$$\left\{\overline{(\alpha, 1_E)((F,E))} : (F,E) \in SS(X)_E\right\}$$

and

$$\left\{\overline{(\gamma, 1_E)((F,E))} : (F,E) \in SS(X)_E\right\}$$

constitute soft closed bases for the soft compact spaces $(\alpha X, E)$ and $(\gamma X, E)$, respectively. Hence $(f, 1_E)$ is a soft homeomerphism of $(\alpha X, E)$ and $(\gamma X, E)$. $\square$

Definition 3.15. A soft extension $\varepsilon = \left((\psi, 1_E), \left(Y, \widetilde{k}, E\right)\right)$ of a soft topological space $(X, \widetilde{c}, E)$ is said to be a soft ideal extension, if

(i) any two distinct soft points of $SS(Y)_E$ have different soft strengths,

(ii) $\left\{\widetilde{k}(\psi, 1_E)((F,E)) : (F,E) \in SS(X)_E\right\}$ is a soft base for the soft closed sets in $\left(Y, \widetilde{k}, E\right)$.

Remark 3.16. A soft topological space $(X, \widetilde{c}, E)$ which admits a soft ideal extension $\varepsilon = \left((\psi, 1_E), \left(Y, \widetilde{k}, E\right)\right)$ is necessarily soft T_0-space.

Proof. Indeed, for any soft points $x_e^1, x_e^2 \widetilde{\in} SS(X)_E$, if $\widetilde{I_{\widetilde{c}}}\left(x_e^1\right) = \widetilde{I_{\widetilde{c}}}\left(x_e^2\right)$, from Proposition 3.11, then $\widetilde{S}\left((\psi, 1_E)\left(x_e^1\right), \varepsilon\right) = \widetilde{S}\left((\psi, 1_E)\left(x_e^2\right), \varepsilon\right)$. Thus $(\psi, 1_E)\left(x_e^1\right) = (\psi, 1_E)\left(x_e^2\right)$. From the Proposition 3.5, $(X, \widetilde{c}, E)$ is a soft T_0-space. $\square$

Corollary 3.17. *Let $(X, \widetilde{c}, E)$ be a soft T_0-space and $\varepsilon = \left((\psi, 1_E), \left(Y, \widetilde{k}, E\right)\right)$ be any soft ideal extension of $(X, \widetilde{c}, E)$. Then the soft ideal extension is also a soft T_0-space.*

Proof. For arbitrary $y_e^1, y_e^2 \widetilde{\in} SS(Y)_E$, let $\widetilde{I_{\widetilde{k}}}\left(y_e^1\right) = \widetilde{I_{\widetilde{k}}}\left(y_e^2\right)$. Then

$$\begin{aligned} \widetilde{S}\left(y_e^1, \varepsilon\right) &= \left\{(F,E) \in SS(X)_E : y_e^1 \widetilde{\notin} \widetilde{k}\left((\psi, 1_E)((F,E))\right)\right\} \\ &= \left\{(F,E) \in SS(X)_E : (\psi, 1_E)((F,E)) \in \widetilde{I_{\widetilde{k}}}\left(y_e^1\right)\right\} \\ &= \left\{(F,E) \in SS(X)_E : (\psi, 1_E)((F,E)) \in \widetilde{I_{\widetilde{k}}}\left(y_e^2\right)\right\} \\ &= \left\{(F,E) \in SS(X)_E : y_e^2 \widetilde{\notin} \widetilde{k}\left((\psi, 1_E)((F,E))\right)\right\} \\ &= \widetilde{S}\left(y_e^2, \varepsilon\right). \end{aligned}$$

This means that $y_e^1 = y_e^2$. From the Proposition 3.5, $\left(Y, \widetilde{k}, E\right)$ is a soft T_0-space. $\square$

Theorem 3.18. *Let $(X, \widetilde{c}, E)$ be a soft T_0-space and $\varepsilon = \left((\psi, 1_E), \left(Y, \widetilde{k}, E\right)\right)$ be a soft ideal extension of $(X, \widetilde{c}, E)$. Then ε is equivalent to $\varepsilon^* = \left((\theta, 1_E), \left(X^*, \widetilde{d}, E\right)\right)$ of $(X, \widetilde{c}, E)$, where (X^*, E) is a suitable collection of soft $\widetilde{c}-$ideals on $(X, \widetilde{c}, E)$ containing all the soft free ideals on $SS(X)_E$.*

Proof. Let $(X^*, E) = \left\{\widetilde{S}(y_e, \varepsilon) : y_e \widetilde{\in} SS(Y)_E\right\}$ be a collection of the strength system of ε on $(X, \widetilde{c}, E)$. It is clear that (X^*, E) is a soft set of soft $c-$ideals on $(X, \widetilde{c}, E)$ containing every soft free ideal on the same. We first give a soft closure operator on (X^*, E), together with a soft mapping

$$(\Omega, 1_E) : (X, \widetilde{c}, E) \to \left(X^*, \widetilde{d}, E\right)$$

by the formula $(\Omega, 1_E)(x_e) = \widetilde{I}_{\widetilde{c}}(x_e)$, $x_e \widetilde{\in} SS(X)_E$. From the Proposition 3.5, we obtain that $(\Omega, 1_E)$ is a soft one to one mapping. For any $(F, E) \in SS(X)_E$, we get a set

$$(F, E)^c = \left\{\widetilde{I} \in SS(X^*)_E : (F, E) \notin \widetilde{I}\right\}.$$

Firstly, we prove that

$$B = \{(F, E)^c : (F, E) \in SS(X)_E\}$$

is a soft base for the soft closed sets with respect to some soft topology on (X^*, E). It is clear that $\Phi^c = \Phi$.

Now, let $(F, E)^c, (G, E)^c \in B$. Then

$$\begin{aligned}\left[(F, E) \widetilde{\cup} (G, E)\right]^c &= \left\{\widetilde{I} \in SS(X^*)_E : (F, E) \widetilde{\cup} (G, E) \notin \widetilde{I}\right\} \\ &= \left\{\widetilde{I} \in SS(X^*)_E : (F, E) \notin \widetilde{I} \text{ or } (G, E) \notin \widetilde{I}\right\} \\ &= \left\{\widetilde{I} \in SS(X^*)_E : (F, E) \notin \widetilde{I}\right\} \widetilde{\cup} \left\{\widetilde{I} \in SS(X^*)_E : (G, E) \notin \widetilde{I}\right\} \\ &= (F, E)^c \widetilde{\cup} (G, E)^c.\end{aligned}$$

Thus, $(F, E)^c \widetilde{\cup} (G, E)^c \in B$.

Let $\widetilde{d}$ be the soft closure operator that associated with this soft topology on (X^*, E). Then for all $\widetilde{\alpha} \widetilde{\in} SS(X^*)_E$, $\widetilde{d}(\widetilde{\alpha})$ is the intersection of all soft closed subsets of (X^*, d^*, E) containing

$$\widetilde{d}(\widetilde{\alpha}) = \widetilde{\cap}\left\{(F, E)^c : \widetilde{\alpha} \widetilde{\in} (F, E)^c \text{ and } (F, E) \in SS(X)_E\right\}.$$

Now we first examine that some of the relations involving $\widetilde{d}$ and $(\Omega, 1_E) : (X, \widetilde{c}, E) \to \left(X^*, \widetilde{d}, E\right)$.

(i) For all $(F, E) \in SS(X)_E$, $(\Omega, 1_E)(\widetilde{c}((F, E))) = (F, E)^c \widetilde{\cap} (\Omega, 1_E)((X, E))$.

Indeed, $x_e \widetilde{\in} \widetilde{c}((F, E)) \Rightarrow (F, E) \notin \widetilde{I}_{\widetilde{c}}(x_e) = (\Omega, 1_E)(x_e) \widetilde{\in} (F, E)^c \widetilde{\cap} ((\Omega, 1_E)((X, E)))$.

(ii) For all $(F, E) \in SS(X)_E$, $(\Omega, 1_E)((F, E)) \widetilde{\subseteq} (F, E)^c$.

Indeed, $(\Omega, 1_E)((F, E)) \widetilde{\subseteq} (\Omega, 1_E)(\widetilde{c}((F, E)))$. From (i), $(\Omega, 1_E)((F, E)) \widetilde{\subseteq} (F, E)^c$ is obtained.

(iii) For all $(F, E) \in SS(X)_E$, $\widetilde{d}((\Omega, 1_E)((F, E))) = (F, E)^c$.

We consider of (ii), it is sufficient to show that whenever $(\Omega, 1_E)((F,E)) \widetilde{\subseteq} (G,E)^c$, for some soft set $(G,E) \in SS(X)_E$, then $(F,E)^c \widetilde{\subseteq} (G,E)^c$.

Suppose $(\Omega, 1_E)((F,E)) \widetilde{\subseteq} (G,E)^c$ for some soft set $(G,E) \in SS(X)_E$. Then

$$\begin{aligned} x_e \widetilde{\in} (F,E) &\Rightarrow (\Omega, 1_E)(x_e) \widetilde{\in} (G,E)^c \\ &\Rightarrow \widetilde{I}_{\widetilde{c}}(x_e) \widetilde{\in} (G,E)^c \\ &\Rightarrow (G,E) \notin \widetilde{I}_{\widetilde{c}}(x_e) \\ &\Rightarrow x_e \widetilde{\in} \widetilde{c}((G,E)). \end{aligned}$$

Thus $(F,E) \widetilde{\subseteq} \widetilde{c}((G,E))$.

Now we prove that the following cases:

Case (a): $(F,E) \widetilde{\subseteq} (G,E) \Rightarrow (F,E)^c \widetilde{\subseteq} (G,E)^c$. Indeed,

$$\widetilde{I} \in (F,E)^c \Rightarrow (F,E) \notin \widetilde{I} \Rightarrow (G,E) \notin \widetilde{I} \Rightarrow \widetilde{\widetilde{I}} \in (G,E)^c.$$

Case (b): For all $(G,E) \in SS(X)_E$, $(\widetilde{c}((G,E)))^c = (G,E)^c$. In fact,

$$\widetilde{I} \in (\widetilde{c}((G,E)))^c \Leftrightarrow \widetilde{c}((G,E)) \notin \widetilde{I}.$$

Since $\widetilde{I}$ is a soft $\widetilde{c}$–ideal, $(G,E) \notin \widetilde{I}$. Then $\widetilde{I} \in (G,E)^c$. Thus $(\widetilde{c}((G,E)))^c = (G,E)^c$. Now,

$$(F,E) \widetilde{\subseteq} \widetilde{c}((G,E)) \Rightarrow (F,E)^c \widetilde{\subseteq} (\widetilde{c}((G,E)))^c = (G,E)^c.$$

So $(F,E)^c \widetilde{\subseteq} \widetilde{d}((\Omega, 1_E)((F,E)))$ and (iii) follows from the definition of $\widetilde{d}$. From (i) and (iii), we have

$$(\Omega, 1_E)(\widetilde{c}((F,E))) = \widetilde{d}((\Omega, 1_E)((F,E))) \widetilde{\cap} (\Omega, 1_E)((X,E))$$

and

$$\widetilde{d}((\Omega, 1_E)((X,E))) = (X,E)^c.$$

Also, we have $(X,E)^c = (X^*, E)$. Indeed,

$$(X,E)^c = \left\{ \widetilde{I} \in (X^*, E) : (X,E) \notin \widetilde{I} \right\}.$$

It is clear that $(X,E)^c \widetilde{\subseteq} (X^*, E)$.

Conversely, $\widetilde{S}(y_e, \varepsilon) \in (X^*, E)$ and for each $y_e \widetilde{\in} SS(Y)_E$, where

$$\widetilde{S}(y_e, \varepsilon) = \left\{ (G,E) \in SS(X)_E : y_e \widetilde{\notin} \widetilde{k}((\psi, 1_E)((F,E))) \right\}.$$

Then we have $(X,E) \notin \widetilde{S}(y_e, \varepsilon)$. Thus, $\widetilde{S}(y_e, \varepsilon) \in (X,E)^c$. So $(X,E)^c = (X^*, E)$.

It is seen that $\left(X^*, \widetilde{d}, E\right)$ is an soft extension of $(X, \widetilde{c}, E)$.

Now, we define a soft mapping

$$(f, 1_E) : \left(Y, \widetilde{k}, E\right) \to \left(X^*, \widetilde{d}, E\right)$$

by the formula $(f, 1_E)(y_e) = \widetilde{S}(y_e, \varepsilon)$. Then for all $x_e \widetilde{\in} SS(X)_E$,

$$F(f, 1_E)((\psi, 1_E)(x_e)) = \widetilde{S}((\psi, 1_E)(x_e), \varepsilon) = \widetilde{I}_{\widetilde{c}}(x_e) = (\Omega, 1_E)(x_e).$$

Thus $(f, 1_E) \circ (\psi, 1_E) = (\Omega, 1_E)$. Now, $(F,E) \in SS(X)_E$. For any $y_e \widetilde{\in} SS(Y)_E$,

$$y_e \widetilde{\in} \widetilde{k}((\psi, 1_E)((F,E))) \Leftrightarrow (F,E) \notin \widetilde{S}(y_e, \varepsilon) = (f, 1_E)(y_e),$$

i.e., $(f,1_E)(y_e)\widetilde{\in}(F,E)^c$. So, the soft bijective mapping

$$(f,1_E):\left(Y,\widetilde{k},E\right)\to\left(X^*,\widetilde{d},E\right)$$

installs a one to one correspondence between the soft closed sets of two spaces. Hence $(f,1_E)$ is a soft homeomorphism of $\left(Y,\widetilde{k},E\right)$ onto $\left(X^*,\widetilde{d},E\right)$. Therefore the proof is completed. □

4. Conclusions

One of the most important theories of topology is ideal topological spaces. This theory is important in sub-branches of mathematics. Here, we defined and explored the notions of $\widetilde{I}_{\widetilde{c}}$soft free ideal and soft $\widetilde{c}-$ ideal. We also investigated soft ideal extension of a given soft topological space.

References

[1] U. Acar, F. Koyuncu and B. Tanay, Soft sets and soft rings, Comput. Math. Appl. 59 (2010) 3458–3463.

[2] H. Aktas and N. Çağman, Soft sets and soft group, Inform. Sci. 177 (2007) 2726–2735.

[3] S. Bayramov and C. Gunduz, Soft locally compact spaces and soft paracompact spaces, Journal of Math. and Sys. Sci. 3 (2013) 122–130.

[4] N. Çağman, S. Karataş and S. Enginoğlu, Soft topology, Comput. Math. Appl. 62 (2011) 351–358.

[5] F. Feng, Y. B. Jun and X. Zhao, Soft semirings, Comput. Math. Appl. 56 (2008) 2621–2628.

[6] C. Gunduz and S. Bayramov, Intuitionistic fuzzy soft modules, Comput. and Math. Appl. 62 (2011) 2480–2486.

[7] C. Gunduz and S. Bayramov, Fuzzy soft modules, International Matematical Forum 6 (11) (2011) 517–527.

[8] S. Hussain and B. Ahmad, Some properties of soft topological spaces, Comput. Math. Appl. 62 (2011) 4058–4067.

[9] D. Jankovic and T. R. Hamlett, New topologies from old via ideals, Amer. Math. Monthly 97 (1990) 295–310.

[10] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, $\gamma-$operation and decompositions of some forms of soft continuity of soft topological spaces via soft ideals, Ann. Fuzzy Math. Inform. 9 (3) (2015) 385–402.

[11] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Soft connectedness via soft ideals, Journal of New results in Science 4 (2014) 90–108.

[12] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Soft ideal theory soft local function and generated soft topological spaces, Appl. Math. Inf. Sci. 8 (4) (2014) 1595–1603.

[13] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Soft regularity and normality based on semi open soft sets and soft ideals, Appl. Math. Inf. Sci. Lett. 3 (2) (2015) 47–55.

[14] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Soft semi compactness via soft ideals,ess via soft ideals, Appl. Math. Inf. Sci. 8 (5) (2014) 2297–2306.

[15] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Soft semi (quasi) Hausdorff spaces via soft ideals, South Asian J. Math. 4 (6) (2014) 265–284.

[16] A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and A. M. Abd El-latif, Supra generalized closed soft sets with respect to an soft ideal in supra soft topological spaces, Appl. Math. Inf. Sci. 8 (4) (2014) 1731–1740.

[17] K. Kuratowski, Topology, vol.1, Academic Press, New York 1989.

[18] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555–562.

[19] P. K. Maji and A. R. Roy, An Application of Soft Sets in a Decision Making Problem, Comput. Math. Appl. 44 (2002) 1077–1083.
[20] W. K. Min, A note on soft topological spaces, Comput. Math. Appl. 62 (2011) 3524–3528.
[21] D. Molodtsov, Soft set theory-first results, Comput. Math. Appl. 37 (1999) 19–31.
[22] M. N. Mukherjee, R. Bishwambhar and S. Ritu, On extensions of topological spaces in terms of ideals, Top. and its App. 154 (2007) 3167–3172.
[23] T. Y. Ozturk and S. Bayramov, Category of chain complexes of soft modules, International Mathematical Forum 7 (20) (2012) 981–992.
[24] T. Y. Ozturk, C. Gunduz and S. Bayramov, Inverse and direct systems of soft modules, Ann. Fuzzy Math. Inform. 5 (1) (2013) 73–85.
[25] M. Shabir and A. Bashir, Some properties of soft topological spaces, Comput. Math. Appl. 62 (2011) 4058–4067.
[26] M. Shabir and M. A. Irfan, Soft ideals and generalized fuzzy ideals in semigroups, New Math. Nat. Comput. 5 (2009) 599–615.
[27] M. Shabir and M. Naz, On soft topological spaces, Comput. Math. Appl. 61 (2011) 1786–1799.
[28] R. Vaidyanathaswamy, Set topology, 2nd ed., Chelsea Publishing Co., New York 1960.
[29] I. Zorlutuna, M. Akdağ, W. K. Min and S. Atmaca, Remarks on soft topological spaces, Ann. Fuzzy Math. Inform. 3 (2) (2012) 171–185.

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