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Fuzzy parameterized fuzzy soft topology with applications

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Abstract. In this paper, we study the concept of fuzzifying soft set called fuzzy parameterized fuzzy soft sets (FPFS-sets) and some results which holds in crisp set theory but does not hold in FPFS-set theory. We study FPFS-topology on FPFS-sets. We introduce closure, interior, frontier and exterior in the context of FPFS-topological spaces. We also discuss some properties of quasi-coincidence and Q-neighborhood for FPFS-sets. Furthermore, we present an application of FPFS-topology to find the aggregate of every FPFS-open set in FPFS-topology by using some algorithms for decision-making.

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1. Introduction

Fuzzy set theory was introduced by Zadeh [43] as a generalization of crisp or classical set theory. Molodtsov [26] proposed the idea of soft set theory. Fuzzy soft set theory has many applications in various fields such as social sciences, physics, engineering, economics, computer science and medical sciences. Akram *et al.* [1, 2, 3, 4] introduced various concepts including Bipolar Fuzzy Soft Lie algebras, Fuzzy soft K-algebras, Fuzzy soft Lie algebras and Fuzzy soft graphs. Ali *et al.* [5] suggested some operations on soft sets which became very useful in the field of soft set theory. Borah and Hazarika [6] studied some properties of mixed FS-topology and applications in Chemistry. Cagman *et al.* [7, 8, 9] proposed soft topology, FPFS-set theory and presented some applications of decision-making problems. Chang [10] studied the notion of fuzzy topological spaces. Chen *et al.* [11] established parameterized reduction of soft sets. Samanta and Das [12, 13, 14] introduced some basic properties of soft real sets and soft real numbers. They also gave the idea of soft elements with soft points in soft sets and discussed soft metric spaces. Feng *et al.* studied soft sets and soft rough sets and they established some applications based on decision-making (See [15, 16, 17, 18]). In [19, 20] Hur *et al.* studied Fuzzy equivalence relations, fuzzy partitions, fuzzy functions and fuzzy partially ordered sets. Jun *et al.* [21] introduced various studied fuzzy subgroups based on fuzzy points. Kharal and Ahmad [22] defined mappings on soft classes, the images of soft sets and the inverse soft images. Maji *et al.* [23, 24, 33] used soft sets theory in decision-making problems and defined some operations on soft sets. Samanta and Majumdar [25] introduced soft groups and discussed the soft images and inverse soft images. Peyghan and Varol [27, 41] gave some intrusting results on FS-topological spaces. Pei and Miao [28] discussed the connection of soft sets with information systems. Riaz *et al.* [29, 30, 31] discussed various concepts including soft σ-algebra, measurable soft set, measurable soft mappings and soft metric spaces. In [32] Rong proposed the countability of soft topology, discussed soft separable and soft Lindölof spaces and prove some important results using these terms. In [34, 35] the idea of soft topology, fuzzy soft point and FS quasi-coincident with Q-neighborhood has studied. Aslihan *et al.* [36, 37] introduced some operations on soft set. They studied various concepts including soft intersection semigroups, ideals and bi-Ideals. Shabir and Naz [38] introduced soft topology and soft topological spaces. In [39] some applications of fuzzy soft relation in decision making problems was presented. In [40] Subhashinin and Sekar used soft pre-open set to define the soft pre-topology, soft pre-sub-maximal and also investigated various interesting properties. Yildirim *et al.* [42] presented the notion of soft ideal for a soft topology and defined soft Ĩ-Baire spaces for a soft ideal topological space as well. Zorlutuna and Çakir [44] proposed soft continuity, soft openness, soft closeness of soft mappings in soft set theory. Zorlutuna and Atmaca [45] introduced the notion of FPFS topological space. Soft set theory and fuzzy soft set theory has studied by many explorers in the last decade (See [7, 8, 12, 13, 15, 23, 26, 38, 41, 45]).

2. Preliminaries

Definition 2.1 ([7, 41])**.** A fuzzy soft set (FS-set) is a mapping $\lambda : R \to \widetilde{P}(X)$ such that $\lambda_A(\zeta) = \phi$, if $\zeta \notin A$, where X is the set of universe and $A \subseteq R$, R is the set of parameters or attributes.

It is denoted as (λ, A) given by

$$(\lambda, A) = \{(\zeta, \lambda_A(\zeta)) : \zeta \in R, \lambda_A(\zeta) \in \widetilde{P}(X)\}.$$

The value $\lambda_A(\zeta)$ is a fuzzy set known as ζ-approximate element of FS-set (λ, A) $\forall\ \zeta \in R$.

The degree of membership of elements is taken in the interval $[0, 1]$.

Definition 2.2 ([7])**.** Let X be the universal set and let R be the set of parameters or attributes. Then the pair $\widetilde{(X, R)}$ represents the family of all FS-sets on X with parameters from R and is known as FS-class.

Example 2.3. Let X={islamabad, lahore, murree, naraan, multan, karachi, rahim yar khan } be a set of some cities of pakistan and let $R = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6\}$ be the set of attributes, where

ζ_1 is the parameter which stands for beautiful,
ζ_2 is the parameter which stands for green surroundings,
ζ_3 is the parameter which stands for having historical places,
ζ_4 is the parameter which stands for nice weather,
ζ_5 is the parameter which stands for clean,
ζ_6 is the parameter which stands for low crime rate.
Let $A = \{\zeta_1, \zeta_2, \zeta_3, \zeta_6\} \subseteq R$, then the FS-set (λ, A) is a mapping $\lambda : R \to \widetilde{P}(X)$, where $\widetilde{P}(X)$ is the collection of all FS-sets of X written as,

$$\begin{aligned}(\lambda, A) = \{&(\zeta_1, \{(\text{islamabad}, 0.7), (\text{lahore}, 0.5), (\text{murree}, 0.8), (\text{naraan}, 0.9),\\ &\quad(\text{multan}, 0.6), (\text{karachi}, 0.4), (\text{rahim yar khan}, 0.4)\}),\\ &(\zeta_2, \{(\text{islamabad}, 0.6), (\text{lahore}, 0.4), (\text{murree}, 0.9), (\text{naraan}, 1),\\ &\quad(\text{multan}, 0.4), (\text{karachi}, 0.2), (\text{rahim yar khan}, 0.1),\\ &(\zeta_3, \{(\text{islamabad}, 0.4), (\text{lahore}, 0.7), (\text{murree}, 0.4), (\text{naraan}, 0.5),\\ &\quad(\text{multan}, 0.8), (\text{karachi}, 0.7), (\text{rahim yar khan}, 0.5),\\ &(\zeta_6, \{(\text{islamabad}, 0.7), (\text{lahore}, 0.6), (\text{murree}, 0.6), (\text{naraan}, 0.7),\\ &\quad(\text{multan}, 0.4), (\text{karachi}, 0.1), (\text{rahim yar khan}, 0.8)\}.\end{aligned}$$

We can also represent the FS-set in tabular form as,

X	ζ_1	ζ_2	ζ_3	ζ_6
islamabad	0.7	0.6	0.4	0.7
lahore	0.5	0.4	0.7	0.6
murree	0.8	0.9	0.4	0.6
naraan	0.9	1	0.5	0.7
multan	0.6	0.4	0.3	0.4
karachi	0.4	0.2	0.7	0.1
rahim yar khan	0.4	0.1	0.5	0.8

Definition 2.4 ([7, 44])**.** A fuzzy parameterized fuzzy soft set (FPFS-set) is a mapping $\gamma : R \to \widetilde{P}(X)$ such that $\gamma_A(\zeta) = \phi$, if $\mu_A(\zeta) = 0$,where X is the initial universe and $A \subseteq R$, R is the set of parameters or attributes. It is denoted as F_A, where

$$F_A = \{(\mu_A(\zeta)/\zeta, \gamma_A(\zeta)) : \zeta \in R, \gamma_A(\zeta) \in \widetilde{P}(X); \mu_A(\zeta), \gamma_A(\xi) \in [0, 1], \xi \in X\}.$$

The value $\gamma_A(\zeta)$ is a fuzzy set known as ζ-element of FPFS-set F_A $\forall\ \zeta \in R$.

Example 2.5. Let a factory wants to fill a place for a highly qualified engineer.There are eight applicants who apply for this job. The set of candidates represented by

$$X = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8\}.$$

The hiring committee consider the set of attributes, $R = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6\}$, where
ζ_1 = hard working, ζ_2 = five years experienced ,
ζ_3 = computer knowledge,
ζ_4 = good speaking,
ζ_5 = punctual and regular,
ζ_6 = friendly.
Each applicant is selected according to the goals and constraint with the help of subset $A = \{0.5/\zeta_1.0.8/\zeta_2, 0.6/\zeta_3, 0.3/\zeta_5\}$.
At last, the panel form the given FPFS-set

$F_A = \{(0.5/\zeta_1, \{0.1/\sigma_3, 0.3/\sigma_5, 0.6/\sigma_7\}), (0.8/\zeta_2, \{0.7/\sigma_1, 0.5/\sigma_4, 0.3/\sigma_8\}), (0.6/\zeta_3, \{0.4/\sigma_3, 0.2/\sigma_5, 1/\sigma_7\}), (0.3/\zeta_5, \{0.4/\sigma_1, 1/\sigma_4, 0.2/\sigma_6\})\}$.
In tabular form, the FPFS-set can be represented as,

X	$0.5/\zeta_1$	$0.8/\zeta_2$	$0.6/\zeta_3$	$0/\zeta_4$	$0.3/\zeta_5$	$0/\zeta_6$
σ_1	0	0.7	0	0	0.4	0
σ_2	0	0	0	0	0	0
σ_3	0.1	0	0.4	0	0	0
σ_4	0	0.5	0	0	1	0
σ_5	0.3	0	0.2	0	0	0
σ_6	0	0	0	0	0.2	0
σ_7	0.6	0	1	0	0	0
σ_8	0	0.3	0	0	0	0

Definition 2.6 ([7, 44]). Let F_A be a FPFS-set over X. If $\lambda_A(\zeta) = \phi \ \forall \ \zeta \in R$, then F_A is called an A-empty FPFS-set. It is represented as F_{ϕ_A}.

If $A = \phi$, then A-empty FPFS-set is called empty FPFS-set denoted as F_ϕ.

Definition 2.7 ([7, 44]). Let F_A be a FPFS-set over X. If $\gamma_A(\zeta) = X$ and $\mu_A(\zeta) = 1$ $\forall \ \zeta \in R$, then F_A is known as A-universal FPFS-set. It is represented as $F_{\widetilde{A}}$.

If $A = R$, then A-universal FPFS-set is said to be universal or absolute FPFS-set written as $F_{\widetilde{R}}$.

Example 2.8. Molodtsov's soft set considered as a special case of FPFS-set, this means we can write every soft set as FPFS-set.

Let (λ, A) be a soft set given as,

$$(\lambda, A) = \{(\zeta_1, \{\sigma_1, \sigma_2\}), (\zeta_2, \{\sigma_2, \sigma_3\})\},$$

where $X = \{\sigma_1, \sigma_2, \sigma_3\}$ and $A =\{\zeta_1, \zeta_2\} \subseteq R = \{\zeta_1, \zeta_2, \zeta_3\}$.
Now we write the soft set (λ, A) in the form of FPFS-set as,

$$(F_A) = \{(1/\zeta_1, \{(\sigma_1, 1), (\sigma_2, 1), (\sigma_3, 0)\}), (1/\zeta_2, \{(\sigma_1, 0), (\sigma_2, 1), (\sigma_3, 1)\})\}.$$

In tabular form,

X	l/ζ_1	$1/\zeta_2$
σ_1	1	0
σ_2	1	1
σ_3	0	1

Definition 2.9 ([44]). Let F_A and F_B be two FPFS-sets. Then F_A is called FPFS-subset of F_B, denoted by $F_A \widetilde{\subseteq} F_B$, if
(i) $\mu_A(\zeta) \leq \mu_B(\zeta)$,
(ii) $\gamma_A(\zeta) \subseteq \gamma_B(\zeta) \ \forall \ \zeta \in R$.

Definition 2.10 ([7, 44]). Let F_A and F_B be two FPFS-sets. The union of two FPFS-sets F_A and F_B, written as $F_A \widetilde{\cup} F_B$, is defined by
$\mu_{A\widetilde{\cup}B}(\zeta) = max\{\mu_A(\zeta), \mu_B(\zeta)\}$, $\gamma_{A\widetilde{\cup}B}(\zeta) = \{\gamma_A(\zeta) \cup \gamma_B(\zeta)\} \forall \ \zeta \in R$.

Definition 2.11 ([7, 44]). Let F_A and F_B be two FPFS-sets. The intersection of two FPFS-sets F_A and F_B, written as $F_A \widetilde{\cap} F_B$ is defined by
$\mu_{A\widetilde{\cap}B}(\zeta) = min\{\mu_A(\zeta), \mu_B(\zeta)\}$, $\gamma_{A\widetilde{\cap}B}(\zeta) = \{\gamma_A(\zeta) \cap \gamma_B(\zeta)\} \forall \ \zeta \in R$.

3. Fuzzy parameterized fuzzy soft topology

Definition 3.1 ([44]). Let $F_{\widetilde{R}}$ be a absolute FPFS-set and FPFS($F_{\widetilde{R}}$) is the family of all FPFS-subsets of $F_{\widetilde{R}}$. Let $\widetilde{\tau}$ be a subfamily of FPFS($F_{\widetilde{R}}$) and $A, B, C \subseteq R$. Then $\widetilde{\tau}$ is known as FPFS-topology on $F_{\widetilde{R}}$, if the given conditions are satisfied:

(i) $F_{\phi}, F_{\widetilde{R}} \widetilde{\in} \widetilde{\tau}$,

(ii) $F_A, F_B \widetilde{\in} \widetilde{\tau}$ then $F_A \widetilde{\cap} F_B \widetilde{\in} \widetilde{\tau}$,

(iii) if $(F_C)_\lambda \widetilde{\in} \widetilde{\tau}$, $\forall\ \lambda \in J$, then $\widetilde{\cup}_{\lambda\in J}(F_C)_\lambda \widetilde{\in} \widetilde{\tau}$.

Members of $\widetilde{\tau}$ are known as FPFS-open sets and FPFS-complement of FPFS-open sets are called FPFS-closed sets.

Example 3.2. Let $X = \{\sigma_1, \sigma_2, \sigma_3\}$ be the set of universe and let $R = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ be the set of attributes. If $A = \{0.3/\zeta_1, 0.5/\zeta_2, 0.7/\zeta_4\} \subseteq R$, $B = \{0.2/\zeta_1, 0.6/\zeta_4\} \subseteq R$ with FPFS-sets,

$$F_A=\{(0.3/\zeta_1, \{0.1/\sigma_1, 0.3/\sigma_2, 0.6/\sigma_3\}), (0.5/\zeta_2, \{0.7/\sigma_1, 0.3/\sigma_2, 0.5/\sigma_3\}), (0.7/\zeta_4, \{0.4/\sigma_1, 1/\sigma_2, 0.2/\sigma_3\})\}$$

and

$$F_B = \{(0.2/\zeta_1, \{0.1/\sigma_1, 0.2/\sigma_2, 0.2/\sigma_3\}), (0.6/\zeta_4, \{0.3/\sigma_1, 0.8/\sigma_2, 0.2/\sigma_3\})\},$$

then $\widetilde{\tau} = \{F_\phi, F_{\widetilde{R}}, F_A, F_B\}$ is a FPFS-topology on X. In fact, $F_B \subseteq F_A$. For

(i)$F_\phi, F_{\widetilde{R}} \in \widetilde{\tau}$.

(ii)$F_\phi \widetilde{\cap} F_A = F_\phi$: $F_\phi \widetilde{\cap} F_B = F_\phi$, $F_\phi \widetilde{\cap} F_{\widetilde{R}} = F_\phi$, $F_{\widetilde{R}} \widetilde{\cap} F_A = F_A$, $F_{\widetilde{R}} \widetilde{\cap} F_B = F_B$, $F_B \widetilde{\cap} F_A = F_B$.

(iii)$F_\phi \widetilde{\cup} F_A = F_A$: $F_\phi \widetilde{\cup} F_B = F_B$, $F_\phi \widetilde{\cup} F_{\widetilde{R}} = F_{\widetilde{R}}$, $F_{\widetilde{R}} \widetilde{\cup} F_A = F_{\widetilde{R}}$, $F_{\widetilde{R}} \widetilde{\cup} F_B = F_{\widetilde{R}}$, $F_B \widetilde{\cup} F_A = F_A$, $F_\phi \widetilde{\cup} F_{\widetilde{R}} \widetilde{\cup} F_A \widetilde{\cup} F_B = F_{\widetilde{R}}$.

This implies that $\widetilde{\tau} = \{F_\phi, F_{\widetilde{R}}, F_A, F_B\}$ is a FPFS-topology on X.

Definition 3.3 ([44]). Let $\widetilde{\tau}_1$ and $\widetilde{\tau}_2$ be two FPFS-topologies on X. If $\widetilde{\tau}_1 \widetilde{\subseteq} \widetilde{\tau}_2$ then $\widetilde{\tau}_1$ is called FPFS-courser or FPFS-weaker and $\widetilde{\tau}_2$ is called FPFS-finer or FPFS-stronger than $\widetilde{\tau}_1$.

Example 3.4. Let $X = \{\sigma_1, \sigma_2, \sigma_3\}$ be the set of universe and let $R = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ be the set of attributes. If $A = \{0.3/\zeta_1, 0.5/\zeta_2, 0.7/\zeta_4\} \subseteq R$, $B = \{0.2/\zeta_1, 0.6/\zeta_4\} \subseteq R$ with the FPFS-sets,

$$F_A=\{(0.3/\zeta_1, \{0.1/\sigma_1, 0.3/\sigma_2, 0.6/\sigma_3\}), (0.5/\zeta_2, \{0.7/\sigma_1, 0.3/\sigma_2, 0.5/\sigma_3\}), (0.7/\zeta_4, \{0.4/\sigma_1, 1/\sigma_2, 0.2/\sigma_3\})\}$$

and

$$F_B = \{(0.2/\zeta_1, \{0.1/\sigma_1, 0.2/\sigma_2, 0.2/\sigma_3\}), (0.6/\zeta_4, \{0.3/\sigma_1, 0.8/\sigma_2, 0.2/\sigma_3\})\},$$

then $\widetilde{\tau}_1 = \{F_\phi, F_{\widetilde{R}}, F_A, \}$ and $\widetilde{\tau}_2 = \{F_\phi, F_{\widetilde{R}}, F_A, F_B\}$ are two FPFS-topologies it is clear that $\widetilde{\tau}_1 \widetilde{\subseteq} \widetilde{\tau}_2$. Thus $\widetilde{\tau}_1$ is called FPFS-courser or FPFS-weaker and $\widetilde{\tau}_2$ is called FPFS-finer or FPFS-stronger than $\widetilde{\tau}_1$.

Definition 3.5 ([44]). The families $\widetilde{\tau}_{indiscrete} = \{F_\phi, F_{\widetilde{R}}\}$ and $\widetilde{\tau}_{discrete} = FPFS(X, R)$ are FPFS-topology on X.

Example 3.6. Let $X = \{\sigma_1, \sigma_2, \sigma_3\}$ be the set of universe and let $R = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ be the set of attributes. If $A = \{0.3/\zeta_1, 0.4/\zeta_2, 0.1/\zeta_3\} \subseteq R$, then $\widetilde{\tau} = \{F_\phi, F_{\widetilde{R}}\}$ is the indiscrete FPFS-topology and $P(F_{\widetilde{R}})$ (power set of $F_{\widetilde{R}}$) is the discrete FPFS-topology on X.

Definition 3.7. Let $\widetilde{\tau}_1$ be a FPFS-topology on (X, R) and $Y \subseteq X$ and let $\widetilde{\tau}_2$ be a FPFS-topology on (Y, E) whose FPFS-open sets can be defined as

$$F_B = F_A \widetilde{\cap} F_{\widetilde{E}},$$

where F_A is the FPFS-open sets of $\widetilde{\tau}_1$ and F_B is the FPFS-open sets of $\widetilde{\tau}_2$, $F_{\widetilde{E}}$ is FPFS-absolute set on (Y, E), and R, E are the set of attributes for X, Y, respectively. Then $\widetilde{\tau}_2$ is known as the FPFS-subspace of $\widetilde{\tau}_1$.

Example 3.8. Let $X = \{\sigma_1, \sigma_2, \sigma_3\}$ be the set of universe and let $R = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ be the set of attributes. If $A = \{0.7/\zeta_1, 0.8/\zeta_3, 0.6/\zeta_4\} \subseteq R$, $B = \{0.5/\zeta_1, 0.3/\zeta_4\} \subseteq R$ with the FPFS-sets,

$$F_A = \{(0.7/\zeta_1, \{0.5/\sigma_1, 0.6/\sigma_2, 0.4/\sigma_3\}), (0.8/\zeta_3, \{0.2/\sigma_1, 0.3/\sigma_2, 0.9/\sigma_3\}), (0.6/\zeta_4, \{0.7/\sigma_1, 0.3/\sigma_2, 0.4/\sigma_3\})\}$$

and

$$F_B = \{(0.5/\zeta_1, \{0.4/\sigma_1, 0.4/\sigma_2, 0.3/\sigma_3\}), (0.3/\zeta_4, \{0.3/\sigma_1, 0.2/\sigma_2, 0.1/\sigma_3\})\},$$

then $\widetilde{\tau}_1 = \{F_\phi, F_{\widetilde{R}}, F_A, F_B\}$ is a FPFS-topology on X.

If $Y = \{\sigma_1, \sigma_2\} \subseteq X$ and $E = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$, then FPFS-absolute set on Y is,

$$F_{\widetilde{E}} = \{(1/\zeta_1, \{1/\sigma_1, 1/\sigma_2\}), (1/\zeta_2, \{1/\sigma_1, 1/\sigma_2\}), (1/\zeta_3, \{1/\sigma_1, 1/\sigma_2\}), (1/\zeta_4, \{1/\sigma_1, 1/\sigma_2\})\}.$$

Where FPFS-open sets for $\widetilde{\tau}_2$ can be calculated as,

$F_{\widetilde{E}} \widetilde{\cap} F_\phi = F_\phi$, $F_{\widetilde{E}} \widetilde{\cap} F_{\widetilde{R}} = F_{\widetilde{E}}$,

$$F_C = F_{\widetilde{E}} \widetilde{\cap} F_A = \{(0.7/\zeta_1, \{0.5/\sigma_1, 0.6/\sigma_2\}), (0.8/\zeta_3, \{0.2/\sigma_1, 0.3/\sigma_2\}), (0.6/\zeta_4, \{0.7/\sigma_1, 0.3/\sigma_2\})\}$$

and

$$F_D = F_{\widetilde{E}} \widetilde{\cap} F_B = \{(0.5/\zeta_1, \{0.4/\sigma_1, 0.4/\sigma_2\}), (0.3/\zeta_4, \{0.3/\sigma_1, 0.2/\sigma_2\})\}.$$

Thus $\widetilde{\tau}_2 = \{F_\phi, F_{\widetilde{E}}, F_C, F_D\}$ is FPFS-topology on Y. So $\widetilde{\tau}_2$ is an FPFS-subspace of $\widetilde{\tau}_1$.

Remark 3.9. (1) Every FPFS-subspace of a discrete FPFS-topological space is always discrete. Similarly, every subspace of indiscrete FPFS-topological space is indiscrete.

(2) A subspace Z of a subspace Y of a FPFS-topological space X is a FPFS-subspace of X.

Definition 3.10 ([44])**.** Let (X, R) be a FPFS-topological space and $F_A \widetilde{\subseteq} (X, R)$ then FPFS-closure of F_A is written as $\overline{F_A}$ which is the FPFS-intersection of all FPFS-closed supersets of F_A.

Clearly $\overline{F_A}$ is the FPFS-smallest closed superset of F_A.

Example 3.11. Let $X = \{\sigma_1, \sigma_2, \sigma_3\}$ be the set of universe and let $R = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ be the set of attributes. If $A = \{0.6/\zeta_1, 0.4/\zeta_2, 0.3/\zeta_3\} \subseteq R$, $B = \{0.2/\zeta_2, 0.1/\zeta_3\} \subseteq R$ with the FPFS-sets,

$$F_A = \{(0.6/\zeta_1, \{0.6/\sigma_1, 0.3/\sigma_2, 0.5/\sigma_3\}), (0.4/\zeta_2, \{0.1/\sigma_1, 0.2/\sigma_2, 0.3/\sigma_3\}), (0.3/\zeta_3, \{0.6/\sigma_1, 0.7/\sigma_2, 0.3/\sigma_3\})\}$$

and

$$F_B = \{(0.2/\zeta_2, \{0.1/\sigma_1, 0.2/\sigma_2, 0.2/\sigma_3\}), (0.1/\zeta_3, \{0.5/\sigma_1, 0.4/\sigma_2, 0.1/\sigma_3\})\},$$

then $\widetilde{\tau} = \{F_\phi, F_{\widetilde{R}}, F_A, F_B\}$ is a FPFS-topology on X. Thus, the closed sets can be calculated as by taking the compliments of FPFS-open sets in $\widetilde{\tau}$, i.e.,

$(F_\phi)^c = F_{\widetilde{R}}$, $(F_{\widetilde{R}})^c = F_\phi$,
$(F_A)^c = \{(0.4/\zeta_1, \{0.4/\sigma_1, 0.7/\sigma_2, 0.5/\sigma_3\}), (0.6/\zeta_2, \{0.9/\sigma_1, 0.8/\sigma_2, 0.7/\sigma_3\}), (0.7/\zeta_3, \{0.4/\sigma_1, 0.3/\sigma_2, 0.7/\sigma_3\})\}$

and

$(F_B)^c = \{(0.8/\zeta_2, \{0.9/\sigma_1, 0.8/\sigma_2, 0.8/\sigma_3\}), (0.9/\zeta_3, \{0.5/\sigma_1, 0.6/\sigma_2, 0.9/\sigma_3\})\}$.

If $C = \{0.2/\zeta_1, 0.4/\zeta_2\} \subseteq R$, then a FPFS-set on X is,

$F_C = \{(0.2/\zeta_1, \{0.3/\sigma_1, 0.5/\sigma_2, 0.4/\sigma_3\}), (0.4/\zeta_2, \{0.7/\sigma_1, 0.6/\sigma_2, 0.4/\sigma_3\})\}$.

Which implies that FPFS-closed supersets of F_C are $(F_A)^c$ and $F_{\widetilde{R}}$ only. Thus $\overline{F_C} = (F_A)^c \widetilde{\cap} F_{\widetilde{R}} = (F_A)^c$.

Definition 3.12 ([44])**.** Let (X, R) be a FPFS-topological space and $F_A \widetilde{\subseteq} (X, R)$. then the interior of F_A is denoted as F_A^o which is the FPFS-union of all FPFS-open subsets of F_A.

It is clear that F_A^o is the FPFS-largest open subset of F_A.

Example 3.13. Let $X = \{\sigma_1, \sigma_2, \sigma_3\}$ be the set of universe and let $R = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ be the set of attributes. If $A = \{0.6/\zeta_1, 0.4/\zeta_2, 0.3/\zeta_3\} \subseteq R$, $B = \{0.2/\zeta_2, 0.1/\zeta_3\} \subseteq R$ with the FPFS-sets,

$F_A = \{(0.6/\zeta_1, \{0.6/\sigma_1, 0.3/\sigma_2, 0.5/\sigma_3\}), (0.4/\zeta_2, \{0.1/\sigma_1, 0.2/\sigma_2, 0.3/\sigma_3\}), (0.3/\zeta_3, \{0.6/\sigma_1, 0.7/\sigma_2, 0.3/\sigma_3\})\}$

and

$F_B = \{(0.2/\zeta_2, \{0.1/\sigma_1, 0.2/\sigma_2, 0.2/\sigma_3\}), (0.1/\zeta_3, \{0.5/\sigma_1, 0.4/\sigma_2, 0.1/\sigma_3\})\}$,

then $\widetilde{\tau} = \{F_\phi, F_{\widetilde{R}}, F_A, F_B\}$ is a FPFS-topology on X.

If $D = \{0.4/\zeta_2, 0.6/\zeta_3\} \subseteq R$, then FPFS-set on X is,

$F_D = \{(0.4/\zeta_2, \{0.5/\sigma_1, 0.3/\sigma_2, 0.4/\sigma_3\}), (0.6/\zeta_3, \{0.6/\sigma_1, 0.4/\sigma_2, 0.9/\sigma_3\})\}$.

This implies that FPFS-open sets contained in F_D are F_B and F_ϕ. Thus $F_D^o = F_B \widetilde{\cup} F_\phi = F_B$.

Definition 3.14 ([44])**.** A FPFS-set F_A is said to be a FPFS-point, denoted by $\zeta(F_A)$, if $A \subseteq R$ is fuzzy singleton, $(\mu(\zeta)/\zeta) \in A$ and $F(\mu(\zeta)/\zeta) = \gamma_{F_A}^{\zeta}(\sigma)$, where

$\gamma_{F_A}^{\zeta}(\sigma) \neq \widetilde{\phi}$ and $F(\mu(\acute{\zeta})/\acute{\zeta}) = \widetilde{\phi}\ \forall\ (\mu(\acute{\zeta})/\acute{\zeta}) \in R - \{(\mu(\zeta)/\zeta)\}$.

Definition 3.15 ([44])**.** A FPFS-point $\zeta(F_A)$ belongs to a FPFS-set F_B, if $\mu_{F_A}(\zeta) \leq \mu_{F_B}(\zeta), \forall\ \zeta \in R$ and $\gamma_{F_A}^{\zeta}(\sigma) \leq \gamma_{F_B}^{\zeta}(\sigma), \forall\ \sigma \in X$. It is denoted as $F_A \widetilde{\in} F_B$.

Example 3.16. Let $X = \{\sigma_1, \sigma_2, \sigma_3\}$ be the set of universe and let $R = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ be the set of attributes. If $A = \{0.3/\zeta_1, 0.7/\zeta_2, 0.9/\zeta_4\} \subseteq R$, $B = \{0.1/\zeta_1\} \subseteq R$ with the FPFS-sets,

$F_A = \{(0.3/\zeta_1, \{0.1/\sigma_1, 0.5/\sigma_2, 0.3/\sigma_3\}), (0.7/\zeta_2, \{0.1/\sigma_1, 0.2/\sigma_2, 0.5/\sigma_3\}), (0.9/\zeta_4, \{0.4/\sigma_1, 0.3/\sigma_2, 0.6/\sigma_3\})\}$

and

$F_B = \{(0.1/\zeta_1, \{0.1/\sigma_1, 0.3/\sigma_2, 0.2/\sigma_3\}), (0/\zeta_2, \{0/\sigma_1, 0/\sigma_2, 0/\sigma_3\}), (0/\zeta_3, \{0/\sigma_1, 0/\sigma_2, 0/\sigma_3\}), (0/\zeta_4, \{0/\sigma_1, 0/\sigma_2, 0/\sigma_3\})\}$

then F_B is called FPFS-point.

Clearly $\mu_{F_B}(\zeta) \leq \mu_{F_A}(\zeta), \forall \zeta \in R$ and $\gamma_{F_B}^{\zeta}(\sigma) \leq \gamma_{F_A}^{\zeta}(\sigma), \forall\ \sigma \in X$. Thus, $F_B \widetilde{\in} F_A$.

Remark 3.17. Every non-empty FPFS-set F_A can be written as the FPFS-union of all the FPFS-points which are in F_A.

Definition 3.18 ([44]). Let $\zeta(F_{A_1})$ and $F_{A_2} \in FPFS(X, R)$. Then $\zeta(F_{A_1})$ is called FPFS quasi-coincident with F_{A_2}, written as $\zeta(F_{A_1})qF_{A_2}$, if

$$\mu_{F_{A_1}}(\zeta) + \mu_{F_{A_2}}(\zeta) > 1, \zeta \in R \text{ and } \gamma^{\zeta}_{F_{A_1}}(\sigma) + \gamma^{\zeta}_{F_{A_2}}(\sigma) > 1, \text{for some } \sigma \in X.$$

If $\zeta(F_{A_1})$ is not FPFS quasi-coincident with F_{A_2}, then we write $\zeta(F_{A_1})\overline{q}F_{A_2}$.

Example 3.19. Let $X = \{\sigma_1, \sigma_2, \sigma_3\}$ be the set of universe and let $R = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ be the set of attributes. If $A_1 = \{0.6/\zeta_1\} \subseteq R$, $A_2 = \{0.7/\zeta_1, 0.8/\zeta_4\} \subseteq R$ with the FPFS-point and FPFS-set respectively,

$$\zeta(F_{A_1}) = \{(0.6/\zeta_1, \{0.6/\sigma_1, 0.7/\sigma_2, 0.8/\sigma_3\})\}$$

and

$$F_{A_2} = \{(0.7/\zeta_1, \{0.6/\sigma_1, 0.5/\sigma_2, 0.8/\sigma_3\}), (0.8/\zeta_4, \{0.7/\sigma_1, 0.8/\sigma_2, 0.9/\sigma_3\})\}$$

then it is clear that $\zeta(F_{A_1})$ is quasi-coincident with F_{A_2}.
As $\mu_{F_{A_1}}(\zeta_1) + \mu_{F_{A_2}}(\zeta_1) > 1, \zeta_1 \in R$ and $\gamma^{\zeta_1}_{F_{A_1}}(\sigma) + \gamma^{\zeta_1}_{F_{A_2}}(\sigma) > 1, \sigma \in X$.

Definition 3.20 ([44]). Let F_{A_1} and $F_{A_2} \in FPFS(X, R)$. Then F_{A_1} is called FPFS quasi-coincident with F_{A_2}, written as $F_{A_1}qF_{A_2}$, if

$$\mu_{F_{A_1}}(\zeta) + \mu_{F_{A_2}}(\zeta) > 1; \zeta \in A_1\widetilde{\cap}A_2 \text{ and } \gamma^{\zeta}_{F_{A_1}}(\sigma) + \gamma^{\zeta}_{F_{A_2}}(\sigma) > 1; \sigma \in X.$$

If F_{A_1} is not FPFS quasi-coincident with F_{A_2}, then we write $F_{A_1}\overline{q}F_{A_2}$.

Example 3.21. Let $X = \{\sigma_1, \sigma_2, \sigma_3\}$ is a universal set and let $R = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ be the set of parameters. If $A_1 = \{0.6/\zeta_1, 0.7/\zeta_2, 0.8/\zeta_3\} \subseteq R$, $A_2 = \{0.5/\zeta_2, 0.6/\zeta_3, 0.7/\zeta_4\}$ $\subseteq R$ with the FPFS-sets,

$$F_{A_1} = \{(0.6/\zeta_1, \{0.6/\sigma_1, 0.7/\sigma_2, 0.9/\sigma_3\}), (0.7/\zeta_2, \{0.7/\sigma_1, 0.8/\sigma_2, 0.6/\sigma_3\}), (0.8/\zeta_3, \{0.6/\sigma_1, 0.7/\sigma_2, 0.9/\sigma_3\})\}$$

and

$$F_{A_2} = \{(0.5/\zeta_2, \{0.6/\sigma_1, 0.5/\sigma_2, 0.6/\sigma_3\}), (0.6/\zeta_3, \{0.7/\sigma_1, 0.5/\sigma_2, 0.8/\sigma_3\}), (0.7/\zeta_4, \{0.6/\sigma_1, 0.9/\sigma_2, 0.8/\sigma_3\})\},$$

then clearly, $\mu_{F_{A_1}}(\zeta) + \mu_{F_{A_2}}(\zeta) > 1, \zeta \in A_1\widetilde{\cap}A_2$ and $\gamma^{\zeta}_{F_{A_1}}(\sigma) + \gamma^{\zeta}_{F_{A_2}}(\sigma) > 1, \sigma \in X$.
Thus, $F_{A_1}qF_{A_2}$.

Theorem 3.22. [44] *If F_A and F_B are FPFS-sets, then*

(1) $F_A\widetilde{\subseteq}F_B \Leftrightarrow F_A\overline{q}F^c_B$,
(2) $F_AqF_B \Rightarrow F_A\widetilde{\cap}F_B \neq F_\phi$,
(3) $F_A\overline{q}F^c_A$,
(4) $F_AqF_B \Leftrightarrow$ *there exists an* $\zeta(F_C)\widetilde{\in}F_A$ *such that* $\zeta(F_C)qF_B$,
(5) $\zeta(F_C)\widetilde{\in}F^c_A \Leftrightarrow \zeta(F_C)\overline{q}F_A$,
(6) $F_A\widetilde{\subseteq}F_B \Rightarrow$ *if* $\zeta(F_C)qF_A$, *then* $\zeta(F_C)qF_B$ $\forall$ $\zeta(F_C)\widetilde{\in}FPFS(X, R)$.

Theorem 3.23. [44] *Let $\{F_{A_i}\}_{i\in\Omega}$ be a family of FPFS-sets over (X, R). Then a FPFS-point $\zeta(F_B)$ is Q-coincident with $\widetilde{\cup}_iF_{A_i}$ if and only if $\zeta(F_B)qF_{A_i}$ for some $i \in \Omega$.*

4. Main results

In this section we present some results which holds in FPFS-set theory but does not hold in crisp set theory. We introduce frontier and exterior in the context of FPFS-topological space. We define Q-neighborhood, adherence point and accumulation point for FPFS-set.

First we present an illustration to show that $\widetilde{\tau}_1\widetilde{\cup}\widetilde{\tau}_2$ may not be a FPFS-topology on X.

Example 4.1. Let $X = \{\sigma_1, \sigma_2, \sigma_3\}$ be the set of universe and let $R = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ be the set of attributes. If $A = \{0.3/\zeta_1, 0.5/\zeta_3\} \subseteq R$, $B = \{0.4/\zeta_2, 0.7/\zeta_4\} \subseteq R$ with the FPFS-sets,

$F_A = \{(0.3/\zeta_1, \{0.3/\sigma_1, 0.5/\sigma_2, 0.6/\sigma_3\}), (0.5/\zeta_3, \{0.6/\sigma_1, 0.3/\sigma_2, 0.2/\sigma_3\})\}$

and

$F_B = \{(0.4/\zeta_2, \{0.1/\sigma_1, 0.2/\sigma_2, 0.5/\sigma_3\}), (0.7/\zeta_4, \{0.3/\sigma_1, 0.2/\sigma_2, 0.1/\sigma_3\})\}$,

then $\widetilde{\tau}_1 = \{F_\phi, F_{\widetilde{R}}, F_A, \}$ and $\widetilde{\tau}_2 = \{F_\phi, F_{\widetilde{R}}, F_B\}$ are two FPFS-topologies on X.

On the other hand, since $F_A, F_B \widetilde{\in} \widetilde{\tau}_1\widetilde{\cup}\widetilde{\tau}_2$ but $F_A\widetilde{\cup}F_B, F_A\widetilde{\cap}F_B \widetilde{\notin} \widetilde{\tau}_1\widetilde{\cup}\widetilde{\tau}_2$, $\widetilde{\tau}_1\widetilde{\cup}\widetilde{\tau}_2 = \{F_\phi, F_{\widetilde{R}}, F_A, F_B\}$ is not a FPFS-topology. But $\widetilde{\tau}_1\widetilde{\cap}\widetilde{\tau}_2 = \{F_\phi, F_{\widetilde{R}}\}$ is a FPFS-topology on X.

Proposition 4.2. *Let $\widetilde{\tau}_1$ and $\widetilde{\tau}_2$ be two FPFS-topologies on X. Then $\widetilde{\tau}_1\widetilde{\cap}\widetilde{\tau}_2$ is a FPFS-topology on (X, R) but $\widetilde{\tau}_1\widetilde{\cup}\widetilde{\tau}_2$ is not necessarily a FPFS-topology on (X, R).*

Proposition 4.3. *If $\{\widetilde{\tau}_\alpha : \alpha \in \Omega\}$ is a family of FPFS-topologies on X, then $\widetilde{\bigcap}_{\alpha\in\Omega}\{\widetilde{\tau}_\alpha : \alpha \in \Omega\}$ is also a FPFS-topology on X.*

Remark 4.4. The members of discrete FPFS-topology are infinite due to infinite subsets of a FPFS-set.

Remark 4.5. In FPFS-set theory the law of contradiction $F_A\widetilde{\cap}F^c_A = F_\phi$ and the law of excluded middle $F_A\widetilde{\cup}F^c_A = F_{\widetilde{R}}$ does not hold in general. Then the collection of FPFS-sets $\{F_\phi, F_{\widetilde{R}}, F_A, F^c_A\}$ is not a FPFS-topology on X.

Example 4.6. Let $X = \{\sigma_1, \sigma_2, \sigma_3\}$ be the set of universe and let $R = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ be the set of attributes. If $A = \{0.4/\zeta_1, 0.7/\zeta_2\} \subseteq R$, then

$F_A = \{(0.4/\zeta_1, \{0.3/\sigma_1, 0.7/\sigma_2, 0.9/\sigma_3\}), (0.7/\zeta_2, \{0.1/\sigma_1, 0.3/\sigma_2, 0.8/\sigma_3\})\}$

and

$F^c_A = \{(0.6/\zeta_1, \{0.7/\sigma_1, 0.3/\sigma_2, 0.1/\sigma_3\}), (0.3/\zeta_2, \{0.9/\sigma_1, 0.7/\sigma_2, 0.2/\sigma_3\})\}$.

Clearly $\widetilde{\tau} = \{F_\phi, F_{\widetilde{R}}, F_A, F^c_A\}$ is not a FPFS-topology on X, because $F_A\widetilde{\cap}F^c_A \notin \widetilde{\tau}$ and $F_A\widetilde{\cup}F^c_A \notin \widetilde{\tau}$.

Definition 4.7. Let F_A be a FPFS-subset of FPFS-topological space (X, R). Then the frontier or boundary of F_A, denoted as $F_r(F_A)$ and is defined as

$$F_r(F_A) = \overline{F_A}\widetilde{\cap}\overline{F^c_A}.$$

Example 4.8. Let $X = \{\sigma_1, \sigma_2, \sigma_3\}$ be the set of universe and let $R = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ be the set of attributes. If $A = \{0.6/\zeta_1, 0.4/\zeta_2, 0.3/\zeta_3\} \subseteq R$, $B = \{0.2/\zeta_2, 0.1/\zeta_3\} \subseteq R$ with the FPFS-sets,

$F_A = \{(0.6/\zeta_1, \{0.6/\sigma_1, 0.3/\sigma_2, 0.5/\sigma_3\}), (0.4/\zeta_2, \{0.1/\sigma_1, 0.2/\sigma_2, 0.3/\sigma_3\}),$
$(0.3/\zeta_3, \{0.6/\sigma_1, 0.7/\sigma_2, 0.3/\sigma_3\})\}$

and

$F_B = \{(0.2/\zeta_2, \{0.1/\sigma_1, 0.2/\sigma_2, 0.2/\sigma_3\}), (0.1/\zeta_3, \{0.5/\sigma_1, 0.4/\sigma_2, 0.1/\sigma_3\})\}$,

then $\widetilde{\tau} = \{F_\phi, F_{\widetilde{R}}, F_A, F_B\}$ is a FPFS-topology on X. Thus, the closed sets can be calculated as by taking the compliments of FPFS-open sets in $\widetilde{\tau}$, i.e.,

$(F_\phi)^c = F_{\widetilde{R}}$, $(F_{\widetilde{R}})^c = F_\phi$,

$(F_A)^c = \{(0.4/\zeta_1, \{0.4/\sigma_1, 0.7/\sigma_2, 0.5/\sigma_3\}), (0.6/\zeta_2, \{0.9/\sigma_1, 0.8/\sigma_2, 0.7/\sigma_3\}),$

$(0.7/\zeta_3, \{0.4/\sigma_1, 0.3/\sigma_2, 0.7/\sigma_3\})\}$

and

$(F_B)^c = \{(0.8/\zeta_2, \{0.9/\sigma_1, 0.8/\sigma_2, 0.8/\sigma_3\}), (0.9/\zeta_3, \{0.5/\sigma_1, 0.6/\sigma_2, 0.9/\sigma_3\})\}$

If $C = \{0.2/\zeta_1, 0.4/\zeta_2\} \subseteq R$, then a FPFS-set on X is,

$F_C = \{(0.2/\zeta_1, \{0.3/\sigma_1, 0.5/\sigma_2, 0.4/\sigma_3\}), (0.4/\zeta_2, \{0.7/\sigma_1, 0.6/\sigma_2, 0.4/\sigma_3\})\}$.

This shows that the FPFS-closed supersets of F_C are $(F_A)^c$ and $F_{\widetilde{R}}$ only. Thus $\overline{F_C} = (F_A)^c \widetilde{\cap} F_{\widetilde{R}} = (F_A)^c$. On the other hand,

$F_C^c = \{(0.8/\zeta_1, \{0.7/\sigma_1, 0.5/\sigma_2, 0.6/\sigma_3\}), (0.6/\zeta_2, \{0.3/\sigma_1, 0.4/\sigma_2, 0.6/\sigma_3\})\}$.

So $\overline{F_C^c} = F_{\widetilde{R}}$. Hence we obtain $F_r(F_C) = \overline{F_C} \widetilde{\cap} \overline{F_C^c} = (F_A)^c \widetilde{\cap} F_{\widetilde{R}} = (F_A)^c$.

Definition 4.9. Let F_A be a subset of FPFS-topological space (X, R). Then the exterior of F_A, denoted as $Ext(F_A)$ is defined by $Ext(F_A) = (\overline{F_A})^c$.

Example 4.10. Let $X = \{\sigma_1, \sigma_2, \sigma_3\}$ be the set of universe and let $R = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ be the set of attributes. If $A = \{0.6/\zeta_1, 0.4/\zeta_2, 0.3/\zeta_3\} \subseteq R$, $B = \{0.2/\zeta_2, 0.1/\zeta_3\} \subseteq R$ with the FPFS-sets,

$F_A = \{(0.6/\zeta_1, \{0.6/\sigma_1, 0.3/\sigma_2, 0.5/\sigma_3\}), (0.4/\zeta_2, \{0.1/\sigma_1, 0.2/\sigma_2, 0.3/\sigma_3\}),$
$(0.3/\zeta_3, \{0.6/\sigma_1, 0.7/\sigma_2, 0.3/\sigma_3\})\}$

and

$F_B = \{(0.2/\zeta_2, \{0.1/\sigma_1, 0.2/\sigma_2, 0.2/\sigma_3\}), (0.1/\zeta_3, \{0.5/\sigma_1, 0.4/\sigma_2, 0.1/\sigma_3\})\}$,

then $\widetilde{\tau} = \{F_\phi, F_{\widetilde{R}}, F_A, F_B\}$ is a FPFS-topology on X.

On the other hand, the closed FPFS-sets are,

$(F_\phi)^c = F_{\widetilde{R}}$, $(F_{\widetilde{R}})^c = F_\phi$,

$(F_A)^c = \{(0.4/\zeta_1, \{0.4/\sigma_1, 0.7/\sigma_2, 0.5/\sigma_3\}), (0.6/\zeta_2, \{0.9/\sigma_1, 0.8/\sigma_2, 0.7/\sigma_3\}),$
$(0.7/\zeta_3, \{0.4/\sigma_1, 0.3/\sigma_2, 0.7/\sigma_3\})\}$

and

$(F_B)^c = \{(0.8/\zeta_2, \{0.9/\sigma_1, 0.8/\sigma_2, 0.8/\sigma_3\}), (0.9/\zeta_3, \{0.5/\sigma_1, 0.6/\sigma_2, 0.9/\sigma_3\})\}$.

If $D = \{0.4/\zeta_2, 0.6/\zeta_3\} \subseteq R$, then FPFS-set on X is,

$F_D = \{(0.4/\zeta_2, \{0.5/\sigma_1, 0.3/\sigma_2, 0.4/\sigma_3\}), (0.6/\zeta_3, \{0.6/\sigma_1, 0.4/\sigma_2, 0.9/\sigma_3\})\}$.

This shows that FPFS-open sets contained in F_D are F_B and F_ϕ. Then $F_D^o = F_B \widetilde{\cup} F_\phi = F_B$. Thus $\overline{F_D} = F_{\widetilde{R}}$. So $F_r(F_D) = \overline{F_D} \widetilde{\cap} \overline{F_D^c}$ and thus $\overline{F_D^c} = F_B^c$. Hence $F_r(F_D) = F_{\widetilde{R}} \widetilde{\cap} F_B^c = F_B^c$. Therefore $Ext(F_D) = (\overline{F_D})^c = F_\phi$.

Remark 4.11. We present some results about closure, interior, frontier and exterior of a FPFS-set and show investigate some results which hold in crisp set theory but do not hold in FPFS-set theory with the help of some examples. Since the law of contradiction and the law of Excluded middle does not hold in FPFS-set theory. This leads the following theorem.

Theorem 4.12. *If F_A, F_B, F_C, F_D are FPFS-sets, then*

(1) $(F_A^o)^c = \overline{(F_A^c)}$,
(2) $(\overline{F_A})^c = (F_A^c)^o$,
(3) $(F_A)^o \neq F_A - \overline{F_A^c}$,
(4) $Ext(F_A^c) = F_A^o$,
(5) $Ext(F_A) = (F_A^c)^o$,
(6) $Ext(F_A) \widetilde{\cup} F_r(F_A) \widetilde{\cup} F_A^o \neq F_{\widetilde{R}}$,
(7) $F_r(F_A) = F_r(F_A^c)$,
(8) $F_A^o \widetilde{\cap} F_r(F_A) \neq F_\phi$,

(9) $\overline{F_A} \neq F_A \widetilde{\cup} F_r(F_A)$,
(10) $\overline{F_A} \neq F_A^o \widetilde{\cup} F_r(F_A)$.

Proof. By Example 4.10, we observe that $Ext(F_D) = F_\phi$, $F_r(F_D) = F_B^c$, $F_D^o = F_B$ and $\overline{F_D} = F_{\widetilde{R}}$.

(1) and (2) hold by [44].

(3) $(F_A)^o \neq F_A - \overline{F_A^c}$, because $F_A - F_B \neq F_A \widetilde{\cap} F_B^c$.

(4) Clearly, $Ext(F_A^c) = (\overline{F_A^c})^c$. Then $Ext(F_A^c) = [(F_A^c)^c]^o$. Thus $Ext(F_A^c) = F_A^o$.

(5) Clearly, $Ext(F_A) = (\overline{F_A})^c$. Then $Ext(F_A) = (F_A^c)^o$.

(6) Clearly, $Ext(F_A) \widetilde{\cup} F_r(F_A) \widetilde{\cup} F_A^o \neq F_{\widetilde{R}}$. Then by Example 4.10, we observe that $F_\phi \widetilde{\cup} F_B^c \widetilde{\cup} F_B \neq F_{\widetilde{R}}$.

(7) Clearly, $F_r(F_A^c) = \overline{(F_A^c)} \widetilde{\cap} \overline{[(F_A^c)]^c}$. Then $F_r(F_A^c) = \overline{(F_A^c)} \widetilde{\cap} \overline{(F_A)} = F_r(F_A)$.

(8) Clearly, $F_A^o \widetilde{\cap} F_r(F_A) \neq F_\phi$. Then by Example 4.10, we observe that $F_B^c \widetilde{\cap} F_B \neq F_\phi$.

(9) Clearly, $\overline{F_A} \neq F_A \widetilde{\cup} F_r(F_A)$. Then by Example 4.10, we see that $F_{\widetilde{R}} \neq F_D \widetilde{\cup} F_B^c$.

(10) Clearly, $\overline{F_A} \neq F_A^o \widetilde{\cup} F_r(F_A)$. Then by Example 4.10, we see that $F_{\widetilde{R}} \neq F_B \widetilde{\cup} F_B^c$. □

In [6, 35] Sanjay and Borah introduced the idea of fuzzy soft point and FS quasi-coincident with Q-neighborhood. The concept of FPFS-point and quasi-neighborhood was introduced by Idris in [44]. We extend these concepts in FPFS-set theory and prove some important results for it.

Remark 4.13. Every non-empty FPFS-set F_A can be written as the FPFS-union of all the FPFS-points which are in F_A.

Definition 4.14. A FPFS-set F_{A_1} is called Q-neighborhood of $\zeta(F_{A_2})$, if there exists $F_B \widetilde{\in} \tau$ such that $\zeta(F_{A_2}) q F_B$ and $F_B \widetilde{\subseteq} F_{A_1}$.

Example 4.15. Let $X = \{\sigma_1, \sigma_2, \sigma_3\}$ be the set of universe and let $R = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ be the set of attributes. If $B_1 = \{0.6/\zeta_1, 0.7/\zeta_2, 0.8/\zeta_4\} \subseteq R$, $B_2 = \{0.5/\zeta_1, 0.6/\zeta_4\} \subseteq R$ with the FPFS-sets,

$$F_{B_1} = \{(0.6/\zeta_1, \{0.7/\sigma_1, 0.9/\sigma_2, 0.9/\sigma_3\}), (0.7/\zeta_2, \{0.6/\sigma_1, 0.9/\sigma_2, 0.8/\sigma_3\}), (0.8/\zeta_4, \{0.7/\sigma_1, 0.8/\sigma_2, 0.9/\sigma_3\})\}$$

and

$$F_{B_2} = \{(0.5/\zeta_1, \{0.6/\sigma_1, 0.9/\sigma_2, 0.7/\sigma_3\}), (0.6/\zeta_4, \{0.7/\sigma_1, 0.8/\sigma_2, 0.8/\sigma_3\})\},$$

Then $\tau = \{F_\phi, F_{\widetilde{R}}, F_{B_1}, F_{B_2}\}$ is an FPFS-topology on X.

If $\zeta(F_{A_2}) = \{(0.6/\zeta_1, \{0.7/\sigma_1, 0.8/\sigma_2, 0.6/\sigma_3\})\}$ is a FPFS-point, then it is clear that $\zeta(F_{A_2}) q F_{B_2}$ $and F_{B_2} \in \tau$. Thus $\mu_{F_{A_2}}(\zeta) + \mu_{F_{B_2}}(\zeta) > 1, \zeta \in R$ and $\gamma^{\zeta}_{F_{A_2}}(\sigma) + \gamma^{\zeta}_{F_{B_2}}(\sigma) > 1, \sigma \in X$.

If $F_{A_1} = \{(0.7/\zeta_1, \{0.8/\sigma_1, 0.9/\sigma_2, 0.7/\sigma_3\}), (0.7/\zeta_4, \{0.7/\sigma_1, 0.9/\sigma_2, 0.8/\sigma_3\}), (0.6/\zeta_3, \{0.6/\sigma_1, 0.5/\sigma_2, 0.7/\sigma_3\})\}$, then $F_{B_2} \widetilde{\subseteq} F_{A_1}$ and $\zeta(F_{A_2}) q F_{B_2}$. Thus by definition, F_{A_1} is quasi-neighborhood of $\zeta(F_{A_2})$.

Theorem 4.16. *$F_B \widetilde{\subseteq} F_{A_1}$ if and only if F_B and $F_{A_1}^c$ are not quasi-coincident. In particular, $\zeta(F_{A_2}) \widetilde{\in} F_B$ if and only if $\zeta(F_{A_2})$ is not quasi-coincident with F_B^c.*

Proof. This follow from the fact:

$$F_B\widetilde{\subseteq}F_{A_1} \Leftrightarrow \mu_B(\zeta) \leq \mu_{A_1}(\zeta), \zeta \in R \text{ and } \gamma^{\zeta}_B(\sigma) \leq \gamma^{\zeta}_{A_1}(\sigma), \sigma \in X$$
$$\Leftrightarrow \mu_B(\zeta) + \mu_{A_1^c}(\zeta) = \mu_B(\zeta) + 1 - \mu_{A_1}(\zeta) \leq 1.$$

Then $\gamma^{\zeta}_B(\sigma) + \gamma^{\zeta}_{A_1^c}(\sigma) = \gamma^{\zeta}_B(\sigma) + 1 - \gamma^{\zeta}_{A_1}(\sigma) \leq 1$, for $\zeta \in R$ and $\sigma \in X$.

Thus $F_B\widetilde{\subseteq}F_{A_1}$. So $F_B\overline{q}F^c_{A_1}$.

Similarly, $\zeta(F_{A_2})\widetilde{\in}F_B$. Then $(F_{A_2})\overline{q}F^c_B$. □

Theorem 4.17. *Let U_ζ be the collection of FPFS Q-neighborhoods of a FPFS point $\zeta(F_A)$ in a FPFS-topological space τ.*

(1) *If $F_B\widetilde{\in}U_\zeta$, then $\zeta(F_A)$ is quasi-coincident with F_B.*

(2) *If $F_{B_1}\widetilde{\in}U_\zeta$ and $F_{B_1}\widetilde{\subseteq}F_{B_2}$, then $F_{B_2}\widetilde{\in}U_\zeta$.*

(3) *If $F_{B_1}\widetilde{\in}U_\zeta$, then there exists $F_{B_2}\widetilde{\in}U_\zeta$ such that $F_{B_2}\widetilde{\subseteq}F_{B_3}$ and $F_{B_3}\widetilde{\in}U_d$ for every FPFS-point $d(F_A)$ which is quasi-coincident with F_{B_2}.*

Proof. (1) Suppose that $F_B\widetilde{\in}U_\zeta$. Then by definition, there exists $I_C\widetilde{\in}\tau$ such that
$$\zeta(F_A)qI_C \text{ and } I_C\widetilde{\subseteq}F_B.$$
Thus $\mu_{F_A}(\zeta) + \mu_{I_C}(\zeta) > 1, \zeta \in R$ and $\gamma^{\zeta}_{F_A}(\sigma) + \gamma^{\zeta}_{I_C}(\sigma) > 1, \sigma \in X$.
Again $\mu_{I_C}(\zeta) \leq \mu_{F_B}(\zeta), \zeta \in R$ and $\gamma^{\zeta}_{I_C}(\sigma) \leq \gamma^{\zeta}_{F_B}(\sigma), \sigma \in X$
So,
$$\mu_{F_A}(\zeta) + \mu_{F_B}(\zeta) \geq \mu_{F_A}(\zeta) + \mu_{I_C}(\zeta) > 1\zeta \in R$$
and
$$\gamma^{\zeta}_{F_A}(\sigma) + \gamma^{\zeta}_{F_B}(\sigma) \geq \gamma^{\zeta}_{F_A}(\sigma) + \gamma^{\zeta}_{I_C}(\sigma) > 1, \sigma \in X.$$
This shows that $\zeta(F_A)$ is quasi-coincident with F_B.

(2) Suppose that $F_{B_1}\widetilde{\in}U_\zeta$. Then by definition, there exists $I_C\widetilde{\in}\tau$ such that
$$\zeta(F_A)qI_C \text{ and } I_C\widetilde{\subseteq}F_{B_1}.$$
Thus $\mu_{F_A}(\zeta) + \mu_{I_C}(\zeta) > 1, \zeta \in R$ and $\gamma^{\zeta}_{F_A}(\sigma) + \gamma^{\zeta}_{I_C}(\sigma) > 1, \sigma \in X$.
Again $\mu_{I_C}(\zeta) \leq \mu_{F_{B_1}}(\zeta), \zeta \in R$ and $\gamma^{\zeta}_{I_C}(\sigma) \leq \gamma^{\zeta}_{F_{B_1}}(\sigma), \sigma \in X$
Given is that $F_{B_1}\widetilde{\subseteq}F_{B_2}$. Then by definition of FPFS-subset,
$$\mu_{B_1}(\zeta) \leq \mu_{B_2}(\zeta), \zeta \in R \tag{4.1}$$
and
$$\gamma^{\zeta}_{B_1}(\sigma) \leq \gamma^{\zeta}_{B_2}(\sigma), \sigma \in X. \tag{4.2}$$
Since $\zeta(F_A)qI_C$, now only we have to show that $I_C\widetilde{\subseteq}F_{B_2}$. Since
$$\mu_{I_C}(\zeta) \leq \mu_{F_{B_1}}(\zeta), \zeta \in R \tag{4.3}$$
and
$$\gamma^{\zeta}_{I_C}(\sigma) \leq \gamma^{\zeta}_{F_{B_1}}(\sigma), \sigma \in X. \tag{4.4}$$
Comparing (4.1),4.2),(4.4) and (4.4),
$$\mu_{I_C}(\zeta) \leq \mu_{F_{B_1}}(\zeta) \leq \mu_{B_2}(\zeta), \zeta \in R$$
and
$$\gamma^{\zeta}_{I_C}(\sigma) \leq \gamma^{\zeta}_{F_{B_1}}(\sigma) \leq \gamma^{\zeta}_{B_2}(\sigma), \sigma \in X.$$
Thus, $F_{B_2}\widetilde{\in}U_\zeta$.

(3) Suppose that $F_{B_1}\widetilde{\in}U_\zeta$. Then, there exists $F_{B_2}\widetilde{\in}\tau$ such that $\zeta(F_A)qF_{B_2}$ and $F_{B_2}\widetilde{\subseteq}F_{B_1}$. Thus , there exists $F_{B_2}\widetilde{\in}U_\zeta$ such that $\zeta(F_A)qF_{B_2}$ and $F_{B_2}\widetilde{\subseteq}F_{B_1}$.

Let $d(F_A)$ be any FPFS-point which is Q-coincident with F_{B_2}. Then $F_{B_2}\widetilde{\in}U_d$. □

Theorem 4.18. *Intersection of two Q-neighborhoods of FPFS-point $\zeta(F_A)$ is a Q-neighborhood.*

Proof. Let F_{A_1} and F_{A_2} be two Q-neighborhoods of a FPFS-point $\zeta(F_A)$. Then by definition for F_{A_1}, there exists some $I_{C_1}\widetilde{\in}\tau$ such that $\zeta(F_A)qI_{C_1}$ and $I_{C_1}\widetilde{\subseteq}F_{A_1}$. Thus
$\mu_{F_A}(\zeta)+\mu_{I_{C_1}}(\zeta)>1;\zeta\in R,$
$\gamma^{\zeta}_{F_A}(\sigma)+\gamma^{\zeta}_{I_{C_1}}(\sigma)>1;\sigma\in X$
and
$\mu_{I_{C_1}}(\zeta)\leq\mu_{F_{A_1}}(\zeta);\zeta\in R,$
$\gamma^{\zeta}_{I_{C_1}}(\sigma)\leq\gamma^{\zeta}_{F_{A_1}}(\sigma);\sigma\in X.$

Similarly, for Q-neighborhood F_{A_2}, there exists some $I_{C_2}\widetilde{\in}\tau$ such that $\zeta(F_A)qI_{C_2}$ and $I_{C_2}\widetilde{\subseteq}F_{A_2}$. Then
$\mu_{F_A}(\zeta)+\mu_{I_{C_2}}(\zeta)>1;\zeta\in R,$
$\gamma^{\zeta}_{F_A}(\sigma)+\gamma^{\zeta}_{I_{C_2}}(\sigma)>1;\sigma\in X$
and
$\mu_{I_{C_2}}(\zeta)\leq\mu_{F_{A_2}}(\zeta);\zeta\in R,$
$\gamma^{\zeta}_{I_{C_2}}(\sigma)\leq\gamma^{\zeta}_{F_{A_2}}(\sigma);\sigma\in X.$
Since F_{A_1} and F_{A_2} both are FPFS-sets, their intersection is also an FPFS-set.

Suppose $F_{A_1}\widetilde{\cap}F_{A_2}=F_{A_3}$. Then

$$\mu_{F_{A_3}}(\zeta)=min\{\mu_{F_{A_1}}(\zeta),\mu_{F_{A_2}}(\zeta)\}$$

and

$$\gamma^{\zeta}_{F_{A_3}}(\sigma)=min\{\gamma^{\zeta}_{F_{A_1}}(\sigma),\gamma^{\zeta}_{F_{A_2}}(\sigma)\}.$$

Since $I_{C_1}\widetilde{\subseteq}F_{A_1}$ and $I_{C_2}\widetilde{\subseteq}F_{A_2}$, $I_{C_1}\widetilde{\cap}I_{C_2}\widetilde{\subseteq}F_{A_1}\widetilde{\cap}F_{A_2}$.

If $I_{C_1}\widetilde{\cap}I_{C_2}=I_{C_3}$, then $I_{C_3}\widetilde{\subseteq}F_{A_3}$. Thus

$$\mu_{I_{C_3}}(\zeta)\leq\mu_{F_{A_3}}(\zeta),\zeta\in R$$

and

$$\gamma^{\zeta}_{I_{C_3}}(\sigma)\leq\gamma^{\zeta}_{F_{A_3}}(\sigma),\sigma\in X.$$

Since $\zeta(F_A)qI_{C_1}$ and $\zeta(F_A)qI_{C_2}$, $\zeta(F_A)q[I_{C_1}\cap I_{C_2}]=I_{C_3}$, where
$\mu_{I_{C_3}}(\zeta)=min\{\mu_{I_{C_1}}(\zeta),\mu_{I_{C_2}}(\zeta)\},$
$\gamma^{\zeta}_{I_{C_3}}(\sigma)=min\{\gamma^{\zeta}_{I_{C_1}}(\sigma),\gamma^{\zeta}_{I_{C_2}}(\sigma)\}$
and
$\mu_{F_A}(\zeta)+\mu_{I_{C_3}}(\zeta)>1,\zeta\in R,$
$\gamma^{\zeta}_{F_A}(\sigma)+\gamma^{\zeta}_{I_{C_3}}(\sigma)>1,\sigma\in X$
with $I_{C_3}\widetilde{\subseteq}F_{A_3}$. So, F_{A_3} is a Q-neighborhood of $\zeta(F_A)$. □

Theorem 4.19. *A FPFS-point $\zeta(F_B)\widetilde{\in}\overline{F_A}$ if and only if Q-neighborhood of $\zeta(F_B)$ is Q-coincident with F_A.*

Proof. $\zeta(F_B)\widetilde{\in}\overline{F_A}$ if and only if every closed set F_C containing F_A contains $\zeta(F_B)$, i.e., $\zeta(F_B)\widetilde{\in}F_C$. On the other hand,
$\mu_{F_B}(\zeta)\leq\mu_{F_C}(\zeta),\zeta\in R$
and

$\gamma^{\zeta}_{F_B}(\sigma) \leq \gamma^{\zeta}_{F_C}(\sigma), \sigma \in X.$
Then
$\zeta(F_B)\widetilde{\in}\overline{F_A}$ if and only if for all closed sets $F_A\widetilde{\subseteq}F_C$,
$1-\mu_{F_B}(\zeta) \geq 1-\mu_{F_C}(\zeta), \zeta \in R$
and
$1-\gamma^{\zeta}_{F_B}(\sigma) \geq 1-\gamma^{\zeta}_{F_C}(\sigma), \sigma \in X.$
Thus $\zeta(F_B)\widetilde{\in}\overline{F_A}$ if and only if for any FPFS-open set $I_C\widetilde{\subseteq}F^c_A$, we have
$\mu_{I_C}(\zeta) \leq 1-\mu_{F_B}(\zeta), \zeta \in R$
and
$\gamma^{\zeta}_{I_C}(\sigma) \leq 1-\gamma^{\zeta}_{F_B}(\sigma), \sigma \in X.$
In other words, for every FPFS-open set I_C satisfying
$\mu_{I_C}(\zeta) > 1-\mu_{F_B}(\zeta), \zeta \in R$
and
$\gamma^{\zeta}_{I_C}(\sigma) > 1-\gamma^{\zeta}_{F_B}(\sigma), \sigma \in X,$
I_C is not contained in F^c_A.
Again I_C is not contained in F^c_A if and only if I_C is Q-coincident with F_A. So we proved that $\zeta(F_B)\widetilde{\in}\overline{F_A}$ if and only if every open Q-neighborhood I_C of $\zeta(F_B)$, which is evidently equivalent to what we want to prove. □

Definition 4.20. A FPFS-point $\zeta(F_A)$ is called an adherence point of FPFS-set F_B, if every FPFS Q-neighborhood of $\zeta(F_A)$ is a Q-coincident with F_B.

Theorem 4.21. *Every FPFS-point of F_A is an adherence point of F_A.*

Proof. Let $\zeta(F_B)$ be an arbitrary FPFS-point of F_A. Then $\zeta(F_B)\widetilde{\in}F_A$. Thus

$$\mu_{F_B}(\zeta) \leq \mu_{F_A}(\zeta), \zeta \in R \tag{4.5}$$

and

$$\gamma^{\zeta}_{F_B}(\sigma) \leq \gamma^{\zeta}_{F_A}(\sigma), \sigma \in X. \tag{4.6}$$

Suppose that F_C be a Q-neighborhood of $\zeta(F_B)$. Then by definition, there exists $F_D\widetilde{\in}\tau$ such that $\zeta(F_B)qF_D$ and $F_D\widetilde{\subseteq}F_C$. Thus

$$\mu_{F_B}(\zeta) + \mu_{F_D}(\zeta) > 1, \zeta \in R, \tag{4.7}$$

$$\gamma^{\zeta}_{F_B}(\sigma) + \gamma^{\zeta}_{F_D}(\sigma) > 1, \sigma \in X \tag{4.8}$$

and

$$\mu_{F_D}(\zeta) \leq \mu_{F_C}(\zeta), \zeta \in R, \tag{4.9}$$

$$\gamma^{\zeta}_{F_D}(\sigma) \leq \gamma^{\zeta}_{F_C}(\sigma), \sigma \in X. \tag{4.10}$$

Adding (4.5),(4.6) and (4.9),(4.10) using (4.7),(4.8), we get
$\mu_{F_C}(\zeta) + \mu_{F_A}(\zeta) \geq \mu_{F_D}(\zeta) + \mu_{F_B}(\zeta) > 1; \zeta \in R$
and
$\gamma^{\zeta}_{F_C}(\sigma) + \gamma^{\zeta}_{F_A}(\sigma) \geq \gamma^{\zeta}_{F_D}(\sigma) + \gamma^{\zeta}_{F_B}(\sigma) > 1; \sigma \in X.$
So F_CqF_A. Hence, F_C being a Q-neighborhood of $\zeta(F_B)$ is Q-coincident with F_A. Therefore $\zeta(F_B)$ is adherence point of F_A. □

Definition 4.22. A FPFS-point $\zeta(F_A)$ is called limit point of a FPFS-set F_B, if $\zeta(F_A)$ is an adherence point of F_B, and every FPFS Q-neighborhood of $\zeta(F_A)$ and F_B are Q-coincident at some FPFS-point different from ζ, and $\zeta(F_A)\widetilde{\in}F_B$.

The FPFS-union of all accumulation points of a FPFS-set F_B is called the derived set of F_B denoted as F_B^d.

Theorem 4.23. $\overline{F_A} = F_A\widetilde{\cup}F_A^d$.

Proof. Let $\Omega = \{\zeta(F_B)$ is an adherent point of$F_A\}$. Then by theorem
"A FPFS-point $\zeta(F_B)\widetilde{\in}\overline{F_A}$ if and only if Q-neighborhood of $\zeta(F_B)$ is Q-coincident with F_A",
$\overline{F_A} = \widetilde{\cup}\Omega$. Thus $\zeta(F_B)\widetilde{\in}\Omega$ if and only if either $\zeta(F_B)\widetilde{\in}F_A$ or $\zeta(F_B)\widetilde{\in}F_A^d$. So $\overline{F_A} = \widetilde{\cup}\Omega = F_A\widetilde{\cup}F_A^d$. □

Corollary 4.24. *A FPFS-set F_A is closed if and only if F_A contains all of its accumulation points.*

Proof. Let F_A be a FPFS-set. Then by Theorem 4.23, $\overline{F_A} = F_A\widetilde{\cup}F_A^d$.
Thus F_A is closed
$\Leftrightarrow \overline{F_A} = F_A$
$\Leftrightarrow \overline{F_A} = F_A\widetilde{\cup}F_A^d = F_A$
$\Leftrightarrow F_A^d\widetilde{\subseteq}F_A$
$\Leftrightarrow F_A$ contains all of its accumulation points. □

5. Applications of FPFS-topology to decision-making

Example 5.1. Assume that a committee wants to fill a position for scholarship. There are seven candidates which form the set of alternatives,
$X = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7\}$, where
σ_1 = Ahmed Ali,
σ_2 = Hafsa,
σ_3 = Asma,
σ_4 = Mohsin,
σ_5 = Saniya,
σ_6 = Haniya,
σ_7 = Fatima.
The panel (committee) consider the set of attributes $R = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}$. The parameters $\zeta_i (i = 1, 2, 3, 4, 5)$ stands for,
ζ_1 = needy,
ζ_2 = intelligent,
ζ_3 = best result percentage,
ζ_4 = intrusted in higher education,
ζ_5 = hard working.
The panel consists of two members A and B after some discussion each applicant is evaluated from point of view of the goals and the constraint according to the chosen subsets by member-1 and member-2 respectively
$A = \{0.6/\zeta_3, 0.8/\zeta_4, 0.7/\zeta_5\}$ and $B = \{0.3/\zeta_3, 0.6/\zeta_4\}$ of R.
We here use the algorithm for FPFS-sets which is used by Cagman in [7].

Step 1: After a discussion the members of committee construct FPFS-sets F_A and F_B over X given by
F_A={$(0.6/\zeta_3, \{0.3/\sigma_2, 0.4/\sigma_3, 0.7/\sigma_4, 0.2/\sigma_6\}), (0.8/\zeta_4, \{0.3/\sigma_3, 0.5/\sigma_5, 0.7/\sigma_6, 0.9/\sigma_7\}),$
$(0.7/\zeta_5, \{0.1/\sigma_1, 0.3/\sigma_2, 0.7/\sigma_3, 0.6/\sigma_4, 1/\sigma_5\})\}$ and
F_B={$(0.3/\zeta_3, \{0.2/\sigma_2, 0.3/\sigma_3, 0.5/\sigma_4, 0.2/\sigma_6\}), (0.6/\zeta_4, \{0.2/\sigma_3, 0.4/\sigma_5, 0.4/\sigma_7\})\}$.
In tabular form, the FPFS-set F_A can be written as

X	$0.6/\zeta_3$	$0.8/\zeta_4$	$0.7/\zeta_5$
σ_1	0	0	0.1
σ_2	0.3	0	0.3
σ_3	0.4	0.3	0.7
σ_4	0.7	0	0.6
σ_5	0	0.5	1
σ_6	0.2	0.7	0
σ_7	0	0.9	0

In tabular form, the FPFS-set F_B can be written as,

X	$0.3/\zeta_3$	$0.4/\zeta_4$
σ_1	0	0
σ_2	0.2	0
σ_3	0.3	0.2
σ_4	0.5	0
σ_5	0	0.4
σ_6	0.2	0
σ_7	0	0.4

Step 2: Now we make here a FPFS-topology as

$$\widetilde{\tau} = \{F_\phi, F_{\widetilde{R}}, F_A, F_B\},$$

where F_ϕ and $F_{\widetilde{R}}$ are FPFS-empty and FPFS-absolute sets, respectively.

Step 3: Now we find the aggregate fuzzy set by using the formula,

$$F_{A^*} = \{\mu_{F_{A^*}}(\sigma)/\sigma : \sigma \in X\},$$

where

$$\mu_{F_{A^*}}(\sigma) = \Sigma_{\zeta\in R}\ \mu_A(\zeta)\ \gamma_A(\sigma)/|R|.$$

Then

$$F_{A^*} = \{0.014/\sigma_1, 0.078/\sigma_2.0.194/\sigma_3, 0.168/\sigma_4, 0.220/\sigma_5, 0.136/\sigma_6, 0.144/\sigma_7\}.$$

Similarly, we can also find the aggregate fuzzy set for F_B given as,

$$F_{B^*} = \{0/\sigma_1, 0.012/\sigma_2, 0.042/\sigma_3, 0.03/\sigma_4, 0.048/\sigma_5, 0.012/\sigma_6, 0.048/\sigma_7\}.$$

The aggregate fuzzy set of F_ϕ and $F_{\widetilde{R}}$ given respectively, as

$$F_\phi = \{0/\sigma_1, 0/\sigma_2, 0/\sigma_3, 0/\sigma_4, 0/\sigma_5, 0/\sigma_6, 0/\sigma_7\}$$

and

$$F_{\widetilde{R}} = \{1/\sigma_1, 1/\sigma_2, 1/\sigma_3, 1/\sigma_4, 1/\sigma_5, 1/\sigma_6, 1/\sigma_7\}.$$

Step 4: Now we find the final decision set by adding F_{A^*} and F_{B^*} only because there is no need to add the aggregate fuzzy sets of F_ϕ and $F_{\widetilde{R}}$. Then

$$\mu_{F_{A^*}+F_{B^*}}(\sigma) = \mu_{A^*}(\sigma) + \mu_{B^*}(\sigma) - [\mu_{A^*}(\sigma) * \mu_{B^*}(\sigma)].$$

This shows that

$F_{A^*}+F_{B^*} = \{0.014/\sigma_1, 0.089/\sigma_2, 0.2278/\sigma_3, 0.1929/\sigma_4, 0.2574/\sigma_5, 0.1463/\sigma_6, 0.1850/\sigma_7\}$.

Step 5: Finally the largest membership grate can be chosen by $max\ \mu_{F_{A^*}+F_{B^*}}(\sigma) = 0.2574$. Which shows that the applicant σ_5 has the greatest membership degree, which implies that Saniya is selected for the scholarship.

Example 5.2. We introduce here another algorithm for FPFS-set in decision making problem which is modified form of algorithm for FS-set in [9].

Now we solve the above example by using modified algorithm of FPSS-decision making method.

Step 1: After a discussion the members of committee construct FPFS-sets F_A and F_B over X given by

F_A=$\{(0.6/\zeta_3, \{0.3/\sigma_2, 0.4/\sigma_3, 0.7/\sigma_4, 0.2/\sigma_6\}), (0.8/\zeta_4, \{0.3/\sigma_3, 0.5/\sigma_5, 0.7/\sigma_6, 0.9/\sigma_7\}), (0.7/\zeta_5, \{0.1/\sigma_1, 0.3/\sigma_2, 0.7/\sigma_3, 0.6/\sigma_4, 1/\sigma_5\})\}$ and

F_B=$\{(0.3/\zeta_3, \{0.2/\sigma_2, 0.3/\sigma_3, 0.5/\sigma_4, 0.2/\sigma_6\}), (0.6/\zeta_4, \{0.2/\sigma_3, 0.4/\sigma_5, 0.4/\sigma_7\})\}$.

In tabular form, the FPFS-set F_A can be written as

X	$0.6/\zeta_3$	$0.8/\zeta_4$	$0.7/\zeta_5$
σ_1	0	0	0.1
σ_2	0.3	0	0.3
σ_3	0.4	0.3	0.7
σ_4	0.7	0	0.6
σ_5	0	0.5	1
σ_6	0.2	0.7	0
σ_7	0	0.9	0

In tabular form, the FPFS-set F_B can be written as

X	$0.3/\zeta_3$	$0.4/\zeta_4$
σ_1	0	0
σ_2	0.2	0
σ_3	0.3	0.2
σ_4	0.5	0
σ_5	0	0.4
σ_6	0.2	0
σ_7	0	0.4

Step 2: Now we make here a FPFS-topology as

$$\widetilde{\tau} = \{F_\phi, F_{\widetilde{R}}, F_A, F_B\},$$

where F_ϕ and $F_{\widetilde{R}}$ are FPFS-empty and FPFS-absolute sets respectively.

Step 3: The cardinal is computed by the formula,

$$cF_A = \{\mu_{cF_R(\zeta)}/\zeta : \zeta \in A\},$$

where $\mu_{cF_R(\zeta)} = \Sigma_{\sigma \in X}\ \mu_A(\zeta)\ \gamma_A(\sigma)/|X|$.
Then

$$cF_A = \{0.137/\zeta_3, 0.274/\zeta_4, 0.27/\zeta_5\}.$$

Similarly, the cardinal for F_B is

$$cF_B = \{0.0514/\zeta_3, 0.0857/\zeta_4\}.$$

The cardinal for F_ϕ and $F_{\widetilde{R}}$, respectively as

$$cF_\phi = \{0/\zeta_1, 0/\zeta_2, 0/\zeta_3, 0/\zeta_4, 0/\zeta_5\}$$

and

$$cF_{\widetilde{R}} = \{1/\zeta_1, 1/\zeta_2, 1/\zeta_3, 1/\zeta_4, 1/\zeta_5\}.$$

Step 4: We use here this formula to find the aggregate fuzzy set,

$$|R| * M_{F_A^*} = M_{F_A} * M_{cF_A}^t, \tag{5.1}$$

where M_{F_A}, M_{cF_A} and $M_{F_A^*}$ are representation matrices of F_A, cF_A and F_A^*, respectively. Then we find out the matrix of F_A^* by using (5.1),

$$M_{F_A^*} = 1/5 \begin{bmatrix} 0 & 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0.3 & 0 & 0.3 \\ 0 & 0 & 0.4 & 0.3 & 0.7 \\ 0 & 0 & 0.7 & 0 & 0.6 \\ 0 & 0 & 0 & 0.5 & 1 \\ 0 & 0 & 0.2 & 0.7 & 0 \\ 0 & 0 & 0 & 0.9 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.137 \\ 0.274 \\ 0.27 \end{bmatrix} = \begin{bmatrix} 0.0054 \\ 0.0244 \\ 0.0652 \\ 0.0515 \\ 0.0814 \\ 0.0438 \\ 0.0493 \end{bmatrix}$$

that means,

$$F_A^* = \{0.0054/\sigma_1, 0.0244/\sigma_2, 0.0652/\sigma_3, 0.0515/\sigma_4, 0.0814/\sigma_5, 0.0438/\sigma_6, 0.0493/\sigma_7\}.$$

Similarly, we can find the aggregate for F_B calculated as,

$$M_{F_B^*} = 1/5 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0.3 & 0.2 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.0514 \\ 0.0857 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.00205 \\ 0.00651 \\ 0.00514 \\ 0.00685 \\ 0.00205 \\ 0.00685 \end{bmatrix}$$

that means,

$$F_B^* = \{0/\sigma_1, 0.00205/\sigma_2, 0.00651/\sigma_3, 0.00514/\sigma_4, 0.00685/\sigma_5, 0.00205/\sigma_6, 0.00685/\sigma_7\}.$$

Step 5: Now we find the final decision set by adding F_{A^*} and F_{B^*} only because there is no need to add the aggregate fuzzy sets of F_ϕ and $F_{\widetilde{R}}$.Then

$$\mu_{F_{A^*}+F_{B^*}}(\sigma) = \mu_{A^*}(\sigma) + \mu_{B^*}(\sigma) - [\mu_{A^*}(\sigma) * \mu_{B^*}(\sigma)].$$

This shows that

$$\begin{aligned} &F_{A^*} + F_{B^*} \\ &= \{0.0054/\sigma_1, 0.0263/\sigma_2, 0.0712/\sigma_3, 0.0563/\sigma_4, 0.0876/\sigma_5, 0.0457/\sigma_6, 0.0558/\sigma_7\}. \end{aligned}$$

Step 6: In the last, we choose the greatest degree of membership by

$$max\ \mu_{F_{A^*}+F_{B^*}}(\sigma) = 0.0876.$$

Which shows that the applicant σ_5 has the greatest membership degree, so Saniya is selected for the scholarship.

It is interesting to note that both algorithms used in above two applications yields the same result.

6. Conclusion

In this paper we define FPFS-sets and FPFS-topology with some examples. We study adherence point and accumulation point for FPFS-set which help us to proof some important results. We present an interesting application of FPFS-topology to the decision-making with some algorithms. To make the result better we modify some algorithms which will helpful and beneficial for the researchers in their research work on FS-set, FPFS-set theory and FPFS-topology. We hope that the results investigated in this paper make a significant and technically sound contribution in the field of FPFS-set theory.

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References

[1] M. Akram, Bipolar Fuzzy Soft Lie algebras, Quasigroups and Related Systems 21 (2013) 11–18.
[2] M. Akram, N. O. Al-Shehrie and R. S. Alghamdi, Fuzzy soft K-algebras, Utilitas Mathematica 90 (2013) 307–325.
[3] M. Akram, B. Davvaz and F. Feng, Fuzzy soft Lie algebras, Journal of multivalued valued and soft computing 24 (5-6) (2015) 501–520.
[4] M. Akram and S. Nawaz, Fuzzy soft graphs with applications, Journal of Intelligent and Fuzzy Systems 30 (6) (2016) 3619–3632.
[5] M. I. Ali, F. Feng, X. Y. Liu, W. K. Min and M. Shabir, On some new operations in soft set theory, Computers and Mathematics with Applications 57 (2009) 15471553.
[6] M. J. Borah and B. Hazarika Some results of mixed fuzzy soft topology and applications in Chemistry, Ann. Fuzzy Math. Inform. 12 (1) (2016) 129–138.
[7] N. Çağman, F. Çitak and S. Enginoglu, Fuzzy parameterized fuzzy soft set theory and its applications, Turkish Journal of Fuzzy Systems 1 (1) (2010) 21–35.
[8] N. Çağman, S. Karataş and S. Enginoglu, Soft topology, Computers and Mathematics with Applications 62 (2011) 351-358.
[9] N. Çağman,S. Enginoglu and F. Çitak , Fuzzy soft set theory and its applications, Iranian Journal of Fuzzy Systems 8 (8) (2011) 137–147.
[10] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.
[11] D. Chen, E. C. C. Tsang, D. S. Yueng and X. Wang, The parametrization reduction of soft sets and its applications, Computers and Mathematics with Applications 49 (2005) 757–763.
[12] S. Das and S. K. Samanta, Soft real sets, soft real numbers and their properties, J. Fuzzy Math. 20 (3) (2012) 551–576.
[13] S. Das and S. K. Samanta, Soft metric, Ann. Fuzzy Math. Inform. 6 (1) (2013) 77–94.
[14] S. Das and S. K. Samanta, On soft metric spaces, J. Fuzzy Math. 21 (3) (2013) 707–734.
[15] F. Feng, M. Akram, B. Davvaz, V. L. Fotea, Attribute analysis of information systems based on elementary soft implications, Knowledge-Based Systems 70 (2014) 281–292.
[16] F. Feng, Y. B. Jun, X. Liu and L. Li, An adjustable approach to fuzzy soft set based decision making, Journal of Computational and Applied Mathematics 234 (1) (2010) 10–20.
[17] F. Feng, C. Li, B. Davvaz and M. I. Ali, Soft sets combined with fuzzy sets and rough sets: a tentative approach, Soft Computing 14 (9) (2010) 899–911.

[18] F. Feng, X. Y. Liu, V. Leoreanu-Fotea and Y.B. Jun, Soft sets and soft rough sets, Inform. Sci. 181 (2011) 1125–1137.

[19] K. Hur, H. W. Kang and K. C. Lee, Fuzzy equivalence relations and fuzzy partitions, Honam Math. J. 28 (3) (2006) 291–315.

[20] K. Hur, H. M. Jung and W. R. Lee, Fuzzzy Functions and Fuzzy Partially Ordered Sets, International Journal of Fuzzy Logic and Intelligent Systems 9 (4) (2009) 285–293.

[21] Y. B. Jun, M. S. Kang and C. H. Park, Fuzzy subgroups based on fuzzy points, Communications of the Korean Mathematical Society 26 (3) (2011) 349–371.

[22] A. Kharal and B. Ahmad, Mappings on soft classes, New Mathematics and Natural Computation 7 (3) (2011) 471–482.

[23] P. K. Maji, R. Biswas and A. R. Roy, An application of soft sets in decision making problem, Computer and Mathematics with Applications 44 (2002) 1077–1083.

[24] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, Computers and Mathematics with Applications 45 (2003) 555–562.

[25] P. Majumdar and S. K. Samanta, On soft mappings, Computer and Mathematics with Applications 60 (2010) 2666–2672.

[26] D. Molodtsov, Soft set theory-first results, Computers and Mathematics with Applications 37 (1999) 19–31.

[27] E. Peyghan, B. Samadi and A. Tayebi, Some results related to soft topological spaces, Facta Universitatis 29 (4) (2014) 325–336.

[28] D. Pie and D. Miao, From soft sets to information systems, Granular Computing, IEEE International Conference 2 (2005) 617–622.

[29] M. Riaz and K. Naeem, Novel Concepts of Soft Sets with Applications, Ann. Fuzzy Math. Inform. (in press) (2016).

[30] M. Riaz and K. Naeem, Measurable Soft Mappings, Punjab University Journal of mathematics 48 (2) (2016) 19–34.

[31] M. Riaz and Z. Fatima, Certain properties of soft metric spaces, J. Fuzzy Math. (in press) (2016).

[32] W. Rong, The countabilities of soft topological spaces, World Accedmy of Science, Engineering and Technology 6 (2012) 784–787.

[33] A. R. Roy and P. K. Maji, A fuzzy soft set theoretic approach to decision-making problems, Journal of Computational and Applied Mathematics 203 (2007) 412–418.

[34] S. Roy and T. K. Samanta, A note on a soft topological space, Punjab University Journal of Mathematics 46 (1) (2014) 19–24.

[35] S. Roy and T. K. Samanta, An introduction to open and closed sets on fuzzy soft topological spaces, Ann. Fuzzy Math. Inform. 6 (2) (2013) 425–431.

[36] A. Sezgin and A. O. Atagun, On operations of soft set, Computers and Mathematics with Applications 61 (2011) 1457–1467.

[37] A. Sezgin, N. Cagman, A. O. Atagun, M. I. Ali and E. Turkmen, Soft intersection semigroups, ideals and bi-Ideals; A new Aapplication on semigroup theory I, Filomat 29 (5) (2015) 917–946.

[38] M. Shabir and M. Naz, On soft topological spaces, Computers and Mathematics with Applications 61 (2011) 1786–1799.

[39] D. K. Sut, An application of fuzzy soft relation in decision making problems, International Journal of Mathematics Trends and Technology 3 (2) (2012) 50–53.

[40] J. Subhashinin and C. Sekar, Related properties of soft dense and soft pre open sets in a soft topological spaces, International Journal of Innovative and Applied Research 2 (9) (2014) 34–38.

[41] B. P. Varol and H. Aygun, Fuzzy soft topology, Hacettepe Journal of Mathematics and Statistics 41 (3) (2012) 407–419.

[42] E. D. Yildirim, A. Ç. Güler and O. B. Özbakir, On soft $\tilde{I}$-Baire spaces, Ann. Fuzzy Math. Inform. 10 (1) (2015) 109–121.

[43] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

[44] I. Zorlutuna and H. Çakir, On continuity of soft mappings, Applied Mathematics and Information sciences 9 (1) (2015) 403–409.

[45] I. Zorlutuna and S. Atmaca, Fuzzy parametrized fuzzy soft topology, New Trends in Mathematical Sciences 4 (1) (2016) 142–152.

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