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Double-framed soft topological space

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Abstract. In this paper, we describe double-framed soft (DFS) topology over an initial universe with a fixed set of parameters. The notions of DFS open set, DFS closed set, DFS closure, DFS neighborhood are defined and their basic properties are investigated. These notions are illustrated with the help of examples.

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1. Introduction

There are several theories which deals with uncertainty like, theory of fuzzy sets [21], theory of intuitionists fuzzy sets [3], theory of vague sets [8], theory of interval mathematics [4], theory of rough sets [18] and theory of probability [9]. But these theories have their inherent limitations due to the inadequacy of the parametrization tool associated with these theories. No mathematical tool can deal with various kind of uncertainties successfully which can occur while modeling problems in engineering, economics and environmental areas. Moldstove [16] presented the concept of soft sets, that can be seen as a new mathematical tool to deal with uncertainties. He applied the concept of soft sets in various directions like smoothness of function, game theory, theory of measurement and so on. Theoretical work on soft sets have been presented by Maji et al.[14]. They were the first who gave practical application of soft sets in decision making problems [13]. Aktas and Cagman [1] proposed the concept of soft groups and study its various properties. Ali et al. [2] investigated several operations on soft sets. Soft semi groups and soft ideals over a semigroup have been studied by Shabir and Ali [19]. By using t-norm, Aygunoglu and Aygun presented fuzzy soft groups [5].

Topological spaces show up naturally in almost every branch of mathematics. Topological structures depends on the ideas of set theory. Shabir and Naz [20] proposed and discussed the notions of soft topology over an initial universe with a fixed set of parameters. After that different researchers studied the notion of soft topological spaces [6], [10], [22], [7], [15].

In 2012, the notion of double-framed soft sets have been introduced by Jun et al. [11]. They applied it in BCK/BCI algebras and examined many results with uni int concepts. Later on, Jun et al. [12] defined the notion of a (closed) double-framed soft ideals in BCK/BCI algebras. They discussed relation between double-framed soft algebra and double-framed soft ideal. Naz [17] studied algebraic structural properties of double framed soft (DFS) sets. She defined new operations for DFS set and studied their characteristics. She also studied classes of MV-algebras and BCK/BCI-algebras of DFS sets.

The aim of this paper is to study the double-framed soft topological space defined on an initial universe with fixed set of parameters. Some basic notions related to soft sets and double-framed soft sets are given in section 2. In section 3, we present double framed soft topological space. We discuss some basic notions like DFS open set, DFS closed set, DFS closure, DFS neighborhood and verified their basic properties. We also define DFS relative topology. We discuss how from soft topology we obtained DFS topology and vice versa. We discuss the parametrized topologies corresponding to DFS topology.

2. Preliminaries

Let U be an initial universe and E be the set of parameters. Let $P(U)$ be the family of all subsets of U and A, B be non empty subsets of E.

Definition 2.1 ([16])**.** A soft set over U is a pair (F, A), where F is a mapping from A to the family of all subsets of U that is $F : A \longrightarrow P(U)$.

In other words, a soft set is a parametrized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of $\varepsilon-$approximate elements of the soft set (F, A). Clearly soft set is not a set.

Definition 2.2 ([2])**.** A soft set (F, A) is said to be null soft set (with respect to the parameters set A), denoted by Φ_A, if for all $e \in A$, $F(e) = \varnothing$ (null set).

Definition 2.3 ([2])**.** A soft set (F, A) over U is relative whole set (with respect to the parameters set A), denoted by U_A, if for all $e \in A$, $F(e) = U$.

The relative whole soft set with respect to the universe set of parameters E is called the absolute soft set over U.

Definition 2.4 ([2])**.** The relative complement of a soft set (F, A) is denoted by $(F, A)'$ and is defined by $(F, A)' = (F', A)$ where $F' : A \longrightarrow P(U)$ is a mapping given by

$$F'(\alpha) = U - F(\alpha) \text{ for all } \alpha \in A.$$

Definition 2.5 ([17])**.** Let τ be the collection of soft sets over U with set of parameter E. Then τ is said to be a soft topology on U, if it satisfies the following axioms:

(i) $\Phi_E, \widetilde{U}_E$ belongs to τ.

(ii) (2) The union of any number of soft sets in τ belongs to τ.

(iii) (3) The intersection of any two soft sets in τ belongs to τ.
The triplet (U, τ, E) is called a soft topological space over U.

Definition 2.6 ([11])**.** Let U be an initial universe and E be the set of parameters. A double framed pair $\langle(\psi, \eta); A\rangle$ is called a double-framed soft (DFS) set over U, where ψ and η are mappings from A to $P(U)$.

We shall use the notation $A_{(\psi,\eta)}$ to denote a double framed soft set over U.

We shall use DFS set to deonte a double framed soft set.

Definition 2.7 ([11])**.** For two double framed soft sets $A_{(\psi,\eta)}$ and $B_{(\lambda,\mu)}$ over U, we say that $A_{(\psi,\eta)}$ is double framed soft subset of $B_{(\lambda,\mu)}$,if
(i) $A \subseteq B$,
(ii) $\psi(e) \subseteq \lambda(e)$ and $\mu(e) \subseteq \eta(e)$, for all $e \in A$.
This relation is denoted by $A_{(\psi,\eta)} \sqsubseteq B_{(\lambda,\mu)}$.

Definition 2.8 ([17])**.** For two double framed soft sets $A_{(\psi,\eta)}$ and $B_{(\lambda,\mu)}$ over U, the extended uni-int double framed soft set of $A_{(\psi,\eta)}$ and $B_{(\lambda,\mu)}$ is defined as a double framed soft set $(A \cup B)_{(\psi\tilde{\cup}\lambda,\eta\tilde{\cap}\mu)}$,
where $\psi\tilde{\cup}\lambda : (A \cup B) \longrightarrow P(U)$ defined by

$$e \longrightarrow \begin{cases} \psi(e) \;\; if \; e \in A - B \\ \lambda(e) \;\; if \; e \in B - A \\ \psi(e) \cup \lambda(e) \;\text{ if } e \in A \cap B \end{cases}$$

and $\eta\tilde{\cap}\mu : (A \cup B) \longrightarrow P(U)$, defined by

$$e \longrightarrow \begin{cases} \eta(e) \;\; if \; e \in A - B \\ \mu(e) \;\; if \; e \in B - A \\ \eta(e) \cap \mu(e) \;\text{ if } e \in A \cap B. \end{cases}$$

It is denoted by $A_{(\psi,\eta)} \sqcup_{\varepsilon} B_{(\lambda,\mu)} = (A \cup B)_{(\psi\tilde{\cup}\lambda,\eta\tilde{\cap}\mu)}$. We shall call this extended uni-int double framed soft set as union of double feamed soft sets.

Definition 2.9 ([17])**.** For two double framed soft set $A_{(\psi,\eta)}$ and $B_{(\lambda,\mu)}$ over U, the extended int-uni double framed soft set of $A_{(\psi,\eta)}$ and $B_{(\lambda,\mu)}$ is defined as a double-framed soft set $(A \cup B)_{(\psi\tilde{\cap}\lambda,\eta\tilde{\cup}\mu)}$,
where $\psi\tilde{\cap}\lambda : A \cup B \longrightarrow P(U)$ is defined by

$$e \longrightarrow \begin{cases} \psi(e) \;\; if \; e \in A - B \\ \lambda(e) \;\; if \; e \in B - A \\ \psi(e) \cap \lambda(e) \;\; if \; e \in A \cap B \end{cases}$$

and $\eta\tilde{\cup}\mu : A \cup B \longrightarrow P(U)$ defined by

$$e \longrightarrow \begin{cases} \eta(e) \;\; if \; e \in A - B \\ \mu(e) \;\; if \; e \in B - A \\ \eta(e) \cup \mu(e) \;\; if \; e \in A \cap B. \end{cases}$$

It is denoted by $A_{(\psi,\eta)} \sqcap^{\varepsilon} B_{(\lambda,\mu)} = (A \cup B)_{(\psi\tilde{\cap}\lambda,\eta\tilde{\cup}\mu)}$. We shall call this extended int-uni double framed soft set as intersection of double framed soft sets.

Definition 2.10 ([17]). Let $A_{(\psi,\eta)}$ be a double framed soft set over U. The complement of a double framed soft set $A_{(\psi,\eta)}$ is defined as a double framed soft set $A_{(\psi^c,\eta^c)}$, where

$$\psi^c : A \longrightarrow P(U) \text{ and } \eta^c : A \longrightarrow P(U)$$

are defined as

$$\psi^c(e) = (\psi(e))^c \text{ and } \eta^c(e) = (\eta(e))^c.$$

It is denoted by $A_{(\psi,\eta)^c} \cong A_{(\psi^c,\eta^c)}$.

Definition 2.11 ([17]). A double framed soft set $A_{(\kappa,\phi)}$ over U is said to be a relative absolute double-framed soft set, if

$$\kappa : A \longrightarrow P(U) \text{ and } \phi : A \longrightarrow P(U)$$

are defined as

$$\kappa(e) = U \text{ and } \phi(e) = \varnothing \text{ for all } e \in A.$$

Definition 2.12 ([17]). A double framed soft set $A_{(\phi,\kappa)}$ over U is said to be a relative null double framed soft set, where

$$\kappa : A \longrightarrow P(U) \text{ and } \phi : A \longrightarrow P(U)$$

are defined as

$$\kappa(e) = U \text{ and } \phi(e) = \varnothing \text{ for all } e \in A.$$

3. Double framed soft topological spaces

Througout this section, X will be an initial universe and E be the set of parameters.

Definition 3.1. Let $E_{(\psi,\eta)}$ be a double framed soft set over X and x be an element of X. Then $x \in E_{(\psi,\eta)}$, whenever $x \in \psi(e)$ and $x \notin \eta(e)$, for all $e \in E$. We say that $x \notin E_{(\psi,\eta)}$, if $x \notin \psi(e)$, for some $e \in E$ or $x \in \eta(e)$, for some $e \in E$.

Definition 3.2. Let X be an initial universe and E be the set of parameters. For $x \in X$, we define a DFS set $E_{(x,x^c)}$, where

$$x : E \longrightarrow P(X) \text{ and } x^c : E \longrightarrow P(X)$$

are defined as

$$x(e) = \{x\} \text{ and } x^c(e) = \{x\}^c \text{ for all } e \in E.$$

Proposition 3.3. *Let $E_{(\psi,\eta)}$ and $E_{(\lambda,\mu)}$ be double framed soft sets on X. Then*

(1) $\left(E_{(\psi,\eta)} \sqcup_{\epsilon} E_{(\lambda,\mu)}\right)^c = E_{(\psi,\eta)^c} \sqcap^{\varepsilon} E_{(\lambda,\mu)^c}$.
(2) $\left(E_{(\psi,\eta)} \sqcap^{\varepsilon} E_{(\lambda,\mu)}\right)^c = E_{(\psi,\eta)^c} \sqcup_{\varepsilon} E_{(\lambda,\mu)^c}$.

Proof. (1) By definition of double framed soft sets, $\left(E_{(\psi,\eta)} \sqcup_{\varepsilon} E_{(\lambda,\mu)}\right) = E_{(\psi\cup\lambda,\eta\cap\mu)}$. Then $\left(E_{(\psi,\eta)} \sqcup_{\varepsilon} E_{(\lambda,\mu)}\right)^c = E_{(\psi\cup\lambda,\eta\cap\mu)^c} = E_{((\psi\cup\lambda)^c,(\eta\cap\mu)^c)}$. On one hand,

$$\begin{aligned}
(\psi\cup\lambda)^c(e) &= ((\psi\cup\lambda)(e))^c \\
&= (\psi(e)\cup\lambda(e))^c \\
&= ((\psi(e))^c \cap (\lambda(e))^c) \\
&= (\psi^c(e)\cap\lambda^c(e)) \\
&= (\psi^c\cap\lambda^c)(e).
\end{aligned}$$

Thus $(\psi \cup \lambda)^c = (\psi^c \cap \lambda^c)$ for $e \in E$. Also

$$\begin{aligned}(\eta \cap \mu)^c(e) &= ((\eta \cap \mu)(e))^c \\ &= (\eta(e) \cap \mu(e))^c \\ &= ((\eta(e))^c \cup (\mu(e))^c) \\ &= (\eta^c(e) \cup \mu^c(e)) \\ &= (\eta^c \cup \mu^c)(e).\end{aligned}$$

So

$$(\eta \cap \mu)^c = (\eta^c \cup \mu^c) \text{ for all } e \in E.$$

Hence

$$\begin{aligned}\left(E_{(\psi,\eta)} \sqcup_\varepsilon E_{(\lambda,\mu)}\right)^c &= E_{((\psi\cup\lambda)^c,(\eta\cap\mu)^c)} \\ &= E_{(\psi^c\cap\lambda^c,\eta^c\cup\mu^c)} \\ &= E_{(\psi^c,\eta^c)} \sqcap^\varepsilon E_{(\lambda^c,\mu^c)} \\ &= E_{(\psi,\eta)^c} \sqcap^\varepsilon E_{(\lambda,\mu)^c}.\end{aligned}$$

(2) By definition of double framed soft sets, $E_{(\psi,\mu)} \sqcap^\varepsilon E_{(\lambda,\mu)} = E_{(\psi\cap\lambda,\eta\cup\mu)}$. Then $\left(E_{(\psi,\eta)} \sqcap^\varepsilon E_{(\lambda,\mu)}\right)^c = E_{((\psi\cap\lambda)^c,(\eta\cup\mu)^c)}$. On the other hand,

$$\begin{aligned}(\psi \cap \lambda)^c(e) &= (\psi(e) \cap \lambda(e))^c \\ &= ((\psi(e))^c \cup (\lambda(e))^c) \\ &= (\psi^c(e) \cup \lambda^c(e)) \\ &= (\psi^c \cup \lambda^c)(e).\end{aligned}$$

Thus $(\psi \cap \lambda)^c = \psi^c \cup \lambda^c$. Similarly, $(\eta \cup \mu)^c = \eta^c \cap \mu^c$. So

$$\begin{aligned}\left(E_{(\psi,\eta)} \sqcap^\varepsilon E_{(\lambda,\mu)}\right)^c &= E_{((\psi\cap\lambda)^c,(\eta\cup\mu)^c)} \\ &= E_{(\psi^c\cup\lambda^c,\eta^c\cap\mu^c)} \\ &= E_{(\psi^c,\eta^c)} \sqcup_\varepsilon E_{(\lambda^c,\mu^c)} \\ &= E_{(\psi,\eta)^c} \sqcup_\varepsilon E_{(\lambda,\mu)^c}.\end{aligned}$$

□

Definition 3.4. Let $\widetilde{\Upsilon}$ be the collection of double framed soft (shortly DFS) sets on X, where X is an initial universe and E as a set of parameters. Then $\widetilde{\Upsilon}$ is said to be double framed soft topology over X, if $\widetilde{\Upsilon}$ satisfy the following conditions:

(i) $E_{(\phi,\kappa)}$ and $E_{(\kappa,\phi)}$ belongs to $\widetilde{\Upsilon}$.

(ii) The intersection of two or finite DFS sets in $\widetilde{\Upsilon}$ belongs to $\widetilde{\Upsilon}$.

(iii) The union of arbitrary DFS sets in $\widetilde{\Upsilon}$ belongs to $\widetilde{\Upsilon}$.

In this case, $\left(X, \widetilde{\Upsilon}, E\right)$ is said to be a double framed soft topological space on X.

Example 3.5. Let $X = \mathbb{N}$, $E = \{e_1, e_2\}$ and

$$\widetilde{\Upsilon} = \left\{E_{(\phi,\kappa)},\, E_{(\kappa,\phi)},\, E_{(\psi_1,\eta_1)},\; E_{(\psi_n,\eta_n)} : n \geq 2\right\},$$

where $E_{(\psi_1,\eta_1)}$ is defined as:

$\psi_1 : E \longrightarrow P(X)$

$$e \longrightarrow \begin{cases} \mathbb{N} \text{ if } e = e_1 \\ \varnothing \text{ if } e = e_2, \end{cases}$$

$\eta_1 : E \longrightarrow P(X)$

$$e \longrightarrow \begin{cases} \varnothing \text{ if } e = e_1 \\ \mathbb{N} \text{ if } e = e_2 \end{cases}$$

and $E_{(\psi_n,\eta_n)} : n \geq 2$ is defined as:

$\psi_n : E \longrightarrow P(X)$

$$e \longrightarrow \begin{cases} \{n, n+1, n+2, ...\} \text{ if } e = e_1 \text{ for all } n \geq 2 \\ \varnothing \text{ if } e = e_2 \text{ for all } n \geq 2, \end{cases}$$

$\eta_n : E \longrightarrow P(X)$

$$e \longrightarrow \begin{cases} \{1, 2, 3, ...n-1\} \text{ if } e = e_1 \text{ for all } n \geq 2 \\ \mathbb{N} \text{ if } e = e_2 \text{ for all } n \geq 2. \end{cases}$$

Then $\widetilde{\Upsilon}$ is a DFS topology and $\left(X, \widetilde{\Upsilon}, E\right)$ is a DFS topology space.

Definition 3.6. Let $\left(X, \widetilde{\Upsilon}, E\right)$ be a DFS topological space on X. The elements of $\widetilde{\Upsilon}$ are called DFS open sets.

Definition 3.7. Let $\left(X, \widetilde{\Upsilon}, E\right)$ be a DFS topological space on X. A DFS set $E_{(\delta,\gamma)}$ is said to be DFS closed set, if its compliment $E_{(\delta,\gamma)^c}$ belongs to $\widetilde{\Upsilon}$.

Proposition 3.8. *Let $\left(X, \widetilde{\Upsilon}, E\right)$ be a DFS soft topological space on X. Then*

(1) *$E_{(\phi,\kappa)}$ and $E_{(\kappa,\phi)}$ are DFS closed sets,*
(2) *the arbitrary intersection of DFS closed sets is DFS closed set over X,*
(3) *the union of finite or two DFS closed sets is DFS closed set.*

Proof. Straightforward. □

Example 3.9. Let X be an initial universe and E be the set of parameters. Let $\widetilde{\Upsilon}$ consists of null DFS set $E_{(\phi,\kappa)}$ and absolute DFS set $E_{(\kappa,\phi)}$, i.e., $\widetilde{\Upsilon} = \left\{E_{(\phi,\kappa)}, E_{(\kappa,\phi)}\right\}$. Then $\widetilde{\Upsilon}$ is a DFS topology over X and is called indiscrete DFS topology and the triplet $\left(X, \widetilde{\Upsilon}, E\right)$ is called DFS indiscrete topological space.

Example 3.10. Let X be an initial universe and E be the set of parameters. Let $\widetilde{\Upsilon}$ be the collection of all DFS sets over$X$ with parameter set E. Then $\widetilde{\Upsilon}$ is a DFS topology and is called discrete DFS topology and $\left(X, \widetilde{\Upsilon}, E\right)$ is called discrete DFS topological space.

Theorem 3.11. *Let $\left(X, \widetilde{\Upsilon}, E\right)$ be a DFS topological space over X. Then $\left(X, \widetilde{\Upsilon}_\psi, E\right)$ is a soft topological space, where*

$$\widetilde{\Upsilon}_\psi = \left\{(\psi, E) : E_{(\psi,\eta)} \in \widetilde{\Upsilon}\right\}.$$

Proof. Since $E_{(\phi,\kappa)}$ and $E_{(\kappa,\phi)}$ belongs to $\widetilde{\Upsilon}$, (ϕ, E) and (κ, E) belongs to $\widetilde{\Upsilon}_{\psi}$, where $\phi(e) = \phi$ and $\kappa(e) = X$, for all $e \in E$.

Let $\{(\psi_i, E) : i \in E\}$ be the collection of soft sets in $\widetilde{\Upsilon}_{\psi}$. Then $E_{(\psi_i,\eta_i)} \in \widetilde{\Upsilon}$ which implies that $\sqcup_{\varepsilon} E_{(\psi_i,\eta_i)} = E_{(\cup\psi_i,\cap\eta_i)} \in \widetilde{\Upsilon}$. Thus $\cup_{i\in I}(\psi_i, E)$ belongs to $\widetilde{\Upsilon}_{\psi}$.

Let (ψ_1, E) and (ψ_2, E) be soft sets in $\widetilde{\Upsilon}_{\psi}$. Then $E_{(\psi_1,\eta_1)}$ and $E_{(\psi_2,\eta_2)}$ belongs to $\widetilde{\Upsilon}$. Since $E_{(\psi_1,\eta_1)}$ and $E_{(\psi_2,\eta_2)}$ are DFS open sets, $E_{(\psi_1,\eta_1)} \sqcap^{\varepsilon} E_{(\psi_2,\eta_2)} \in \widetilde{\Upsilon}$. Thus $E_{(\psi_1\cap\psi_2,\eta_1\cup\eta_2)} \in \widetilde{\Upsilon}$. TSo $(\psi_1 \cap \psi_2, E) \in \widetilde{\Upsilon}_{\psi}$. Hence $\widetilde{\Upsilon}_{\psi} = \left\{(\psi, E) : E_{(\psi,\eta)} \in \widetilde{\Upsilon}\right\}$ is a soft topology. □

Theorem 3.12. *Let (X, Υ, E) be a soft topological space over X. Then $\left(X, \hat{\Upsilon}, E\right)$ is a DFS topological space over X, where $\hat{\Upsilon}$ is defined as*

$$\hat{\Upsilon} = \left\{E_{(\psi,\eta)} : \eta(e) = (\psi(e))^c \text{ for all } e \in E \text{ and } (\psi, E) \in \Upsilon\right\}.$$

Proof. As $(\phi, E) \in \Upsilon$, where $\phi(e) = \phi$, for all $e \in E$, define $\kappa = (\phi(e))^c = X$. Then $E_{(\phi,\kappa)} \in \hat{\Upsilon}$. Similarly, $(\kappa, E) \in \Upsilon$ implies that $E_{(\kappa,\phi)} \in \hat{\Upsilon}$.

Let $\left\{E_{(\psi_i,\eta_i)} : i \in I\right\}$ be collection of DFS sets in $\hat{\Upsilon}$. Then $(\psi_i, E) \in \Upsilon$ and $\eta_i(e) = (\psi_i(e))^c$. Since Υ is soft topology, $\cup_{i\in I}(\psi_i, E) \in \Upsilon$. On one hand, $(\cup\psi_i)^c(e) = \cap\psi_i^c(e) = \cap(\psi_i(e))^c$. Thus $(\cup\psi_i)^c(e) = \cap\eta_i(e)$. So $E_{(\cup\psi_i,\cap\eta_i)} = \sqcup_{\varepsilon} E_{(\psi_i,\eta_i)} \in \hat{\Upsilon}$. Hence $\sqcup_{\varepsilon} E_{(\psi_i,\eta_i)}$ belongs to $\hat{\Upsilon}$.

Let $E_{(\psi_1,\eta_1)}$ and $E_{(\psi_2,\eta_2)} \in \hat{\Upsilon}$. Then $(\psi_1, E), (\psi_2, E) \in \Upsilon$, $\eta_1(e) = (\psi_1(e))^c$ and $\eta_2(e) = (\psi_2(e))^c$. Since (ψ_1, E) and (ψ_2, E) are soft open sets, $(\psi_1, E) \cap (\psi_2, E) = (\psi_1 \cap \psi_2, E) \in \Upsilon$. On the other hand, $(\psi_1 \cap \psi_2)^c(e) = \psi_1^c(e) \cup \psi_2^c(e)$. Thus $(\psi_1 \cap \psi_2)^c(e) = ((\psi_1(e))^c \cup (\psi_2(e))^c) = (\eta_1(e) \cup \eta_2(e))$. So $(\psi_1 \cap \psi_2)^c(e) = (\eta_1 \cup \eta_2)(e)$. Hence $E_{(\psi_1\cap\psi_2,\eta_1\cup\eta_2)} = E_{(\psi_1,\eta_1)} \sqcap^{\varepsilon} E_{(\psi_2,\eta_2)} \in \hat{\Upsilon}$. Therefore $\hat{\Upsilon}$ is DFS topology and $\left(X, \hat{\Upsilon}, E\right)$ is DFS topological space. □

Proposition 3.13. *Let $\left(X, \widetilde{\Upsilon}, E\right)$ be a DFS topological space and*

$$\widetilde{\Upsilon}_{e_\psi} = \left\{\psi(e) \mid E_{(\psi,\eta)} \in \widetilde{\Upsilon}\right\},$$

then for each parameter $e \in E$, $\widetilde{\Upsilon}_{e_\psi}$ form a topology on X.

Proof. Straightforward. □

From Theorem 3.11 and Proposition 3.13 we observe that corresponding to first component of DFS sets in $\widetilde{\Upsilon}$ we have a topology $\widetilde{\Upsilon}_{e_\psi}$ over X corresponding to each parameter $e \in E$. Thus from a DFS topology we get a family of parametrized topologies on X.

Remark 3.14. (1) In DFS topology, we can not obtained parametrized topology corresponding to second component of DFS sets.

(2) Converse of the Proposition 3.13 is not hold.

Example 3.15. Let $X^* = \mathbb{R}$, E* be the set of parameters and

$$\widetilde{\Upsilon} = \left\{E^*_{(\phi,\kappa)}, E^*_{(\kappa,\phi)}, E^*_{(\psi_i,\eta_i)} : i \in \mathbb{N}\right\},$$

where $E^*_{(\psi_i,\eta_i)}$ is defined as

$$\psi_i\left(e^*\right)=\mathbb{R}\ ,\ \eta_i\left(e^*\right)=\{1,2,3...i\}.$$

If $i,j\in\mathbb{N}$ and $j\succeq i$, then $E^*_{(\psi_i,\eta_i)}\sqcup_\varepsilon E^*_{(\psi_j,\eta_j)}=E^*_{(\psi_i,\eta_i)}$ and $E^*_{(\psi_i,\eta_i)}\sqcap_\varepsilon E^*_{(\psi_j,\eta_j)}=E^*_{(\psi_j,\eta_j)}$. Thus $\left(\mathbb{R},E^*,\widetilde{\Upsilon}\right)$ is DFS topology over $\mathbb{R}$. Now consider,

$$\widetilde{\Upsilon}_{\eta_e}=\left\{\varnothing,\mathbb{R},E^*_{\eta_i}=\{1,2,3...i\}:i\in\mathbb{N}\right\}.$$

As $\underset{i\in\mathbb{N}}{\cup}E_{\psi_i}=\mathbb{N}$ but $\underset{i\in\mathbb{N}}{\cup}E_{\psi_i}=\mathbb{N}\notin\widetilde{\Upsilon}_{\eta_e}$, $\left(\widetilde{\Upsilon}_{\eta_e},\mathbb{R}\right)$ is not a topological space.

Example 3.16. Let $X=\{x_1,x_2,x_3\}$, $E=\{e_1,e_2\}$ and

$$\widetilde{\Upsilon}=\left\{E_{(\phi,\kappa)},E_{(\kappa,\phi)},E_{(\psi_1,\eta_1)},E_{(\psi_2,\eta_2)},E_{(\psi_3,\eta_3)}\right\},$$

where $E_{(\psi_1,\eta_1)},E_{(\psi_2,\eta_2)},E_{(\psi_3,\eta_3)},E_{(\psi_4,\eta_4)}$ are DFS set on X defined as:

$\psi_1:E\longrightarrow P\left(X\right)$

$$e\longrightarrow\begin{cases}\{x_1\}\ \text{if } e=e_1\\ \{x_2,x_3\}\,\text{if} e=e_2,\end{cases}$$

$\eta_1:E\longrightarrow P\left(X\right)$

$$e\longrightarrow\begin{cases}\{x_1,x_3\}\ \text{if } e=e_1,\\ \phi\text{ if } e=e_2\end{cases}$$

$\psi_2:E\longrightarrow P\left(X\right)$

$$e\longrightarrow\begin{cases}\{x_2\}\ \text{if } e=e_1\\ \{x_2\}\,\text{if} e=e_2,\end{cases}$$

$\eta_2:E\longrightarrow P\left(X\right)$

$$e\longrightarrow\begin{cases}\{x_1\}\ \text{if } e=e_1\\ \{x_1,x_2\}\,\text{if} e=e_2,\end{cases}$$

$\psi_3:E\longrightarrow P\left(X\right)$

$$e\longrightarrow\begin{cases}\{x_1,x_2\}\ \text{if } e=e_1\\ \{x_3\}\,\text{if} e=e_2\end{cases}$$

and

$\eta_3:E\longrightarrow P\left(X\right)$

$$e\longrightarrow\begin{cases}\{x_1\}\ \text{if } e=e_1\\ \phi\ \text{if} e=e_2.\end{cases}$$

Then $\widetilde{\Upsilon}$ is not a topology as

$$E_{(\psi_1,\eta_1)}\sqcup_\varepsilon E_{(\psi_2,\eta_2)}=E_{(\delta,\gamma)}\notin\widetilde{\Upsilon},$$

where $\delta:E\longrightarrow P\left(X\right)$

$$e\longrightarrow\begin{cases}\{x_1,x_2\}\ \text{if } e=e_1\\ \{x_2,x_3\}\,\text{if } e=e_2\end{cases}$$

and $\gamma:E\longrightarrow P\left(X\right)$

$$e\longrightarrow\begin{cases}\{x_1\}\ \text{if } e=e_1\\ \phi\text{ if } e=e_2.\end{cases}$$

But

$$\widetilde{\Upsilon}_{(e_1)_\psi} = \{\phi, X, \{x_1\}, \{x_2\}, \{x_1, x_2\}\}$$

and

$$\widetilde{\Upsilon}_{(e_2)_\psi} = \{\phi, X, \{x_2\}, \{x_3\}, \{x_2, x_3\}\}$$

are topologies on X.

Proposition 3.17. *Let* $\left(X, \widetilde{\Upsilon}_1, E\right)$ *and* $\left(X, \widetilde{\Upsilon}_2, E\right)$ *be two double-framed soft topological spaces on* X*. Then the intersection* $\left(X, \widetilde{\Upsilon}_1 \cap \widetilde{\Upsilon}_2, E\right)$ *is also a DFS topological spaces on* X*.*

Proof. Straightforward. □

Remark 3.18. The union of two DFS topological spaces need not to be a DFS topological space.

Example 3.19. Let $X = \{x_1, x_2, x_3\}$, $E = \{e_1, e_2\}$,

$$\widetilde{\Upsilon}_1 = \left\{E_{(\phi,\kappa)}, E_{(\kappa,\phi)}, E_{(\alpha_1,\beta_1)}, E_{(\alpha_2,\beta_2)}\right\}$$

and

$$\Upsilon_2 = \left\{E_{(\phi,\kappa)}, E_{(\kappa,\phi)}, E_{(\psi,\eta)}\right\},$$

where $E_{(\alpha_1,\beta_1)}, E_{(\alpha_2,\beta_2)}, E_{(\psi,\eta)}$ are DFS open set illustrated as:

Example 3.20. Consider $E_{(\alpha_1,\beta_1)}, E_{(\alpha_2,\beta_2)}$ and $E_{(\psi,\eta)}$ defined as follows:

$\alpha_1 : E \longrightarrow P(X)$

$$e \longrightarrow \begin{cases} \{x_1\} \text{ if } e = e_1 \\ \{x_1\} \text{ if } e = e_2, \end{cases}$$

$\beta_1 : E \longrightarrow P(X)$

$$e \longrightarrow \begin{cases} \{x_1, x_2\} \text{ if } e = e_1 \\ \{x_2, x_3\} \text{ if } e = e_2, \end{cases}$$

$\alpha_2 : E \longrightarrow P(X)$

$$e \longrightarrow \begin{cases} \{x_1\} \text{ if } e = e_1 \\ \{x_1\} \text{ if } e = e_2, \end{cases}$$

$\beta_1 : E \longrightarrow P(X)$

$$e \longrightarrow \begin{cases} \{x_2\} \text{ if } e = e_1 \\ \{x_2\} \text{ if } e = e_2, \end{cases}$$

$\psi : E \longrightarrow P(X)$

$$e \longrightarrow \begin{cases} \{x_2, x_3\} \text{ if } e = e_1 \\ \{x_1\} \text{ if } e = e_2 \end{cases}$$

and

$\eta : E \longrightarrow P(X)$

$$e \longrightarrow \begin{cases} \{x_1, x_3\} \text{ if } e = e_1 \\ \phi \text{ if } e = e_2. \end{cases}$$

Then Υ_1 and Υ_2 are DFS topologies on X. But

$$\Upsilon_1 \sqcup_\varepsilon \Upsilon_2 = \left\{E_{(\phi,\kappa)}, E_{(\kappa,\phi)}, E_{(\alpha_1,\beta_1)}, E_{(\alpha_2,\beta_2)}, E_{(\psi,\eta)}\right\}$$

is not a DFS topology as $E_{(\alpha_1,\beta_1)} \sqcup_\varepsilon E_{(\psi,\eta)} = E_{(\alpha_1\cup\psi,\beta_1\cap\eta)}$, where
$\alpha_1 \cup \psi : E \longrightarrow P(X)$

$$e \longrightarrow \begin{cases} X \text{ if } e = e_1 \\ \{x_1\} \text{ if } e = e_2 \end{cases}$$

and
$\beta_1 \cap \eta : E \longrightarrow P(X)$

$$e \longrightarrow \begin{cases} \{x_1\} \text{ if } e = e_1 \\ \phi \text{ if } e = e_2. \end{cases}$$

We can see that $E_{(\alpha_1,\beta_1)} \sqcup_\varepsilon E_{(\psi,\eta)} \notin \Upsilon_1 \sqcup_\varepsilon \Upsilon_2$. Thus $\Upsilon_1 \sqcup_\varepsilon \Upsilon_2$ is not a DFS topology. So $(X, \Upsilon_1 \sqcup_\varepsilon \Upsilon_2, E)$ is not a DFS topological space.

Definition 3.21. Let $\left(X, \widetilde{\Upsilon}, E\right)$ be a DFS topological space over X and $E_{(\psi,\eta)}$ be a DFS set over X with E as a set of parameters. Then the closure of $E_{(\psi,\eta)}$ is the intersection of all DFS closed superset of $E_{(\psi,\eta)}$ and is denoted by $\overline{E_{(\psi,\eta)}}$.

Of course $\overline{E_{(\psi,\eta)}}$ is the smallest DFS closed set containing $E_{(\psi,\eta)}$.

Theorem 3.22. *Let $\left(X, \widetilde{\Upsilon}, E\right)$ be a DFS topological space over X and $E_{(\psi,\eta)}$ and $E_{(\gamma,\delta)}$ be DFS sets over X. Then*

(1) $\overline{E_{(\phi,\kappa)}} = E_{(\phi,\kappa)}$ and $\overline{E_{(\kappa,\phi)}} = E_{(\kappa,\phi)}$,
(2) $E_{(\psi,\eta)} \sqsubseteq \overline{E_{(\psi,\eta)}}$,
(3) $E_{(\psi,\eta)}$ is DFS closed set if and only if $\overline{E_{(\psi,\eta)}} = E_{(\psi,\eta)}$,
(4) $\overline{\overline{E_{(\psi,\eta)}}} = \overline{E_{(\psi,\eta)}}$,
(5) $E_{(\psi,\eta)} \sqsubseteq E_{(\gamma,\delta)}$ implies that $\overline{E_{(\psi,\eta)}} \sqsubseteq \overline{E_{(\gamma,\delta)}}$,
(6) $\overline{E_{(\psi,\eta)} \sqcup_\varepsilon E_{(\gamma,\delta)}} = \overline{E_{(\psi,\eta)}} \sqcup_\varepsilon \overline{E_{(\gamma,\delta)}}$,
(7) $\overline{E_{(\psi,\eta)} \sqcap^\varepsilon E_{(\gamma,\delta)}} \sqsubseteq \overline{E_{(\psi,\eta)}} \sqcap^\varepsilon \overline{E_{(\gamma,\delta)}}$.

Proof. (1) The proof is obvious.

(2) By definition, the proof is obvious.

(3) Let $E_{(\psi,\eta)}$ be a DFS closed set over X. Then $E_{(\psi,\eta)}$ is itself a DFS closed set on X containing $E_{(\psi,\eta)}$. Thus $E_{(\psi,\eta)}$ is the smallest DFS closed set containing $E_{(\psi,\eta)}$, i.e., $\overline{E_{(\psi,\eta)}} = E_{(\psi,\eta)}$.

(4) Proof follows from (3).

(5) Let $E_{(\psi,\eta)} \sqsubseteq E_{(\gamma,\delta)}$. Then DFS closed super sets which contains $E_{(\gamma,\delta)}$ will also contain $E_{(\psi,\eta)}$. This means that the DFS closed super sets of $E_{(\gamma,\delta)}$ are also DFS closed super sets of $E_{(\psi,\eta)}$. Thus the intersection of DFS closed super sets of $E_{(\psi,\eta)}$ is contained in the intersection of DFS closed super sets of $E_{(\gamma,\delta)}$, i.e., $\overline{E_{(\psi,\eta)}} \sqsubseteq \overline{E_{(\gamma,\delta)}}$.

(6) Since $E_{(\psi,\eta)} \sqsubseteq E_{(\psi,\eta)} \sqcup_\varepsilon E_{(\gamma,\delta)}$ and $E_{(\gamma,\delta)} \sqsubseteq E_{(\psi,\eta)} \sqcup_\varepsilon E_{(\gamma,\delta)}$, by (5),
$\overline{E_{(\psi,\eta)}} \sqsubseteq \overline{E_{(\psi,\eta)} \sqcup_\varepsilon E_{(\gamma,\delta)}}$ and $\overline{E_{(\gamma,\delta)}} \sqsubseteq \overline{E_{(\psi,\eta)} \sqcup_\varepsilon E_{(\gamma,\delta)}}$.
Thus $\overline{E_{(\psi,\eta)}} \sqcup_\varepsilon \overline{E_{(\gamma,\delta)}} \sqsubseteq \overline{E_{(\psi,\eta)} \sqcup_\varepsilon E_{(\gamma,\delta)}}$.

For reverse inclusion, as $E_{(\psi,\eta)} \sqsubseteq \overline{E_{(\psi,\eta)}}$ and $E_{(\gamma,\delta)} \sqsubseteq \overline{E_{(\gamma,\delta)}}$,

$$E_{(\psi,\eta)} \sqcup_\varepsilon E_{(\gamma,\delta)} \sqsubseteq \overline{E_{(\psi,\eta)}} \sqcup_\varepsilon \overline{E_{(\gamma,\delta)}}.$$

Since the union of DFS closed sets $\overline{E_{(\psi,\eta)}} \sqcup_\varepsilon \overline{E_{(\gamma,\delta)}}$ is DFS closed set, by (5),

$$\overline{E_{(\psi,\eta)} \sqcup_\varepsilon E_{(\gamma,\delta)}} \sqsubseteq \overline{E_{(\psi,\eta)}} \sqcup_\varepsilon \overline{E_{(\gamma,\delta)}}.$$

(7) Since $E_{(\psi,\eta)} \sqcap^\varepsilon E_{(\gamma,\delta)} \sqsubseteq E_{(\psi,\eta)}$ and $E_{(\psi,\eta)} \sqcap^\varepsilon E_{(\gamma,\delta)} \sqsubseteq E_{(\gamma,\delta)}$,
$\overline{E_{(\psi,\eta)} \sqcap^\varepsilon E_{(\gamma,\delta)}} \sqsubseteq \overline{E_{(\psi,\eta)}}$ and $\overline{E_{(\psi,\eta)} \sqcap^\varepsilon E_{(\gamma,\delta)}} \sqsubseteq \overline{E_{(\gamma,\delta)}}$.
Then $\overline{E_{(\psi,\eta)} \sqcap^\varepsilon E_{(\gamma,\delta)}} \sqsubseteq \overline{E_{(\psi,\eta)}} \sqcap^\varepsilon \overline{E_{(\gamma,\delta)}}$.

□

Definition 3.23. Let $\left(X, \widetilde{\Upsilon}, E\right)$ be a DFS topological space on X with parameters set E, and $E_{(\psi,\eta)}$ be a DFS set on X. We define a DFS Set $E_{\left(\overline{\psi},\eta\right)}$ associated with $E_{(\psi,\eta)}$, where $\overline{\psi} : E \longrightarrow P(X)$ is defined as $\overline{\psi}(e) = \overline{\psi(e)}$ and $\overline{\psi(e)}$ is closure of $\psi(e)$ in $\left(X, \widetilde{\Upsilon}_{e_\psi}\right)$.

Proposition 3.24. *Let $\left(X, \widetilde{\Upsilon}, E\right)$ be a DFS topological space on X and $E_{(\psi,\eta)}$ be a DFS set. Then $E_{\left(\overline{\psi},\eta\right)} \sqsubseteq \overline{E_{(\psi,\eta)}}$.*

Proof. Let $\overline{E_{(\psi,\eta)}} = E_{(\delta,\gamma)}$. Then by definition, $E_{(\psi,\eta)} \sqsubseteq E_{(\delta,\gamma)}$ implies that $\psi(e) \subseteq \delta(e)$ and $\gamma(e) \subseteq \eta(e)$, for all $e \in E$, where $\delta(e)$ is a closed set in $\left(X, \widetilde{\Upsilon}_{e_\psi}\right)$. Since $\overline{\psi(e)}$ is the smallest closed set containing $\psi(e)$, $\overline{\psi(e)} \subseteq \delta(e)$. Thus $E_{\left(\overline{\psi},\eta\right)} \sqsubseteq \overline{E_{(\psi,\eta)}}$. □

Corollary 3.25. *Let $\left(X, \widetilde{\Upsilon}, E\right)$ be a DFS topological space on X and $E_{(\psi,\eta)}$ be a DFS set on X. Then $E_{\left(\overline{\psi},\eta\right)} = \overline{E_{(\psi,\eta)}}$ if and only if $E_{\left(\overline{\psi},\eta\right)^c} \in \widetilde{\Upsilon}$.*

Proof. Assume that $E_{\left(\overline{\psi},\eta\right)} = \overline{E_{(\psi,\eta)}}$. Then $E_{\left(\overline{\psi},\eta\right)}$ is DFS closed set implies that $E_{\left(\overline{\psi},\eta\right)^c} \in \widetilde{\Upsilon}$.

Conversely, if $E_{\left(\overline{\psi},\eta\right)^c} \in \widetilde{\Upsilon}$, then $E_{\left(\overline{\psi},\eta\right)}$ is DFS closed set containing $E_{(\psi,\eta)}$, i.e., $E_{(\psi,\eta)} \sqsubseteq E_{\left(\overline{\psi},\eta\right)}$. Thus $\overline{E_{(\psi,\eta)}} \sqsubseteq E_{\left(\overline{\psi},\eta\right)}$, as $\overline{E_{(\psi,\eta)}}$ is the smallest DFS closed set containing $E_{(\psi,\eta)}$. From Proposition 3.24, we have $E_{\left(\overline{\psi},\eta\right)} \sqsubseteq \overline{E_{(\psi,\eta)}}$. So $E_{\left(\overline{\psi},\eta\right)} = \overline{E_{(\psi,\eta)}}$. □

Definition 3.26. Let $\left(X, \widetilde{\Upsilon}, E\right)$ be a DFS topological space on X and $x \in X$ and $E_{(\alpha,\beta)}$ be a DFS set over X. Then x is known to be interior point of $E_{(\alpha,\beta)}$, if there exist a DFS open set $E_{(\psi,\eta)}$ such that

$$x \in E_{(\psi,\eta)} \sqsubseteq E_{(\alpha,\beta)}.$$

Definition 3.27. Let $\left(X, \widetilde{\Upsilon}, E\right)$ be a DFS topological space on X and $E_{(\alpha,\beta)}$ be a DFS set over X. Then $E_{(\alpha,\beta)}$ is called DFS neighborhood of $x \in X$, if there exist a DFS open set $E_{(\psi,\eta)}$ such that $x \in E_{(\psi,\eta)} \sqsubseteq E_{(\alpha,\beta)}$.

Proposition 3.28. *Let $\left(X, \widetilde{\Upsilon}, E\right)$ be a DFS topological space on X and $E_{(\psi,\eta)}$ be a DFS set on X and x be an element of X. If x is an interior point of $E_{(\psi,\eta)}$, then x is also an interior point of $\psi(e)$, for all $e \in E$ in $\left(X, \widetilde{\Upsilon}_{e_\psi}\right)$.*

Proof. Let x be an interior point of $E_{(\psi,\eta)}$. Then by definition, there exist a DFS open set $E_{(\psi_1,\eta_1)}$ such that $x \in E_{(\psi_1,\eta_1)} \sqsubseteq E_{(\psi,\eta)}$. This means that $x \in \psi_1(e)$ and $x \notin \eta_1(e)$, for all $e \in E$ and $\psi_1(e) \subseteq \psi(e)$ and $\eta(e) \subseteq \eta_1(e)$, for all $e \in E$. Thus $x \in \psi_1(e) \subseteq \psi(e)$, for all $e \in E$, where $\psi_1(e)$ is an open set in $\left(X, \widetilde{\Upsilon}_{e_\psi}\right)$. So x is an interior point of $\psi(e)$, for all $e \in E$ in $\left(X, \widetilde{\Upsilon}_{e_\psi}\right)$. □

Proposition 3.29. *Let $\left(X, \widetilde{\Upsilon}, E\right)$ be a DFS topological space on X. Then*

(1) *every $x \in X$ has a DFS neighborhood,*

(2) *$E_{(\psi,\eta)} \sqcap^{\varepsilon} E_{(\lambda,\mu)}$ is a DFS neighborhood of some $x \in X$, if $E_{(\psi,\eta)}$ and $E_{(\lambda,\mu)}$ are DFS neighborhoods of x,*

(3) *if $E_{(\psi,\eta)} \sqsubseteq E_{(\lambda,\mu)}$ and $E_{(\psi,\eta)}$ is DFS neighborhood of any $x \in X$, then $E_{(\lambda,\mu)}$ is also a DSF neighborhood of x.*

Proof. (1) For every $x \in X$, $x \in E_{(\kappa,\phi)}$. Since $E_{(\kappa,\phi)}$ is DFS open set on X, $x \in E_{(\kappa,\phi)} \sqsubseteq E_{(\kappa,\phi)}$. Then $E_{(\kappa,\phi)}$ is a DFS neighborhood of x.

(2) Let $E_{(\psi,\eta)}$ and $E_{(\lambda,\mu)}$ be two DFS neighborhoods of x. Then by definition, there exist DFS open sets $E_{(\psi_1,\eta_1)}$ and $E_{(\lambda_1,\mu_1)}$ such that

$$x \in E_{(\psi_1,\eta_1)} \sqsubseteq E_{(\psi,\eta)} \text{ and } x \in E_{(\lambda_1,\mu_1)} \sqsubseteq E_{(\lambda,\mu)}.$$

Thus $x \in E_{(\psi_1,\eta_1)} \sqcap^{\varepsilon} E_{(\lambda_1,\mu_1)} \sqsubseteq E_{(\psi,\eta)} \sqcap^{\varepsilon} E_{(\lambda,\mu)}$, since $E_{(\psi_1,\eta_1)}, E_{(\lambda_1,\mu_1)} \in \widetilde{\Upsilon}$. So $E_{(\psi_1,\eta_1)} \sqcap^{\varepsilon} E_{(\lambda_1,\mu_1)} \in \widetilde{\Upsilon}$. Hence $E_{(\psi,\eta)} \sqcap^{\varepsilon} E_{(\lambda,\mu)}$ is DFS neighborhood of x.

(3) Suppose $E_{(\psi,\eta)} \sqsubseteq E_{(\lambda,\mu)}$ and $E_{(\psi,\eta)}$ is DFS neighborhood of any $x \in X$. Then by definition. there exist a DFS open set $E_{(\psi_1,\eta_1)}$ such that $x \in E_{(\psi_1,\eta_1)} \sqsubseteq E_{(\psi,\eta)}$. Since $E_{(\psi,\eta)} \sqsubseteq E_{(\lambda,\mu)}$, $x \in E_{(\psi_1,\eta_1)} \sqsubseteq E_{(\psi,\eta)} \sqsubseteq E_{(\lambda,\mu)}$. Thus $x \in E_{(\psi_1,\eta_1)} \sqsubseteq E_{(\lambda,\mu)}$. So $E_{(\lambda,\mu)}$ is also a DFS neighborhood of x.

□

Proposition 3.30. *Let $\left(X, \widetilde{\Upsilon}, E\right)$ be a DFS topological space on X and $E_{(\psi,\eta)}$ be a DFS set over X with set of parameter E. Then $E_{(\psi,\eta)}$ is a DFS neighborhood of each $x \in \cap_{e\in E}\psi(e)$, if $x \notin \eta(e)$, for all $e \in E$.*

Proof. Let $x \in \cap_{e\in E}\psi(e)$ imply that $x \in \psi(e)$, for all $e \in E$. Then by hypothesis, $x \in E_{(\psi,\eta)} \sqsubseteq E_{(\psi,\eta)}$. Thus $E_{(\psi,\eta)}$ is a DFS neighborhood of x. □

Definition 3.31. Let $\left(X, \widetilde{\Upsilon}, E\right)$ be a DFS topological space on X and Y be a non empty subset of X. Then

$$\widetilde{\Upsilon}_Y = \left\{E_{\left(\widehat{\psi},\widehat{\eta}\right)} : E_{(\psi,\eta)} \in \widetilde{\Upsilon} \text{ such that } \widehat{\psi}(e) = \psi(e) \cap Y \text{ and } \widehat{\eta}(e) = \eta(e) \cap Y\right\}$$

is called DFS relative topology on X and $\left(Y, \widetilde{\Upsilon}_Y, E\right)$ is known as DFS subspace of $\left(X, \widetilde{\Upsilon}, E\right)$. In fact, $\widetilde{\Upsilon}_Y$ is a DFS topology on Y.

Example 3.32. If $\left(X, \widetilde{\Upsilon}, E\right)$ is DFS discrete topological space, then any DFS subspace of $\left(X, \widetilde{\Upsilon}, E\right)$ is DFS discrete topological space.

Example 3.33. If $\left(X, \widetilde{\Upsilon}, E\right)$ is DFS indiscrete topological space, then any DFS subspace of $\left(X, \widetilde{\Upsilon}, E\right)$ is DFS indiscrete topological space.

Proposition 3.34. *Let* $\left(X, \widetilde{\Upsilon}, E\right)$ *be a DFS topological space on* X *and* Y *be a non empty subset of* X. *Then* $\left(Y, \widetilde{\Upsilon}_{e_{\widehat{\psi}}}\right)$ *is a subspace of* $\left(X, \widetilde{\Upsilon}_{e_{\psi}}\right)$.

Proof. Since $\left(Y, \widetilde{\Upsilon}_Y, E\right)$ is DFS topological space on Y, $\left(Y, \widetilde{\Upsilon}_{e_{\widehat{\psi}}}\right)$ is a topological space, for every $e \in E$. On the other hand,

$$\begin{aligned} \widetilde{\Upsilon}_{e_{\widehat{\psi}}} &= \left\{\widehat{\psi}\left(e\right) : E_{\left(\widehat{\psi},\widehat{\eta}\right)} \in \widetilde{\Upsilon}_Y\right\} \\ &= \left\{\psi\left(e\right) \cap Y : E_{\left(\psi,\eta\right)} \in \widetilde{\Upsilon}\right\} \\ &= \left\{\psi\left(e\right) \cap Y : \psi\left(e\right) \in \widetilde{\Upsilon}_{e_{\psi}}\right\}. \end{aligned}$$

Then $\left(Y, \widetilde{\Upsilon}_{e_{\widehat{\psi}}}\right)$ is a subspace of $\left(X, \widetilde{\Upsilon}\right)$. □

Definition 3.35. Let $E_{\left(\widehat{\psi},\widehat{\eta}\right)}$ be a DFS open set in $\widetilde{\Upsilon}_Y$. Then $E_{\left(\widehat{\psi},\widehat{\eta}\right)^c}$ is closed in $\widetilde{\Upsilon}_Y$, where $\widehat{\psi}^c\left(e\right) = Y \cap \left(\psi\left(e\right)\right)^c$ and $\widehat{\eta}^c\left(e\right) = Y \cap \left(\eta\left(e\right)\right)^c$.

4. Conclusion

In this paper, we have studied double-framed soft topological space. We have defined DFS open set, DFS closed set, DFS closure, DFS neighborhood , and investigated their basic properties. We also discussed how from soft topology we obtained DFS topology and vice verca.

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