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Pairwise soft separation axioms in soft bitopological spaces

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Abstract. The motivation of the current paper is to define and study some soft separation axioms in soft bitopological spaces in terms of pairwise softness namely, pairwise soft T_0^*, pairwise soft T_1^*, pairwise soft T_2^*, and pairwise soft R_1^*. Characterizations and properties of these soft separation axioms have been obtained. Moreover, we study the implications of these types of soft separation axioms in soft and crisp cases. Finally, we show that these properties are hereditary.

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1. Introduction

In 1999, Molodtsov [14] introduced the soft set theory as a new mathematical tool for dealing with uncertainties inherent in many of real world problems. This theory is a relatively new approach to discuss vagueness and uncertainties. It is getting popularity among the researchers and a good number of papers is being published every year. The main characteristic of soft set theory is that it is free of the difficulties in the theories of probability, fuzzy set, interval-valued fuzzy set and other theories. Some important applications of soft sets are in decision making, data mining, medical diagnosis and complete (incomplete) information systems. Pei and Miao [18] showed that the soft set is a simple information system. Topological structure of soft sets also was studied by many of authors see [1, 4, 5, 6, 15, 16, 17, 19, 22]. Soft separation axioms in soft topological spaces studied in some papers (see, for example, [4, 16]). Ittanagi [2] introduced the concept of soft bitopological space and studied some types of soft separation axioms for soft bitopological spaces from his point of view. Recently, Kandil et al. [8] introduced the concept of generalized pairwise closed soft sets and the associated pairwise soft separation axioms namely, PSR_0^* and $PST_{\frac{1}{2}}^*$.

The present paper is a continuation of [8], [9] and [10]. We introduce some pairwise soft separation axioms in soft bitopological spaces namely, PST_0^*, PST_1^*, PSR_1^*, and PST_2^* and study some of their properties. We show that these axioms are hereditary properties. Moreover, we studied the implications of these types of soft separation axioms in soft and crisp cases.

2. Preliminaries

In this section, we briefly review some concepts and some related topics of soft sets, soft topological spaces and soft bitopological spaces which are needed to used in current paper.

Definition 2.1 ([16])**.** A pair (F, E) is called a soft set over X, where F is a mapping given by $F : E \to P(X)$. A soft set can also be defined by the set of ordered pairs

$$(F, E) = \{(e, F(e)) : e \in E, \ \ F : E \to P(X)\}.$$

From now on, $SS(X)_E$ denotes the family of all soft sets over X with a fixed set of parameters E.

Definition 2.2 ([16])**.** For two soft sets (F, E), $(G, E) \in SS(X)_E$, (F, E) is called a soft subset of (G, E), denoted by $(F, E)\tilde{\subseteq}(G, E)$, if $F(e) \subseteq G(e)$, $\forall e \in E$. In this case, (G, E) is called a soft superset of (F, E).

Definition 2.3 ([16])**.** The union of the soft sets (F, E) and (G, E), denoted by $(F, E)\tilde{\cup}(G, E)$, is the soft set (H, E) which defined as $H(e) = F(e) \cup G(e)$, $\forall e \in E$.

Definition 2.4 ([16])**.** The intersection of the soft sets (F, E) and (G, E), denoted by $(F, E)\tilde{\cap}(G, E)$, is the soft set (M, E) which defined as $M(e) = F(e) \cap G(e)$, $\forall e \in E$.

Definition 2.5 ([16])**.** The complement of the soft set (F, E), denoted by $(F, E)^c$, is defined as $(F, E)^c = (F^c, E)$, where $F^c : E \to P(X)$ is a mapping given by $F^c(e) = X \setminus F(e)$, $\forall\, e \in E$.

Definition 2.6 ([16])**.** The difference of the soft sets (F, E) and (G, E), denoted by $(F, E) \setminus (G, E)$, is the soft set (H, E), where for all $e \in E$, $H(e) = F(e) \setminus G(e)$.

Clearly, $(F, E) \setminus (G, E) = (F, E)\tilde{\cap}(G, E)^c$.

Definition 2.7 ([16])**.** A soft set (F, E) is called a null soft set, denoted by $(\tilde{\phi}, E)$, if $F(e) = \phi$ for all $e \in E$. Moreover, a soft set (F, E) is called an absolute soft set, denoted by $(\tilde{X}, E)$, if $F(e) = X$, $\forall e \in E$.

Clearly, we have $(\tilde{\phi}, E)^c = (\tilde{X}, E)$ and $(\tilde{X}, E)^c = (\tilde{\phi}, E)$.

For more details about the properties of the union, the intersection and the complement of soft sets, you can see [4, 7, 18, 20, 22].

Definition 2.8 ([1, 15, 20])**.** A soft set $(F, E) \in SS(X)_E$ is called a soft point in $(\tilde{X}, E)$, if there exist $x \in X$ and $e \in E$ such that $F(e) = \{x\}$ and $F(e') = \phi$ for each $e' \in E \setminus \{e\}$. This soft point is denoted by (x_e, E) or x_e, i.e.,
$x_e : E \to P(X)$ is a mapping defined by

$$x_e(a) = \begin{cases} \{x\} & if \ \ e = a, \\ \phi & if \ \ e \neq a \end{cases} \qquad \text{for all } a \in E.$$

The set of all soft points in $(\tilde{X}, E)$ is denoted by $\xi(X)_E$.

Definition 2.9 ([20]). A soft point (x_e, E) is said to be belonging to the soft set (G, E), denoted by $x_e \tilde{\in} (G, E)$, if $x_e(e) \subseteq G(e)$, i.e., $\{x\} \subseteq G(e)$.

Clearly, $x_e \tilde{\in} (G, E)$ if and only if $(x_e, E) \tilde{\subseteq} (G, E)$.

Definition 2.10 ([20]). A two soft points x_{e_1}, y_{e_2} over X are said to be equal, if $x = y$ and $e_1 = e_2$.

Thus, $x_{e_1} \neq y_{e_2}$ iff $x \neq y$ or $e_1 \neq e_2$.

Proposition 2.11. [20] *The union of any collection of soft points can be considered as a soft set and every soft set can be expressed as a union of all soft points belonging to it, i.e.,* $(G, E) = \bigcup\{(x_e, E) : x_e \tilde{\in} (G, E)\}$.

Proposition 2.12. [20] *Let* (G, E), (H, E) *be two soft sets over* X. *Then*
(1) $x_e \tilde{\in} (G, E) \Leftrightarrow x_e \tilde{\notin} (G, E)^c$,
(2) $x_e \tilde{\in} (G, E) \tilde{\cup} (H, E) \Leftrightarrow x_e \tilde{\in} (G, E)$ *or* $x_e \tilde{\in} (H, E)$,
(3) $x_e \tilde{\in} (G, E) \tilde{\cap} (H, E) \Leftrightarrow x_e \tilde{\in} (G, E)$ *and* $x_e \tilde{\in} (H, E)$,
(4) $(G, E) \tilde{\subseteq} (H, E) \Leftrightarrow [x_e \tilde{\in} (G, E) \Rightarrow x_e \tilde{\in} (H, E)]$.

Definition 2.13 ([19]). Let η be a collection of soft sets over a universe X with a fixed set of parameters E. Then, $\eta \subseteq SS(X)_E$ is called a soft topology on X if it satisfies the following axioms:
(i) $(\tilde{X}, E), (\tilde{\phi}, E) \in \eta$, where $\tilde{\phi}(e) = \phi$ and $\tilde{X}(e) = X$, $\forall e \in E$,
(ii) the union of any number of soft sets in η belongs to η,
(iii) The intersection of any two soft sets in η belongs to η.

The triple (X, η, E) is called a soft topological space. Any member of η is said to be an open soft set. A soft set (F, E) over X is said to be a closed soft set in (X, η, E), if its complement $(F, E)^c$ is an open soft set in (X, η, E).

Definition 2.14 ([16]). A soft topological space (X, η, E) is said to be a soft T_0[briefly, ST_0], if for each $x_\alpha,\ y_\beta \in \xi(X)_E$ with $x_\alpha \neq y_\beta$, there exists $(G, E) \in \eta$ such that

$$x_\alpha \tilde{\in} (G, E),\ y_\beta \tilde{\notin} (G, E) \text{ or } y_\beta \tilde{\in} (G, E),\ x_\alpha \tilde{\notin} (G, E).$$

Definition 2.15 ([11]). A soft set (G, E) in a soft topological space (X, η, E) is called a generalized closed soft set [briefly, g-closed soft set], if $scl_\eta(G, E) \tilde{\subseteq} (H, E)$, whenever

$$(G, E) \tilde{\subseteq} (H, E) \text{ and } (H, E) \in \eta.$$

Definition 2.16 ([11]). A soft topological space (X, η, E) is called a soft $T_{\frac{1}{2}}$ [briefly, $ST_{\frac{1}{2}}$], if every g-closed soft set is a closed soft set.

Theorem 2.17 ([8]). *A soft topological space* (X, η, E) *is a soft* $T_{\frac{1}{2}}$ *if and only if every soft point either open soft set or closed soft set.*

Definition 2.18 ([16]). A soft topological space (X, η, E) is said to be a soft T_1[briefly, ST_1], if for each $x_\alpha,\ y_\beta \in \xi(X)_E$ with $x_\alpha \neq y_\beta$, there exist $(G, E), (H, E) \in \eta$ such that $x_\alpha \tilde{\in} (G, E)$, $y_\beta \tilde{\notin} (G, E)$ and $y_\beta \tilde{\in} (H, E)$, $x_\alpha \tilde{\notin} (H, E)$.

Theorem 2.19 ([16]). *A soft topological space* (X, η, E) *is a soft* T_1 *if and only if every soft point is a closed soft set.*

Definition 2.20 ([16]). A soft topological space (X, η, E) is said to be a soft T_2[briefly, ST_2], if for each $x_\alpha,\ y_\beta \in \xi(X)_E$ with $x_\alpha \neq y_\beta$, there exist $(G, E), (H, E) \in \eta$ such that $x_\alpha \tilde{\in} (G, E)$, $y_\beta \tilde{\in} (H, E)$ and $(G, E) \tilde{\cap} (H, E) = (\tilde{\phi}, E)$.

For more details about the properties of the soft topological space, you can see [1, 4, 5, 6, 13, 15, 16, 17, 19, 21, 22].

Definition 2.21 ([12]). A triple (X, τ_1, τ_2) is called a bitopological space [briefly, bts], where $\tau_1,\ \tau_2$ are arbitrary topologies on X. The collection τ_{12} is a supra topology on X, where

$$\tau_{12} = \{G \subseteq X : G = G_1 \cup G_2; G_i \in \tau_i, i = 1, 2\}.$$

Definition 2.22 ([3]). Let (X, τ_1, τ_2) be a bts. Then (X, τ_1, τ_2) is called:

(i) PT_0, if for every $x, y \in X,\ x \neq y$, there exists $G \in \tau_{12}$ such that
$x \in G,\ y \notin G$ or $x \notin G,\ y \in G$,

(ii) PT_1, if for every $x, y \in X,\ x \neq y$, there exist $G, H \in \tau_{12}$ such that
$x \in G,\ y \notin G$ and $x \notin H,\ y \in H$,

(iii) PT_2, if for every $x, y \in X,\ x \neq y$, there exist $G, H \in \tau_{12}$ such that
$x \in G,\ y \in H$ and $G \cap H = \phi$,

(iv) PR_0, if $x \in cl_{12}\{y\} \Rightarrow y \in cl_{12}\{x\}$,

(v) PR_1, if $cl_{12}\{x\} \neq cl_{12}\{y\}$ there exist $G, H \in \tau_{12}$ such that
$cl_{12}\{x\} \subseteq G,\ cl_{12}\{y\} \subseteq H$ and $G \cap H = \phi$.

Definition 2.23 ([2]). A quadrable system (X, η_1, η_2, E) is called a soft bitopological space [briefly, sbts], where $\eta_1,\ \eta_2$ are arbitrary soft topologies on X and E be a set of parameters.

Definition 2.24 ([9]). Let (X, η_1, η_2, E) be a sbts.

(i) A soft set (G, E) over X is said to be a pairwise open soft set in (X, η_1, η_2, E) [briefly, p-open soft set], if there exist an open soft set (G_1, E) in η_1 and an open soft set (G_2, E) in η_2 such that $(G, E) = (G_1, E) \tilde{\cup} (G_2, E)$.

(ii) A soft set (G, E) over X is said to be a pairwise closed soft set in (X, η_1, η_2, E) [briefly, p-closed soft set], if its complement is a p-open soft set in (X, η_1, η_2, E).

The family of all p-open (p-closed) soft sets in sbts (X, η_1, η_2, E) is denoted by η_{12} (η_{12}^c), respectively, i.e., $\eta_{12} = \{(G, E) \in SS(X)_E : (G, E) = (G_1, E) \tilde{\cup} (G_2, E) : (G_i, E) \in \eta_i, i = 1, 2\}$. Moreover, η_{12} is a supra soft topology on X.

Definition 2.25 ([9]). Let (X, η_1, η_2, E) be a sbts and let $(G, E) \in SS(X)_E$. The pairwise soft closure of (G, E), denoted by $scl_{12}(G, E)$, is the intersection of all p-closed soft super sets of (G, E), i.e.,

$$scl_{12}(G, E) = \tilde{\bigcap} \{(F, E) \in \eta_{12}^c : (G, E) \tilde{\subseteq} (F, E)\}.$$

Clearly, $scl_{12}(G, E)$ is the smallest p-closed soft set containing (G, E).

Definition 2.26 ([9]). Let (X, η_1, η_2, E) be a sbts and let $(G, E) \in SS(X)_E$. The pairwise soft interior of (G, E), denoted by $sint_{12}(G, E)$, is the union of all p-open soft subsets of (G, E), i.e.,

$$sint_{12}(G, E) = \tilde{\bigcup} \{(H, E) \in \eta_{12} : (H, E) \tilde{\subseteq} (G, E)\}.$$

Clearly, $sint_{12}(G, E)$ is the largest p-open soft set contained in (G, E).

For more details about pairwise soft closure (interior) operator see [9].

Definition 2.27 ([9])**.** Let (X, η_1, η_2, E) be a sbts and let $(G, E) \in SS(X)_E$. The pairwise soft kernel of (G, E) [briefly, $sker_{12}(G, E)$], is the intersection of all p-open soft supersets of (G, E), i.e.,

$$sker_{12}(G, E) = \tilde{\bigcap}\{(H, E) \in \eta_{12} : (G, E)\tilde{\subseteq}(H, E)\}.$$

Theorem 2.28 ([9])**.** *Let (X, η_1, η_2, E) be a sbts and let $(G, E), (H, E) \in SS(X)_E$. Then*

(1) $sker_{12}(\tilde{X}, E) = (\tilde{X}, E)$ *and* $sker_{12}(\tilde{\phi}, E) = (\tilde{\phi}, E)$,
(2) $(G, E)\tilde{\subseteq} sker_{12}(G, E)$,
(3) $(G, E)\tilde{\subseteq}(H, E) \ \Rightarrow \ sker_{12}(G, E)\tilde{\subseteq} sker_{12}(H, E)$,
(4) *if* $(G, E) \in \eta_{12}$, *then* $sker_{12}(G, E) = (G, E)$,
(5) $sker_{12}[sker_{12}(G, E)] = sker_{12}(G, E)$,
(6) $sker_{12}[\tilde{\bigcap}\{(H_i, E) : i \in \Delta\}]\tilde{\subseteq}\tilde{\bigcap}\{sker_{12}(H_i, E) : i \in \Delta\}$,
(7) $sker_{12}[\tilde{\bigcup}\{(G_i, E) : i \in \Delta\}] = \tilde{\bigcup}\{sker_{12}(G_i, E) : i \in \Delta\}$.

Definition 2.29 ([9])**.** A soft set (G, E) is said to be a pairwise Λ- soft set in a sbts (X, η_1, η_2, E) [briefly, $p\Lambda$-soft set], if $sker_{12}(G, E) = (G, E)$.

Theorem 2.30 ([9])**.** *Let (X, η_1, η_2, E) be a sbts. Then, The family of all $p\Lambda$-soft sets is a soft topology on X. This soft topology, we denoted by $\eta_{p\Lambda}$. The triple $(X, \eta_{p\Lambda}, E)$ is the soft topological space associated to the sbts (X, η_1, η_2, E). Members of $\eta_{p\Lambda}$ are called $p\Lambda$-open soft sets. A soft set (G, E) in a sbts (X, η_1, η_2, E) is called a $p\Lambda$-closed soft set, if its complement is a $p\Lambda$-open soft set. We denote the family of all $p\Lambda$-closed soft set by $\eta^c_{p\Lambda}$.*

Theorem 2.31 ([9])**.** *Let (X, η_1, η_2, E) be a sbts. Then,*

$$\eta_1 \cup \eta_2 \subseteq \eta_{12} \subseteq \eta_{p\Lambda} \subseteq SS(X)_E.$$

Definition 2.32 ([9])**.** A soft set (G, E) is said to be a pairwise λ-closed soft set in a sbts (X, η_1, η_2, E) [briefly, $p\lambda$-closed soft set] if $(G, E) = (F, E)\tilde{\cap}(H, E)$, where (F, E) is a p-closed soft set and (H, E) is a $p\Lambda$-soft set.

The family of all $p\lambda$-closed soft sets we denoted by $P\lambda CS(X, \eta_1, \eta_2)_E$.

Theorem 2.33 ([9])**.** *Let (X, η_1, η_2, E) be a sbts. Then*

(1) *every p-closed (p-open)soft set is a $p\lambda$-closed soft set,*
(2) *every $p\Lambda$-open soft set is a $p\lambda$-closed soft set.*

Definition 2.34 ([8])**.** Let (X, η_1, η_2, E) be a sbts. A soft set (G, E) is said to be a gp-closed soft set, if $scl_{12}(G, E)\tilde{\subseteq}(H, E)$, whenever $(G, E)\tilde{\subseteq}(H, E)$ and (H, E) is a p-open soft set.

Theorem 2.35 ([8])**.** *Let (X, η_1, η_2, E) be a sbts and $(G, E) \in SS(X)_E$. Then every p-closed soft set is a gp-closed soft set.*

Definition 2.36 ([8])**.** A sbts (X, η_1, η_2, E) is said to be a pairwise soft $T^*_{\frac{1}{2}}$ [briefly, $PST^*_{\frac{1}{2}}$], if every gp-closed soft set is a p-closed soft set.

Theorem 2.37 ([8])**.** *Let (X, η_1, η_2, E) be a sbts. Then*

(X, η_1, η_2, E) is a $PST^*_{\frac{1}{2}}$ iff every soft point is either p-open soft set or p-closed soft set.

Definition 2.38 ([8]). A sbts (X, η_1, η_2, E) is said to be a pairwise soft R_0^* [briefly, PSR_0^*], if $x_\alpha \tilde{\in} scl_{12}(y_\beta, E) \Rightarrow y_\beta \tilde{\in} scl_{12}(x_\alpha, E)$, where $x_\alpha, y_\beta \in \xi(X)_E$.

Theorem 2.39 ([8]). *A sbts (X, η_1, η_2, E) is a PSR_0^* iff every soft point is a gp-closed soft set.*

Theorem 2.40 ([8]). *A sbts (X, η_1, η_2, E) is a PSR_0^* iff $x_\alpha \tilde{\in} scl_{12}(y_\beta, E) \Rightarrow scl_{12}(x_\alpha, E) = scl_{12}(y_\beta, E)$.*

Theorem 2.41. [8]*] Let (X, η_1, η_2, E) be a sbts. Then, (X, η_1, η_2, E) is a PSR_0^* if and only if $scl_{12}(x_\alpha, E) \tilde{\cap} scl_{12}(y_\beta, E) \neq (\tilde{\phi}, E) \Rightarrow scl_{12}(x_\alpha, E) = scl_{12}(y_\beta, E)$.*

Theorem 2.42 ([8]). *A sbts (X, η_1, η_2, E) is a PSR_0^* iff $pscl_{12}(x_\alpha, E) = psker_{12}(x_\alpha, E)$, $\forall\, x_\alpha \in \xi(X)_E$.*

3. Pairwise soft separation axioms

Definition 3.1. A sbts (X, η_1, η_2, E) is said to be a pairwise soft T_0^*[briefly, PST_0^*], if for each $x_\alpha,\ y_\beta \in \xi(X)_E$ with $x_\alpha \neq y_\beta$, there exists $(G, E) \in \eta_{12}$ such that

$$x_\alpha \tilde{\in} (G, E),\ y_\beta \tilde{\notin} (G, E) \text{ or } y_\beta \tilde{\in} (G, E),\ x_\alpha \tilde{\notin} (G, E).$$

Example 3.2. Let $X = \{x, y\}$, $E = \{e_1, e_2\}$ and let

$$\eta_1 = \{(\tilde{\phi}, E), (\tilde{X}, E), (G_1, E), (G_2, E)\},$$
$$\eta_2 = \{(\tilde{\phi}, E), (\tilde{X}, E), (H_1, E), (H_2, E)\},$$

where

$$(G_1, E) = \{(e_1, \{x\}), (e_2, \phi)\},\ (G_2, E) = \{(e_1, X), (e_2, \{x\})\},$$
$$(H_1, E) = \{(e_1, \{y\}), (e_2, \phi)\},\ (H_2, E) = \{(e_1, X), (e_2, \{y\})\}.$$

Then, (X, η_1, η_2, E) is a sbts. Thus,

$$\eta_{12} = \{(\tilde{\phi}, E), (\tilde{X}, E), (G_1, E), (G_2, E), (H_1, E), (H_2, E), (P, E)\},$$

where

$(P, E) = \{(e_1, X), (e_2, \phi)\}$. It is clear that (X, η_1, η_2, E) is a PST_0^*.

Example 3.3. Let $X = \{x, y, z\}$, $E = \{e_1, e_2\}$ and

$$\eta_1 = \{(\tilde{\phi}, E), (\tilde{X}, E), (G_1, E), (G_2, E)\},$$
$$\eta_2 = \{(\tilde{\phi}, E), (\tilde{X}, E), (H, E)\},$$

where

$$(G_1, E) = \{(e_1, \{x\}), (e_2, \{x, y\})\},$$
$$(G_2, E) = \{(e_1, \{x\}), (e_2, \{y\})\},$$
$$(H, E) = \{(e_1, \{x, y\}), (e_2, \{z\})\}.$$

Then, (X, η_1, η_2, E) is a sbts. Thus,

$$\eta_{12} = \{(\tilde{\phi}, E), (\tilde{X}, E), (G_1, E), (G_2, E), (H, E), (P_1, E), (P_2, E)\},$$

where

$$(P_1, E) = \{(e_1, \{x, y\}), (e_2, X)\},$$
$$(P_2, E) = \{(e_1, \{x, y\}), (e_2, \{y, z\})\}.$$

So, (X, η_1, η_2, E) is not PST_0^* because $y_{e_1} \neq z_{e_2}$ and there is no p-open soft set contains y_{e_1} but not contains z_{e_2} or contains z_{e_2} but not contains y_{e_1}.

Theorem 3.4. *Let (X, η_1, η_2, E) be a sbts. The following statements are equivalent:*

(1) *(X, η_1, η_2, E) is a PST_0^*,*
(2) *$scl_{12}(x_\alpha, E) \neq scl_{12}(y_\beta, E)$, $\forall\, x_\alpha,\ y_\beta \in \xi(X)_E$, $x_\alpha \neq y_\beta$,*
(3) *$sker_{12}(x_\alpha, E) \neq sker_{12}(y_\beta, E)$, $\forall\, x_\alpha,\ y_\beta \in \xi(X)_E$, $x_\alpha \neq y_\beta$.*

Proof. (1) $\Rightarrow$ (2): Let $x_\alpha,\ y_\beta \in \xi(X)_E$ such that $x_\alpha \neq y_\beta$. Then, by (1), there exists $(G,E) \in \eta_{12}$ such that $x_\alpha \tilde{\in} (G,E)$, $y_\beta \tilde{\notin} (G,E)$. It follows that $x_\alpha \tilde{\notin} (G,E)^c$, $y_\beta \tilde{\in} (G,E)^c$. Thus, $scl_{12}(y_\beta,E) \tilde{\subseteq} (G,E)^c$. Since $x_\alpha \tilde{\notin} (G,E)^c$, $x_\alpha \tilde{\notin} scl_{12}(y_\beta,E)$. So, $scl_{12}(x_\alpha,E) \neq scl_{12}(y_\beta,E)$[for $x_\alpha \tilde{\in} scl_{12}(x_\alpha,E)$]. Hence, (2) holds.

(2) $\Rightarrow$ (3) : Let $x_\alpha,\ y_\beta \in \xi(X)_E$ such that $x_\alpha \neq y_\beta$. Then, by (2), $scl_{12}(x_\alpha,E) \neq scl_{12}(y_\beta,E)$. Thus, there exists a soft point z_e such that $z_e \tilde{\in} scl_{12}(x_\alpha,E)$ and $z_e \tilde{\notin} scl_{12}(y_\beta,E)$ or $z_e \tilde{\notin} scl_{12}(x_\alpha,E)$ and $z_e \tilde{\in} scl_{12}(y_\beta,E)$.

We claim that $z_e \tilde{\in} scl_{12}(x_\alpha,E)$ and $z_e \tilde{\notin} scl_{12}(y_\beta,E)$. Since $z_e \tilde{\notin} scl_{12}(y_\beta,E)$, there exists a p-closed soft set (F,E) such that $y_\beta \tilde{\in} (F,E)$ and $z_e \tilde{\notin} (F,E)$, or equivalently, $y_\beta \tilde{\notin} (F,E)^c$ and $z_e \tilde{\in} (F,E)^c$. So, $x_\alpha \tilde{\in} (F,E)^c$ because if $x_\alpha \tilde{\notin} (F,E)^c$, then $x_\alpha \tilde{\in} (F,E)$ which implies that $scl_{12}(x_\alpha,E) \tilde{\subseteq} (F,E)$ which contradicts with $z_e \tilde{\notin} (F,E)$. Now, since $x_\alpha \tilde{\in} (F,E)^c$, $sker_{12}(x_\alpha,E) \tilde{\subseteq} (F,E)^c$. It follows that $y_\beta \tilde{\notin} sker_{12}(x_\alpha,E)$ but $y_\beta \tilde{\in} sker_{12}(y_\beta,E)$, then $sker_{12}(x_\alpha,E) \neq sker_{12}(y_\beta,E)$. Hence, (3) holds.

(3) $\Rightarrow$ (1) : Let $x_\alpha,\ y_\beta \in \xi(X)_E$ such that $x_\alpha \neq y_\beta$. Then, by (3), $sker_{12}(x_\alpha,E) \neq sker_{12}(y_\beta,E)$. Thus, there exists a soft point z_e such that $z_e \tilde{\in} sker_{12}(x_\alpha,E)$ and $z_e \tilde{\notin} sker_{12}(y_\beta,E)$ or $z_e \tilde{\notin} sker_{12}(x_\alpha,E)$ and $z_e \tilde{\in} sker_{12}(y_\beta,E)$.

We claim that $z_e \tilde{\in} sker_{12}(x_\alpha,E)$ and $z_e \tilde{\notin} sker_{12}(y_\beta,E)$, then there exists $(G,E) \in \eta_{12}$ such that $y_\beta \tilde{\in} (G,E)$ but $z_e \tilde{\notin} (G,E)$. So, $x_\alpha \tilde{\notin} (G,E)$, for if $x_\alpha \tilde{\in} (G,E)$, then $sker_{12}(x_\alpha,E) \tilde{\subseteq} (G,E)$ which contradicts with $z_e \tilde{\notin} (G,E)$]. Hence, there exists $(G,E) \in \eta_{12}$ such that $y_\beta \tilde{\in} (G,E)$ and $x_\alpha \tilde{\notin} (G,E)$. Therefore, (X,η_1,η_2,E) is a PST_0^*. □

Lemma 3.5. *Let (X,η_1,η_2,E) be a sbts. Then,*

(X,η_1,η_2,E) is a PST_0^* if and only if for all $x_\alpha, y_\beta \in \xi(X)_E$, $x_\alpha \neq y_\beta$, there exists $(G,E) \in \eta_{12} \cup \eta_{12}^c$ such that $x_\alpha \tilde{\in} (G,E)$ and $y_\beta \tilde{\notin} (G,E)$.

Proof. Straightforward. □

Theorem 3.6. *Let (X,η_1,η_2,E) be a sbts. Then,*

(X,η_1,η_2,E) is a PST_0^* if and only if every soft point $x_e \in \xi(X)_E$ is a $p\lambda$-closed soft set.

Proof. $(\Rightarrow)$: Let $x_e \in \xi(X)_E$. Then, by Lemma 3.5, for each $y_\alpha \in \xi(X)_E$ such that $y_\alpha \neq x_e$, there exists $(G,E) \in \eta_{12} \cup \eta_{12}^c$ such that $x_e \tilde{\in} (G,E)$ and $y_\alpha \tilde{\notin} (G,E)$.

We set

$$(M,E) = \tilde{\bigcap}\{(G,E) \in \eta_{12} : x_e \tilde{\in} (G,E), y_\alpha \tilde{\notin} (G,E)\}$$

and

$$(N,E) = \tilde{\bigcap}\{(F,E) \in \eta_{12}^c : x_e \tilde{\in} (F,E), y_\alpha \tilde{\notin} (F,E)\}.$$

Then,

$$\begin{aligned}
&sker_{12}(M,E)\\
&= sker_{12}\tilde{\bigcap}\{(G,E) \in \eta_{12} : x_e \tilde{\in} (G,E), y_\alpha \tilde{\notin} (G,E)\}\\
&\tilde{\subseteq} \tilde{\bigcap}\{sker_{12}(G,E) \in \eta_{12} : x_e \tilde{\in} (G,E), y_\alpha \tilde{\notin} (G,E)\}\text{[by Theorem 2.28 (6)]}\\
&= \tilde{\bigcap}\{(G,E) \in \eta_{12} : x_e \tilde{\in} (G,E), y_\alpha \tilde{\notin} (G,E)\} \text{ [by Theorem 2.28 (4)]}\\
&= (M,E).
\end{aligned}$$

Thus, $sker_{12}(M,E) \tilde{\subseteq} (M,E)$. So, (M,E) is a $p\Lambda$-soft set. Also, it is clear that (N,E) is a p-closed soft set. Consequently, $(N,E) \tilde{\cap} (M,E)$ is a $p\lambda$-closed soft set. Now, if

$z_\beta \tilde{\in} (N, E) \tilde{\cap} (M, E)$, then $z_\beta \tilde{\in} (N, E)$ and $z_\beta \tilde{\in} (M, E)$. It follows that $z_\beta \neq y_\alpha$ $\forall y_\alpha$, $y_\alpha \neq x_e$. Hence, $z_\beta = x_e$ and thus $(N, E) \tilde{\cap} (M, E) = (x_e, E)$. Therefore, (x_e, E) is a $p\lambda$-closed soft set.

Conversely, let $x_\alpha \in \xi(X)_E$. Then , by hypothesis, $(x_\alpha, E) = (F, E) \tilde{\cap} (G, E)$, where (F, E) is a p-closed soft set and (G, E) is a $p\Lambda$-soft set. For each $y_\beta \in \xi(X)_E$ such that $x_\alpha \neq y_\beta$, we have $y_\beta \tilde{\notin} (F, E) \tilde{\cap} (G, E)$ implies $y_\beta \tilde{\notin} (F, E)$ or $y_\beta \tilde{\notin} (G, E)$. Here we have two cases:

Case 1: If $y_\beta \tilde{\notin} (F, E)$, then $y_\beta \tilde{\in} (F, E)^c$ and $x_\alpha \tilde{\notin} (F, E)^c$.

Case 2: If $y_\beta \tilde{\notin} (G, E)$, then $y_\beta \tilde{\notin} sker_{12}(G, E)$. Thus, there exists $(M, E) \in \eta_{12}$ such that $(G, E) \tilde{\subseteq} (M, E)$ and $y_\beta \tilde{\notin} (M, E)$. But $x_\alpha \tilde{\in} (G, E)$. So $x_\alpha \tilde{\in} (M, E)$.

From both cases, we have (X, η_1, η_2, E) is a PST_0^*. □

Theorem 3.7. *If (X, η_1, E) or (X, η_2, E) is an ST_0, then (X, η_1, η_2, E) is PST_0^*.*

Proof. It follows from the fact $\eta_i \subseteq \eta_{12}$, $i = 1, 2$. □

Remark 3.8. The converse of Theorem 3.7 is not true in general which shown in the Example 3.2. It is clear that (X, η_1, E) is not ST_0 because $x_{e_2} \neq y_{e_1}$ and there is no open soft set in η_1 which contains one of points but not contains the other.

Similarly, (X, η_2, E) is not ST_0.

Theorem 3.9. *A sbts (X, η_1, η_2, E) is PST_0^* if and only if $(X, \eta_{p\Lambda}, E)$ is an ST_0.*

Proof. The first direction is immediate from the fact that $\eta_{12} \subseteq \eta_{p\Lambda}$. To prove the inverse direction, let $x_\alpha, y_\beta \in \xi(X)_E$ with $x_\alpha \neq y_\beta$. Then, by hypothesis, there exists a $p\Lambda$-soft set $(G, E) \in \eta_{p\Lambda}$ such that $x_\alpha \tilde{\in} (G, E)$, $y_\beta \tilde{\notin} (G, E)$(say). Thus, $y_\beta \tilde{\notin} sker_{12}(G, E)$. So, there exists a p-open soft set $(H, E) \in \eta_{12}$ such that

$$(G, E) \tilde{\subseteq} (H, E) \text{ and } y_\beta \tilde{\notin} (H, E).$$

Hence, $(H, E) \in \eta_{12}$, $x_\alpha \tilde{\in} (H, E)$ and $y_\beta \tilde{\notin} (H, E)$. Therefore, (X, η_1, η_2, E) is a PST_0^*. □

Theorem 3.10. *Every $PST_{\frac{1}{2}}^*$ is a PST_0^*.*

Proof. Let (X, η_1, η_2, E) be a $PST_{\frac{1}{2}}^*$ and let x_α, $y_\beta \in \xi(X)_E$ such that $x_\alpha \neq y_\beta$. Then, $y_\beta \tilde{\notin} (x_\alpha, E)$. Now, since x_α is a soft point in sbts (X, η_1, η_2, E), it follows that, by Theorem 2.37, (x_α, E) is either p-closed soft set or p-open soft set.

If (x_α, E) is a p-closed soft set, then $(x_\alpha, E)^c$ is a p-open soft set, i.e., $(x_\alpha, E)^c \in \eta_{12}$ and $y_\beta \tilde{\in} (x_\alpha, E)^c$, $x_\alpha \tilde{\notin} (x_\alpha, E)^c$.

If (x_α, E) is a p-open soft set, then $(x_\alpha, E) \in \eta_{12}$ and $y_\beta \tilde{\notin} (x_\alpha, E)$, $x_\alpha \tilde{\in} (x_\alpha, E)$. Thus, (X, η_1, η_2, E) is a PST_0^*. □

Remark 3.11. The converse of Theorem 3.10 is not true in general which shown in the following example.

Example 3.12. Let $X = \{x, y\}$, $E = \{e_1, e_2\}$ and let

$$\eta_1 = \{(\tilde{\phi}, E), (\tilde{X}, E), (G_1, E), (G_2, E)\},\ \eta_2 = \{(\tilde{\phi}, E), (\tilde{X}, E), (H, E)\},$$

where

$(G_1, E) = \{(e_1, \{x\}), (e_2, \{y\})\}$,
$(G_2, E) = \{(e_1, \{y\}), (e_2, \{x\})\}$,

$(H, E) = \{(e_1, \{x\}), (e_2, \{x\})\}$.
Then, (X, η_1, η_2, E) is a sbts. Consequently,
$\eta_{12} = \{(\tilde{\phi}, E), (\tilde{X}, E), (G_1, E), (G_2, E), (H, E), (P_1, E), (P_2, E)\}$,
where $(P_1, E) = \{(e_1, \{x\}), (e_2, X)\}$, $(P_2, E) = \{(e_1, X), (e_2, \{x\})\}$.
It is clear that (X, η_1, η_2, E) is a PST_0^*. Since the soft point $(x_{e_1}, E) = \{(e_1, \{x\}), (e_2, \phi)\}$ is neither p-open soft set nor p-closed soft set, by Theorem 2.37, (X, η_1, η_2, E) is not $PST^*_{\frac{1}{2}}$.

Theorem 3.13. *If (X, η_1, η_2, E) is $PST^*_{\frac{1}{2}}$, then $(X, \eta_{p\Lambda}, E)$ is an $ST_{\frac{1}{2}}$.*

Proof. Straightforward. □

Remark 3.14. The converse of Theorem 3.13 is not true in general which shown in the following example.

Example 3.15. In Example 3.12, we deduced that (X, η_1, η_2, E) is not $PST^*_{\frac{1}{2}}$. The family of all soft points is $\xi(X)_E = \{x_{e_1}, x_{e_2}, y_{e_1}, y_{e_2}\}$. Now, $sker_{12}(x_{e_1}, E) = (G_1, E)\tilde{\cap}(H, E) = (x_{e_1}, E)$. Then, (x_{e_1}, E) is an open soft set in $\eta_{p\Lambda}$. Also, $sker_{12}(x_{e_2}, E) = (G_2, E)\tilde{\cap}(H, E) = (x_{e_2}, E)$. Thus, (x_{e_2}, E) is an open soft set in $\eta_{p\Lambda}$. Since $(P_1, E) \in \eta_{12}$, $(P_1, E) \in \eta_{p\Lambda}$ which implies that $(P_1, E)^c$ is a closed soft set in $\eta_{p\Lambda}$. But $(P_1, E)^c = (y_{e_1}, E)$. So (y_{e_1}, E) is a closed soft set in $\eta_{p\Lambda}$.

Similarly, $(P_2, E)^c = (y_{e_2}, E)$ it follows that (y_{e_2}, E) is a closed soft set in $\eta_{p\Lambda}$. Consequently, every soft point either open soft set or closed soft set in $\eta_{p\Lambda}$. Hence, $(X, \eta_{p\Lambda}, E)$ is a $ST_{\frac{1}{2}}$[by Theorem 2.17].

Definition 3.16. A sbts (X, η_1, η_2, E) is said to be a pairwise soft T_1^* [briefly PST_1^*], if for each $x_\alpha,\ y_\beta \in \xi(X)_E$ with $x_\alpha \neq y_\beta$, there exist (G, E), $(H, E) \in\ \eta_{12}$ such that $x_\alpha \tilde{\in} (G, E)$, $y_\beta \tilde{\notin} (G, E)$ and $y_\beta \tilde{\in} (H, E)$, $x_\alpha \tilde{\notin} (H, E)$.

Theorem 3.17. *Let (X, η_1, η_2, E) be a sbts. Then (X, η_1, η_2, E) is a PST_1^* if and only if every soft point over X is a p-closed soft set.*

Proof. ($\Rightarrow$): Let $x_\alpha \in \xi(X)_E$. Then by hypothesis, for each $y_\beta \in \xi(X)_E$ with $x_\alpha \neq y_\beta$, there exist (G, E), $(H, E) \in\ \eta_{12}$ such that $x_\alpha \tilde{\in} (G, E)$, $y_\beta \tilde{\notin} (G, E)$ and $y_\beta \tilde{\in} (H, E)$, $x_\alpha \tilde{\notin} (H, E)$. It follows that $scl_{12}(x_\alpha, E) \tilde{\subseteq} (H, E)^c$. Thus, $y_\beta \tilde{\notin} scl_{12}(x_\alpha, E)$ for all y_β with $x_\alpha \neq y_\beta$. So, $scl_{12}(x_\alpha, E) = (x_\alpha, E)$. Hence, every soft point over X is a p-closed soft set.

($\Leftarrow$): Obvious. □

Example 3.18. Let Z be the set of all integers numbers and E be a nonempty set of parameters. We denote $Z^-(Z^+)$ for the set of all negative (nonnegative) integers, respectively. Let

$$\eta_1 = \{(\tilde{Z}, E), (\tilde{\phi}, E)\} \bigcup \{(G, E) \in SS(Z)_E : G^c(e) \text{ is finite subset of } Z^+\ \forall\ e \in E\},$$

and

$$\eta_2 = \{(\tilde{Z}, E), (\tilde{\phi}, E)\} \bigcup \{(H, E) \in SS(Z)_E : H^c(e) \text{ is finite subset of } Z^-\ \forall\ e \in E\}.$$

It is easy to verify that η_1 and η_2 are soft topologies over Z. Then, (Z, η_1, η_2, E) is a sbts. Now, let $x_e \in \xi(Z)_E$. Then

$$x_e(a) = \begin{cases} \{x\} & if \ \ e = a \\ \phi & if \ \ e \neq a \end{cases} \quad \text{, for all } a \in E.$$

Thus, if $x \in Z^+$, then $(x_e, E)^c \in \eta_1$. If $x \in Z^-$, then $(x_e, E)^c \in \eta_2$. Consequently, for every soft point (x_e, E) in $(\tilde{Z}, E)$, we have $(x_e, E)^c \in \eta_1 \cup \eta_2 \subseteq \eta_{12}$. So, every soft point is a p-closed soft set. Hence, (Z, η_1, η_2, E) is a PST_1^*.

Theorem 3.19. *Let (X, η_1, η_2, E) be a sbts. If (X, η_1, η_2, E) is a PST_1^*,then every soft set is a $p\Lambda$-soft set, i.e., $\eta_{p\Lambda} = SS(X)_E$ is the discrete soft topology on X.*

Proof. Let (G, E) be an arbitrary soft set over X. From Theorem 2.28(2), we have $(G, E)\tilde{\subseteq} sker_{12}(G, E)$. Now, if $x_e\tilde{\notin}(G, E)$, then $(G, E)\tilde{\subseteq}(x_e, E)^c$ implies $sker_{12}(G, E)$ $\tilde{\subseteq} sker_{12}(x_e, E)^c$. Since (X, η_1, η_2, E) is a PST_1^*, $(x_e, E)^c$ is a p-open soft superset of (G, E). Thus, $sker_{12}(G, E)\tilde{\subseteq}(x_e, E)^c$. It follows that $x_e\tilde{\notin} sker_{12}(G, E)$. So, $sker_{12}(G, E)\tilde{\subseteq}(G, E)$. Hence, $sker_{12}(G, E) = (G, E)$. Therefore, (G, E) is a $p\Lambda$-soft set, i.e., $\eta_{p\Lambda} = SS(X)_E$. □

Theorem 3.20. *Every PST_1^* is a $PST_{\frac{1}{2}}^*$.*

Proof. It is immediate from Theorem 3.17 and Theorem 2.37. □

Corollary 3.21. *Every PST_1^* is a PST_0^*.*

Remark 3.22. The converse of Theorem 3.20 is not true in general which shown in the following example.

Example 3.23. From Example 3.2, we get
$\eta_{12}^c = \{(\tilde{\phi}, E), (\tilde{X}, E), (G_1, E)^c, (G_2, E)^c, (H_1, E)^c, (H_2, E)^c, (P, E)^c\}$,
where
$(G_1, E)^c = \{(e_1, \{y\}), (e_2, X)\}$, $(G_2, E)^c = \{(e_1, \phi), (e_2, \{y\})\}$,
$(H_1, E)^c = \{(e_1, \{x\}), (e_2, X)\}$, $(H_2, E)^c = \{(e_1, \phi), (e_2, \{x\})\}$,
$(P, E)^c = \{(e_1, \phi), (e_2, X)\}$.
Since a soft set (G, E) over X characterized by a function $G : E \to P(X)$, $\mid SS(X)_E \mid = \mid P(X) \mid^{|E|} = 2^{|X|.|E|}$. Then, in present example we have $\mid SS(X)_E \mid =$ 16. We set $SS(X)_E = \eta_{12}^c \bigcup \{(F_i, E) : i = 1, ..., 9\}$,
where
$(F_1, E) = \{(e_1, \{x\}), (e_2, \{x\})\}$, $(F_2, E) = \{(e_1, \{x\}), (e_2, \{y\})\}$,
$(F_3, E) = \{(e_1, \{x\}), (e_2, \phi)\}$, $(F_4, E) = \{(e_1, \{y\}), (e_2, \{y\})\}$,
$(F_5, E) = \{(e_1, \{y\}), (e_2, \{x\})\}$, $(F_6, E) = \{(e_1, \{y\}), (e_2, \phi)\}$,
$(F_7, E) = \{(e_1, X), (e_2, \{x\})\}$, $(F_8, E) = \{(e_1, X), (e_2, \{y\})\}$,
$(F_9, E) = \{(e_1, X), (e_2, \phi)\}$.
It is easy to verify that (F_i, E) is not gp-closed soft set, $i = 1, ..., 9$. It follows that, every gp-closed soft set is a p-closed soft set. Consequently, (X, η_1, η_2, E) is a $PST_{\frac{1}{2}}^*$. It is clear that (x_{e_1}, E) is a soft point but it is not p-closed soft set. Thus, (X, η_1, η_2, E) is not a PST_1^*.

Theorem 3.24. *If (X, η_1, E) or (X, η_2, E) is an ST_1, then (X, η_1, η_2, E) is a PST_1^*.*

Proof. Straightforward. □

Remark 3.25. The converse of Theorem 3.24 is not true in general which shown in the following example.

Example 3.26. In Example 3.18, we have (X, η_1, η_2, E) is a PST_1^*. If x is a positive integer and $e \in E$, then (x_e, E) is a soft point in $(\tilde{Z}, E)$ but it is not closed soft set in η_2, because $x_e(e) = \{x\} \nsubseteq Z^-$. Thus, by Theorem 2.19, (X, η_2, E) is not ST_1.

Similarly, if y is a negative integer and $a \in E$, then (y_a, E) is a soft point in $(\tilde{Z}, E)$ but it is not closed soft set in η_1, because $y_a(a) = \{y\} \nsubseteq Z^+$. Thus, (X, η_1, E) is not ST_1. Consequently, (X, η_1, E) and (X, η_2, E) are not ST_1. Though, we have (X, η_1, η_2, E) is a PST_1^*.

Theorem 3.27. *If (X, η_1, η_2, E) is a PST_1^*, then it is a PSR_0^*.*

Proof. Let (X, η_1, η_2, E) be a PST_1^*. Then, every soft point (x_e, E) is a p-closed soft set, it follows that every soft point (x_e, E) is a gp-closed soft sets [by Theorem 2.35]. Therefore, by Theorem 2.39, (X, η_1, η_2, E) is a PSR_0^*. □

Remark 3.28. The converse of Theorem 3.27 is not true in general which shown in the following example.

Example 3.29. Let $X = \{a, b\}$, $E = \{e_1, e_2\}$ and let

$$\eta_1 = \{(\tilde{\phi}, E), (\tilde{X}, E), (G, E)\},\ \eta_2 = \{(\tilde{\phi}, E), (\tilde{X}, E), (H, E)\},$$

where

$$(G, E) = \{(e_1, \{a\}), (e_2, \{b\})\},\ (H, E) = \{(e_1, \{b\}), (e_2, \{a\})\}.$$

Then, (X, η_1, η_2, E) is a sbts. Thus, $\eta_{12} = \{(\tilde{\phi}, E), (\tilde{X}, E), (G, E), (H, E)\}$.
Moreover, $\eta_{12}^c = \{(\tilde{\phi}, E), (\tilde{X}, E), (G, E), (H, E)\}$. It is clear that

$$scl_{12}(a_{e_1}, E) = scl_{12}(b_{e_2}, E) = (G, E) \text{ and } scl_{12}(a_{e_2}, E) = scl_{12}(b_{e_1}, E) = (H, E).$$

Thus, $a_{e_1} \tilde{\in} scl_{12}(b_{e_2}, E)$ and $b_{e_2} \tilde{\in} scl_{12}(a_{e_1}, E)$, $b_{e_1} \tilde{\in} scl_{12}(a_{e_2}, E)$ and $a_{e_2} \tilde{\in} scl_{12}(b_{e_1}, E)$.
Also, we have

$$a_{e_1} \tilde{\notin} scl_{12}(b_{e_1}, E),\ b_{e_1} \tilde{\notin} scl_{12}(a_{e_1}, E) \text{ and } a_{e_2} \tilde{\notin} scl_{12}(b_{e_2}, E),\ b_{e_2} \tilde{\notin} scl_{12}(a_{e_2}, E).$$

So, (X, η_1, η_2, E) is a PSR_0^*. It is clear that (a_{e_1}, E) is a soft point but it is not p-closed soft set. Hence, (X, η_1, η_2, E) is not PST_1^*.

Theorem 3.30. *A sbts (X, η_1, η_2, E) is a PST_1^* if and only if it is PST_0^* and PSR_0^*.*

Proof. The first direction is immediate from the Corollary 3.21 and Theorem 3.27. To prove the inverse direction, let $x_e \in \xi(X)_E$. Suppose that $y_\alpha \tilde{\in} scl_{12}(x_e, E)$. Then by PSR_0^* property, $scl_{12}(y_\alpha, E) = scl_{12}(x_e, E)$ [by Theorem 2.40]. By PST_0^* property, we have $(y_\alpha, E) = (x_e, E)$ [by Theorem 3.4 (2)]. Thus, $y_\alpha \tilde{\in} (x_e, E)$. So, $scl_{12}(x_e, E) = (x_e, E)$. Hence, (X, η_1, η_2, E) is a PST_1^*. □

Theorem 3.31. *If (X, η_1, η_2, E) is PST_1^* , then $(X, \eta_{p\Lambda}, E)$ is an ST_1.*

Proof. It is immediate from Theorem 3.19. □

Definition 3.32. A sbts (X, η_1, η_2, E) is said to be a pairwise soft R_1^* [briefly, PSR_1^*], if for each $x_\alpha,\ y_\beta \in \xi(X)_E$, $scl_{12}(x_\alpha, E) \neq scl_{12}(y_\beta, E)$, there exist (G, E), $(H, E) \in \eta_{12}$ such that $scl_{12}(x_\alpha, E) \tilde{\subseteq} (G, E)$, $scl_{12}(y_\beta, E) \tilde{\subseteq} (H, E)$ and $(G, E) \tilde{\cap} (H, E) = (\tilde{\phi}, E)$.

Theorem 3.33. *If (X, η_1, η_2, E) is a PSR_1^*, then it is PSR_0^*.*

Proof. Let $scl_{12}(x_\alpha, E) \neq scl_{12}(y_\beta, E)$. Then, by hypothesis, there exists (G, E), $(H, E) \in\ \eta_{12}$ such that

$scl_{12}(x_\alpha, E)\tilde{\subseteq}(G, E)$, $scl_{12}(y_\beta, E)\tilde{\subseteq}(H, E)$ and $(G, E)\tilde{\cap}(H, E) = (\tilde{\phi}, E)$.
It follows that $scl_{12}(x_\alpha, E)\tilde{\cap}scl_{12}(y_\beta, E)\tilde{\subseteq}(G, E)\tilde{\cap}(H, E) = (\tilde{\phi}, E)$. Thus,

$$scl_{12}(x_\alpha, E)\tilde{\cap}scl_{12}(y_\beta, E) = (\tilde{\phi}, E).$$

So, by Theorem 2.41, (X, η_1, η_2, E) is a PSR_0^*. □

Remark 3.34. The converse of Theorem 3.33 is not true in general which shown in the following example.

Example 3.35. Let X be an infinite universe and let E be a set of parameters. Then $\eta_{co} \subseteq SS(X)_E$ is a soft topology on X , where
$\eta_{co} = \{(\tilde{\phi}, E)\} \bigcup \{(G, E) \in SS(X)_E : G^c(e)$ is finite $\forall\ e \in E\}$ [see Proposition 3.1 in [16]]. Thus, $(X, \eta_{co}, \eta_{co}, E)$ is a PST_1^*. So it is PSR_0^*. [here $\eta_{12} = \eta_{co}$]. Now, for any two non-null soft set (G, E), (F, E) in η_{co}, $G^c(e)$, $F^c(e)$ are finite for all $e \in E$. It follows that $G^c(e) \cup F^c(e)$ is finite, for all $e \in E$. So, $(G, E)^c\tilde{\cup}(F, E)^c$ is a finite soft set . On the other hand,
$(G, E)\tilde{\cap}(F, E) = (\tilde{X}, E) \setminus (G, E)^c\tilde{\cap}(\tilde{X}, E) \setminus (F, E)^c = (\tilde{X}, E) \setminus [(G, E)^c\tilde{\cup}(F, E)^c]$.
Hence, $(G, E)\tilde{\cap}(F, E) \neq (\tilde{\phi}, E)$. Therefore, $(X, \eta_{co}, \eta_{co}, E)$ is not PSR_1^*.

Definition 3.36. A sbts (X, η_1, η_2, E) is said to be a pairwise soft T_2^* [briefly, PST_2^*], if $\forall\ x_\alpha,\ y_\beta \in \xi(X)_E$, $x_\alpha \neq y_\beta$, there exist (O_{x_α}, E), $(O_{y_\beta}, E) \in\ \eta_{12}$ such that $(O_{x_\alpha}, E)\tilde{\cap}(O_{y_\beta}, E) = (\tilde{\phi}, E)$, where (O_{x_α}, E) means that $x_\alpha\tilde{\in}(O_{x_\alpha}, E)$.

Example 3.37. For any soft topologyη on X, we have $(X, \eta, SS(X)_E, E)$ is a PST_2^*.

Theorem 3.38. *Every PST_2^* is a PST_1^*.*

Proof. Immediate. □

Remark 3.39. The converse of Theorem 3.38 is not true in general as shown in Example 3.35.

Theorem 3.40. *If (X, η_1, E) or (X, η_2, E) is a ST_2, then (X, η_1, η_2, E) is a PST_2^*.*

Proof. It is clear from the fact $\eta_1, \eta_2 \subseteq \eta_{12}$. □

Theorem 3.41. *(X, η_1, η_2, E) is a PST_2^* if and only if it is PST_0^* and PSR_1^*.*

Proof. It is easy to prove that every PST_2^* is a PST_0^*. Let $scl_{12}(x_\alpha, E) \neq scl_{12}(y_\beta, E)$. Then, $x_\alpha \neq y_\beta$. Thus, there exist $(G, E), (H, E) \in \eta_{12}$ such that
$x_\alpha\tilde{\in}(G, E)$, $y_\beta\tilde{\in}(H, E)$ and $(G, E)\tilde{\cap}(H, E) = (\tilde{\phi}, E)$.
It follows that, by Theorem 2.28, $sker_{12}(x_\alpha, E)\tilde{\subseteq}(G, E)$ and $sker_{12}(y_\beta, E)\tilde{\subseteq}(H, E)$. So, by Theorems 2.42 and 3.27,
$scl_{12}(x_\alpha, E)\tilde{\subseteq}(G, E)$, $scl_{12}(y_\beta, E)\tilde{\subseteq}(H, E)$ and $(G, E)\tilde{\cap}(H, E) = (\tilde{\phi}, E)$.
Hence, (X, η_1, η_2, E) is a PSR_1^*.

Conversely, let $x_\alpha \neq y_\beta$. Then, by given and Theorem 3.4, $scl_{12}(x_\alpha, E) \neq scl_{12}(y_\beta, E)$. Thus, by hypothesis there exist $(G, E), (H, E) \in \eta_{12}$ such that
$scl_{12}(x_\alpha, E)\tilde{\subseteq}(G, E)$, $scl_{12}(y_\beta, E)\tilde{\subseteq}(H, E)$ and $(G, E)\tilde{\cap}(H, E) = (\tilde{\phi}, E)$.
It follows that $x_\alpha\tilde{\in}(G, E)$, $y_\beta\tilde{\in}(H, E)$ and $(G, E)\tilde{\cap}(H, E) = (\tilde{\phi}, E)$.
So, (X, η_1, η_2, E) is a PST_2^*. □

Corollary 3.42. *Let* (X, η_1, η_2, E) *be a sbts. The following diagram holds:*

$$\begin{array}{ccccccc} PST_2^* & \Rightarrow & PST_1^* & \Rightarrow & PST_{\frac{1}{2}}^* & \Rightarrow & PST_0^*. \\ \Downarrow & & \Downarrow & & & & \\ PSR_1^* & \Rightarrow & PSR_0^* & & & & \end{array}$$

Theorem 3.43. *Let* (X, η_1, η_2, E) *be a sbts. Then*
(1) *if* (X, η_1, η_2, E) *is a* PST_i^*, *then* (X, η_1^e, η_2^e) *is a* PT_i, $i = 0, \frac{1}{2}, 1, 2$, *for all* $e \in E$,
(2) *if* (X, η_1, η_2, E) *is a* PSR_i^*, *then* (X, η_1^e, η_2^e) *is a* PR_i, $i = 0, 1$, *for all* $e \in E$.

Proof. (1) We shall prove the theorem at $i = 0$ and the others are similar. Let $x, y \in X$ such that $x \neq y$. Then $x_e \neq y_e \ \forall \ e \in E$. Since (X, η_1, η_2, E) is a PST_0^*, there exists $(G, E) \in \eta_{12}$ such that $x_e \tilde{\in} (G, E)$, $y_e \tilde{\notin} (G, E)$ or $x_e \tilde{\notin} (G, E)$, $y_e \tilde{\in} (G, E)$. We claim that $x_e \tilde{\in} (G, E)$, $y_e \tilde{\notin} (G, E)$, then $x \in G(e)$ and $y \notin G(e)$. Now, since $(G, E) \in \eta_{12}$, $G(e) \in \eta_{12}^e$. Thus, for all $x, y \in X$ such that $x \neq y$, there exists $G(e) \in \eta_{12}^e$ such that $x \in G(e)$, $y \notin G(e)$ or $x \notin G(e)$, $y \in G(e)$. So, $(X, \eta_1^e, \eta_2^e, E)$ is a PT_0.
(2) At $i = 0$, see Theorem 5.7 in [8]. Similarly, At $i = 1$. □

Theorem 3.44. *Let* (X, η_1, η_2, E) *be a sbts and* $Y \subseteq X$. *Then* $(Y, \eta_{1Y}, \eta_{2Y}, E)$ *is a sbts on* Y. *Moreover,* $\eta_{1Y2Y} = \eta_{12Y}$, *where*
$\eta_{12Y} = \{(\tilde{Y}, E) \tilde{\cap} (G, E) : (G, E) \in \eta_{12}\}$,
and
$\eta_{1Y2Y} = \{(H, E) \in SS(Y)_E : (H, E) = (H_1, E) \tilde{\cup} (H_2, E), (H_i, E) \in \eta_{iY}, i = 1, 2\}$.

Proof. Since (X, η_1, η_2, E) is a sbts, (X, η_1, E) and (X, η_2, E) are soft topological spaces on X. Since $Y \subseteq X$, (Y, η_{1Y}, E) and (Y, η_{2Y}, E) are soft topologies on Y [see Definition 27 in [19]]. Consequently, $(Y, \eta_{1Y}, \eta_{2Y}, E)$ is a sbts on Y.
Now, Since (X, η_1, η_2, E) is a sbts, (X, η_{12}, E) is a supra soft topological space. Then, (Y, η_{12Y}, E) is a supra soft topological space on Y.
Now, let $(G, E) \in \eta_{12Y}$. Then, there exists $(H, E) \in \eta_{12}$ such that
$(G, E) = (\tilde{Y}, E) \tilde{\cap} (H, E)$
$= (\tilde{Y}, E) \tilde{\cap} [(H_1, E) \tilde{\cup} (H_2, E)]$, $(H_1, E) \in \eta_1$ and $(H_2, E) \in \eta_2$
$= [(\tilde{Y}, E) \tilde{\cap} (H_1, E)] \tilde{\cup} [(\tilde{Y}, E) \tilde{\cap} (H_2, E)]$.
Since $(\tilde{Y}, E) \tilde{\cap} (H_1, E) \in \eta_{1Y}$ and $(\tilde{Y}, E) \tilde{\cap} (H_2, E) \in \eta_{2Y}$,
$[(\tilde{Y}, E) \tilde{\cap} (H_1, E)] \tilde{\cup} [(\tilde{Y}, E) \tilde{\cap} (H_2, E)] \in \eta_{1Y2Y}$. Thus, $(G, E) \in \eta_{1Y2Y}$. So, $\eta_{12Y} \subseteq \eta_{1Y2Y}$.
By similar way, we can prove that $\eta_{1Y2Y} \subseteq \eta_{12Y}$. □

Theorem 3.45. *Let* (X, η_1, η_2, E) *be a sbts and let* $Y \subseteq X$. *Then*
(1) *if* (X, η_1, η_2, E) *is a* PST_i^*, *then* $(Y, \eta_{1Y}, \eta_{2Y}, E)$ *is a* PST_i^*, $i = 0, \frac{1}{2}, 1, 2$,
(2) *if* (X, η_1, η_2, E) *is a* PSR_i^*, *then* $(Y, \eta_{1Y}, \eta_{2Y}, E)$ *is a* PSR_i^*, $i = 0, 1$.

Proof. Straightforward. □

4. Conclusion

. Soft sets play a very important role in general and soft topology and they are now the research topics of many topologists worldwide. Some important applications of soft sets are in decision making, data mining, medical diagnosis and complete (incomplete) information systems, etc. Indeed a significant theme in general, soft topology and real analysis concerns the variously modified forms of continuity, separation axioms etc. by utilizing crisp and soft sets. The concept of a soft bitopological space was introduced by Ittanagi [2]. Kandil et al.[9] introduced some structures of soft bitopological space (X, η_1, η_2, E). In this paper, we introduced and studied some classes of soft bitopological spaces, namely, PST_0^*, PST_1^*, PST_2^* and PSR_1^* spaces. Characterizations of these spaces are obtained. Moreover, we studied the implications of these types of soft separation axioms in soft and crisp cases. Finally, we showed that these soft separation axioms are hereditary properties. The future work is to introduce the relation between the family of all information system with set of subjects X and the family of all soft bitopologies on X itself, also we will give some application in decision making by utilizing properties of soft bitopological space.

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