

Classifying fuzzy subgroups of the abelian group $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$ for distinct primes $p_1, p_2, \dots, p_n$

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ABSTRACT. In this paper, we classify the fuzzy subgroups of the group $G = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$ where $p_1, p_2, \dots, p_n$ are distinct primes and $n \in \mathbb{Z}^+$. We develop an algorithm for counting the distinct fuzzy subgroups of the group G using the criss-cut counting technique. This is achieved by using the maximal chains of subgroups of G and the equivalence relation given by Murali and Makamba in their research papers on equivalent fuzzy subgroups.

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1. INTRODUCTION

The group $G = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$, for distinct primes $p_1, p_2, \dots, p_n$ and $n \in \mathbb{Z}^+$, is a cyclic group since it is isomorphic to the cyclic group $\mathbb{Z}_{p_1 p_2 \cdots p_n}$. We will often use $p_1 p_2 \cdots p_n$ to denote the group $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$. Fuzzy group theory has a lot of application in different fields ranging from computer science, electrical engineering to medicine. Fuzzy subgroups are usually uncountably many, even for finite groups. In the recent past, many researchers have been classifying fuzzy subgroups. The classification problem in fuzzy subgroups has grown to be a branch that has many interesting challenges, since there are different notions of equivalence. In [6], Murali and Makamba saw the importance of classifying fuzzy subgroups when they wrote: “one of the most interesting problems in fuzzy group theory is to classify fuzzy subgroups up to some unique invariants of the underlying group”.

The concept of a fuzzy set was first introduced by Zadeh [17] in 1965 and thereafter Rosenfeld [10] followed by introducing the concepts of fuzzy subgroupoids and fuzzy subgroups. Fuzzy subgroups have recently been studied by, among others, [4, 5, 8, 13], thus extending the work done by the earlier authors like Das in [1] and Sherwood

in [14]. R. Sulaiman and A.G. Ahmad [15] worked on the the particular case of the group $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4} \times \mathbb{Z}_{p_5} \times \mathbb{Z}_{p_6}$ where $p_1, p_2, \dots, p_6$ are distinct primes. Using the equivalence relation of [15], the authours [2] and [3] generalized these results to the group $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \dots \times \mathbb{Z}_{p_n}$. M. Tărnăuceanu in [16], using a different equivalence relation from [15], also worked on fuzzy subgroups of $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \dots \times \mathbb{Z}_{p_n}$. A. Sehgal, S. Sehgal and P. K. Sharma in [12], used the equivalence relation of [16] to count the fuzzy subgroups of the p -group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$. The authors [11] and [3], also worked on the number of fuzzy subgroups of a finite cyclic group using the equivalence relation of [16]. Murali and Makamba in [4] gave a different equivalence relation from both [15] and [16], which proved to be stronger than the other two. Thus their equivalence gives more distinct (non-equivalent) fuzzy subgroups for most finite groups.

We begin by giving some fundamental concepts, definitions and propositions that will be necessary in this paper. The number of maximal chains of subgroups of the finite abelian group $G = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \dots \times \mathbb{Z}_{p_n}$ has already been established by, among others, O. Ndiweni [8]. Using the equivalence relation as defined by Murali and Makamba in [4] and their criss-cut counting technique [7], we then classify the fuzzy subgroups of the finite abelian group $G = p_1 p_2 \dots p_n$ and $n \in \mathbb{Z}^+$. In [5], Murali and Makamba worked on the equivalence classes of fuzzy subgroups of G but using a different counting approach called the cross-cut counting in [7]. However, they did not completely give the total number of distinct fuzzy subgroups for the group G , but merely focussed on certain types of keychains. Our goal in this paper is therefore to extend their work, using the criss-cut counting technique to establish the number of distinct fuzzy subgroups for the group G .

Note that we often use the term maximal chains of G to mean maximal chains of subgroups of G .

2. PRELIMINARIES

Since our counting is anchored on maximal chains, we look at some important concepts related to the maximal chains of a group G . A chain of subgroups of G is maximal if no new subgroups can be inserted in the chain.

O. Ndiweni in [8], worked on the number of maximal chains of finite abelian groups and gave the following results in Theorem 2.1.

Theorem 2.1 ([8]). *The group $\mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \dots \times \mathbb{Z}_{p_m^{n_m}}$, p_i 's distinct primes*

and $n_i' \in \mathbb{Z}^+$, has $\frac{(n_1 + n_2 + \dots + n_m)!}{n_1! n_2! \dots n_m!} = \frac{\left(\sum_{i=1}^m n_i\right)!}{\prod_{i=1}^m n_i!}$ maximal chains.

From Theorem 2.1, we can deduce that the group $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \dots \times \mathbb{Z}_{p_n}$ has $\frac{(1+1+\dots+1)!}{1!1!\dots1!}$ maximal chains where $(1+1+\dots+1)$ and $(1!1!\dots1!)$ are respectively a sum and product of n terms. We state this as

Proposition 2.2. *The group $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \dots \times \mathbb{Z}_{p_n}$ has $n!$ maximal chains.*

Let $I = [0, 1]$ be the unit interval of real numbers with the usual ordering and let X be a non-empty set. A fuzzy subset of X is characterized by a function $\mu : X \rightarrow I$.

μ is called the membership function and $\mu(x)$ is the degree of membership of the element x to the fuzzy subset of X defined by μ .

Definition 2.3. Let X be a non-empty set. The support of μ , denoted by $\text{supp}(\mu)$, is defined as $\text{supp}(\mu) = \{x \in X : \mu(x) > 0\}$.

Example 2.4. Let $X = \{65, 70, 82, 94, 76, 88\}$ be the set of Calculus I test scores for six University of Fort Hare students. Define a fuzzy set μ on X as

$$\{(65, 0), (70, 0.52), (82, 0), (94, 0.68), (76, 0.76), (88, 0.9)\}.$$

Then $\text{supp}(\mu) = \{70, 94, 76, 88\}$.

Definition 2.5 ([4]). Two fuzzy subsets μ and ν of X are said to be equivalent, denoted $\mu \sim \nu$, if and only if

- (i) for all $x, y \in X$, $\mu(x) > \mu(y)$ if and only if $\nu(x) > \nu(y)$,
- (ii) $\mu(x) = 0$ if and only if $\nu(x) = 0$.

Clearly this relation is an equivalence relation on I^X and it coincides with equality of sets when restricted to 2^X .

Remark 2.6. The condition $\mu(x) = 0 \iff \nu(x) = 0$ implies that the supports of μ and ν are equal.

Proposition 2.7 ([4]). If $\mu \sim \nu$, then $|\text{Im}(\mu)| = |\text{Im}(\nu)|$.

Definition 2.8 ([10]). Let G be a group. A fuzzy subset μ of G is said to be a fuzzy subgroup of G , if, for all $x, y \in G$,

- (i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$,
- (ii) $\mu(x^{-1}) \geq \mu(x)$.

Murali and Makamba in [6] worked on the number of distinct fuzzy subgroups of the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ obtaining the results in Propositions 2.9 and Theorem 2.10. Note that two fuzzy subgroups of a group G are distinct, if they are non-equivalent in terms of the Murali and Makamba equivalence.

Proposition 2.9 ([6]). The number of distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$ is

$$\left[2^{n+1+1} \sum_{r=0}^1 2^{-r} \binom{n}{r} \binom{1}{r} \right] - 1, \quad n \geq 1$$

More generally

Theorem 2.10 ([6]). The number of distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ is

$$\left[2^{n+m+1} \sum_{r=0}^m 2^{-r} \binom{n}{r} \binom{m}{r} \right] - 1, \quad n \geq m.$$

3. DISTINCT FUZZY SUBGROUPS OF THE GROUP $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$

To achieve our objective of counting the fuzzy subgroups of $p_1 p_2 \cdots p_n$, we list all maximal chains of each group first, thereafter the number of distinct fuzzy subgroups is computed using the criss-cut method of [7] and [9]. We briefly explain the technique below.

Remark 3.1. The order of listing our maximal subgroup chains does not matter and so does not alter the number of distinct fuzzy subgroups. Thus we can start the counting from any chain in the list and proceed in any order. Therefore we number our chains here according to the order in which we consider the chains in our counting.

Now let G be a group. From the list of the maximal subgroup chains, suppose our first chain is

$$0 \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n = G, \quad (3.1)$$

By [4], the chain (3.1) contributes $2^{n+1} - 1$ distinct fuzzy subgroups of G . Let our next maximal chain be

$$0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n = G, \quad (3.2)$$

such that for some i , $J_i \neq H_i$ where $i \in \{1, 2, \dots, n-1\}$. This new subgroup J_i is called a distinguishing factor of the maximal chain. The number of distinct fuzzy subgroups of G contributed by the chain (3.2) is given by Proposition 3.2.

Proposition 3.2 ([9]). *The number of distinct fuzzy subgroups of G contributed by a maximal subgroup chain with a distinguishing factor is equal to $\frac{2^{n+1}}{2} = 2^n$ for $n \geq 2$.*

Suppose in our counting process, we encounter a maximal subgroup chain $0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = G$, such that $\{K_i, K_j\}$, $i \neq j$, is a pair of subgroups in this chain. Moreover, suppose as a pair, the two subgroups have not appeared in any previous chain. We say such a chain has a new pair or distinguishing pair.

Proposition 3.3 ([9]). *In the process of our counting distinct fuzzy subgroups, a maximal subgroup chain that has no single distinguishing factor but has a distinguishing pair, contributes $\frac{2^{n+1}}{2^2} = 2^{n-1}$ new distinct fuzzy subgroups of G for $n \geq 4$.*

A new triple of subgroups in a maximal chain is called a *distinguishing triple* and such a chain contributes $\frac{2^{n+1}}{2^3}$ new distinct fuzzy subgroups. This counting argument continues inductively and can be generalised in Proposition 3.4.

Proposition 3.4. *In the process of counting distinct fuzzy subgroups, if a maximal subgroup chain of length $n+1$, other than the first chain, has no distinguishing $(m-1)$ -tuple, but has a new m -tuple of subgroups that has not been used as a distinguishing m -tuple previously, then that chain contributes $\frac{2^{n+1}}{2^m}$ new distinct fuzzy subgroups of G , $n+1 > m$.*

For further details of the counting technique, see [9]. Example 3.5 illustrates the criss-cut counting technique.

Example 3.5. For $i = 1, 2, 3$, we have the groups $\mathbb{Z}_{p_1}$, $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$ and $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3}$, for distinct primes p_1, p_2, p_3 , whose maximal chains are respectively listed below. The powers of 2 on the extreme right are the numbers of distinct fuzzy subgroups

contributed by the maximal chains.

$$\begin{array}{lll}
 & & p_1 p_2 p_3 \supseteq p_1 p_2 \supseteq p_1 \supseteq 0 : 2^4 - 1 \\
 & & p_1 p_2 p_3 \supseteq p_1 p_2 \supseteq p_2 \supseteq 0 : 2^3 \\
 p_1 \supseteq 0 : 2^2 - 1 & p_1 p_2 \supseteq p_1 \supseteq 0 : 2^3 - 1 & p_1 p_2 p_3 \supseteq p_1 p_3 \supseteq p_1 \supseteq 0 : 2^3 \\
 & p_1 p_2 \supseteq p_2 \supseteq 0 : 2^2 & p_1 p_2 p_3 \supseteq p_1 p_3 \supseteq p_3 \supseteq 0 : 2^3 \\
 & & p_1 p_2 p_3 \supseteq p_2 p_3 \supseteq p_2 \supseteq 0 : 2^3 \\
 & & p_1 p_2 p_3 \supseteq p_2 p_3 \supseteq p_3 \supseteq 0 : 2^2.
 \end{array}$$

Then $\mathbb{Z}_{p_1}$, $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$ and $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3}$ have respectively $2^2 - 1$, $2^3 - 1 + 2^2$ and $(2^4 - 1) + 4 \cdot 2^3 + 1 \cdot 2^2$ distinct fuzzy subgroups.

For $i = 4$, the group $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4}$ has the maximal chains and contributed distinct fuzzy subgroups as listed below.

$$\begin{array}{ll}
 p_1 p_2 p_3 p_4 \supseteq p_1 p_2 p_3 \supseteq p_1 p_2 \supseteq p_1 \supseteq 0 : 2^5 - 1 & p_1 p_2 p_3 p_4 \supseteq p_1 p_3 p_4 \supseteq p_1 p_3 \supseteq p_1 \supseteq 0 : 2^4 \\
 p_1 p_2 p_3 p_4 \supseteq p_1 p_2 p_3 \supseteq p_1 p_2 \supseteq p_2 \supseteq 0 : 2^4 & p_1 p_2 p_3 p_4 \supseteq p_1 p_3 p_4 \supseteq p_1 p_3 \supseteq p_3 \supseteq 0 : 2^3 \\
 p_1 p_2 p_3 p_4 \supseteq p_1 p_2 p_3 \supseteq p_1 p_3 \supseteq p_1 \supseteq 0 : 2^4 & p_1 p_2 p_3 p_4 \supseteq p_1 p_3 p_4 \supseteq p_1 p_4 \supseteq p_1 \supseteq 0 : 2^3 \\
 p_1 p_2 p_3 p_4 \supseteq p_1 p_2 p_3 \supseteq p_1 p_3 \supseteq p_3 \supseteq 0 : 2^4 & p_1 p_2 p_3 p_4 \supseteq p_1 p_3 p_4 \supseteq p_1 p_4 \supseteq p_4 \supseteq 0 : 2^3 \\
 p_1 p_2 p_3 p_4 \supseteq p_1 p_2 p_3 \supseteq p_2 p_3 \supseteq p_2 \supseteq 0 : 2^4 & p_1 p_2 p_3 p_4 \supseteq p_1 p_3 p_4 \supseteq p_3 p_4 \supseteq p_3 \supseteq 0 : 2^4 \\
 p_1 p_2 p_3 p_4 \supseteq p_1 p_2 p_3 \supseteq p_2 p_3 \supseteq p_3 \supseteq 0 : 2^3 & p_1 p_2 p_3 p_4 \supseteq p_1 p_3 p_4 \supseteq p_3 p_4 \supseteq p_4 \supseteq 0 : 2^3 \\
 p_1 p_2 p_3 p_4 \supseteq p_1 p_2 p_4 \supseteq p_1 p_2 \supseteq p_1 \supseteq 0 : 2^4 & p_1 p_2 p_3 p_4 \supseteq p_2 p_3 p_4 \supseteq p_2 p_3 \supseteq p_2 \supseteq 0 : 2^4 \\
 p_1 p_2 p_3 p_4 \supseteq p_1 p_2 p_4 \supseteq p_1 p_2 \supseteq p_2 \supseteq 0 : 2^3 & p_1 p_2 p_3 p_4 \supseteq p_2 p_3 p_4 \supseteq p_2 p_3 \supseteq p_3 \supseteq 0 : 2^3 \\
 p_1 p_2 p_3 p_4 \supseteq p_1 p_2 p_4 \supseteq p_1 p_4 \supseteq p_1 \supseteq 0 : 2^4 & p_1 p_2 p_3 p_4 \supseteq p_2 p_3 p_4 \supseteq p_2 p_4 \supseteq p_2 \supseteq 0 : 2^3 \\
 p_1 p_2 p_3 p_4 \supseteq p_1 p_2 p_4 \supseteq p_1 p_4 \supseteq p_4 \supseteq 0 : 2^4 & p_1 p_2 p_3 p_4 \supseteq p_2 p_3 p_4 \supseteq p_2 p_4 \supseteq p_4 \supseteq 0 : 2^3 \\
 p_1 p_2 p_3 p_4 \supseteq p_1 p_2 p_4 \supseteq p_2 p_4 \supseteq p_2 \supseteq 0 : 2^4 & p_1 p_2 p_3 p_4 \supseteq p_2 p_3 p_4 \supseteq p_3 p_4 \supseteq p_3 \supseteq 0 : 2^3 \\
 p_1 p_2 p_3 p_4 \supseteq p_1 p_2 p_4 \supseteq p_2 p_4 \supseteq p_4 \supseteq 0 : 2^3 & p_1 p_2 p_3 p_4 \supseteq p_2 p_3 p_4 \supseteq p_3 p_4 \supseteq p_4 \supseteq 0 : 2^2.
 \end{array}$$

So $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4}$ has $(2^5 - 1) + 11 \cdot 2^4 + 11 \cdot 2^3 + 2^2$ distinct fuzzy subgroups.

A similar approach gives us the following number of distinct fuzzy subgroups for $i = 5, 6, \dots, 10$;

$$\begin{array}{ll}
 i = 5: & \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_5}: (2^6 - 1) + 26 \cdot 2^5 + 66 \cdot 2^4 + 26 \cdot 2^3 + 2^2. \\
 i = 6: & \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_6}: (2^7 - 1) + 57 \cdot 2^6 + 302 \cdot 2^5 + 302 \cdot 2^4 + 57 \cdot 2^3 + 2^2. \\
 i = 7: & \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_7}: (2^8 - 1) + 120 \cdot 2^7 + 1191 \cdot 2^6 + 2416 \cdot 2^5 + 1191 \cdot 2^4 + 120 \cdot 2^3 + 2^2. \\
 i = 8: & \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_8}: 2^9 - 1 + 247 \cdot 2^8 + 4293 \cdot 2^7 + 15653 \cdot 2^6 + 15653 \cdot 2^5 + \\
 & 4293 \cdot 2^4 + 247 \cdot 2^3 + 2^2. \\
 i = 9: & \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_9}: 2^{10} - 1 + 502 \cdot 2^9 + 14608 \cdot 2^8 + 88234 \cdot 2^7 + 156190 \cdot 2^6 \\
 & + 88234 \cdot 2^5 + 14608 \cdot 2^4 + 502 \cdot 2^3 + 2^2. \\
 i = 10: & \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_{10}}: 2^{11} - 1 + 1013 \cdot 2^{10} + 47840 \cdot 2^9 + 455192 \cdot 2^8 + 1310354 \cdot 2^7 + \\
 & 1310354 \cdot 2^6 + 455195 \cdot 2^5 + 47840 \cdot 2^4 + 1013 \cdot 2^3 + 2^2.
 \end{array}$$

These results are presented in Table 3.1. From this table, we observe that the number of distinct fuzzy subgroups of the group $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \dots \times \mathbb{Z}_{p_n}$ has n terms of powers of 2. The first term is $2^{n+1} - 1$, where $n + 1$ is the length of each maximal subgroup chain of G and the last term is 2^2 . We get a fascinating pattern from

the respective coefficients of $2^{n+1} - 1, 2^n, 2^{n-2}, \dots, 2^2$ for $n = 1, 2, \dots, 8$ in the expression for the number of distinct fuzzy subgroups of G .

TABLE 3.1. Fuzzy subgroups of $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \dots \times \mathbb{Z}_{p_n}$

n	Number of fuzzy subgroups
1	$3 = 2^2 - 1$
2	$11 = (2^3 - 1) + 2^2$
3	$51 = (2^4 - 1) + 4 \cdot 2^3 + 2^2$
4	$299 = (2^5 - 1) + 11 \cdot 2^4 + 11 \cdot 2^3 + 2^2$
5	$2163 = (2^6 - 1) + 26 \cdot 2^5 + 66 \cdot 2^4 + 26 \cdot 2^3 + 2^2$
6	$18731 = (2^7 - 1) + 57 \cdot 2^6 + 302 \cdot 2^5 + 302 \cdot 2^4 + 57 \cdot 2^3 + 2^2$
7	$189171 = (2^8 - 1) + 120 \cdot 2^7 + 1191 \cdot 2^6 + 2416 \cdot 2^5 + 1191 \cdot 2^4 + 120 \cdot 2^3 + 2^2$
8	$2186603 = 2^9 - 1 + 247 \cdot 2^8 + 4293 \cdot 2^7 + 15653 \cdot 2^6 + 15653 \cdot 2^5 + 4293 \cdot 2^4 + 247 \cdot 2^3 + 2^2$
$\vdots$	$\vdots$
k	$2^{k+1} + [(k-1)t_1 + 2t_2] \cdot 2^k + [(k-2)t_2 + 3t_3] \cdot 2^{k-1} - 1 + \dots + [(2t_{k-2} + (k-1)t_{k-1}) \cdot 2^3 + 2^2$

This pattern gives a form of the Pascal triangle, which we present in Figure 3.1 and simply call it the Pascal triangle. The sum of the i -th row of the triangle gives the number of maximal chains of the group $\mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_i}$. The terms of this Pascal triangle can be calculated from Figure 3.2. This helps us to investigate a pattern of getting the coefficients in each column (each position). The first and last terms of the triangle are each equal to 1, and indeed the triangle is symmetric about the vertical line consisting of 1, 4, 66. We explain the algorithm used to generate the Figure 3.1. We consider the group $G_6 = \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_6}$ to explain how the number of distinct fuzzy subgroups can be obtained using the previous level group $G_5 = \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_5}$. The second term of G_6 in Figure 3.1 is given by $5 \cdot 1 + 2 \cdot 26$, where $5 = 6 - 1$, 1 and 26 are the first and second terms of G_5 respectively while 2 represents the position of the term. The second term of $G_7 = \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_7}$ is $6 \cdot 1 + 2 \cdot 57$. Similarly $6 = 7 - 1$, 1 and 57 are the first and second terms of G_6 respectively while 2 is similarly the position of the term. This pattern can be seen in the other second terms for $\mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_i}$ as shown in Figure 3.2. So generally, the second term of $\mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_n}$ can be expressed as $(n-1)t_1 + 2t_2$, where $t_1 = 1$ and t_2 are the first and the second terms respectively of $G_{n-1} = \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_{n-1}}$ in Figure 3.1.

For the third term of G_6 , we have $4 \cdot 26 + 3 \cdot 66$ where $4 = 6 - 2$, 26 and 66 are the second and the third terms of G_5 in Figure 3.1 while 3 represents the position of the term (in this case, the third term). A similar trend is seen in G_7 with $5 \cdot 57 + 3 \cdot 302$ implying $5 = 7 - 2$, 57 and 302 the second and third terms of G_6 and 3 the position. Therefore, the general formula for the third term of $\mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_n}$ is $(n-2)t_2 + 3t_3$ where t_2 and t_3 are the second and the third terms of G_{n-1} . Continuing this pattern, we see the k -th term of $G = \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_n}$ is given by $[n - (k-1)]t_{k-1} + kt_k$ where t_{k-1} and t_k are the $(k-1)$ -st and k -th terms respectively of $G_{n-1} = \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_{n-1}}$.

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FIGURE 3.1. Pascal's triangle for coefficients of terms of fuzzy subgroups of $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$

n	2^{nd} coef	3^{rd} coef	4^{th} coef
3	$4 = 2 \cdot 1 + 2 \cdot 1$		
4	$11 = 3 \cdot 1 + 2 \cdot 4$	$11 = 2 \cdot 4 + 3 \cdot 1$	
5	$26 = 4 \cdot 1 + 2 \cdot 11$	$66 = 3 \cdot 11 + 3 \cdot 11$	$26 = 2 \cdot 11 + 4 \cdot 1$
6	$57 = 5 \cdot 1 + 2 \cdot 26$	$302 = 4 \cdot 26 + 3 \cdot 66$	$302 = 3 \cdot 66 + 4 \cdot 26$
7	$120 = 6 \cdot 1 + 2 \cdot 57$	$1191 = 5 \cdot 57 + 3 \cdot 302$	$2416 = 4 \cdot 302 + 4 \cdot 302$
⋮	⋮	⋮	⋮

FIGURE 3.2. Coefficient columns of $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$

4. CONCLUSION

This paper has discussed the number of distinct fuzzy subgroups of the group $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$. With the discovery of the triangle presented in Figure 3.1, one can find the number of distinct fuzzy subgroups for a given cyclic group G . As further research, one would want to write an algorithm that automatically generates the triangle for any given n , and add it into existing computer algebra systems like GAP, Magma and MuPAD in MATLAB. This will form a basis for another paper in the future.

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