Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 13, No. 2, (February 2017), pp. 213–229
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Several concepts of continuity in fuzzy m-space

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Received 31 May 2016; Accepted 25 July 2016

Abstract. In this paper, we first introduce some open and closed sets in fuzzy m-space and give some interrelations between them. Afterwards, different types of continuity between fuzzy m-spaces have been introduced and characterized and also found the mutual relationships among themselves. Again different types of fuzzy m-compact spaces, fuzzy m-s-closed space and fuzzy s-Urysohn space are introduced and have shown that images of different types of fuzzy m-compact spaces under the functions defined in Section 4 are fuzzy m-s-closed space.

2010 AMS Classification: 54A40, 54C99

Keywords: Fuzzy m_{I^X}-e-open, Fuzzy m_{I^X}-e^*-open, Fuzzy m_{I^X}-a-open, Fuzzy m-compact set, Fuzzy s-Urysohn space, Fuzzy (e^*, r)-continuous function.

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1. Introduction

In [1], the notion of fuzzy minimal structure (in the sense of Lowen) has been introduced as follows : A family $\mathcal{M}$ of fuzzy sets in X is said to be a fuzzy minimal structure on X if $\alpha 1_X \in \mathcal{M}$ for every $\alpha \in [0,1]$. A more general version of fuzzy minimal structure (in the sense of Chang) are introduced in [3, 6] as follows : A family $\mathcal{F}$ of fuzzy sets in X is a fuzzy minimal structure on X if $0_X \in \mathcal{F}$ and $1_X \in \mathcal{F}$. In this paper we use the notion of fuzzy minimal structure in the sense of Chang.

2. Preliminaries

In 1965, Zadeh introduced the notion of fuzzy set [8] A which is a mapping from a non-empty set X into the closed interval $[0,1]$, i.e., $A \in I^X$. The support [7] of a fuzzy set A, denoted by $suppA$ and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. The fuzzy set with the singleton support $\{x\} \subseteq X$ and the value t $(0 < t \leq 1)$ will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 respectively in X. The complement [8] of a fuzzy set A in X is denoted by $1_X \setminus A$

and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for each $x \in X$. For any two fuzzy sets A, B in X, $A \leq B$ means $A(x) \leq B(x)$, for all $x \in X$ [8] while AqB means A is quasi-coincident (q-coincident, for short) [7] with B, i.e., there exists $x \in X$ such that $A(x) + B(x) > 1$. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not{q} B$ respectively. For a fuzzy point x_α and a fuzzy set A in X, $x_\alpha \in A$ means $x_\alpha \leq A$, i.e., $A(x) \geq \alpha$.

3. Some Different Types of Open and Closed Sets in Fuzzy m-Space

Let X be a non-empty set and $m_{I^X} \subseteq I^X$. Then m_{I^X} is said to be a fuzzy minimal structure [3, 6] on X if $0_X, 1_X \in m_{I^X}$. The members of m_{I^X} are called fuzzy m_{I^X}-open sets and the complement of a fuzzy m_{I^X}-open set is called fuzzy m_{I^X}-closed set. The pair (X, m_{I^X}) is called fuzzy m-space.

Definition 3.1 ([2]). Let X be a non-empty set and m_{I^X}, a fuzzy minimal structure on X. For $A \in I^X$, the fuzzy m_{I^X}-closure and fuzzy m_{I^X}-interior of A, denoted by m_{I^X}-clA and m_{I^X}-$intA$ respectively, are defined as follows :

$$m_{I^X}\text{-}clA = \bigwedge\{F : A \leq F, 1_X \setminus F \in m_{I^X}\}$$
$$m_{I^X}\text{-}intA = \bigvee\{D : D \leq A, D \in m_{I^X}\}.$$

It can be observed that given a fuzzy minimal structure m_{I^X} on X, if $A \in I^X$, the m_{I^X}-$intA$ may not be an element of m_{I^X}.

Proposition 3.2 ([2]). *Let X be a non-empty set and m_{I^X}, a fuzzy minimal structure on X. Then for any $A \in I^X$, a fuzzy point $x_\alpha \in m_{I^X}$-clA iff for any $U \in m_{I^X}$ with $x_\alpha qU$, UqA.*

Lemma 3.3 ([2]). *Let X be a non empty set and m_{I^X}, a fuzzy minimal structure on X. For $A, B \in I^X$, the following hold:*
(1) *$A \leq B$ which implies that m_{I^X}-$intA \leq m_{I^X}$-$intB$, m_{I^X}-$clA \leq m_{I^X}$-clB.*
(2) *m_{I^X}-$cl0_X = 0_X$, m_{I^X}-$cl1_X = 1_X$, m_{I^X}-$int0_X = 0_X$, m_{I^X}-$int1_X = 1_X$.*
(3) *m_{I^X}-$intA \leq A \leq m_{I^X}$-clA.*
(4) *m_{I^X}-$clA = A$ if $1_X \setminus A \in m_{I^X}$, m_{I^X}-$intA = A$, if $A \in m_{I^X}$.*
(5) *m_{I^X}-$cl(1_X \setminus A) = 1_X \setminus m_{I^X}$-$intA$, m_{I^X}-$int(1_X \setminus A) = 1_X \setminus m_{I^X}$-$clA$.*
(6) *m_{I^X}-$cl(m_{I^X}$-$clA) = m_{I^X}$-clA, m_{I^X}-$int(m_{I^X}$-$intA) = m_{I^X}$-$intA$.*

It is clear from Lemma 3.3 that

Theorem 3.4. *Let (X, m_{I^X}) be a fuzzy m-space and $A, B \in I^X$. Then the following statements are true:*
(1) *m_{I^X}-$clA \bigvee m_{I^X}$-$clB \leq m_{I^X}$-$cl(A \bigvee B)$.*
(2) *m_{I^X}-$int(A \bigwedge B) \leq m_{I^X}$-$intA \bigwedge m_{I^X}$-$intB$.*

We now introduce the following definitions.

Definition 3.5. Let (X, m_{I^X}) be a fuzzy m-space. $A \in I^X$ is said to be fuzzy
(i) m_{I^X}-regular open, if $A = m_{I^X}$-$int(m_{I^X}$-$clA)$,
(ii) m_{I^X}-semiopen, if $A \leq m_{I^X}$-$cl(m_{I^X}$-$intA)$,
(iii) m_{I^X}-α-open, if $A \leq m_{I^X}$-$int(m_{I^X}$-$cl(m_{I^X}$-$intA))$,
(iv) m_{I^X}-β-open, if $A \leq m_{I^X}$-$cl(m_{I^X}$-$int(m_{I^X}$-$clA))$,
(v) m_{I^X}-preopen, if $A \leq m_{I^X}$-$int(m_{I^X}$-$clA)$.

The complements of the above mentioned fuzzy sets are called their respective closed sets.

The infimum of all fuzzy m_{I^X}-semiclosed (resp., fuzzy m_{I^X}-α-closed, fuzzy m_{I^X}-preclosed) sets containing a fuzzy set A in X is called fuzzy m_{I^X}-semiclosure (resp., fuzzy m_{I^X}-α-closure, fuzzy m_{I^X}-preclosure) of A and is denoted by m_{I^X}-$sclA$ (resp., m_{I^X}-αclA, m_{I^X}-$pclA$).

We denote by m_{I^X}-$RO(X)$ (resp., m_{I^X}-$RC(X)$, m_{I^X}-$SO(X)$, m_{I^X}-$\alpha O(X)$, m_{I^X}-$PO(X)$, m_{I^X}-$\beta O(X)$) the family of all fuzzy m_{I^X}-regular open (resp., fuzzy m_{I^X}-regular closed, fuzzy m_{I^X}-semiopen, fuzzy m_{I^X}-α-open, fuzzy m_{I^X}-preopen, fuzzy m_{I^X}-β-open) sets in X.

Definition 3.6. Let (X, m_{I^X}) be a fuzzy m-space and $A \in I^X$. A fuzzy point x_α in X is said to be fuzzy m_{I^X}-θ-semicluster point of A, if m_{I^X}-$clUqA$ for every fuzzy m_{I^X}-semiopen set U with $x_\alpha qU$. The union of all fuzzy m_{I^X}-θ-semicluster points of A is called fuzzy m_{I^X}-θ-semiclosure of A and is denoted by m_{I^X}-θ-$sclA$.

$A(\in I^X)$ is said to be fuzzy m_{I^X}-θ-semiclosed if $A = m_{I^X}$–θ-$sclA$. The complement of a fuzzy m_{I^X}-θ-semiclosed set is called fuzzy m_{I^X}-θ-semiopen.

Definition 3.7. Let (X, m_{I^X}) be a fuzzy m-space and $A \in I^X$. The m_{I^X}-r-kernel of A, denoted by m_{I^X}-r-Ker$A$, is defined as follows:

$$m_{I^X}\text{-}r\text{-Ker}A = \bigwedge\{U : U \in m_{I^X}\text{-}RO(X), A \leq U\}.$$

Definition 3.8. Let (X, m_{I^X}) be a fuzzy m-space and $A \in I^X$. The fuzzy m_{I^X}-δ-closure and fuzzy m_{I^X}-δ-interior of A, denoted by m_{I^X}-δclA and m_{I^X}-$\delta intA$ resp., are defined by

$$m_{I^X}\text{-}\delta clA = \{x_\alpha \in X : Aqm_{I^X}\text{-}int(m_{I^X}\text{-}clU), \text{ for all } U \in m_{I^X} \text{ with } x_\alpha qU\},$$
$$m_{I^X}\text{-}\delta intA = \bigvee\{W : W \in m_{I^X}\text{-}RO(X), W \leq A\}.$$

It is clear from Definition 3.8 that

Theorem 3.9. *Let (X, m_{I^X}) be a fuzzy m-space and $A \in I^X$. The following statements are true:*

(1) *If $A \leq B$, then m_{I^X}-$\delta clA \leq m_{I^X}$-δclB.*
(2) *If $A \leq B$, then m_{I^X}-$\delta intA \leq m_{I^X}$-$\delta intB$.*
(3) *m_{I^X}-$\delta intA \leq m_{I^X}$-$intA \leq m_{I^X}$-$clA \leq m_{I^X}$-δclA*
(4) *$1_X \setminus m_{I^X}$-$\delta intA = m_{I^X}$-$\delta cl(1_X \setminus A)$.*
(5) *m_{I^X}-$\delta int(1_X \setminus A) = 1_X \setminus m_{I^X}$-$\delta clA$.*

Definition 3.10. Let (X, m_{I^X}) be a fuzzy m-space and $A \in I^X$. Then A is said to be fuzzy

(i) m_{I^X}-δ-open (resp., m_{I^X}-δ-closed), if $A = m_{I^X}$-$\delta intA$ (resp., $A = m_{I^X}$-δclA),
(ii) m_{I^X}-δ-preopen, if $A \leq m_{I^X}$- $int(m_{I^X}$-$\delta clA)$,
(iii) m_{I^X}-δ-semiopen, if $A \leq m_{I^X}$-$cl(m_{I^X}$-$\delta intA)$.

The complements of the above mentioned fuzzy sets are called their respective closed sets.

The collection of all fuzzy m_{I^X}-δ-open (resp., fuzzy m_{I^X}-δ-preopen, fuzzy m_{I^X}-δ-semiopen) sets is denoted by m_{I^X}-$\delta O(X)$ (resp., m_{I^X}-$\delta PO(X)$, m_{I^X}-$\delta SO(X)$).

The collection of all fuzzy m_{I^X}-δ-closed (resp., fuzzy m_{I^X}-δ-preclosed, fuzzy m_{I^X}-δ-semiclosed) sets is denoted by m_{I^X}-$\delta C(X)$ (resp., m_{I^X}-$\delta PC(X)$, m_{I^X}-$\delta SC(X)$).

Definition 3.11. Let (X, m_{I^X}) be a fuzzy m-space and $A \in I^X$. Then A is said to be fuzzy (i) m_{I^X}-e-open, if $A \leq m_{I^X}$-$cl(m_{I^X}$-$\delta intA) \bigvee m_{I^X}$-$int(m_{I^X}$-$\delta clA)$,
(ii) m_{I^X}-e^*-open, if $A \leq m_{I^X}$-$cl(m_{I^X}$-$int(m_{I^X}$-$\delta clA))$,
(iii) m_{I^X}-a-open, if $A \leq m_{I^X}$-$int(m_{I^X}$-$cl(m_{I^X}$-$\delta intA))$.

The complements of the above mentioned sets are called their respective closed sets.

The collection of all fuzzy m_{I^X}-e-open (resp., fuzzy m_{I^X}-e^*-open, fuzzy m_{I^X}-a-open) sets is denoted by m_{I^X}-$eO(X)$ (resp., m_{I^X}-$e^*O(X)$, m_{I^X}-$aO(X)$).

The collection of all fuzzy m_{I^X}-e-closed (resp., fuzzy m_{I^X}-e^*-closed, fuzzy m_{I^X}-a-closed) sets is denoted by m_{I^X}-$eC(X)$ (resp., m_{I^X}-$e^*C(X)$, m_{I^X}-$aC(X)$).

The above definitions show the following relationships.

Example 3.12. Let $X = \{a, b\}, m_{I^X} = \{0_X, 1_X, A, B\}$ where $A(a) = 0.4, A(b) = 0.6, B(a) = 0.6, B(b) = 0.4$. Then (X, m_{I^X}) is a fuzzy m-space. Clearly $m_{I^X} = m_{I^X}$-$RO(X)$. Consider the fuzzy set C defined by $C(a) = C(b) = 0.6$. Now

$$C = \bigvee\{U \in I^X : U \in m_{I^X}\text{-}RO(X), U \leq C\} = m_{I^X}\text{-}\delta intC.$$

Then C is fuzzy m_{I^X}-δ-open in X, but $C \notin m_{I^X}$ as well as $C \notin m_{I^X}$-$RO(X)$.

Example 3.13. Let $X = \{a, b\}, m_{I^X} = \{0_X, 1_X, A\}$ where $A(a) = A(b) = 0.6$. Then (X, m_{I^X}) is a fuzzy m-space. Then $A \in m_{I^X}$ but $A \notin m_{I^X}$-$\delta O(X)$. Again $A \in m_{I^X}$-$\alpha O(X)$.

Example 3.14. Let $X = \{a, b\}, m_{I^X} = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.6$. Then (X, m_{I^X}) is a fuzzy m-space. Consider the fuzzy set B defined by $B(a) = B(b) = 0.6$. Then m_{I^X}-$int(m_{I^X}$-$cl(m_{I^X}$-$intB)) = 1_X \geq B$. Thus $B \in m_{I^X}$-$\alpha O(X)$, but $B \notin m_{I^X}$.

Example 3.15. Let $X = \{a, b\}, m_{I^X} = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4$. Then (X, m_{I^X}) is a fuzzy m-space. Consider the fuzzy set B defined by $B(a) = B(b) = 0.5$. Then m_{I^X}-$cl(m_{I^X}$-$intB) = 1_X \setminus A \geq B$. Thus $B \in m_{I^X}$-$SO(X)$. But m_{I^X}-$int(m_{I^X}$-$cl(m_{I^X}$-$intB)) = A < B$. So $B \notin m_{I^X}$-$\alpha O(X)$.

Again m_{I^X}-$cl(m_{I^X}$-$int(m_{I^X}$-$clB)) = 1_X \setminus A \geq B$. Then $B \in m_{I^X}$-$\beta O(X)$, but m_{I^X}-$int(m_{I^X}$-$clB) = A < B$. Thus $B \notin m_{I^X}$-$PO(X)$.

Example 3.16. Let $X = \{a, b\}, m_{I^X} = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4$. Then (X, m_{I^X}) is a fuzzy m-space. Now m_{I^X}-$\delta O(X) = \{0_X, 1_X, A\}$. Consider the fuzzy set B defined by $B(a) = 0.5, B(b) = 0.3$. Then m_{I^X}-$int(m_{I^X}$-$\delta clB) = m_{I^X}$-$int(1_X \setminus A) = A \geq B$. Thus $B \in m_{I^X}$-$\delta PO(X)$, but $B \notin m_{I^X}$.

Again m_{I^X}-$cl(m_{I^X}$-$int(m_{I^X}$-$\delta clB)) = 1_X \setminus A > B$. Then $B \in m_{I^X}$-$e^*O(X)$, but m_{I^X}-$cl(m_{I^X}$-$intB) = 0_X \ngeq B$. Thus $B \notin m_{I^X}$-$SO(X)$. Also m_{I^X}-$int(m_{I^X}$-$clB) = A > B$. So $B \in m_{I^X}$-$PO(X)$, but m_{I^X}-$int(m_{I^X}$-$cl(m_{I^X}$-$intB)) = 0_X \ngeq B$. Hence $B \notin m_{I^X}$-$\alpha O(X)$.

Example 3.17. Consider Example 3.16 and the fuzzy set C defined by $C(a) = C(b) = 0.5$. Then m_{I^X}-$int(m_{I^X}$-$\delta clC) = m_{I^X}$-$int(1_X \setminus A) = A < C$. Thus $C \notin m_{I^X}$-$\delta PO(X)$. But m_{I^X}-$cl(m_{I^X}$-$\delta intC) = m_{I^X}$-$clA = 1_X \setminus A > C$. So $C \in m_{I^X}$-$eO(X)$.

Again m_{I^X}-$int(m_{I^X}$-$cl(m_{I^X}$-$intC)) = A < C$. Then $C \notin m_{I^X}$-$\alpha O(X)$. But m_{I^X}-$cl(m_{I^X}$-$\delta intC) > C$. Thus $C \in m_{I^X}$-$\delta SO(X)$. Also $C \notin m_{I^X}$-$\delta O(X)$ and m_{I^X}-$int(m_{I^X}$-$cl(m_{I^X}$-$\delta intC)) = A < C$. So $C \notin m_{I^X}$-$aO(X)$.

Example 3.18. Let $X = \{a, b\}, m_{I^X} = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = 0.5, B(b) = 0.7$. Then (X, m_{I^X}) is a fuzzy m-space. Here m_{I^X}-$\delta O(X) = \{0_X, 1_X, A\}$. Consider the fuzzy set C defined by $C(a) = 0.5, C(b) = 0.3$. Then m_{I^X}-$int(m_{I^X}$-$\delta clC) = m_{I^X}$-$int(1_X \setminus A) = A \geq C$. Thus $C \in m_{I^X}$-$\delta PO(X)$, but m_{I^X}-$int(m_{I^X}$-$clC) = 0_X \not\geq C$. So $C \notin m_{I^X}$-$PO(X)$.

Now m_{I^X}-$int(m_{I^X}$-$\delta clC) = A > C$. Then $C \in m_{I^X}$-$eO(X)$.

Again m_{I^X}-$cl(m_{I^X}$-$int(m_{I^X}$-$\delta clC)) = 1_X \setminus A > C$. Then $C \in m_{I^X}$-$e^*O(X)$, but m_{I^X}-$cl(m_{I^X}$-$int(m_{I^X}$-$clC)) = m_{I^X}$-$cl(m_{I^X}$-$int(1_X \setminus B)) = 0_X \not\geq C$. Thus $C \notin m_{I^X}$-$\beta O(X)$. Also m_{I^X}-$int(m_{I^X}$-$cl(m_{I^X}$-$intC)) = 0_X \not\geq C$. So $C \notin m_{I^X}$-$\alpha O(X)$.

Example 3.19. Consider Example 3.12 and the fuzzy set D defined by $D(a) = 0.6, D(b) = 0.61$. Then m_{I^X}-$\delta intD = \bigvee\{U \in m_{I^X}$-$RO(X) : U \leq D\} = C$ and thus m_{I^X}-$int(m_{I^X}$-$cl(m_{I^X}$-$\delta intD)) = 1_X \geq D$. So $D \in m_{I^X}$-$aO(X)$, but $D \notin m_{I^X}$-$\delta O(X)$.

Theorem 3.20. *Let (X, m_{I^X}) be a fuzzy m-space. Then the following statements are true:*

(1) *The union of any collection of fuzzy m_{I^X}-e^*-open sets is fuzzy m_{I^X}-e^*-open.*
(2) *The union of any collection of fuzzy m_{I^X}-e-open sets is fuzzy m_{I^X}-e-open.*
(3) *The union of any collection of fuzzy m_{I^X}-a-open sets is fuzzy m_{I^X}-a-open.*

Proof. Let $\{G_\alpha : \alpha \in \Lambda\}$ be any collection of fuzzy m_{I^X}-e^*-open sets. Then for any $\alpha \in \Lambda$, $G_\alpha \leq m_{I^X}$-$cl(m_{I^X}$-$int(m_{I^X}$-$\delta clG_\alpha))$. Also, $G_\alpha \leq \bigvee_{\alpha\in\Lambda} G_\alpha$. Thus,

$$\begin{aligned} G_\alpha &\leq m_{I^X}\text{-}cl(m_{I^X}\text{-}int(m_{I^X}\text{-}\delta clG_\alpha)) \\ &\leq m_{I^X}\text{-}cl(m_{I^X}\text{-}int(m_{I^X}\text{-}\delta cl(\bigvee_{\alpha\in\Lambda} G_\alpha))), \end{aligned}$$

for all $\alpha \in \Lambda$. So $\bigvee_{\alpha\in\Lambda} G_\alpha \leq m_{I^X}$-$cl(m_{I^X}$-$int(m_{I^X}$-$\delta cl(\bigvee_{\alpha\in\Lambda} G_\alpha)))$. Hence $\bigvee_{\alpha\in\Lambda} G_\alpha$ is a fuzzy m_{I^X}-e^*-open.

The proofs of (2) and (3) are same as that of (1). □

Definition 3.21. Let (X, m_{I^X}) be a fuzzy m-space and $A \in I^X$. Then fuzzy m_{I^X}-e-closure (resp., fuzzy m_{I^X}-e^*-closure, fuzzy m_{I^X}-a-closure) of A, denoted by m_{I^X}-e-clA (resp., m_{I^X}-e^*-clA, m_{I^X}-a-clA), is defined by

$$m_{I^X}\text{-}e\text{-}clA = \bigwedge\{F \in I^X : A \leq F, 1_X \setminus F \in m_{I^X}\text{-}eO(X)\}$$
$$(\text{resp., } m_{I^X}\text{-}e^*\text{-}clA = \bigwedge\{F \in I^X : A \leq F, 1_X \setminus F \in m_{I^X}\text{-}e^*O(X)\},$$
$$m_{I^X}\text{-}a\text{-}clA = \bigwedge\{F \in I^X : A \leq F, 1_X \setminus F \in m_{I^X}\text{-}aO(X)\}).$$

Lemma 3.22. *Let (X, m_{I^X}) be a fuzzy m-space. Then the following statements hold:*

(1) *For any fuzzy point x_α and any $U \in I^X$, $x_\alpha \in m_{I^X}$-e^*-$clU \Rightarrow$ for any $V \in m_{I^X}$-$e^*O(X)$ with $x_\alpha qV$, VqU.*

(2) *For any two fuzzy sets U, V where $V \in m_{I^X}$-$e^*O(X)$, $U \not{q} V \Rightarrow m_{I^X}$-$e^*$-$clU \not{q} V$.*

(3) *For any $A \in m_{I^X}$, m_{I^X}-$sclA = m_{I^X}$-$int(m_{I^X}$-$clA)$.*

(4) *For any $A \in m_{I^X}$-$RO(X), m_{I^X}$-θ-$sclA = A$.*

(5) *For any $A \in m_{I^X}$-$\beta O(X), m_{I^X}$-$clA = m_{I^X}$-αclA.*

(6) *For any $A \in m_{I^X}$-$SO(X), m_{I^X}$-$clA = m_{I^X}$-$pclA$.*

(7) *For any* $A \in m_{I^X}$, m_{I^X}*-scl*$A = m_{I^X}$*-*θ*-scl*A.

Proof. (1) Let $V \in m_{I^X}$-$e^*O(X)$ with $x_\alpha qV$. Then $V(x)+\alpha > 1$. Thus $x_\alpha \notin 1_X \setminus V$ which is m_{I^X}-e^*-closed in X. So $U \not\leq 1_X \setminus V$. Hence there exists $y \in X$ such that $U(y) > 1 - V(y)$. Therefore UqV.

(2) If possible, let m_{I^X}-e^*-$clUqV$, but $U \not{q} V$. Then there exists $x \in X$ such that $(m_{I^X}\text{-}e^*\text{-}clU)(x)+V(x) > 1$. Thus $V(x)+t > 1$, where $t = (m_{I^X}\text{-}e^*\text{-}clU)(x)$. So $x_t \in m_{I^X}$-e^*-clU, where $x_tqV, V \in m_{I^X}$-$e^*O(X)$. By definition, VqU, a contradiction.

(3) We first prove that m_{I^X}-$clA = m_{I^X}$-$cl(m_{I^X}$-$sclA)$, for $A \in m_{I^X}$.

Now $A \leq m_{I^X}$-$sclA \leq m_{I^X}$-clA. Then m_{I^X}-$clA \leq m_{I^X}$-$cl(m_{I^X}$-$sclA) \leq m_{I^X}$-clA. Thus m_{I^X}-$clA = m_{I^X}$-$cl(m_{I^X}$-$sclA)$. Since infimum of any two fuzzy m_{I^X}-semiclosed sets in a fuzzy m-space is fuzzy m_{I^X}-semiclosed, m_{I^X}-$sclA$ is fuzzy m_{I^X}-semiclosed in X. So m_{I^X}-$int(m_{I^X}$-$cl(m_{I^X}$-$sclA)) \leq m_{I^X}$-$sclA$ and so by above,

(3.22.1) $\quad m_{I^X}$-$int(m_{I^X}$-$clA) \leq m_{I^X}$-$sclA$.

To prove the converse, let $x_\alpha \notin m_{I^X}$-$int(m_{I^X}$-$clA)$. Then $[m_{I^X}\text{-}int(m_{I^X}\text{-}clA)](x) < \alpha$. Thus $x_\alpha q(1_X \setminus m_{I^X}\text{-}int(m_{I^X}\text{-}clA)) = m_{I^X}\text{-}cl(m_{I^X}\text{-}int(1_X \setminus A))$. Since $A \in m_{I^X}$, $A \leq m_{I^X}$-$int(m_{I^X}$-$clA)$. Thus

(3.22.2) $\quad A \not{q} m_{I^X}\text{-}cl(m_{I^X}\text{-}int(1_X \setminus A))$.

Now $m_{I^X}\text{-}cl(m_{I^X}\text{-}int(m_{I^X}\text{-}cl(m_{I^X}\text{-}int(1_X \setminus A)))) \geq m_{I^X}\text{-}cl(m_{I^X}\text{-}int(1_X \setminus A))$. Thus $m_{I^X}\text{-}cl(m_{I^X}\text{-}int(1_X \setminus A)) \in m_{I^X}\text{-}SO(X)$. So by (3.22.2), $x_\alpha \notin m_{I^X}$-$sclA$. Consequently,

(3.22.3) $\quad m_{I^X}$-$sclA \leq m_{I^X}$-$int(m_{I^X}$-$clA)$.

Combining (3.22.1) and (3.22.3), we get the result.

(4) It is obvious that $A \leq m_{I^X}$-θ-$sclA$. To prove the converse, let $x_\alpha \in m_{I^X}$-θ-$sclA$, but $x_\alpha \notin A$. Then $A(x) < \alpha$. Thus $x_\alpha q(1_X \setminus A) = m_{I^X}\text{-}cl(m_{I^X}\text{-}int(1_X \setminus A))$. So $1_X \setminus A \in m_{I^X}$-$SO(X)$. Also,

(3.22.4)
$$\begin{aligned} m_{I^X}\text{-}cl(1_X \setminus A) &= m_{I^X}\text{-}cl(m_{I^X}\text{-}cl(m_{I^X}\text{-}int(1_X \setminus A))) \\ &= m_{I^X}\text{-}cl(m_{I^X}\text{-}int(1_X \setminus A)) = 1_X \setminus A. \end{aligned}$$

As $x_\alpha \in m_{I^X}$-θ-$sclA, m_{I^X}$-$cl(1_X \setminus A)qA$. So $(1_X \setminus A)qA$ by (3.22.4) which is absurd. Hence m_{I^X}-θ-$sclA \leq A$, for $A \in m_{I^X}$-$RO(X)$.

(5) Clearly, m_{I^X}-$\alpha clA \leq m_{I^X}$-clA. To prove the converse, let $x_\alpha \in m_{I^X}$-clA, where $A \in m_{I^X}$-$\beta O(X)$. Then $A \leq m_{I^X}\text{-}cl(m_{I^X}\text{-}int(m_{I^X}\text{-}clA))$. Thus

(3.22.5) $\quad m_{I^X}\text{-}int(m_{I^X}\text{-}cl(m_{I^X}\text{-}int(1_X \setminus A))) \leq 1_X \setminus A$.

Let $U \in m_{I^X}$-$\alpha O(X)$ with $x_\alpha qU \leq m_{I^X}\text{-}int(m_{I^X}\text{-}cl(m_{I^X}\text{-}intU))$. Then $x_\alpha q(m_{I^X}\text{-}int(m_{I^X}\text{-}cl(m_{I^X}\text{-}intU)))$. Then there exists $V \in m_{I^X}$ such that $x_\alpha qV \leq m_{I^X}\text{-}cl(m_{I^X}\text{-}intU)$. So VqA. Hence

(3.22.6) $\quad V = m_{I^X}\text{-}intV \leq m_{I^X}\text{-}int(m_{I^X}\text{-}cl(m_{I^X}\text{-}intU))qA$.

If possible, let $U \not{q} A$. Then $U \leq 1_X \setminus A$. Thus by (3.22.5),

$m_{I^X}\text{-}int(m_{I^X}\text{-}cl(m_{I^X}\text{-}intU)) \leq m_{I^X}\text{-}int(m_{I^X}\text{-}cl(m_{I^X}\text{-}int(1_X \setminus A))) \leq 1_X \setminus A$. So $m_{I^X}\text{-}int(m_{I^X}\text{-}cl(m_{I^X}\text{-}intU)) \not{q} A$. This is contradicts (3.22.6).

(6) It is similar to that of (5).

(7) It is clear that m_{I^X}-$sclA \leq m_{I^X}$-θ-$sclA$, for any $A \in I^X$. To prove the converse, let $x_\alpha \in m_{I^X}$-θ-$sclA$, but $x_\alpha \notin m_{I^X}$-$sclA$. Then there exists $U \in m_{I^X}$-$SO(X)$ with $x_\alpha qU$, $U \not{q} A \Rightarrow U \leq 1_X \setminus A$. Thus $clU \leq cl(1_X \setminus A) = 1_X \setminus A$. So $clU \not{q} A$. This is a contradiction, as $x_\alpha \in m_{I^X}$-θ-$sclA$. □

4. Continuous functions in fuzzy m-space

In this section, a new class of continuous functions between fuzzy m-spaces are introduced and characterized and found the mutual relationships among themselves.

Definition 4.1. Let (X, m_{I^X}) and (Y, m_{I^Y}) be two fuzzy m-spaces and $f : X \to Y$ be a function between fuzzy m-spaces. Then f is called fuzzy

(i) contra R-map, if $f^{-1}(A) \in m_{I^X}\text{-}RC(X)$ for all $A \in m_{I^Y}\text{-}RO(Y)$,
(ii) (δ, r)-continuous, if $f^{-1}(A) \in m_{I^X}\text{-}\delta C(X)$ for all $A \in m_{I^Y}\text{-}RO(Y)$,
(iii) (δ-semi, r)-continuous, if $f^{-1}(A) \in m_{I^X}\text{-}\delta SC(X)$ for all $A \in m_{I^Y}\text{-}RO(Y)$,
(iv) (δ-pre, r)-continuous, if $f^{-1}(A) \in m_{I^X}\text{-}\delta PC(X)$ for all $A \in m_{I^Y}\text{-}RO(Y)$,
(v) (e^*, r)-continuous, if $f^{-1}(A) \in m_{I^X}\text{-}e^*C(X)$ for all $A \in m_{I^Y}\text{-}RO(Y)$,
(vi) (e, r)-continuous, if $f^{-1}(A) \in m_{I^X}\text{-}eC(X)$ for all $A \in m_{I^Y}\text{-}RO(Y)$,
(vii) (a, r)-continuous, if $f^{-1}(A) \in m_{I^X}\text{-}aC(X)$ for all $A \in m_{I^Y}\text{-}RO(Y)$.

Theorem 4.2. *Let (X, m_{I^X}) and (Y, m_{I^Y}) be two fuzzy m-spaces and $f : X \to Y$ be a function between fuzzy m-spaces. Then the following statements are true:*

(1) *If f is fuzzy (δ, r)-continuous, then f is (a, r)-continuous.*
(2) *If f is fuzzy (a, r)-continuous, then f is fuzzy (δ-semi, r)-continuous.*
(3) *If f is fuzzy (a, r)-continuous, then f is fuzzy (δ-pre, r)-continuous.*
(4) *If f is fuzzy (δ-semi, r)-continuous, then f is fuzzy (e, r)-continuous.*
(5) *If f is fuzzy (δ-pre, r)-continuous, then f is fuzzy (e, r)-continuous.*
(6) *If f is fuzzy (e, r)-continuous, then f is fuzzy (e^*, r)-continuous.*

Proof. (1) Let $A \in m_{I^Y}\text{-}RO(Y)$. Then $f^{-1}(A) \in m_{I^X}\text{-}\delta C(X)$. Thus $m_{I^X}\text{-}\delta cl(f^{-1}(A)) = f^{-1}(A)$. Now

$$\begin{aligned} & m_{I^X}\text{-}cl(m_{I^X}\text{-}int(m_{I^X}\text{-}\delta cl(f^{-1}(A)))) \\ &= m_{I^X}\text{-}cl(m_{I^X}\text{-}int(f^{-1}(A))) \\ &\le m_{I^X}\text{-}cl(f^{-1}(A)) \\ &\le m_{I^X}\text{-}\delta cl(f^{-1}(A)) \\ &= f^{-1}(A). \end{aligned}$$

Then $f^{-1}(A) \in m_{I^X}\text{-}aC(X)$. Thus f is fuzzy (a, r)-continuous.

(2) The proof follows from the fact that $A \in m_{I^X}\text{-}aC(X) \Rightarrow A \in m_{I^X}\text{-}\delta SC(X)$.
(3) The proof follows from the fact that $A \in m_{I^X}\text{-}aO(X) \Rightarrow A \in m_{I^X}\text{-}\delta PO(X)$.
(4) The proof follows from the fact that $A \in m_{I^X}\text{-}\delta SC(X) \Rightarrow A \in m_{I^X}\text{-}eC(X)$.
(5) The proof follows from the fact that $A \in m_{I^X}\text{-}\delta PC(X) \Rightarrow A \in m_{I^X}\text{-}eC(X)$.
(6) The proof follows from the fact that $A \in m_{I^X}\text{-}eC(X) \Rightarrow A \in m_{I^X}\text{-}e^*C(X)$. □

But the converses are not true, in general, follow from the following examples.

Example 4.3. Fuzzy (a, r)-continuity $\not\Rightarrow$ fuzzy (δ, r)-continuity.
Let $X = \{a, b\}$, $m_{I^X} = \{0_X, 1_X, A, B, C, D\}$, $m'_{I^X} = \{0_X, 1_X, E\}$, where $A(a) = 0.4, A(b) = 0.55, B(a) = 0.5, B(b) = 0.45, C(a) = 0.45, C(b) = 0.55, D(a) = 0.55,$ $D(b) = 0.4, E(a) = E(b) = 0.5$. Then (X, m_{I^X}) and (X, m'_{I^X}) are fuzzy m-spaces. Thus $m_{I^X}\text{-}\delta O(X) = \{0_X, 1_X, B, C, T\}$, where $T(a) = 0.5, T(b) = 0.55$.

Consider the identity function $i : (X, m_{I^X}) \rightarrow (X, m'_{I^X})$. Now $E \in m'_{I^X}$-$RO(X)$. Then $i^{-1}(E) = E$. Thus m_{I^X}-$cl(m_{I^X}$-$int(m_{I^X}$-$\delta clE)) = m_{I^X}$-$cl(m_{I^X}$-$int(1_X \setminus B)) = B < E$. So $E \in m_{I^X}$-$aC(X)$, but m_{I^X}-$\delta clE = 1_X \setminus B \neq E$. Hence $E \notin m_{I^X}$-$\delta C(X)$.

Example 4.4. Fuzzy (δ-semi, r)-continuity $\not\Rightarrow$ fuzzy (a, r)-continuity.
Let $X = \{a, b\}$, $m_{I^X} = \{0_X, 1_X, A\}$, $m'_{I^X} = \{0_X, 1_X, C\}$, where $A(a) = 0.5, A(b) = 0.4, C(a) = C(b) = 0.5$. Then (X, m_{I^X}) and (X, m'_{I^X}) are fuzzy m-spaces.

Consider the identity function $i : (X, m_{I^X}) \rightarrow (X, m'_{I^X})$. Now $C \in m'_{I^X}$-$RO(X)$. $i^{-1}(C) = C = 1_X \setminus C \in m_{I^X}$-$\delta SO(X)$, but $1_X \setminus C \notin m_{I^X}$-$aO(X)$. Thus $C \in m_{I^X}$-$\delta SC(X)$, but $C \notin m_{I^X}$-$aC(X)$.

Example 4.5. Fuzzy (δ-pre, r)-continuity $\not\Rightarrow$ fuzzy (a, r)-continuity.
Let $X = \{a, b\}$, $m_{I^X} = \{0_X, 1_X, A\}$, $m'_{I^X} = \{0_X, 1_X, B\}$, where $A(a) = 0.5, A(b) = 0.4, B(a) = 0.5, B(b) = 0.3$. Then (X, m_{I^X}) and (X, m'_{I^X}) are fuzzy m-spaces.

Consider the identity function $i : (X, m_{I^X}) \rightarrow (X, m'_{I^X})$. Now m_{I^X}-$\delta O(X) = \{0_X, 1_X, A\}$ and m'_{I^X}-$\delta O(X) = \{0_X, 1_X, B\}$. Now $B \in m'_{I^X}$-$RO(X)$. $i^{-1}(B) = B$. Then m_{I^X}-$cl(m_{I^X}$-$\delta intB) = 0_X < B$. Thus $B \in m_{I^X}$-$\delta PC(X)$, but m_{I^X}-$cl(m_{I^X}$-$int(m_{I^X}$-$\delta clB)) = m_{I^X}$-$cl(m_{I^X}$-$int(m_{I^X}$-$clB)) = m_{I^X}$-$cl(m_{I^X}$-$int(1_X \setminus A)) = m_{I^X}$-$clA = 1_X \setminus A \not\leq B$. So $B \notin m_{I^X}$-$aC(X)$.

Example 4.6. Fuzzy (e, r)-continuity $\not\Rightarrow$ fuzzy (δ-semi, r)-continuity.
Consider Example 4.5. Here $B \in m'_{I^X}$-$RO(X)$. $i^{-1}(B) = B$. Then m_{I^X}-$int(m_{I^X}$-$\delta clB) = m_{I^X}$-$int(1_X \setminus A) = A \not\leq B$. Thus $B \notin m_{I^X}$-$\delta SC(X)$, but m_{I^X}-$int(m_{I^X}$-$\delta clB) \bigwedge m_{I^X}$-$cl(m_{I^X}$-$\delta intB) = A \bigwedge 0_X = 0_X < B$. So $B \in m_{I^X}$-$eC(X)$.

Example 4.7. Fuzzy (e, r)-continuity $\not\Rightarrow$ fuzzy (δ-pre, r)-continuity.
Consider Example 4.4. Here m_{I^X}-$cl(m_{I^X}$-$\delta intC) = m_{I^X}$-$clA = 1_X \setminus A \not\leq C$. Thus $C \notin m_{I^X}$-$\delta PC(X)$. But m_{I^X}-$int(m_{I^X}$-$\delta clC) \bigwedge m_{I^X}$-$cl(m_{I^X}$-$\delta intC) = A \bigwedge (1_X \setminus A) = A < C$. So $C \in m_{I^X}$-$eC(X)$.

Example 4.8. Fuzzy (e^*, r)-continuity $\not\Rightarrow$ fuzzy (e, r)-continuity.
Let $X = \{a, b\}, m_{I^X} = \{0_X, 1_X, A, B\}, m'_{I^X} = \{0_X, 1_X, C\}$, where $A(a) = 0.5, A(b) = 0.6, B(a) = B(b) = 0.4, C(a) = 0.4, C(b) = 0.5$. Then (X, m_{I^X}) and (X, m'_{I^X}) are fuzzy m-spaces.

Consider the identity function $i : (X, m_{I^X}) \rightarrow (X, m'_{I^X})$. Here m_{I^X}-$\delta O(X) = \{0_X, 1_X, A, B\}$. Now $C \in m'_{I^X}$-$RO(X)$. $i^{-1}(C) = C$. Now m_{I^X}-$int(m_{I^X}$-$cl(m_{I^X}$-$\delta intC)) = m_{I^X}$-$int(m_{I^X}$-$clB) = m_{I^X}$-$int(1_X \setminus A) = B < C$. Then $C \in m_{I^X}$-$e^*C(X)$. But m_{I^X}-$int(m_{I^X}$-$\delta clC) \bigwedge m_{I^X}$-$cl(m_{I^X}$-$\delta intC) = m_{I^X}$-$int(1_X \setminus B) \bigwedge m_{I^X}$-$clB = A \bigwedge (1_X \setminus A) = 1_X \setminus A \not\leq C$. Thus $C \notin m_{I^X}$-$eC(X)$.

Definition 4.9. Let (X, m_{I^X}) and (Y, m_{I^Y}) be two fuzzy m-spaces and $f : X \rightarrow Y$ be a function between fuzzy m-spaces. Then f is said to be fuzzy

(i) e^*-continuous, if $f^{-1}(A) \in m_{I^X}$-$e^*O(X)$, for all $A \in m_{I^Y}$,
(ii) almost-e^*-continuous, if $f^{-1}(A) \in m_{I^X}$-$e^*O(X)$, for all $A \in m_{I^Y}$-$RO(Y)$,
(iii) almost-e-continuous, if $f^{-1}(A) \in m_{I^X}$-$eO(X)$, for all $A \in m_{I^Y}$-$RO(Y)$,
(iv) almost-a-continuous, if $f^{-1}(A) \in m_{I^X}$-$aO(X)$, for all $A \in m_{I^Y}$-$RO(Y)$.

Theorem 4.10. *Let (X, m_{I^X}) and (Y, m_{I^Y}) be two fuzzy m-spaces and $f : X \to Y$ be a function between fuzzy m-spaces. Then the following statements hold:*

(1) *If f is fuzzy e^*-continuous, then f is fuzzy almost-e^*-continuous.*
(2) *If f is fuzzy almost-e-continuous, then f is fuzzy almost-e^*-continuous.*
(3) *If f is fuzzy almost-a-continuous, then f is fuzzy almost-e-continuous.*

Proof. The proof is obvious. □

But the converses are not true, in general, follow from the next examples.

Example 4.11. Fuzzy almost-e^*-continuity $\not\Rightarrow$ fuzzy e^*-continuity.
Let $X = \{a, b\}, m_{I^X} = \{0_X, 1_X, A, B\}, m'_{I^X} = \{0_X, 1_X, E, F\}$ where $A(a) = 0.4$, $A(b) = 0.6, B(a) = 0.6, B(b) = 0.4, E(a) = E(b) = 0.4, F(a) = 0.5, F(b) = 0.45$. Then (X, m_{I^X}) and (X, m'_{I^X}) are fuzzy m-spaces. Now m_{I^X}-$\delta O(X) = \{0_X, 1_X, A, B, C\}$, where $C(a) = C(b) = 0.6$, m'_{I^X}-$RO(X) = \{0_X, 1_X, F\}$.

Now consider the identity function $i : (X, m_{I^X}) \to (X, m'_{I^X})$. Then $i^{-1}(F) = F$. Now m_{I^X}-$cl(m_{I^X}$-$int(m_{I^X}$-$\delta clF)) = 1_X > F$. Then $F \in m_{I^X}$-$e^*O(X)$. But $i^{-1}(E) = E$, m_{I^X}-$cl(m_{I^X}$-$int(m_{I^X}$-$\delta clE)) = m_{I^X}$-$cl(m_{I^X}$-$int(1_X \setminus C)) = m_{I^X}$-$cl0_X = 0_X \not\geq E \Rightarrow E \notin m_{I^X}$-$e^*O(X)$. Thus i is fuzzy almost-e^*-continuous but not fuzzy e^*-continuous.

Example 4.12. Fuzzy almost-e^*-continuity $\not\Rightarrow$ fuzzy almost-e-continuity.
Let $X = \{a, b\}, m_{I^X} = \{0_X, 1_X, A\}, m'_{I^X} = \{0_X, 1_X, B, C\}$, where $A(a) = 0.5, A(b) = 0.4, B(a) = 0.4, B(b) = 0.6, C(a) = 0.6, C(b) = 0.4$. Then (X, m_{I^X}) and (X, m'_{I^X}) are fuzzy m-spaces. Now m_{I^X}-$\delta O(X) = m_{I^X}, m'_{I^X}$-$RO(X) = m'_{I^X}$.

Now consider the identity function $i : (X, m_{I^X}) \to (X, m'_{I^X})$. Then $i^{-1}(B) = B$. Now m_{I^X}-$cl(m_{I^X}$-$int(m_{I^X}$-$\delta clB)) = 1_X \setminus A > B$. Then $B \in m_{I^X}$-$e^*O(X)$, $i^{-1}(C) = C, m_{I^X}$-$cl(m_{I^X}$-$int(m_{I^X}$-$\delta clC)) = 1_X > C$. Thus $C \in m_{I^X}$-$e^*O(X)$. So i is fuzzy almost-e^*-continuous. But m_{I^X}-$cl(m_{I^X}$-$\delta intB) \bigvee m_{I^X}$-$int(m_{I^X}$-$\delta clB) = 0_X \bigvee A = A \not\geq B$. Hence $B \notin m_{I^X}$-$eO(X)$. Therefore i is not fuzzy almost e-continuous.

Example 4.13. Fuzzy almost-e-continuity $\not\Rightarrow$ fuzzy almost-a-continuity.
Let $X = \{a, b\}, m_{I^X} = \{0_X, 1_X, A\}, m'_{I^X} = \{0_X, 1_X, C\}$, where $A(a) = 0.5, A(b) = 0.4, C(a) = C(b) = 0.5$. Then (X, m_{I^X}) and (X, m'_{I^X}) are fuzzy m-spaces. Now m_{I^X}-$\delta O(X) = \{0_X, 1_X, A\}$, m'_{I^X}-$RO(X) = m'_{I^X}$.

Consider the identity function $i : (X, m_{I^X}) \to (X, m'_{I^X})$. Now $i^{-1}(C) = C$, m_{I^X}-$cl(m_{I^X}$-$\delta intC) \bigvee m_{I^X}$-$int(m_{I^X}$-$\delta clC) = (1_X \setminus A) \bigvee A = 1_X \setminus A > C$. Then $C \in m_{I^X}$-$eO(X)$. Thus i is fuzzy almost-e-continuous. But m_{I^X}-$int(m_{I^X}$-$cl(m_{I^X}$-$\delta intC)) = m_{I^X}$-$int(m_{I^X}$-$clA) = A < C$. So $C \notin m_{I^X}$-$aO(X)$. Hence i is not fuzzy almost-e-continuous.

Definition 4.14. A fuzzy m-space (X, m_{I^X}) is said to be fuzzy m_{I^X}-extremally disconnected, if the fuzzy m_{I^X}-closure of all fuzzy m_{I^X}-interior of a fuzzy set in X is fuzzy m_{I^X}-open.

Example 4.15. Let $X = \{a, b\}, m_{I^X} = \{0_X, 1_X, A\}$, where $A(a) = A(b) = 0.5$. Then (X, m_{I^X}) is a fuzzy m-space. Now m_{I^X}-$clA = A \in m_{I^X} \Rightarrow X$ is fuzzy m_{I^X}-extremally disconnected.

Theorem 4.16. *Let (X, m_{I^X}) and (Y, m_{I^Y}) be two fuzzy m-spaces and $f : X \to Y$ be a function. If (Y, m_{I^Y}) is a fuzzy m_{I^Y}-extremally disconnected, then f is fuzzy (e^*, r)-continuous (resp., fuzzy (e, r)-continuous, fuzzy (a, r)-continuous) iff f is fuzzy almost-e^*-continuous (resp., fuzzy almost-e-continuous, fuzzy almost-a-continuous).*

Proof. First suppose that f is fuzzy (e^*, r)-continuous. Let $U(\in I^Y) \in m_{I^Y}\text{-}RO(Y)$. Then $U = m_{I^Y}\text{-}int(m_{I^Y}\text{-}clU)$. As Y is fuzzy m_{I^Y}-extremally disconnected, $m_{I^Y}\text{-}clU \in m_{I^Y}$ and so $U \in m_{I^Y}$ as well as $1_Y \setminus U \in m_{I^Y}$, i.e., U is fuzzy m_{I^Y}-open as well as fuzzy m_{I^Y}-closed and so $U = m_{I^Y}\text{-}cl(m_{I^Y}\text{-}intU)$, i.e., $U \in m_{I^Y}\text{-}RC(Y)$. As f is fuzzy (e^*, r)-continuous, $f^{-1}(U) \in m_{I^X}\text{-}e^*O(X)$. Then f is fuzzy almost-e^*-continuous.

Conversely, suppose that f is fuzzy almost-e^*-continuous and let $W \in m_{I^Y}\text{-}RC(Y)$. Since Y is fuzzy m_{I^Y}-extremally disconnected, $W \in m_{I^Y}\text{-}RO(Y)$. By hypothesis, $f^{-1}(W) \in m_{I^X}\text{-}e^*O(X) \Rightarrow f$ is fuzzy (e^*, r)-continuous.

The other two cases are similar to that of first case. □

Definition 4.17. A fuzzy m-space (X, m_{I^X}) is said to be fuzzy

(i) $m_{I^X}\text{-}e^*\text{-}T_{1/2}$-space, if all fuzzy $m_{I^X}\text{-}e^*$-closed set in X is fuzzy $m_{I^X}\text{-}\delta$-closed in X,

(ii) $m_{I^X}\text{-}e\text{-}T_{1/2}$-space, if all fuzzy $m_{I^X}\text{-}e^*$-closed set in X is fuzzy $m_{I^X}\text{-}e$-closed in X,

(iii) $m_{I^X}\text{-}a\text{-}T_{1/2}$-space, if all fuzzy $m_{I^X}\text{-}e^*$-closed set in X is fuzzy $m_{I^X}\text{-}a$-closed in X.

Example 4.18. Consider Example 4.15. Here (X, m_{I^X}) is a fuzzy $m_{I^X}\text{-}e^*\text{-}T_{1/2}$-space.

Theorem 4.19. *Let (X, m_{I^X}) and (Y, m_{I^Y}) be two fuzzy m-spaces and $f : X \to Y$ be a function. If X is fuzzy $m_{I^X}\text{-}e^*\text{-}T_{1/2}$-space, then the following statements are equivalent:*

(1) *f is fuzzy (e^*, r)-continuous.*
(2) *f is fuzzy (e, r)-continuous.*
(3) *f is fuzzy $(\delta$-semi, $r)$-continuous.*
(4) *f is fuzzy $(\delta$-pre, $r)$-continuous.*
(5) *f is fuzzy (a, r)-continuous.*
(6) *f is fuzzy (δ, r)-continuous.*

Proof. (1) ⇒ (4): Let $W \in m_{I^Y}\text{-}RO(Y)$. By (1), $f^{-1}(W) \in m_{I^X}\text{-}eC(X)$. As X is fuzzy $m_{I^X}\text{-}e^*\text{-}T_{1/2}$-space, $f^{-1}(W) \in m_{I^X}\text{-}\delta C(X)$. Then $f^{-1}(W) \in m_{I^X}\text{-}\delta PC(X)$. Thus f is fuzzy (δ-pre, r)-continuous.

(4) ⇒ (6): Let $W \in m_{I^Y}\text{-}RO(Y)$. By (4), $f^{-1}(W) \in m_{I^X}\text{-}\delta PC(X)$. Then $f^{-1}(W) \in m_{I^X}\text{-}e^*C(X)$. As X is fuzzy $m_{I^X}\text{-}e^*\text{-}T_{1/2}$-space, $f^{-1}(W) \in m_{I^X}\text{-}\delta C(X)$. Thus f is fuzzy (δ, r)-continuous.

(6) ⇒ (5): Let $W \in m_{I^Y}\text{-}RO(Y)$. By (6), $f^{-1}(W) \in m_{I^X}\text{-}\delta C(X)$. Then $f^{-1}(W) = m_{I^X}\text{-}\delta cl(f^{-1}(W))$. Thus, $m_{I^X}\text{-}cl(m_{I^X}\text{-}int(m_{I^X}\text{-}\delta cl(f^{-1}(W)))) \leq m_{I^X}\text{-}cl(m_{I^X}\text{-}\delta cl(f^{-1}(W))) \leq m_{I^X}\text{-}\delta cl(m_{I^X}\text{-}\delta cl(f^1(W))) = m_{I^X}\text{-}\delta cl(f^{-1}(W)) = f^{-1}(W)$. So $f^{-1}(W) \in m_{I^X}\text{-}aC(X)$. Hence f is fuzzy (a, r)-continuous.

(5) ⇒ (3): Let $W \in m_{I^Y}$-$RO(Y)$. By (5), $f^{-1}(W) \in m_{I^X}$-$aC(X)$. Then $1_X \setminus f^{-1}(W) \in m_{I^X}$-$aO(X)$. Thus $1_X \setminus f^{-1}(W) \leq m_{I^X}$-$int(m_{I^X}$-$cl(m_{I^X}$-$\delta int(1_X \setminus f^{-1}(W)))) \leq m_{I^X}$-$cl(m_{I^X}$-$\delta int(1_X \setminus f^{-1}(W)))$. So $1_X \setminus f^{-1}(W) \in m_{I^X}$-$\delta SO(X)$. Hence $f^{-1}(W) \in m_{I^X}$-$\delta SC(X)$. Therefore f is fuzzy (δ-semi, r)-continuous.

(3) ⇒ (2): Let $W \in m_{I^Y}$-$RO(Y)$. By (3), $f^{-1}(W) \in m_{I^X}$-$\delta SC(X)$. Then $f^{-1}(W) \in m_{I^X}$-$eC(X)$ as every fuzzy m_{I^X}-δ-semiclosed set is fuzzy m_{I^X}-e-closed. Therefore, f is fuzzy (e, r)-continuous.

(2) ⇒ (1): Let $W \in m_{I^Y}$-$RO(Y)$. By (2), $f^{-1}(W) \in m_{I^X}$-$eC(X)$. Then $f^{-1}(W) \in m_{I^X}$-$e^*C(X)$ (as every fuzzy m_{I^X}-e-closed set is fuzzy m_{I^X}-e^*-closed). Thus f is fuzzy (e^*, r)-continuous. □

Theorem 4.20. *Let (X, m_{I^X}) and (Y, m_{I^Y}) be two fuzzy m-spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(1) *f is fuzzy (e^*, r)-continuous.*

(2) *$f^{-1}(A) \in m_{I^X}$-$e^*O(X)$, for all $A \in m_{I^Y}$-$RC(Y)$.*

(3) *$f(m_{I^X}$-e^*-$clU) \leq m_{I^Y}$-r-$ker(f(U))$, for all $U \in I^X$.*

(4) *m_{I^X}-e^*-$cl(f^{-1}(A)) \leq f^{-1}(m_{I^Y}$-$r$-$ker(A))$, for all $A \in I^Y$.*

(5) *For each fuzzy point x_α in X and each $A \in m_{I^Y}$-$SO(Y)$ with $f(x_\alpha)qA$, there exists $U \in m_{I^X}$-$e^*O(X)$ with $x_\alpha qU$, $f(U) \leq m_{I^Y}$-clA.*

(6) *$f(m_{I^X}$-e^*-$clP) \leq m_{I^Y}$-θ-$scl(f(P))$, for all $P \in I^X$.*

(7) *m_{I^X}-e^*-$cl(f^{-1}(R)) \leq f^{-1}(m_{I^Y}$-$\theta$-$sclR))$, for all $R \in I^Y$.*

(8) *m_{I^X}-e^*-$cl(f^{-1}(R)) \leq f^{-1}(m_{I^Y}$-$\theta$-$sclR))$, for all $R \in m_{I^Y}$.*

(9) *m_{I^X}-e^*-$cl(f^{-1}(R)) \leq f^{-1}(m_{I^Y}$-$sclR))$, for all $R \in m_{I^Y}$.*

(10) *m_{I^X}-e^*-$cl(f^{-1}(R)) \leq f^{-1}(m_{I^Y}$-$int(m_{I^Y}$-$clR))$, for all $R \in m_{I^Y}$.*

(11) *For each fuzzy point x_α in X and each $A \in m_{I^Y}$-$SO(Y)$ with $f(x_\alpha) \in A$, there exists $U \in m_{I^X}$-$e^*O(X)$ such that $x_\alpha \in U$ and $f(U) \leq m_{I^Y}$-clA.*

(12) *$f^{-1}(A) \leq m_{I^X}$-e^*-$int(f^{-1}(m_{I^Y}$-$clA))$, for all $A \in m_{I^Y}$-$SO(Y)$.*

(13) *$f^{-1}(m_{I^Y}$-$int(m_{I^Y}$-$clA)) \in m_{I^X}$-$e^*C(X)$, for all $A \in m_{I^Y}$.*

(14) *$f^{-1}(m_{I^Y}$-$cl(m_{I^Y}$-$intF)) \in m_{I^X}$-$e^*O(X)$, for all $1_X \setminus F \in m_{I^Y}$.*

(15) *$f^{-1}(m_{I^Y}$-$clU) \in m_{I^X}$-$e^*O(X)$, for all $U \in m_{I^Y}$-$\beta O(Y)$.*

(16) *$f^{-1}(m_{I^Y}$-$clU) \in m_{I^X}$-$e^*O(X)$, for all $U \in m_{I^Y}$-$SO(Y)$.*

(17) *$f^{-1}(m_{I^Y}$-$int(m_{I^Y}$-$clU)) \in m_{I^X}$-$e^*C(X)$, for all $U \in m_{I^Y}$-$PO(Y)$.*

(18) *$f^{-1}(m_{I^Y}$-$\alpha clU) \in m_{I^X}$-$e^*O(X)$, for all $U \in m_{I^Y}$-$\beta O(Y)$.*

(19) *$f^{-1}(m_{I^Y}$-$pclU) \in m_{I^X}$-$e^*O(X)$, for all $U \in m_{I^Y}$-$SO(Y)$.*

(20) *m_{I^X}-e^*-$cl(f^{-1}(R)) \leq f^{-1}(m_{I^Y}$-$\theta$-$sclR))$, for all $R \in m_{I^Y}$-$SO(Y)$.*

(21) *m_{I^X}-e^*-$cl(f^{-1}(R)) \leq f^{-1}(m_{I^Y}$-$\theta$-$sclR))$, for all $R \in m_{I^Y}$-$PO(Y)$.*

(22) *m_{I^X}-e^*-$cl(f^{-1}(R)) \leq f^{-1}(m_{I^Y}$-$\theta$-$sclR))$, for all $R \in m_{I^Y}$-$\beta O(Y)$.*

Proof. (1) ⇒ (2): Let $W \in m_{I^Y}$-$RC(Y)$. Then $1_Y \setminus W \in m_{I^Y}$-$RO(Y)$. By (1), $f^{-1}(1_Y \setminus W) = 1_X \setminus f^{-1}(W) \in m_{I^X}$-$e^*C(X)$. Thus $f^{-1}(W) \in m_{I^X}$-$e^*O(X)$.

(2) ⇒ (1): Let $W \in m_{I^Y}$-$RO(Y)$. Then $1_Y \setminus W \in m_{I^Y}$-$RC(Y)$. By (2), $f^{-1}(1_Y \setminus W) = 1_X \setminus f^{-1}(W) \in m_{I^X}$-$e^*O(X)$. Thus $f^{-1}(W) \in m_{I^X}$-$e^*C(X)$.

(2) ⇒ (3): Let $U \in I^X$ and suppose that y_α be a fuzzy point in Y with $y_\alpha \notin m_{I^Y}$-r-ker$(f(U))$. Then there exists $V \in m_{I^Y}$-$RO(Y)$ such that $f(U) \leq V$ and $y_\alpha \notin V$. Thus $V(y) < \alpha$. So $y_\alpha q(1_Y \setminus V) \in m_{I^Y}$-$RC(Y)$ and $1_Y \setminus f(U) \geq 1_Y \setminus V$. Hence $f(U) \not q(1_Y \setminus V)$. Therefore $U \not q f^{-1}(1_Y \setminus V)$. By (2), $f^{-1}(1_Y \setminus V) = 1_X \setminus f^{-1}(V) \in$

m_{I^X}-$e^*O(X)$. By Lemma 3.22(2), m_{I^X}-e^*-$clU \not q(1_X \setminus f^{-1}(U))$. Then m_{I^X}-e^*-$clU \leq f^{-1}(V)$, i.e., $f(m_{I^X}$-e^*-$clU) \leq V$. Thus $1_Y \setminus f(m_{I^X}$-e^*-$clU) \geq 1_Y \setminus V$. So $1 - f(m_{I^X}$-e^*-$clU)(y) > 1 - V(y) > 1 - \alpha$, i.e., $\alpha > f(m_{I^X}$-e^*-$clU)(y)$. Hence $y_\alpha \notin f(m_{I^X}$-$e^*$-$clU)$. Therefore, $f(m_{I^X}$-e^*-$clU) \leq m_{I^Y}$-r-ker$(f(U))$.

(3) ⇒ (4): Let $A \in I^Y$. Then $f^{-1}(A) \in I^X$. By (3), $f(m_{I^X}$-e^*-$clf^{-1}(A)) \leq m_{I^Y}$-$r$-ker$(A)$. Then m_{I^X}-e^*-$cl(f^{-1}(A)) \leq f^{-1}(m_{I^Y}$-$r$-ker$(A))$.

(4) ⇒ (1): Let $A \in m_{I^Y}$-$RO(Y)$. By (4), m_{I^X}-e^*-$cl(f^{-1}(A)) \leq f^{-1}(m_{I^Y}$-$r$-ker$(A)) = f^{-1}(A)$. But $f^{-1}(A) \leq m_{I^X}$-e^*-$cl(f^{-1}(A))$ and thus $f^{-1}(A) = m_{I^X}$-e^*-$cl(f^{-1}(A))$. So $f^{-1}(A) \in m_{I^X}$-$e^*C(X)$.

(5) ⇒ (6). Let $P \in I^X$ and x_α be any fuzzy point in X such that $x_\alpha \in m_{I^X}$-e^*-clP and let $G \in m_{I^Y}$-$SO(Y)$ with $f(x_\alpha)qG$. By (5), there exists $U \in m_{I^X}$-$e^*O(X)$ with $x_\alpha qU$, $f(U) \leq m_{I^Y}$-clG. As $x_\alpha \in m_{I^X}$-e^*-clP, by Lemma 3.22(1), UqP and so $f(U)qf(P)$. Then $f(P)qm_{I^Y}$-$clG \Rightarrow f(x_\alpha) \in m_{I^Y}$-$\theta$-$scl(f(P))$. Thus $f(m_{I^X}$-e^*-$clP) \leq m_{I^Y}$-θ-$scl(f(P))$.

(6) ⇒ (7): Let $R \in I^Y$. By (6), $f(m_{I^X}$-e^*-$cl(f^{-1}(R))) \leq m_{I^Y}$-$\theta$-$scl(f(f^{-1}(R))) \leq m_{I^Y}$-$\theta$-$sclR$. Then m_{I^X}-e^*-$cl(f^{-1}(R)) \leq f^{-1}(m_{I^Y}$-$\theta$-$sclR)$.

(7) ⇒ (5): Let x_α be any fuzzy point in X and $A \in m_{I^Y}$-$SO(Y)$ with $f(x_\alpha)qA$. Since, $(m_{I^Y}$-$clA) \not q(1_Y \setminus m_{I^Y}$-$clA)$, by definition $f(x_\alpha) \notin m_{I^Y}$-$\theta$-$scl(1_Y \setminus m_{I^Y}$-$clA)$. Then $x_\alpha \notin f^{-1}(m_{I^Y}$-$\theta$-$scl(1_Y \setminus m_{I^Y}$-$clA))$. By (7), $x_\alpha \notin m_{I^X}$-e^*-$cl(f^{-1}(1_Y \setminus m_{I^Y}$-$clA))$. Thus there exists $U \in m_{I^X}$-$e^*O(X)$ with $x_\alpha qU$, $U \not q f^{-1}(1_Y \setminus m_{I^Y}$-$clA)$. So $f(U) \not q(1_Y \setminus m_{I^Y}$-$clA)$. Hence $f(U) \leq m_{I^Y}$-clA.

(7) ⇒ (8): Let $A \in m_{I^Y}$. By (7), m_{I^X}-e^*-$cl(f^{-1}(A)) \leq f^{-1}(m_{I^Y}$-$\theta$-$sclA)$.

(8) ⇒ (9): It follows from Lemma 3.22(7).

(9) ⇒ (10): It follows from Lemma 3.22(3).

(10) ⇒ (1): Let $A \in m_{I^Y}$-$RO(Y)$. By (10), m_{I^X}-e^*-$cl(f^{-1}(A)) \leq f^{-1}(m_{I^Y}$-$int(m_{I^Y}$-$clA)) = f^{-1}(A)$.Then $f^{-1}(A) \in m_{I^X}$-$e^*C(X)$. Thus f is fuzzy (e^*, r)-continuous.

(1) ⇒ (10): Let $A \in m_{I^Y}$. Then m_{I^Y}-$int(m_{I^Y}$-$clA) \in m_{I^Y}$-$RO(Y)$. By (1), $f^{-1}(m_{I^Y}$-$int(m_{I^Y}$-$clA)) \in m_{I^X}$-$e^*C(X)$. Thus

$$m_{I^X}\text{-}e^*\text{-}cl(f^{-1}(A)) \leq m_{I^X}\text{-}e^*\text{-}cl(f^{-1}(m_{I^Y}\text{-}int(m_{I^Y}\text{-}clA))) = f^{-1}(m_{I^Y}\text{-}int(m_{I^Y}\text{-}clA)).$$

(10) ⇒ (9): It follows from lemma 3.22(3).

(9) ⇒ (8): It follows from Lemma 3.22(7).

(7) ⇒ (1): Let $R \in m_{I^Y}$-$RO(Y)$. By (7), m_{I^X}-e^*-$cl(f^{-1}(R)) \leq f^{-1}(m_{I^Y}$-$\theta$-$sclR) = f^{-1}(R)$. Then $f^{-1}(R) \in m_{I^X}$-$e^*C(X)$. Thus f is fuzzy (e^*, r)-continuous.

(5) ⇒ (12): Let $A \in m_{I^Y}$-$SO(Y)$ and x_α be any fuzzy point in X such that $x_\alpha qf^{-1}(A)$. Then $f(x_\alpha)qA$. By (5), there exists $U \in m_{I^X}$-$e^*O(X)$ such that $x_\alpha qU$, $f(U) \leq m_{I^Y}$-clA. Thus $x_\alpha qU \leq f^{-1}(m_{I^Y}$-$clA)$. So $x_\alpha qm_{I^X}$-e^*-$int(f^{-1}(m_{I^Y}$-$clA))$, as m_{I^X}-e^*-$int(f^{-1}(m_{I^Y}$-$clA))$ is the union of all fuzzy m_{I^X}-e^*-open sets in X contained in $f^{-1}(m_{I^Y}$-$clA)$. Hence $f^{-1}(A) \leq m_{I^X}$-e^*-$int(f^{-1}(m_{I^Y}$-$clA))$.

(12) ⇒ (5): Let x_α be any fuzzy point in X and $A \in m_{I^Y}$-$SO(Y)$ with $f(x_\alpha)qA$. Then $x_\alpha qf^{-1}(A) \leq m_{I^X}$-$e^*$-$int(f^{-1}(m_{I^Y}$-$clA))$ (by (12)) implies there exists $U \in m_{I^X}$-$e^*O(X)$ with $x_\alpha qU$, $U \leq f^{-1}(m_{I^Y}$-$clA)$. Thus $f(U) \leq m_{I^Y}$-clA.

(11) ⇒ (12): Let $A \in m_{I^Y}$-$SO(Y)$ and x_α be any fuzzy point in X such that $x_\alpha \in f^{-1}(A)$. Then $f(x_\alpha) \in A$. By (11), there exists $U \in m_{I^X}$-$e^*O(X)$ with $x_\alpha \in U$

and $f(U) \leq m_{I^Y}\text{-}clA$. Thus $U \leq f^{-1}(m_{I^Y}\text{-}clA)$. So $x_\alpha \in m_{I^X}\text{-}e^*\text{-}int(f^{-1}(m_{I^Y}\text{-}clA))$. Hence $f^{-1}(A) \leq m_{I^X}\text{-}e^*\text{-}int(f^{-1}(m_{I^Y}\text{-}clA))$.

(12) $\Rightarrow$ (11): Let x_α be any fuzzy point in X and $A \in m_{I^Y}\text{-}SO(Y)$ with $f(x_\alpha) \in A$. Then $x_\alpha \in f^{-1}(A) \leq m_{I^X}\text{-}e^*\text{-}int(f^{-1}(m_{I^Y}\text{-}clA))$ (by (12)) implies there exists $U \in m_{I^X}\text{-}e^*O(X)$ with $x_\alpha \in U$ and $U \leq f^{-1}(m_{I^Y}\text{-}clA)$. Thus $f(U) \leq m_{I^Y}\text{-}clA$.

(1) $\Rightarrow$ (13): Let $A \in m_{I^Y}$. Then $m_{I^Y}\text{-}int(m_{I^Y}\text{-}clA) \in m_{I^Y}\text{-}RO(Y)$. By (1), $f^{-1}(m_{I^Y}\text{-}int(m_{I^Y}\text{-}clA)) \in m_{I^X}\text{-}e^*C(X)$.

(13) $\Rightarrow$ (1): Let $A \in m_{I^Y}\text{-}RO(Y)$. Then $A \in m_{I^Y}$. By (13), $f^{-1}(A) = f^{-1}(m_{I^Y}\text{-}int(m_{I^Y}\text{-}clA)) \in m_{I^X}\text{-}e^*C(X)$.

(12) $\Rightarrow$ (2): Let $F \in m_{I^Y}\text{-}RC(Y)$. Then $F \in m_{I^Y}\text{-}SO(Y)$. By (12), $f^{-1}(F) \leq m_{I^X}\text{-}e^*\text{-}int(f^{-1}((m_{I^Y}\text{-}clF)) = m_{I^X}\text{-}e^*\text{-}int(f^{-1}(F))$.

(2) $\Rightarrow$ (14): Let $F \in m_{I^Y}\text{-}RC(Y)$. By (2), $f^{-1}(F) \in m_{I^X}\text{-}e^*O(X)$. But $f^{-1}(F) = f^{-1}(m_{I^Y}\text{-}cl(m_{I^Y}\text{-}intF))$. Then $f^{-1}(m_{I^Y}\text{-}cl(m_{I^Y}\text{-}intF)) \in m_{I^X}\text{-}e^*O(X)$.

(14) $\Rightarrow$ (2): Let $F \in m_{I^Y}\text{-}RC(Y)$. By (14), $f^{-1}(F) = f^{-1}(m_{I^Y}\text{-}cl(m_{I^Y}\text{-}intF)) \in m_{I^X}\text{-}e^*O(X)$.

(2) $\Rightarrow$ (15): Let $U \in m_{I^Y}\text{-}\beta O(Y)$. Then $U \leq m_{I^Y}\text{-}cl(m_{I^Y}\text{-}int(m_{I^Y}\text{-}clU)) \leq m_{I^Y}\text{-}clU$. Thus $m_{I^Y}\text{-}clU \leq m_{I^Y}\text{-}cl(m_{I^Y}\text{-}cl(m_{I^Y}\text{-}int(m_{I^Y}\text{-}clU))) = m_{I^Y}\text{-}cl(m_{I^Y}\text{-}int(m_{I^Y}\text{-}clU)) \leq m_{I^Y}\text{-}cl(m_{I^Y}\text{-}clU) = m_{I^Y}\text{-}clU \Rightarrow m_{I^Y}\text{-}clU = m_{I^Y}\text{-}cl(m_{I^Y}\text{-}int(m_{I^Y}\text{-}clU))$. So $m_{I^Y}\text{-}clU \in m_{I^Y}\text{-}RC(Y)$. Hence by (2), $f^{-1}(m_{I^Y}\text{-}clU) \in m_{I^X}\text{-}e^*O(X)$.

(15) $\Rightarrow$ (16): Since $m_{I^Y}\text{-}SO(Y) \subseteq m_{I^Y}\text{-}\beta O(Y)$, by (15), $f^{-1}(m_{I^Y}\text{-}clU) \in m_{I^X}\text{-}e^*O(X)$, for all $U \in m_{I^Y}\text{-}SO(Y)$.

(16) $\Rightarrow$ (17): Let $U \in m_{I^Y}\text{-}PO(Y)$. Then $U \leq m_{I^Y}\text{-}int(m_{I^Y}\text{-}clU)$. We claim that $m_{I^Y}\text{-}int(m_{I^Y}\text{-}clU) \in m_{I^Y}\text{-}RO(Y)$. Indeed,

$$m_{I^Y}\text{-}int(m_{I^Y}\text{-}clU) \leq m_{I^Y}\text{-}int(m_{I^Y}\text{-}cl(m_{I^Y}\text{-}int(m_{I^Y}\text{-}clU))) \leq m_{I^Y}\text{-}int(m_{I^Y}\text{-}clU).$$

Thus $m_{I^Y}\text{-}int(m_{I^Y}\text{-}clU) = m_{I^Y}\text{-}int(m_{I^Y}\text{-}cl(m_{I^Y}\text{-}int(m_{I^Y}\text{-}clU))) \Rightarrow m_{I^Y}\text{-}int(m_{I^Y}\text{-}clU) \in m_{I^Y}\text{-}RO(Y)$. So $1_Y \setminus m_{I^Y}\text{-}int(m_{I^Y}\text{-}clU) \in m_{I^Y}\text{-}RC(Y)$. Hence $1_Y \setminus m_{I^Y}\text{-}int(m_{I^Y}\text{-}clU) \in m_{I^Y}\text{-}SO(Y)$. By (16), $f^{-1}(m_{I^Y}\text{-}cl(1_Y \setminus m_{I^Y}\text{-}int(m_{I^Y}\text{-}clU))) \in m_{I^X}\text{-}e^*O(X)$. Thus $1_X \setminus f^{-1}(m_{I^Y}\text{-}int(m_{I^Y}\text{-}int(m_{I^Y}\text{-}clU))) = 1_X \setminus f^{-1}((m_{I^Y}\text{-}int(m_{I^Y}\text{-}clU)) \in m_{I^X}\text{-}e^*O(X)$. So $f^{-1}(m_{I^Y}\text{-}int(m_{I^Y}\text{-}clU)) \in m_{I^X}\text{-}e^*C(X)$.

(17) $\Rightarrow$ (1): Let $U \in m_{I^Y}\text{-}RO(Y)$. Then $U \in m_{I^Y}\text{-}PO(Y)$. By (17), $f^{-1}(m_{I^Y}\text{-}int(m_{I^Y}\text{-}clU)) \in m_{I^X}\text{-}e^*C(X)$. Thus $f^{-1}(U) = f^{-1}(m_{I^Y}\text{-}int(m_{I^Y}\text{-}clU)) \in m_{I^X}\text{-}e^*C(X)$. So (1) holds.

(15) $\Leftrightarrow$ (18): The proof follows from Lemma 3.22(5).

(15) $\Leftrightarrow$ (19): The proof follow from Lemma 3.22(6).

(7) $\Rightarrow$ (20): Obvious.

(20) $\Rightarrow$ (8): Let $A \in m_{I^Y}$. Since $m_{I^Y}\text{-}SO(Y) \supseteq m_{I^Y}$, by (20), $m_{I^X}\text{-}e^*\text{-}cl(f^{-1}(A)) \leq f^{-1}(m_{I^Y}\text{-}\theta\text{-}sclA)$.

(7) $\Rightarrow$ (22): Obvious.

(22) $\Rightarrow$ (20): Since $m_{I^Y}\text{-}SO(Y) \subseteq m_{I^Y}\text{-}\beta O(Y)$, the result follows.

(7) $\Rightarrow$ (21). Obvious.

(21) $\Rightarrow$ (8): Since $m_{I^Y} \subseteq m_{I^Y}\text{-}PO(Y)$, the result follows. □

Remark 4.21. In a similar manner we can characterize fuzzy (e, r)-continuous (resp., fuzzy (a, r)-continuous) function by changing e^* by e (resp., by a) in the Theorem 4.20.

5. Fuzzy compact sets and fuzzy s-closed sets in fuzzy m-Space

Definition 5.1 ([5, 4]). Let A be a fuzzy set in X. A collection $\mathcal{U}$ of fuzzy sets in X is called a fuzzy cover of A, if $sup\{U(x) : U \in \mathcal{U}\} = 1$, for each $x \in suppA$. In particular, if $A = 1_X$, we get the definition of fuzzy cover of X.

Definition 5.2 ([5, 4]). A fuzzy cover $\mathcal{U}$ of a fuzzy set A in X is said to have a finite subcover $\mathcal{U}_0$, if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that $\bigcup \mathcal{U}_0 \geq A$, i.e., $\mathcal{U}_0$ is also a fuzzy cover of A. In particular, if $A = 1_X$, we get $\bigcup \mathcal{U}_0 = 1_X$.

Definition 5.3. A fuzzy set A in a fuzzy m-space (X, m_{I^X}) is said to be fuzzy m-compact (resp., fuzzy m-e^*-compact, fuzzy m-e-compact, fuzzy m-a-compact), if every fuzzy covering $\mathcal{U}$ of A by fuzzy m_{I^X}-open (resp., fuzzy m_{I^X}-$e^*O(X)$, fuzzy m_{I^X}-$eO(X)$, fuzzy m_{I^X}-$aO(X)$) sets in X has a finite subcovering $\mathcal{U}_0$ of $\mathcal{U}$. In particular, if $A = 1_X$, we get the definition of fuzzy m-compact (resp., fuzzy m-e^*-compact, fuzzy m-e-compact, fuzzy m-a-compact) space.

Since every fuzzy m_{I^X}-open (resp., fuzzy m_{I^X}-e-open, fuzzy m_{I^X}-a-open) set is fuzzy m_{I^X}-e^*-open (resp., fuzzy m_{I^X}-e^*-open, fuzzy m_{I^X}-e-open), the following theorem is obvious.

Theorem 5.4. *Let (X, m_{I^X}) be a fuzzy m-space and $A \in I^X$.*
(1) *If A is fuzzy m-e^*-compact, then A is fuzzy m-compact.*
(2) *If A is fuzzy m-e^*-compact, then A is fuzzy m-e-compact.*
(3) *If A is fuzzy m-e-compact, then A is fuzzy m-a-compact.*

Definition 5.5. A fuzzy m-space (X, m_{I^X}) is said to be fuzzy m-s-closed, if for every fuzzy covering of X by fuzzy m_{I^X}-regular closed sets in X contains a finite subcovering.

Theorem 5.6. *Let (X, m_{I^X}) and (Y, m_{I^Y}) be two fuzzy m-spaces and $f : X \to Y$ be surjective, fuzzy (e^*, r)-continuous function. If X is fuzzy m-e^*-compact space, then Y is fuzzy m-s-closed space.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a fuzzy covering of Y by fuzzy m_{I^Y}-regular closed sets of Y. As f is fuzzy (e^*, r)-continuous, $\mathcal{V} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ covers X by fuzzy m_{I^X}-e^*-open sets of X. As X is fuzzy m-e^*-compact, there exists a finite subset Λ_0 of Λ such that $1_X = \bigvee_{\alpha \in \Lambda_0} f^{-1}(U_\alpha)$. Then $1_Y = f(\bigvee_{\alpha \in \Lambda_0} f^{-1}(U_\alpha)) = \bigvee_{\alpha \in \Lambda_0} f(f^{-1}(U_\alpha)) \leq \bigvee_{\alpha \in \Lambda_0} U_\alpha$. Thus Y is fuzzy m-s-closed space. □

In a similar manner we can easily state the following two theorems the proof of which are similar to that of Theorem 5.6.

Theorem 5.7. *Let (X, m_{I^X}) and (Y, m_{I^Y}) be two fuzzy m-spaces and $f : X \to Y$ be surjective, fuzzy (e, r)-continuous function. If X is fuzzy m-e-compact space, then Y is fuzzy m-s-closed space.*

Theorem 5.8. *Let (X, m_{I^X}) and (Y, m_{I^Y}) be two fuzzy m-spaces and $f : X \to Y$ be surjective, fuzzy (a, r)-continuous function. If X is fuzzy m-a-compact space, then Y is fuzzy m-s-closed space.*

Theorem 5.9. *Every fuzzy m_{I^X}-e*-closed set A in a fuzzy m-e*-compact space X is fuzzy m-e*-compact.*

Proof. Let A be a fuzzy m_{I^X}-e^*-closed set in a fuzzy m-e^*-compact space X. Let $\mathcal{U}$ be a fuzzy covering of A by fuzzy m_{I^X}-e^*-open sets in X. Then $\mathcal{V} = \mathcal{U} \bigcup (1_X \setminus A)$ is a fuzzy m_{I^X}-e^*-open covering of X. By hypothesis, there exists a finite subcollection $\mathcal{V}_0$ of $\mathcal{V}$ which also covers X. If $\mathcal{V}_0$ contains $1_X \setminus A$, we omit it and get a finite subcovering of A. Consequently, A is fuzzy m-e^*-compact. □

Similarly we can easily state the following two theorems the proof of which are similar to that of Theorem 5.9.

Theorem 5.10. *Every fuzzy m_{I^X}-e-closed set A in a fuzzy m-e-compact space X is fuzzy m-e-compact.*

Theorem 5.11. *Every fuzzy m_{I^X}-a-closed set A in a fuzzy m-a-compact space X is fuzzy m-a-compact.*

Theorem 5.12. *Let (X, m_{I^X}) and (Y, m_{I^Y}) be two fuzzy m-spaces and $f : X \to Y$ be fuzzy e*-continuous function. If A is fuzzy m-e*-compact relative to X, then the image $f(A)$ is fuzzy m-compact relative to Y.*

Proof. Let A be fuzzy m-e^*-compact relative to X and $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a fuzzy covering of $f(A)$ by fuzzy m_{I^Y}-open sets of Y, i.e, $f(A) \leq \bigvee_{\alpha\in\Lambda} U_\alpha$. Then $A \leq f^{-1}(\bigvee_{\alpha\in\Lambda} U_\alpha) = \bigvee_{\alpha\in\Lambda} f^{-1}(U_\alpha)$. Thus $\mathcal{V} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a fuzzy covering of A by fuzzy m_{I^X}-e^*-open sets in X. As A is fuzzy m-e^*-compact relative to X, there exists a finite subcollection $\mathcal{V}_0 = \{f^{-1}(U_{\alpha_i}) : 1 \leq i \leq n\}$ of $\mathcal{V}$ such that $A \leq \bigvee_{i=1}^{n} f^{-1}(U_{\alpha_i})$. So $f(A) \leq f(\bigvee_{i=1}^{n} f^{-1}(U_{\alpha_i})) = \bigvee_{i=1}^{n} f(f^{-1}(U_{\alpha_i})) \leq \bigvee_{i=1}^{n} U_{\alpha_i}$. Hence $\mathcal{U}_0 = \{U_{\alpha_i} : 1 \leq i \leq n\}$ is a finite subcovering of $f(A)$. Therefore the result holds. □

Similarly we can easily state the following two theorems the proof of which are similar to that of Theorem 5.12.

Theorem 5.13. *Let (X, m_{I^X}) and (Y, m_{I^Y}) be two fuzzy m-spaces where (X, m_{I^X}) is fuzzy m_{I^X}-e-$T_{1/2}$-space and $f : X \to Y$ be fuzzy e*-continuous function. If A is fuzzy m-e-compact relative to X, then the image $f(A)$ is fuzzy m-compact relative to Y.*

Theorem 5.14. *Let (X, m_{I^X}) and (Y, m_{I^Y}) be two fuzzy m-spaces where (X, m_{I^X}) is fuzzy m_{I^X}-a-$T_{1/2}$-space and $f : X \to Y$ be fuzzy e*-continuous function. If A is fuzzy m-a-compact relative to X, then the image $f(A)$ is fuzzy m-compact relative to Y.*

Definition 5.15. Let (X, m_{I^X}) be a fuzzy m-space. Then X is said to be fuzzy m_{I^X}-T_2 (resp., fuzzy m_{I^X}-e^*-T_2, fuzzy m_{I^X}-e-T_2, fuzzy m_{I^X}-a-T_2) space, if for each pair of distinct fuzzy points x_α, y_β ; when $x \neq y$, there exist fuzzy m_{I^X}-open (resp., fuzzy m_{I^X}-e^*-open, fuzzy m_{I^X}-e-open, fuzzy m_{I^X}-a-open) sets U_1, U_2, V_1, V_2 in X

such that $x_\alpha \in U_1, y_\beta q V_1$ and $U_1 \not q V_1$ and $x_\alpha q U_2$, $y_\beta \in V_2$ and $U_2 \not q V_2$; when $x = y$, $\alpha < \beta$ (say), there exist fuzzy m_{I^X}-open (resp., fuzzy m_{I^X}-e^*-open, fuzzy m_{I^X}-e-open, fuzzy m_{I^X}-a-open) sets U, V in X such that $x_\alpha \in U, y_\beta q V$ and $U \not q V$.

Definition 5.16. A fuzzy m-space (X, m_{I^X}) is said to be fuzzy s-Urysohn if for each pair of distinct fuzzy points x_α, y_β : when $x \neq y$, there exist fuzzy m_{I^X}-semiopen sets U_1, U_2, V_1, V_2 in X such that $x_\alpha \in U_1, y_\beta q V_1$ and m_{I^X}-$clU_1 \not q m_{I^X}$-clV_1 and $x_\alpha q U_2$, $y_\beta \in V_2$ and m_{I^X}-$clU_2 \not q m_{I^X}$-clV_2 ; when $x = y$, $\alpha < \beta$ (say), there exist fuzzy m_{I^X}-semiopen sets U, V in X such that $x_\alpha \in U, y_\beta q V$ and m_{I^X}-$clU \not q m_{I^X}$-clV.

Theorem 5.17. *Let (X, m_{I^X}) and (Y, m_{I^Y}) be two fuzzy m-spaces and $f : X \to Y$ be injective fuzzy (e^*, r)-continuous function and Y is fuzzy s-Urysohn space. Then X is fuzzy m_{I^X}-e^*-T_2.*

Proof. Let x_α and y_β be two distinct fuzzy points in X where $x \neq y$. Since f is injective, $f(x_\alpha) \neq f(y_\beta)$. Since Y is fuzzy s-Urysohn, there exist fuzzy m_{I^Y}-semiopen sets U_1, U_2, V_1, V_2 in Y such that $f(x_\alpha) \in U_1, f(y_\beta) q V_1$ and m_{I^Y}-$clU_1 \not q m_{I^Y}$-clV_1 and $f(x_\alpha) q U_2$, $f(y_\beta) \in V_2$ and m_{I^Y}-$clU_2 \not q m_{I^Y}$-clV_2. By Theorem 4.20, there exist $W_1, W_2 \in m_{I^X}$-$e^*O(X)$ such that $x_\alpha \in W_1, W_1 \leq f^{-1}(m_{I^Y}\text{-}clU_1), y_\beta q W_2, W_2 \leq f^{-1}(m_{I^Y}\text{-}clV_1)$ or $x_\alpha q W_2, W_2 \leq f^{-1}(m_{Y}\text{-}clU_2), y_\beta \in W_1, W_1 \leq f^{-1}(m_{I^Y}\text{-}clV_2)$. We claim that $W_1 \not q W_2$. Indeed, m_{I^Y}-$clU_1 \not q m_{I^Y}$-clV_1 and m_{I^Y}-$clU_2 \not q m_{I^Y}$-clV_2. Then $f^{-1}(m_{I^Y}\text{-}clU_1) \not q f^{-1}(m_{I^Y}\text{-}clV_1)$ and $f^{-1}(m_{I^Y}\text{-}clU_2) \not q f^{-1}(m_{I^Y}\text{-}clV_2)$.

Similarly, when $x = y$, $\alpha < \beta$ (say), there exist $U_1, U_2 \in m_{I^Y}$-$SO(Y)$ such that $f(x_\alpha) \in U_1, f(y_\beta) q U_2$ and m_{I^Y}-$clU_1 \not q m_{I^Y}$-clU_2. By Theorem 4.20, there exist $W_1, W_2 \in m_{I^X}$-$e^*O(X)$ such that $x_\alpha \in W_1, W_1 \leq f^{-1}(m_{I^Y}\text{-}clU_1), y_\beta q W_2, W_2 \leq f^{-1}(m_{I^Y}\text{-}clU_2)$. Thus as above, $W_1 \not q W_2$. So X is fuzzy m_{I^X}-e^*-T_2-space. □

Similarly we can easily state the following two theorems the proof of which are similar to that of Theorem 5.17

Theorem 5.18. *Let (X, m_{I^X}) and (Y, m_{I^Y}) be two fuzzy m-spaces and $f : X \to Y$ be injective, fuzzy (e, r)-continuous function and Y is fuzzy s-Urysohn space. Then X is fuzzy m_{I^X}-e-T_2.*

Theorem 5.19. . *Let (X, m_{I^X}) and (Y, m_{I^Y}) be two fuzzy m-spaces and $f : X \to Y$ be injective, fuzzy (a, r)-continuous function and Y is fuzzy s-Urysohn space. Then X is fuzzy m_{I^X}-a-T_2.*

Acknowledgements. The author acknowledges the financial support from UGC (Minor Research Project), New Delhi.

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