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r-(i,j)-fuzzy regular semi open sets in smooth bitopological spaces

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Abstract. In this paper, we introduce and study the concept of r-(i,j)-fuzzy regular semi open (closed) sets in smooth bitopological spaces. By using r-(i,j)-fuzzy regular semi open (closed) sets, we define a new fuzzy closure operator namely r-(i,j)-fuzzy regular semi interior (closure) operator. Also, we introduce FP-regular semi continuous and FP-regular semi irresolute mappings. Moreover, we investigate the relationship among FP-regular semi continuous and FP-regular semi irresolute mappings. Finally, we have given some counter examples to show that these types of mappings are not equivalent.

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1. Introduction

The concept of fuzzy sets was introduced by Zadeh [14] in his classical paper. Using the concept of fuzzy sets Chang [1] introduced fuzzy topological spaces and several other authors continued the investigation of such spaces. Chattopadhyay et al. [2] and Ramadan [9] introduced new definition of smooth topological spaces as a generalization of fuzzy topological spaces. Kandil [4] introduced and studied the notion of fuzzy bitopological spaces as a natural generalization of fuzzy topological spaces. Lee et al. [6] introduced the concept of smooth bitopological spaces as a generalization of smooth topological spaces and Kandil's fuzzy bitopological spaces. Mashhour [8], Kerre et. al., [5] and Zahran [15] introduced the notion of fuzzy regular semi open and regular semi closed sets and investigate the relationship among fuzzy regular semi continuity and fuzzy regular semi irresolute mappings. In 2003, Lee [7] introduced the concept of preopen sets in smooth bitopological spaces. In this paper, we introduce and study the concept of r-(i,j)-fuzzy regular semi open (closed) sets in smooth bitopological spaces. By using r-(i,j)-fuzzy regular semi open (closed) sets, we define a new fuzzy closure operator namely r-(i,j)-fuzzy regular semi interior (closure) operator. Also, we generalize the notions of FP-regular semi continuous (open, closed), FP-regualr semi irresolute (irresolute open, irresolute closed) mappings in smooth bitopological spaces. We show that FP-regular continuous mapping is FP-regular semi continuous, however, the converse is not true. Also, we show that FP-regular semi irresolute mapping and FP-regular semi continuous mappings are independent. Therefore, we have given characterizations of all mentioned types of mappings. Finally, we established some counter examples to show that these types are not equivalent.

2. Preliminaries

Throughout this paper, let X be a non-empty set, $I=[0,1]$, $I_0=(0,1]$. A fuzzy set λ of X is a mapping $\lambda : X \to I$, and I^X be the family of all fuzzy sets on X. The complement of a fuzzy set λ is denoted by $\overline{1}-\lambda$. For $\lambda \in I^X,\ \overline{\lambda}(x)=\lambda$ for all $x \in X$. For each $x \in X$ and $t \in I_0$, a fuzzy point x_t is defined by

$$x_t(y)=\begin{cases} t & \text{if } y=x \\ 0 & \text{if } y \neq x. \end{cases}$$

Let $Pt(X)$ be the family of all fuzzy points in X. All other notations and definitions are standard in the fuzzy set theory.

Definition 2.1 ([12])**.** A function $\tau : I^X \to I$ is called a smooth topology on X, if it satisfies the following conditions:

(i) $\tau(\overline{0})=\tau(\overline{1})=1$.
(ii) $\tau(\bigvee_{i\in J}\mu_i) \geq \bigwedge_{i\in J}\tau(\mu_i)$, for any $\{\mu_i : i \in J\} \subseteq I^X$.
(iii) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1)\wedge\tau(\mu_2)$, for all $\mu_1,\ \mu_2 \in I^X$.

The pair (X,τ) is called a smooth topological space. The triple $(X,\ \tau_1,\ \tau_2)$ is called a smooth bitopological spaces (for short, smooth bts), where τ_1 and τ_2 are smooth topologies on X. Throughout this paper, the indices $i,j \in \{1,2\}$.

Theorem 2.2 ([3])**.** *Let (X,τ) be a smooth topological space. Then for each $\lambda \in I^X$ and $r \in I_0$, we define an operator $C_\tau : I^X \times I_0 \to I^X$ as follows: $C_\tau(\lambda,\ r)=\bigwedge\{\mu \in I^X : \lambda \leq \mu,\ \tau(\overline{1}-\mu) \geq r\}$. For $\lambda,\ \mu \in I^X$ and $r,s \in I_0$, the operator C_τ satisfies the following statements:*

(C1) $C_\tau(\overline{0},r)=\overline{0}$.
(C2) $\lambda \leq C_\tau(\lambda,\ r)$.
(C3) $C_\tau(\lambda,\ r)\vee C_\tau(\mu,\ r)=C_\tau(\lambda\vee\mu,\ r)$.
(C4) $C_\tau(\lambda,\ r)\leq C_\tau(\lambda,\ s)$, *if* $r \leq s$.
(C5) $C_\tau(C_\tau(\lambda,\ r),\ r)=C_\tau(\lambda,\ r)$.

Theorem 2.3 ([3])**.** *Let (X,τ) be a smooth topological space. Then for each $\lambda \in I^X$ and $r \in I_0$, we define an operator $I_\tau : I^X \times I_0 \to I^X$ as follows: $I_\tau(\lambda,\ r)=\bigvee\{\mu \in I^X : \mu \leq \lambda,\ \tau(\mu) \geq r\}$. For $\lambda,\ \mu \in I^X$ and $r,s \in I_0$, the operator I_τ satisfies the following statements:*

(I1) $I_\tau(\overline{1},r)=\overline{1}$.
(I2) $I_\tau(\lambda,\ r)\leq\lambda$.

(I3) $I_\tau(\lambda,\ r) \wedge I_\tau(\mu,\ r) = I_\tau(\lambda \wedge \mu,\ r)$.
(I4) $I_\tau(\lambda,\ r) \leq I_\tau(\lambda,\ s)$, *if* $s \leq r$.
(I5) $I_\tau(I_\tau(\lambda,\ r),\ r) = I_\tau(\lambda,\ r)$.
(I6) $I_\tau(\overline{1} - \lambda, r) = \overline{1} - C_\tau(\lambda, r)$ *and* $C_\tau(\overline{1} - \lambda, r) = \overline{1} - I_\tau(\lambda,\ r)$.

Definition 2.4 ([10])**.** Let $(X,\ \tau_1,\ \tau_2)$ be a smooth topological space, $\lambda \in I^X$ and $r \in I_0$. Then a fuzzy set λ is called an:

(i) r-(i,j)-fuzzy regular open (for short, r-(i,j)-fro) if $\lambda = I_{\tau_i}(C_{\tau_j}(\lambda, r), r)$.
(ii) r-(i,j)-fuzzy regular closed (for short, r-(i,j)-frc) if $\lambda = C_{\tau_i}(I_{\tau_j}(\lambda, r), r)$.

Definition 2.5 ([13])**.** Let $(X,\ \tau)$ be a smooth topological space. For $\lambda, \mu \in I^X$ and $r \in I_0$.

(1) The r-fuzzy regular closure of λ, denoted by $RC_\tau(\lambda, r)$, and is defined by $RC_\tau(\lambda, r) = \bigwedge\{\mu \in I^X | \mu \geq \lambda, \mu \text{ is } r\text{-frc }\}$.
(2) The r-fuzzy regular interiror of λ, denoted by $RI_\tau(\lambda, r)$, and is defined by $RI_\tau(\lambda, r) = \bigvee\{\mu \in I^X | \mu \leq \lambda, \mu \text{ is } r\text{-fro }\}$.

Definition 2.6 ([11])**.** Let $(X,\ \tau_1,\ \tau_2)$ be a smooth bts and $\lambda \in I^X,\ r \in I_0$. Then a fuzzy set λ is called an:

(i) r-(i,j)-fuzzy semi open (for short, r-(i,j)-fso) if there exists $\mu \in I^X$ with $\tau_i(\mu) \geq r$ and $\mu \leq \lambda \leq C_{\tau_j}(\mu, r)$, $i, j = 1, 2,\ i \neq j$.

(ii) r-(i,j)-fuzzy semi closed (for short, r-(i,j)-fsc) if there exists $\mu \in I^X$ with $\tau_i(\overline{1} - \mu) \geq r$ and $I_{\tau_j}(\mu, r) \leq \lambda \leq \mu$, $i, j = 1, 2,\ i \neq j$.

3. r-(i,j)-fuzzy regular semi open and r-(i,j)-fuzzy regular semi closed sets

Definition 3.1. Let $(X,\ \tau_1,\ \tau_2)$ be a smooth bts and $\lambda \in I^X,\ r \in I_0$.

(i) λ is called an r-(i,j)-fuzzy regular semi open (for short, r-(i,j)-frso), if there exists r-(i,j)-fro set $\mu \in I^X$ and $\mu \leq \lambda \leq C_{\tau_j}(\mu, r)$, $i, j = 1, 2,\ i \neq j$.

(ii) λ is called an r-(i,j)-fuzzy regular semi closed (for short, r-(i,j)-frsc), if there exists r-(i,j)-frc set $\mu \in I^X$ and $I_{\tau_j}(\mu, r) \leq \lambda \leq \mu$, $i, j = 1, 2,\ i \neq j$.

(iii) The r-(i,j)-fuzzy regular semi interiror of λ, denoted by $RSI_{ij}(\lambda, r)$, is defined by $RSI_{ij}(\lambda, r) = \bigvee\{\mu \in I^X \mid \mu \leq \lambda,\ \mu \text{ is } r\text{-}(i,j)\text{-frso }\}$.

(iv) The r-(i,j)-fuzzy regular semi closure of λ, denoted by $RSC_{ij}(\lambda, r)$ is defined by $RSC_{ij}(\lambda, r) = \bigwedge\{\mu \in I^X \mid \mu \geq \lambda,\ \mu \text{ is } r\text{-}(i,j)\text{-frsc }\}$.

Theorem 3.2. *Let $(X,\ \tau_1,\ \tau_2)$ be a smooth bts. For $\lambda \in I^X,\ r \in I_0$, the following statements are equivalent:*

(1) λ *is* r-(i,j)-*frso.*
(2) $\lambda \leq C_{\tau_j}(I_{\tau_i}(\lambda, r), r)$.
(3) $\overline{1} - \lambda$ *is* r-(i,j)-*frsc.*
(4) $I_{\tau_i}(C_{\tau_j}(\lambda, r), r) \leq (\overline{1} - \lambda, r)$.

Proof. (1)⇒(2): Let λ be an r-(i,j)-frso set. Then there exists an r-(i,j)-fro set μ such that $\mu \leq \lambda \leq C_{\tau_j}(\mu, r)$. Since μ is r-(i,j)-fro set and also since every r-(i,j)-fro set r-τ_i-fuzzy open, $\tau_i(\mu) \geq r$ and by Theorem 2.3, $I_{\tau_i}(\mu, r) = \mu$. Since $\mu \leq \lambda$, we have $\mu = I_{\tau_i}(\mu, r) \leq I_{\tau_i}(\lambda, r)$. Thus $C_{\tau_j}(\mu, r) \leq C_{\tau_j}(I_{\tau_i}(\lambda, r), r)$. So $\lambda \leq C_{\tau_j}(I_{\tau_i}(\lambda, r), r)$.

(2)⇒(1): Put $\mu = I_{\tau_i}(\lambda, r)$ and let μ be a r-(i, j)-fro set. Then by (2), $\lambda \leq C_{\tau_j}(I_{\tau_i}(\lambda, r), r) \leq C_{\tau_j}(\mu, r)$. Thus $\mu = I_{\tau_i}(\lambda, r) \leq \lambda \leq C_{\tau_j}(\mu, r)$. So λ is r-(i,j)-frso set.

(1)⇔(3): It is easily proved from the following:

$\mu \leq \lambda \leq C_{\tau_j}(\mu, r) \Leftrightarrow \overline{1} - C_{\tau_j}(\mu, r) \leq \overline{1} - \lambda \leq \overline{1} - \mu \Leftrightarrow I_{\tau_j}(\overline{1} - \mu, r) \leq \overline{1} - \lambda \leq \overline{1} - \mu$. (By Theorem 2.3(I6)).

(2)⇔(4): It follow immediately by taking the complement of the two sides. □

Theorem 3.3. *Let (X, τ_1, τ_2) be a smooth bts. Then*

(1) *any union of r-(i,j)-frso sets is r-(i,j)-frso set,*
(2) *any intersection of r-(i,j)-frsc sets is r-(i,j)-frsc set.*

Proof. (1) Let $\{\lambda_\alpha : \alpha \in \Gamma\}$ be a family of r-(i,j)-frso. Then for each $\alpha \in \Gamma$, there exists r-(i,j)-fro set $\nu_\alpha \in I^X$ such that

$$\nu_\alpha \leq \lambda_\alpha \leq C_{\tau_j}(\nu_\alpha, r) \tag{3.1}$$

Also, $\nu_\alpha \leq \vee \nu_\alpha$ implies that $C_{\tau_j}(\nu_\alpha, r) \leq C_{\tau_j}(\vee \nu_\alpha, r)$. Then, from (3.1), we have

$$\vee \nu_\alpha \leq \vee \lambda_\alpha \leq C_{\tau_j}(\nu_\alpha, r) \leq C_{\tau_j}(\vee \nu_\alpha, r).$$

Since $\tau_i(\vee \nu_\alpha) \geq \wedge(\nu_\alpha)$ is r-(i,j)-fro set, $\vee \lambda_\alpha$ is r-(i,j)-frso.

(2) It is similar to (1). □

Theorem 3.4. *Let (X, τ_1, τ_2) be smooth bts, $\lambda, \mu \in I^X$ and $r \in I_0$.*

(1) *If λ is r-(i,j)-fro set, then λ is r-(i,j)-frso.*
(2) *If λ is r-(i,j)-frso and $I_{\tau_i}(\lambda, r) \leq \mu \leq C_{\tau_j}(\lambda, r)$, then μ is r-(i,j)-frso.*
(3) *If λ is r-(i,j)-frsc and $I_{\tau_j}(\lambda, r) \leq \mu \leq C_{\tau_i}(\lambda, r)$, then μ is r-(i,j)-frsc.*

Proof. (1) Let λ is r-(i,j)-fro, since $\lambda \leq \lambda \leq C_{\tau_j}(\lambda, r)$. Then λ is r-(i,j)-frso.

(2) Let λ be r-(i,j)-frso such that $I_{\tau_i}(\lambda, r) \leq \mu \leq C_{\tau_j}(\lambda, r)$. Then there exists r-(i,j)-fro set $\nu \in I^X$ such that $\nu \leq \lambda \leq C_{\tau_j}(\nu, r)$. It implies $C_{\tau_j}(\lambda, r) \leq C_{\tau_j}(\nu, r)$ and thus $\mu \leq C_{\tau_j}(\nu, r)$. Also, $\nu \leq I_{\tau_i}(\lambda, r) \leq \mu$. It follows that, $\nu \leq \mu \leq C_{\tau_j}(\nu, r)$. So, μ is r-(i,j)-frso.

(3) It is similar to (2). □

Remark 3.5. Note that every r-(i,j)-fro (resp. r-(i,j)-frso) set is r-(i,j)-frso (resp. r-(i,j)-fso) set. However, the converse is not true as shown in the following example.

Example 3.6. Let $X = \{a, b, c\}$, $\lambda, \mu, \delta \in I^X$ are defined as $\lambda(a) = 0.5, \lambda(b) = 0.5, \lambda(c) = 0.6$; $\mu(a) = 0.4, \mu(b) = 0.5, \mu(c) = 0.6$; $\delta(a) = 0.4, \delta(b) = 0.5, \delta(c) = 0.4$. We define smooth topologies $\tau_1, \tau_2 : I^X \to I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise,} \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \delta, \\ 0 & \text{otherwise.} \end{cases}$$

Then the fuzzy set μ is $\frac{1}{2}$-(1, 2)-fro set, $\mu \leq \lambda \leq C_{\tau_j}(\mu, r)$ and λ is $\frac{1}{2}$-(1, 2)-frso set but not $\frac{1}{2}$-(1, 2)-fro set.

Example 3.7. Let $X = \{a, b, c\}$, $\lambda, \mu, \delta \in I^X$ are defined as $\lambda(a) = 0.4, \lambda(b) = 0.5, \lambda(c) = 0.6$; $\mu(a) = 0.3, \mu(b) = 0.5, \mu(c) = 0.6$; $\delta(a) = 0.6, \delta(b) = 0.5, \delta(c) = 0.5$. We define smooth topologies $\tau_1, \tau_2 : I^X \to I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise,} \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \delta, \\ 0 & \text{otherwise.} \end{cases}$$

Then the fuzzy open set μ, $\mu \leq \lambda \leq C_{\tau_j}(\mu, r)$ and λ is $\frac{1}{2}$-$(1,2)$-fso set but not $\frac{1}{2}$-$(1,2)$-frso set.

Proposition 3.8. *Let (X, τ_1, τ_2) be smooth bts. For $\lambda, \mu \in I^X$ and $r \in I_0$. Then:*
(1) $RSI_{ij}(\overline{1} - \lambda, r) = \overline{1} - RSC_{ij}(\lambda, r)$.
(2) $RSC_{ij}(\overline{1} - \lambda, r) = \overline{1} - RSI_{ij}(\lambda, r)$.
(3) *If* $\lambda \leq \mu$, *then* $RSC_{ij}(\lambda, r) \leq RSC_{ij}(\mu, r)$.
(4) *If* $\lambda \leq \mu$, *then* $RSI_{ij}(\lambda, r) \leq RSI_{ij}(\mu, r)$.

Proof. We prove (1), using Definition 3.1.
$\overline{1} - RSC_{ij}(\lambda, r) = \overline{1} - \bigwedge\{\mu \in I^X \mid \mu \geq \lambda,\ \mu \text{ is } r\text{-}(i,j)\text{-frsc}\}$.
$= \bigvee\{\overline{1} - \mu \in I^X \mid \overline{1} - \mu \leq \overline{1} - \lambda,\ \overline{1} - \mu \text{ is } r\text{-}(i,j)\text{-frso}\}$.
$= RSI_{ij}(\overline{1} - \lambda, r)$.
(2) It is similar to the proof of (1).
(3) It follows from Definition 3.1.
(4) Taking the complement of (3) and (1). □

Theorem 3.9. *Let (X, τ_1, τ_2) be smooth bts. For $\lambda, \mu \in I^X$ and $r \in I_0$. It satisfies the following statements:*
(1) λ *is* r-(i,j)-*frso* $\Leftrightarrow \lambda = RSI_{ij}(\lambda, r)$.
(2) λ *is* r-(i,j)-*frsc* $\Leftrightarrow \lambda = RSC_{ij}(\lambda, r)$.
(3) $RSC_{ij}(\overline{0}, r) = \overline{0}$ *and* $RSI_{ij}(\overline{1}, r) = \overline{1}$.
(4) $RI_{\tau_i}(\lambda, r) \leq RSI_{ij}(\lambda, r) \leq \lambda \leq RSC_{ij}(\lambda, r) \leq RC_{\tau_i}(\lambda, r)$.
(5) $RSC_{ij}(\lambda, r) \vee RSC_{ij}(\mu, r) \leq RSC_{ij}(\lambda \vee \mu, r)$.
(6) $RC_{\tau_i}(RSC_{ij}(\lambda, r), r) = RSC_{ij}(RC_{\tau_i}(\lambda, r), r) = RC_{\tau_i}(\lambda, r)$.

Proof. (1) Let λ be r-(i,j)-frso. Then

$$RSI_{ij}(\lambda, r) = \vee\{\rho \in I^X : \rho \leq \lambda,\ \rho \text{ is } r\text{ -}(i,j)\text{-frso}\} = \lambda.$$

Conversely, let $\lambda = RSI_{ij}(\lambda, r)$. Since $RSI_{ij}(\lambda, r)$ is the arbitrary union of r-(i,j)-frso, λ is r-(i,j)-frso.
(2) It is similar to (1).
(3) It is easily obtained from Definition 3.1.
(4) Clearly, $RI_{\tau_i}(\lambda, r) = \bigvee\{\rho \in I^X : \rho \leq \lambda,\ \rho \text{ is } r\text{ -}(i,j)\text{-fro}\}$
$\leq \bigvee\{\rho \in I^X : \rho \leq \lambda, \rho \text{ is } r\text{ -}(i,j)\text{-frso}\}$
$= RSI_{ij}(\lambda, r)$.
Then $RI_{\tau_i}(\lambda, r) \leq RSI_{ij}(\lambda, r)$. On one hand,

$$RSC_{ij}(\lambda, r) = \bigwedge\{\rho \in I^X : \rho \geq \lambda,\ \rho \text{ is } r\text{ -}(i,j)\text{-frsc}\} \leq RC_{\tau_i}(\lambda, r).$$

Finally, we have

$$RI_{\tau_i}(\lambda, r) \leq RSI_{ij}(\lambda, r) \leq \lambda \leq RSC_{ij}(\lambda, r) \leq RC_{\tau_i}(\lambda, r).$$

(5) Since, $\mu \leq \mu \vee \rho,\ \rho \leq \mu \vee \rho$. Then by Proposition 3.8(3), we have

$$RSC_{ij}(\mu, r) \leq RSC_{ij}(\mu \vee \rho, r) \text{ and } RSC_{ij}(\rho, r) \leq RSC_{ij}(\mu \vee \rho, r).$$

Thus, $\quad RSC_{ij}(\mu, r) \vee RSC_{ij}(\rho, r) \leq RSC_{ij}(\mu \vee \rho, r)$.

(6) Since $RC_{\tau_i}(\lambda, r)$ is r-(i,j)-frsc set, then

$$RSC_{ij}(RC_{\tau_i}(\lambda, r), r) = RC_{\tau_i}(\lambda, r) \tag{3.2}$$

Now it remains to prove only the relation:

$$RC_{\tau_i}(RSC_{ij}(\lambda, r), r) = RC_{\tau_i}(\lambda, r).$$

Since, $\lambda \leq RSC_{ij}(\lambda, r)$, $RC_{\tau_i}(\lambda, r) \leq RC_{\tau_i}(RSC_{ij}(\lambda, r)$. It remains to prove: $RC_{\tau_i}(RSC_{ij}(\lambda, r), r) \leq RC_{\tau_i}(\lambda, r)$. Let the contrary, that is, $RC_{\tau_i}(RSC_{ij}(\lambda, r), r) \nleq RC_{\tau_i}(\lambda, r)$. Then $RC_{\tau_i}(RSC_{ij}(\lambda, r), r) > RC_{\tau_i}(\lambda, r)$. Thus, there exists r-(i,j)-frc set $\rho \in I^X$, $\rho \geq \lambda$ such that

$$RC_{\tau_i}(\lambda, r)(x) < \rho(x) < RC_{\tau_i}(RSC_{ij}(\lambda, r), r)(x). \tag{3.3}$$

Since $\lambda \leq \rho$, $RSC_{ij}(\lambda, r) \leq RSC_{ij}(\rho, r)$
$$= RSC_{ij}(RC_{\tau_i}(\lambda, r), r)$$
$$= RC_{\tau_i}(\rho, r).$$

So, $RSC_{ij}(\lambda, r) \leq RC_{\tau_i}(\rho, r)$ and this implies $RC_{\tau_i}(RSC_{ij}(\lambda, r), r) \leq RC_{\tau_i}(\lambda, r)$ which contradicts to the relation (3.3). Hence the result holds. □

4. Fuzzy pairwise regular semi continuous mappings

Definition 4.1. Let (X, τ_1, τ_2) and (Y, η_1, η_2) be smooth bts's. Let $f : X \to Y$ be a mapping. Then f is said to be:

(i) FP-regular continuous, if $f^{-1}(\mu)$ is r-(i,j)-fro for each $\mu \in I^Y$, $\eta_i(\mu) \geq r$, $i, j = 1, 2$, $i \neq j$.

(ii) FP-regular semi irresolute (resp. FP-regular semi continuous), if $f^{-1}(\mu)$ is r-(i,j)-frso for each r-(i,j)-frso set $\mu \in I^Y$ (resp. $\mu \in I^Y$, $\eta_i(\mu) \geq r$), $i, j = 1, 2$, $i \neq j$.

(iii) FP-regular semi irresolute open (resp. FP-regular semi open), if $f(\lambda)$ is r-(i,j)-frso in Y for each r-(i,j)-frso set $\lambda \in I^X$ (resp. $\lambda \in I^X$, $\tau_i(\lambda) \geq r$).

(iv) FP-regular semi irresolute closed (resp. FP-regular semi closed), if $f(\lambda)$ is r-(i,j)-frsc in Y for each r-(i,j)-frsc set $\lambda \in I^X$ (resp. $\lambda \in I^X$, $\tau_i(\overline{1} - \lambda) \geq r$).

(v) FP-regular semi irresolute homeomorphism iff f is bijective, f and f^{-1} are FP-regular semi irresolute.

Remark 4.2. (1) Every FP-regular continuous mapping is FP-regular semi continuous. However, the converse is not true as shown in the following example.

(2) FP-regular semi irresolute mappings and FP-regular semi continuous mappings are independent.

Example 4.3. Let $X = \{a, b, c\} = Y$, $\lambda_1, \lambda_2 \in I^X$, $\lambda_3, \lambda_4 \in I^Y$ are defined as $\lambda_1(a) = 0.4, \lambda_1(b) = 0.5, \lambda_1(c) = 0.6$; $\lambda_2(a) = 0.4, \lambda_2(b) = 0.5, \lambda_2(c) = 0.4$; $\lambda_3(a) = 0.5, \lambda_3(b) = 0.5, \lambda_3(c) = 0.6$; $\lambda_4(a) = 0.4, \lambda_4(b) = 0.5, \lambda_4(c) = 0.6$. We define smooth topologies $\tau_1, \tau_2, \eta_1, \eta_2 : I^X \to I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & \text{otherwise,} \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\eta_1(\lambda)=\begin{cases}1 & \text{if } \lambda\in\{\overline{0},\overline{1}\},\\ \frac{1}{2} & \text{if } \lambda=\lambda_3,\\ 0 & \text{otherwise,}\end{cases}\qquad \eta_2(\lambda)=\begin{cases}1 & \text{if } \lambda\in\{\overline{0},\overline{1}\},\\ \frac{1}{2} & \text{if } \lambda=\lambda_4,\\ 0 & \text{otherwise.}\end{cases}$$

For $r=\frac{1}{2}$, $\eta_1(\lambda_3)\geq r$ and there exist a $\frac{1}{2}$-$(1,2)$-fro set λ_1 such that $\lambda_1\leq\lambda_3\leq C_{\tau_j}(\lambda_1,r)$, $f^{-1}(\lambda_3)$ is $\frac{1}{2}$-$(1,2)$-frso set in X and also $\eta_2(\lambda_4)\geq r$, there exist a $\frac{1}{2}$-$(2,1)$-fro set λ_2 such that $\lambda_2\leq\lambda_4\leq C_{\tau_j}(\lambda_2,r)$, $f^{-1}(\lambda_4)$ is $\frac{1}{2}$-$(2,1)$-frso set in X, but $f^{-1}(\lambda_3)$ is not $\frac{1}{2}$-$(1,2)$-fro set in X. Then the identity mapping f is FP-regular semi continuous but not FP-regular continuous.

Example 4.4. Let $X=\{a,b,c\}=Y$, $\lambda_1,\lambda_2\in I^X$, $\lambda_3,\lambda_4,\lambda_5,\lambda_6\in I^Y$ are defined as $\lambda_1(a)=0.4,\lambda_1(b)=0.5,\lambda_1(c)=0.6$; $\lambda_2(a)=0.4,\lambda_2(b)=0.5,\lambda_2(c)=0.4$; $\lambda_3(a)=0.4,\lambda_3(b)=0.3,\lambda_3(c)=0.6$; $\lambda_4(a)=0.4,\lambda_4(b)=0.5,\lambda_4(c)=0.4$; $\lambda_5(a)=0.5,\lambda_5(b)=0.5,\lambda_5(c)=0.6$; $\lambda_6(a)=0.5,\lambda_6(b)=0.5,\lambda_6(c)=0.4$. We define smooth topologies $\tau_1,\tau_2,\eta_1,\eta_2: I^X\to I$ as follows:

$$\tau_1(\lambda)=\begin{cases}1 & \text{if } \lambda\in\{\overline{0},\overline{1}\},\\ \frac{1}{2} & \text{if } \lambda=\lambda_1,\\ 0 & \text{otherwise,}\end{cases}\qquad \tau_2(\lambda)=\begin{cases}1 & \text{if } \lambda\in\{\overline{0},\overline{1}\},\\ \frac{1}{2} & \text{if } \lambda=\lambda_2,\\ 0 & \text{otherwise,}\end{cases}$$

$$\eta_1(\lambda)=\begin{cases}1 & \text{if } \lambda\in\{\overline{0},\overline{1}\},\\ \frac{1}{2} & \text{if } \lambda=\lambda_3,\\ 0 & \text{otherwise,}\end{cases}\qquad \eta_2(\lambda)=\begin{cases}1 & \text{if } \lambda\in\{\overline{0},\overline{1}\},\\ \frac{1}{2} & \text{if } \lambda=\lambda_4,\\ 0 & \text{otherwise.}\end{cases}$$

For $r=\frac{1}{2}$, the fuzzy set λ_5 is r-η_1-frso set in Y, $f^{-1}(\lambda_5)$ is r-$(1,2)$-frso set in X and also the fuzzy set λ_6 is r-η_2-frso set in Y, $f^{-1}(\lambda_6)$ is r-$(2,1)$-frso set in X. But $\eta_1(\lambda_3)\geq r$ and there exists no r-(1, 2)-fro set μ such that $\mu\leq\lambda\leq C_{\tau_j}(\mu,r)$. Then the identity mapping f is FP-regular semi irresolute but not FP-regular semi continuous.

Example 4.5. We define smooth topologies $\tau_1,\tau_2,\eta_1,\eta_2: I^X\to I$ as follows:

$$\tau_1(\lambda)=\begin{cases}1 & \text{if } \lambda\in\{\overline{0},\overline{1}\},\\ \frac{1}{2} & \text{if } \lambda=\overline{0.3},\\ \frac{1}{2} & \text{if } \lambda=\overline{0.4},\\ \frac{1}{2} & \text{if } \lambda=\overline{0.6},\\ 0 & \text{otherwise,}\end{cases}\qquad \tau_2(\lambda)=\begin{cases}1 & \text{if } \lambda\in\{\overline{0},\overline{1}\},\\ \frac{1}{2} & \text{if } \lambda=\overline{0.3},\\ \frac{1}{2} & \text{if } \lambda=\overline{0.6},\\ 0 & \text{otherwise,}\end{cases}$$

$$\eta_1(\lambda)=\begin{cases}1 & \text{if } \lambda\in\{\overline{0},\overline{1}\},\\ \frac{1}{2} & \text{if } \lambda=\overline{0.3},\\ 0 & \text{otherwise,}\end{cases}\qquad \eta_2(\lambda)=\begin{cases}1 & \text{if } \lambda\in\{\overline{0},\overline{1}\},\\ \frac{1}{2} & \text{if } \lambda=\overline{0.4},\\ 0 & \text{otherwise.}\end{cases}$$

For $r=\frac{1}{2}$, $\eta_1(\overline{0.3})\geq r$, there exist a r-$(1,2)$-fro set $\overline{0.3}$ in X such that $\overline{0.3}\leq\overline{0.3}\leq C_{\tau_j}(\overline{0.3})=\overline{0.4}$ and $\eta_2(\overline{0.4})\geq r$, there exist a r-$(2,1)$-fro set $\overline{0.3}$ in X such that $\overline{0.3}\leq\overline{0.4}\leq C_{\tau_j}(\overline{0.3})=\overline{0.4}$. But the fuzzy set $\overline{0.5}$ is r-$(1,2)$-frso set in Y, there exists no r-$(1,2)$-fro set μ in X such that $\mu\leq\overline{0.5}\leq C_{\tau_j}(\mu)$. Then the identity mapping $id:(X,\ \tau_1,\ \tau_2)\to(Y,\ \eta_1,\ \eta_2)$ is FP-regular semi continuous but not FP-regular semi irresolute.

Theorem 4.6. *Let $(X,\ \tau_1,\ \tau_2)$ and $(Y,\ \eta_1,\ \eta_2)$ be smooth bts's. Let $f : X \to Y$ be a mapping. The following statements are equivalent:*

(1) *f is FP-regular semi irresolute.*
(2) *For each r-(i,j)-frsc set $\mu \in I^Y$, $f^{-1}(\mu)$ is r-(i,j)-frsc set in X.*
(3) *$f(RSC_{ij}(\lambda,r)) \leq RSC_{ij}(f(\lambda),r)$, for each $\lambda \in I^X$ and $r \in I_0$.*
(4) *$RSC_{ij}(f^{-1}(\mu),r) \leq f^{-1}(RSC_{ij}(\mu,r))$, for each $\mu \in I^Y$ and $r \in I_0$.*
(5) *$f^{-1}(RSI_{ij}(\mu,r)) \leq RSI_{ij}(f^{-1}(\mu),r)$, for each $\mu \in I^Y$ and $r \in I_0$.*

Proof. (1)⇔(2): It is obvious.

(2)⇒(3): Suppose (2) holds and let $f(RSC_{ij}(\lambda,r)) \nleq RSC_{ij}(f(\lambda),r)$, for some $\lambda \in I^X$ and $r \in I_0$. So there exists $y \in Y,\ t \in (0,1]$ such that

$$f(RSC_{ij}(\lambda,r))(y) > t > RSC_{ij}(f(\lambda),r)(y).$$

If $f^{-1}(y) = \varnothing$, then it is a contradiction, because $f(RSC_{ij}(\lambda,r))(y) = \overline{0}$. Thus, if $f^{-1}(y) \neq \varnothing$, then there exists $x \in f^{-1}(y)$ such that

$$f(RSC_{ij}(\lambda,r))(y) \geq RSC_{ij}(\lambda,r)(x) > t > RSC_{ij}(f(\lambda),r)(f(x)). \tag{4.1}$$

Also, $RSC_{ij}(f(\lambda),r)(f(x)) < t$, implies that there exists r-(i,j)-frsc set μ with $f(\lambda) = \mu$ such that $RSC_{ij}(f(\lambda),r)(f(x)) \leq \mu(f(x)) < t$. Moreover, $f(\lambda) \leq \mu$ implies $\lambda \leq f^{-1}(\mu)$. From (2), $f^{-1}(\mu)$ is r-(i,j)-frsc. So $RSC_{ij}(\lambda,r) \leq f^{-1}(\mu)$ and this implies:

$$RSC_{ij}(\lambda,r)(x) \leq f^{-1}(\mu)(x) = \mu(f(x)) < t. \tag{4.2}$$

From relations (4.1) and (4.2) we see that

$$RSC_{ij}(\lambda,r)(x) > t \text{ and } RSC_{ij}(\lambda,r)(x) \leq t$$

which is a contradiction. Hence the result holds.

(3)⇒(4): Put $\lambda = f^{-1}(\mu)(\mu \in I^Y)$ and apply (3), we have

$$f(RSC_{ij}(f^{-1}(\mu,r)) \leq RSC_{ij}(f(f^{-1}(\mu),r)) \leq RSC_{ij}(\mu,r).$$

Then, $RSC_{ij}(f^{-1}(\mu),r) \leq f^{-1}(RSC_{ij}(\mu,r))$.

(4)⇒(5): It follows immediately by taking the complement of (4).

(5)⇒(1): Suppose (5) holds and let μ be r-(i,j)-frso in Y. By (5)

$$f^{-1}(RSI_{ij}(\mu,r)) \leq RSI_{ij}(f^{-1}(\mu),r).$$

Then, $f^{-1}(\mu) \leq RSI_{ij}(f^{-1}(\mu),r)$. But $RSI_{ij}(f^{-1}(\mu),r) \leq f^{-1}(\mu)$.
Thus $RSI_{ij}(f^{-1}(\mu),r) = f^{-1}(\mu)$. So, $f^{-1}(\mu)$ is r-(i,j)-frso. Hence f is FP-regular semi irresolute. □

Theorem 4.7. *Let $(X,\ \tau_1,\ \tau_2)$ and $(Y,\ \eta_1,\ \eta_2)$ be smooth bts's. Let $f : X \to Y$ be a mapping. The following statements are equivalent:*

(1) *f is FP-regular semi continuous.*
(2) *$f(RSC_{ij}(\lambda,r)) \leq RC_{\eta_i}(f(\lambda),r)$, for each $\lambda \in I^X$.*
(3) *$RSC_{ij}(f^{-1}(\mu),r) \leq f^{-1}(RC_{\eta_i}(\mu,r))$, for each $\mu \in I^Y$.*
(4) *$f^{-1}(RI_{\eta_i}(\mu,r)) \leq RSI_{ij}(f^{-1}(\mu),r)$, for each $\mu \in I^Y$.*

Proof. (1)⇒(2): Suppose (1) holds and let $\lambda \in I^X$. Assume that there exists $\lambda \in I^X$ and $r \in I_0$ such that $f(RSC_{ij}(\lambda, r)) \nleq RC_{\eta_i}(f(\lambda), r)$. Then there exists $y \in Y,\ t \in (0, 1]$ such that $f(RSC_{ij}(\lambda, r))(y) > t > RC_{\eta_i}(f(\lambda), r)(y)$.

If $f^{-1}(y) = \varnothing$, then $f(RSC_{ij}(\lambda, r))(y) = \overline{0}$. Thus, there exists $x \in f^{-1}(y)$ such that

$$f(RSC_{ij}(\lambda, r))(y) > RSC_{ij}(\lambda, r))(x) > t > RC_{\eta_i}(f(\lambda), r)(y) \tag{4.3}$$

Since $RC_{\eta_i}(f(\lambda), r)(y) < t$, there exists r-(i, j)-frc set μ such that $f(\lambda) = \mu$. So, $RC_{\eta_i}(f(\lambda), r)(y) \leq \mu$ and this implies $RC_{\eta_i}(f(\lambda), r)(f(x)) \leq \mu(f(x)) < t$. Moreover, $f(\lambda) \leq \mu$ implies $\lambda \leq f^{-1}(\mu)$. Hence, $\overline{1} - \lambda \geq \overline{1} - f^{-1}(\mu)$. Since $f^{-1}(\overline{1} - \mu)$ is r-(i, j)-frso, from (1),

$f^{-1}(\overline{1} - \mu) \leq RSI_{ij}(\overline{1} - \lambda, r)$ or $\overline{1} - RSI_{ij}(\overline{1} - \lambda, r) \leq \overline{1} - f^{-1}(\overline{1} - \mu)$.

Hence, $RSC_{ij}(\lambda, r) \leq f^{-1}(\mu)$. This gives

$$RSC_{ij}(\lambda, r)(x) \leq f^{-1}(\mu)(x) = \mu(f(x)) < t.$$

But the last relation contradicts the relation (4.3) above. Therefore, the result in (2) is true.

(2)⇒(3): Take $\lambda = f^{-1}(\mu)(\mu \in I^Y)$ and apply (2), we have the required result.

(3)⇒(4): Taking the complement of (3), we have the result.

(4)⇒(1): Suppose (4) holds and let $\mu \in I^Y$ be a r-(i, j)-fro set. Since every r-(i, j)-fro set is r-(i)-fuzzy open, $\eta_i(\mu) \geq r$. Then by (4),

$f^{-1}(RI_{\eta_i}(\mu, r)) \leq RSI_{ij}(f^{-1}(\mu), r)$ or $f^{-1}(\mu) \leq RSI_{ij}(f^{-1}(\mu), r)$,

since $\eta_i(\mu) \geq r$. But $RSI_{ij}(f^{-1}(\mu), r) \leq f^{-1}(\mu)$. Thus, $f^{-1}(\mu) = RSI_{ij}(f^{-1}(\mu), r)$. So $f^{-1}(\mu)$ is r-(i, j)-frso. Hence f is FP-regular semi continuous. □

Theorem 4.8. *Let $(X,\ \tau_1,\ \tau_2)$ and $(Y,\ \eta_1,\ \eta_2)$ be smooth bts's. Let $f : X \to Y$ be a mapping. The following statements are equivalent:*

(1) *A map f is FP-regular semi irresolute open.*

(2) *$f(RSI_{ij}(\lambda, r)) \leq RSI_{ij}(f(\lambda), r)$, for each $\lambda \in I^X$ and $r \in I_0$.*

(3) *$RSI_{ij}(f^{-1}(\mu), r) \leq f^{-1}(RSI_{ij}(\mu, r))$, for each $\mu \in I^Y$ and $r \in I_0$.*

(4) *For any $\lambda \in I^Y$ and any r-(i, j)-frso set $\mu \in I^X$ such that $\mu \geq f^{-1}(\lambda)$, there exists a r-(i, j)-frsc set $\rho \in I^Y$ with $\lambda \leq \rho$ such that $f^{-1}(\rho) \leq \lambda$.*

Proof. (1)⇒(2): Since $RSI_{ij}(\lambda, r) \leq \lambda$, $RSI_{ij}(\lambda, r) \leq f(\lambda)$. By (1), $f(RSI_{ij}(\lambda, r)) \leq RSI_{ij}(f(\lambda), r)$.

(2)⇒(3): Suppose (2) holds. Take $\lambda = f^{-1}(\mu),\ \lambda \in I^Y$ and apply (2).

(3)⇒(4): Let $\mu \in I^X$ be a r-(i, j)-frsc set. Since $f^{-1}(\lambda) \leq \mu$,

$$\overline{1} - \mu \leq \overline{1} - f^{-1}(\lambda) = f^{-1}(\overline{1} - \lambda.$$

It follows, $RSI_{ij}(\overline{1} - \mu, r) \leq \overline{1} - \mu \leq RSI_{ij}(f^{-1}(\overline{1} - \lambda), r)$. By (3),

$$\overline{1} - \mu \leq RSI_{ij}(f^{-1}(\overline{1} - \lambda), r) \leq f^{-1}(RSI_{ij}(\overline{1} - \lambda, r)).$$

It implies

$$\mu \geq \overline{1} - f^{-1}(RSI_{ij}(\overline{1} - \lambda, r)) = f^{-1}(\overline{1} - RSI_{ij}(\overline{1} - \lambda, r)) = f^{-1}(RSC_{ij}(\lambda, r)).$$

Then, $\mu \geq f^{-1}(RSC_{ij}(\lambda, r))$. Take $\rho = RSC_{ij}(\lambda, r))$. Then ρ is r-(i, j)-frsc such that $\mu \geq f^{-1}(\rho)$ and $\rho \geq \lambda$. Thus the result holds.

(4)⇒(1): Let ω be r-(i,j)-frso in X. Put $\lambda = \overline{1} - f(\omega)$ and $\mu = \overline{1} - \omega$. It is easy to see that $\mu \geq f^{-1}(\lambda)$. By part (4), there exists r-(i,j)-frsc set $\rho \in I^Y$ such that $\rho \geq \lambda$ and $\mu \geq f^{-1}(\rho)$ or $\overline{1} - \omega \geq f^{-1}(\rho)$. It implies $\omega \leq \overline{1} - f^{-1}(\rho) = f^{-1}(\overline{1} - \rho)$. Thus $f(w) \leq ff^{-1}(\overline{1} - \rho) \leq \overline{1} - \rho$. On the other hand, $\lambda \leq \rho$, $f(\omega) = \overline{1} - \lambda$ implies $f(\omega) = \overline{1} - \lambda \geq \overline{1} - \rho$. Finally, we have $f(\omega) = \overline{1} - \rho$. So $f(\omega)$ is r-(i,j)-frso. Hence f is FP-regular semi irresolute open. □

Theorem 4.9. *Let $(X,\ \tau_1,\ \tau_2)$ and $(Y,\ \eta_1,\ \eta_2)$ be smooth bts's. Let $f : X \to Y$ be a mapping. The following statements are equivalent:*

(1) *f is FP-regular semi open.*
(2) *$f(I_{\tau_i}(\lambda, r)) \leq RSI_{ij}(f(\lambda), r)$, for each $\lambda \in I^X$.*
(3) *$I_{\tau_i}(f^{-1}(\mu), r) \leq f^{-1}(RSI_{ij}(\mu, r))$, for each $\mu \in I^Y$.*
(4) *For any $\lambda \in I^Y$ and any $\mu \in I^X$ such that $\tau_i(\overline{1} - \mu) \geq r$, $\mu \geq f^{-1}(\lambda)$, there exists r-(i,j)-frsc set $\rho \in I^Y$ with $\lambda \leq \rho$ and $\mu \geq f^{-1}(\rho)$.*

Proof. (1)⇒(2): Since $I_{\tau_i}(\lambda, r) \leq \lambda$, $(\lambda \in I^X)$, $f(I_{\tau_i}(\lambda, r)) \leq f(\lambda)$. But, by (1), $f(I_{\tau_i}(\lambda, r))$ is r-(i,j)-frso. Then $f(I_{\tau_i}(\lambda, r)) \leq RSI_{ij}(f(\lambda), r)$.

(2)⇒(3): Suppose (2) holds. Put $\lambda = f^{-1}(\mu)$, $\mu \in I^Y$ and apply (2), we have

$$f(I_{\tau_i}(f^{-1}(\mu), r)) \leq RSI_{ij}(ff^{-1}(\mu), r) \leq RSI_{ij}(\mu, r).$$

Then, $I_{\tau_i}(f^{-1}(\mu), r) \leq f^{-1}(RSI_{ij}(\mu, r))$.

(3)⇒(4): Suppose (3) holds and let $\lambda \in I^Y$ and $\mu \in I^X$ such that $\tau_i(\overline{1} - \mu) \geq r$ and $\mu \geq f^{-1}(\lambda)$. Since $\overline{1} - \mu \leq \overline{1} - f^{-1}(\lambda) = f^{-1}(\overline{1} - \lambda)$,

$$I_{\tau_i}(\overline{1} - \mu) = \overline{1} - \mu \leq I_{\tau_i}(f^{-1}(\overline{1} - \lambda), r) \text{ or } \overline{1} - \mu \leq f^{-1}(RSI_{ij}(\overline{1} - \lambda, r)).$$

Then, $\mu \geq \overline{1} - f^{-1}(RSI_{ij}(\overline{1} - \lambda, r)) = f^{-1}(RSC_{ij}(\lambda, r))$. Take $\rho = RSC_{ij}(\lambda, r)$. Thus ρ is r-(i,j)-frsc and $\rho \geq \lambda$ such that $\mu \geq f^{-1}(\rho)$.

(4)⇒(1): Let ω be a fuzzy set such that $\omega \in I^X$ and $\tau_i(\omega) \geq r$. Put $\mu = \overline{1} - \omega$ and $\lambda = \overline{1} - f(\omega)$. Then $\mu \geq f^{-1}(\lambda)$. Thus there exists r-(i,j)-frsc set ρ such that $\rho \geq \lambda$ and $\mu \geq f^{-1}(\rho)$. So $\overline{1} - \omega \geq f^{-1}(\rho)$ implies $\omega \leq f^{-1}(\overline{1} - \rho)$. Thus $f(w) \leq \overline{1} - \rho$. Also, $\lambda \leq \rho$, $f(\omega) = \overline{1} - \lambda \geq \overline{1} - \rho$. So $f(\omega) = \overline{1} - \rho$. Hence $f(\omega)$ is r-(i,j)-frso. Therefore f is FP-regular semi open. □

Theorem 4.10. *Let $(X,\ \tau_1,\ \tau_2)$ and $(Y,\ \eta_1,\ \eta_2)$ be smooth bts's. Let $f : X \to Y$ be a mapping. The following statements are equivalent:*

(1) *f is FP-regular semi irresolute closed.*
(2) *$f(RSC_{ij}(\lambda, r)) \geq RSC_{ij}(f(\lambda), r)$, for each $\lambda \in I^X$.*

Proof. (1)⇒(2): Let $\lambda \in I^X$. Since $\lambda \leq RSC_{ij}(\lambda, r)$, $f(\lambda) \leq f(RSC_{ij}(\lambda, r))$. It implies $RSC_{ij}(f(\lambda), r) \leq f(RSC_{ij}(\lambda, r))$.

(2)⇒(1): Suppose (2) holds and $\lambda \in I^X$ such that λ is r-(i,j)-frsc. Then $RSC_{ij}(f(\lambda), r) \leq f(\lambda)$. But $f(\lambda) \leq RSC_{ij}(f(\lambda), r)$. Thus $f(\lambda)$ is r-(i,j)-frsc. So f is FP-regular semi irresolute closed. □

Theorem 4.11. *Let $(X,\ \tau_1,\ \tau_2)$ and $(Y,\ \eta_1,\ \eta_2)$ be smooth bts's. Let $f : X \to Y$ be a mapping. The following statements are equivalent:*

(1) *f is FP-regular semi closed.*
(2) *$RSC_{ij}(f(\lambda), r)) \leq f(RC_{\tau_i}(\lambda, r))$, for each $\lambda \in I^X$.*

Proof. (1)⇒(2): Suppose (1) holds and let $\lambda \in I^X$. Since $\lambda \leq RC_{\tau_i}(\lambda, r)$, $f(\lambda) \leq f(RC_{\tau_i}(\lambda, r))$. Then, $RSC_{ij}(f(\lambda), r) \leq f(RC_{\tau_i}(\lambda, r))$.

(2)⇒(1): Suppose (2) holds and let $\lambda \in I^X$ be a r-(i,j)-frc set. Then $RSC_{ij}(f(\lambda), r) \leq f(RC_{\tau_i}(\lambda, r))$. It implies $RSC_{ij}(f(\lambda), r) \leq f(\lambda)$. But $RSC_{ij}(f(\lambda), r) \geq f(\lambda)$. Thus, $f(\lambda) = RSC_{ij}(f(\lambda), r)$. So $f(\lambda)$ is r-(i,j)-frsc. Hence f is FP-regular semi closed. □

Theorem 4.12. *Let $(X,\ \tau_1,\ \tau_2)$ and $(Y,\ \eta_1,\ \eta_2)$ be smooth bts's. Let $f : X \to Y$ is bijective. Then:*

(1) *f is FP-regular semi irresolute closed iff $f^{-1}(RSC_{ij}(\mu, r)) \leq RSC_{ij}(f^{-1}(\mu), r)$ for each $\mu \in I^Y$ and $r \in I_0$.*

(2) *f is FP-regular semi irresolute closed iff f is FP-regular semi irresolute open.*

Proof. (1)(⇒): Let f be FP-regular semi irresolute closed. From Theorem 4.10, we have

$$f(RSC_{ij}(\lambda, r)) \geq RSC_{ij}(f(\lambda), r)),\ \lambda \in I^X.$$

Let $\mu \in I^Y$ and put $\lambda = f^{-1}(\mu)$, we have

$$f(RSC_{ij}(f^{-1}(\mu), r)) \geq RSC_{ij}(ff^{-1}(\mu), r) = RSC_{ij}(\mu, r).$$

It implies $RSC_{ij}(f^{-1}(\mu), r) \geq f^{-1}(RSC_{ij}(\mu, r))$.

(⇐): On the other hand, let the condition is satisfied and let $\mu \in I^X$ such that μ is r-(i,j)-frsc. Then $f(\mu) \in I^Y$. Apply the condition, we have

$$RSC_{ij}(f^{-1}f(\mu, r)) \geq f^{-1}(RSC_{ij}(f(\mu), r), r).$$

It implies that $RSC_{ij}(\mu, r) \geq f^{-1}(RSC_{ij}(f(\mu), r), r)$. Then, $f(RSC_{ij}(\mu, r)) \geq RSC_{ij}(f(\mu), r)$. Thus by Theorem 4.10, f is FP-regular semi irresolute closed.

(2) Apply Theorem 4.10 and taking the complement, we have the required result. □

From Theorems 4.6, 4.8, 4.10, 4.12, we obtain the following Theorem.

Theorem 4.13. *Let $(X,\ \tau_1,\ \tau_2)$ and $(Y,\ \eta_1,\ \eta_2)$ be smooth bts's. Let $f : X \to Y$ be a mapping. The following statements are equivalent:*

(1) *f is FP-regular semi irresolute homeomorphism.*
(2) *f is FP-regular semi irresolute and FP-regular semi irresolute open.*
(3) *f is FP-regular semi irresolute and FP-regular semi irresolute closed.*
(4) *$f(RSI_{ij}(\lambda, r)) = RSI_{ij}(f(\lambda), r)$, for each $\lambda \in I^X,\ r \in I_0$.*
(5) *$f(RSC_{ij}(\lambda, r)) = RSC_{ij}(f(\lambda), r)$, for each $\lambda \in I^X,\ r \in I_0$.*
(6) *$RSI_{ij}(f^{-1}(\mu), r) = f^{-1}(RSI_{ij}(\mu, r))$.*
(6) *$RSC_{ij}(f^{-1}(\mu), r) = f^{-1}(RSC_{ij}(\mu, r))$, $\mu \in I^Y,\ r \in I_0$.*

Note that the composition of two FP-regular semi irresolute mappings is FP-regular semi irresolute. In general, the composition of two FP-regular semi continuous mappings is not FP-regular semi continuous.

Example 4.14. We define smooth topologies $\tau_1, \tau_2, \eta_1, \eta_2, \sigma_1, \sigma_2 : I^X \to I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \overline{0.3}, \\ \frac{1}{2} & \text{if } \lambda = \overline{0.4}, \\ \frac{1}{2} & \text{if } \lambda = \overline{0.6}, \\ 0 & \text{otherwise}, \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \overline{0.3}, \\ \frac{1}{2} & \text{if } \lambda = \overline{0.6}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\eta_1(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \overline{0.3}, \\ 0 & \text{otherwise}, \end{cases} \qquad \eta_2(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \overline{0.4}, \\ 0 & \text{otherwise}. \end{cases}$$

$$\sigma_1(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \overline{0.5}, \\ 0 & \text{otherwise}, \end{cases} \qquad \sigma_2(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \overline{0.4}, \\ \frac{1}{2} & \text{if } \lambda = \overline{0.5}, \\ 0 & \text{otherwise}. \end{cases}$$

For $r = \frac{1}{2}$, $\eta_1(\overline{0.3}) \geq r$, there exist a r-$(1,2)$-fro set $\overline{0.3}$ in X such that $\overline{0.3} \leq \overline{0.3} \leq C_{\tau_j}(\overline{0.3}) = \overline{0.4}$ and $\eta_2(\overline{0.4}) \geq r$, there exist a r-$(2,1)$-fro set $\overline{0.3}$ in X such that $\overline{0.3} \leq \overline{0.4} \leq C_{\tau_j}(\overline{0.3}) = \overline{0.4}$. $\sigma_1(\overline{0.5}) \geq r$, there exist a r-$(1,2)$-fro set $\overline{0.3}$ in Y such that $\overline{0.3} \leq \overline{0.5} \leq C_{\tau_j}(\overline{0.3}) = \overline{0.6}$ and $\sigma_2(\overline{0.5}) \geq r$, there exist a r-$(2,1)$-fro set $\overline{0.4}$ in Y such that $\overline{0.4} \leq \overline{0.5} \leq C_{\tau_j}(\overline{0.4}) = \overline{0.6}$. But $\sigma_1(\overline{0.5}) \geq r$, there exists no r-$(1,2)$-fro set μ in X such that $\mu \leq \overline{0.5} \leq C_{\tau_j}(\mu)$. Then the identity mapping $id : (X,\ \tau_1,\ \tau_2) \to (Y,\ \eta_1,\ \eta_2)$ and $id : (Y,\ \eta_1,\ \eta_2) \to (Z,\ \sigma_1,\ \sigma_2)$ are FP-regular semi continuous but $id : (X,\ \tau_1,\ \tau_2) \to (Z,\ \sigma_1,\ \sigma_2)$ is not FP-regular semi continuous.

5. Conclusions

In this paper, we have introduced r-(i,j)-fuzzy regular semi open (closed) sets in smooth bitopological spaces and studied some of its properties. By using r-(i,j)-fuzzy regular semi open (closed) sets, we have defined a new fuzzy closure operator namely r-(i,j)-fuzzy regular semi interior (closure) operator. Also, we have introduced FP-regular semi continuous and FP-regular semi irresolute mappings. Moreover, we have investigated the relationship among FP-regular semi continuous and FP-regular semi irresolute mappings. Finally, we have given some counter examples to show that these types of mappings are not equivalent.

These results will help to extent the some generalized open sets and hence it will help to improve smooth bitopological spaces. In future some generalized open sets can be prepared using the results.

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References

[1] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.

[2] K. C. Chattopadhyay, R. N. Hazra and S. K. Samanta, Gradation of openness, Fuzzy Sets and Systems 49 (2) (1992) 237–242.
[3] K. C. Chattopadhyay and S. K. Samanta, Fuzzy topology, Fuzzy Sets and Systems 54 (1993) 207–212.
[4] A. Kandil, Biproximities and fuzzy bitopological spaces, Simon Stevin 63 (1989) 45–66.
[5] E. E. Kerre and M. A. Fath Alla, On fuzzy regular semi open sets and αS^*-closed fuzzy topological spaces, The Journal of Fuzzy Mathematics 11 (1) (2003) 225–235.
[6] E. P. Lee, Y. B. Im and H. Han, Semiopen sets on smooth bitopological spaces, Far East J. Math. Sci. 3 (2001) 493–521.
[7] E. P. Lee, Preopen sets in smooth bitopological spaces, Commun. Korean Math. Soc. 18 (3) (2003) 521–532.
[8] A. S. Mashhour, M. H. Ghanim and M. A. Fath Alla, α-Separation axioms and α-compactness in fuzzy topological spaces, Rocky Mount. J. Math. 6 (1986) 591–600.
[9] A. A. Ramadan, Smooth topological spaces, Fuzzy Sets and Systems 48 (1992) 371–375.
[10] A. A. Ramadan, S. E. Abbas and A. A. A. El-Latif, On fuzzy bitopological spaces in Sostak's sense (II), Commun. Korean. Math. Soc. 25 (3) (2010) 457–475.
[11] S. A. El-Sheikh, On smooth fuzzy bitopological spaces, J. Fuzzy Math. 16 (1) (2008) 15–34.
[12] A. P. Šostak, On a fuzzy topological structure, Rend. Circ. Matem. Palermo Ser II. 11 (1986) 89–103.
[13] A. Vadivel and E. Elavarasan, Applications of r-generalized regular fuzzy closed sets, Ann. Fuzzy Math. Inform. (Accepted).
[14] L. A. Zadeh, Fuzzy sets, Information and Control. 8 (1965) 338–353.
[15] A. M. Zahran, Fuzzy regular semi open sets and αS-closed spaces, J. Fuzzy Math. 2 (4) (1997) 579–586.

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