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Representation of labeling tree based on $m-$ polar fuzzy sets

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Abstract. We introduce the concept of $m-$polar fuzzy labeling tree G_p^ω generated by $m-$polar fuzzy spanning subgraph S_p^ω and investigate some of its properties. We present the concept of bipartite $m-$polar fuzzy labeling graphs. Furthermore, we present an algorithm for finding an $m-$polar fuzzy spanning subgraph S_p^ω of an $m-$polar fuzzy labeling tree G_p^ω.

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1. Introduction

In 1965, Zadeh[12] introduced the mathematical frame work to discuss the phenomena of vagueness and uncertainty in real life systems. It is expressed with the comfort of membership function valued in the real unit interval $[0, 1]$. In 1994, Zhang [13] extended the concept of fuzzy sets and introduced the concept of bipolar fuzzy sets whose membership degrees range belong to interval $[-1, 1]$. The membership degree 0 of an element means the element is inconsequent to the analogous property, the membership degree $(0, 1]$ revels that the element fascinate the assertive property where as the membership degree $[-1, 0)$ revels that the element fascinates the converse property. But sometimes modeling in actual world investigations often contain multi-agent, multi-attribute, multi-objects, multi-index, multi-polar information or uncertainty rather than a single bit. With the analysis to classical, fuzzy and bipolar fuzzy models an m-polar fuzzy model give more efficiency and more preciseness, extensibility and accuracy. Chen *et al.* [7] introduced the notion of m-polar fuzzy set as a generalization of bipolar fuzzy set and showed that bipolar fuzzy sets and 2-polar fuzzy sets are cryptomorphic mathematical notions.
Based on Zadeh's fuzzy relations [12] Kauffmann defined in [8] a fuzzy graph. Rosenfeld [11] described the structure of fuzzy graphs obtaining analogs of several graph

theoretical concepts. Bhattachariya [5] discussed the connectivity ideas between fuzzy cut nodes and fuzzy bridges named as some remarks on fuzzy graph. Buhtani and Rosenfeld [6] introduced the concept of strong arcs in fuzzy graphs. Nagoorgani and Rajalaxami [9, 10] worked on the properties of fuzzy labeling graphs and introduced the idea of fuzzy labeling tree. Akram et al.[1, 2, 3, 4] has initiated several concepts, including bipolar fuzzy graphs, m-polar fuzzy graphs, certain metrics in m-polar fuzzy graphs. In this article, we present the concept of $m-$polar fuzzy labeling tree G_p^ω generated by $m-$polar fuzzy spanning subgraph S_p^ω and interrogate some of its properties. We precede the concept of bipartite $m-$polar fuzzy labeling graphs. Furthermore, we present an algorithm for finding $m-$polar fuzzy spanning subgraph S_p^ω of an $m-$polar fuzzy labeling tree G_p^ω.

2. Labeling tree based on $m-$polar fuzzy set

Definition 2.1 ([7])**.** An $m-$polar fuzzy set in a universe X is a function $C : X \to [0,1]^m$. The degree of membership of each element $x \in X$ is denoted by $C(x) = (P_1 \circ C(x), P_2 \circ C(x), \dots, P_m \circ C(x))$, where $P_i \circ C : [0,1]^m \to [0,1]$ is the $i-$th projection mapping.

Note that $[0,1]^m$ (mth-power of [0, 1]) is considered as a poset with the point-wise order $\leq$, where m is an arbitrary ordinal number (we make an appointment that $m = \{n | n < m\}$ when $m > 0$), $\leq$ is defined by $x \leq y \Leftrightarrow p_i(x) \leq p_i(y)$ for each $i \in m$ ($x, y \in [0,1]^m$), and $P_i : [0,1]^m \to [0,1]$ is the $i-$th projection mapping ($i \in m$). $\mathbf{0} = (0,0,\dots,0)$ is the smallest value in $[0,1]^m$ and $\mathbf{1} = (1,1,\dots,1)$ is the greatest value in $[0,1]^m$.

Definition 2.2 ([3])**.** Let C be an $m-$polar fuzzy set in a universe X. An $m-$polar fuzzy relation $D = (P_1 \circ D, P_2 \circ D, \dots, P_m \circ D)$ on C is a mapping $D : X \times X \to [0,1]^m$ such that, $D(xy) \leq \inf\{C(x), C(y)\}$, for all $x, y \in X$, that is, for all $x, y \in X$ and for each $1 \leq i \leq m$, $P_i \circ D(xy) \leq \inf\{P_i \circ C(x), P_i \circ C(y)\}$, where $P_i \circ C(x)$ denotes the $i-$th degree of membership of the element x and $P_i \circ D(xy)$ denotes the $i-$th degree of membership of the relation $xy \in E$.

Definition 2.3 ([3, 7])**.** An $m-$polar fuzzy graph $G = (C, D)$ on a nonempty set X is a pair of functions $C : X \to [0,1]^m$ and $D : X \times X \to [0,1]^m$ such that for all $x, y \in X$, $D(uv) \leq \inf\{C(u), C(v)\}$, i.e., $P_i \circ D(uv) \leq \inf\{P_i \circ C(u), P_i \circ C(v)\}$, $1 \leq i \leq m$. We call C is an $m-$polar fuzzy vertex set of G and D is an $m-$polar fuzzy edge set of G. Note that $P_i \circ D(uv) = 0$ for all $uv \in \widetilde{X}^2 - E$, $1 \leq i \leq m$. D is called an $m-$polar fuzzy relation on C. An $m-$polar fuzzy relation D on C is called symmetric if $P_i \circ D(uv) = P_i \circ D(vu)$ for all $u, v \in X$.

Definition 2.4 ([3])**.** An m-polar fuzzy path $P = x - y$ is a sequence of distinct vertices $x = x_1, x_2, \cdots, x_n = y$ such that for all j there exists at least one i such that, $P_i \circ (x_j x_{j+1}) > 0$.

Definition 2.5. An edge $P_i \circ D(xy)$ where $1 \leq i \leq m$ is called an $m-$polar fuzzy bridge of $G = (C, D)$, if its extraction shorten the strength of connectedness between some other pair of vertices in G.

Definition 2.6. A vertex y is an $m-$polar fuzzy cut vertex of $G = (C, D)$, if its extraction shorten the strength of connectedness between some other pair of vertices in G.

Definition 2.7. A vertex x is an $m-$polar fuzzy end vertex of $G = (C, D)$, if there is absolutely one strong neighbor in G associated with this vertex.

Definition 2.8. An arc $P_i \circ D(xy)$, where $1 \leq i \leq m$, of an $m-$polar fuzzy graph is called strong arc if its weight is as great as the strength of connectedness of its $m-$polar fuzzy end nodes.

Definition 2.9. An $m-$polar fuzzy strong path is a path consisting of all $m-$polar fuzzy strong arcs.

Definition 2.10. An $m-$polar fuzzy path $P = x - y$ is said to be strongest $m-$polar fuzzy path, if its strength equals to its connectedness.

Example 2.11. Consider a 3-polar fuzzy graph G as shown in Fig. 1.

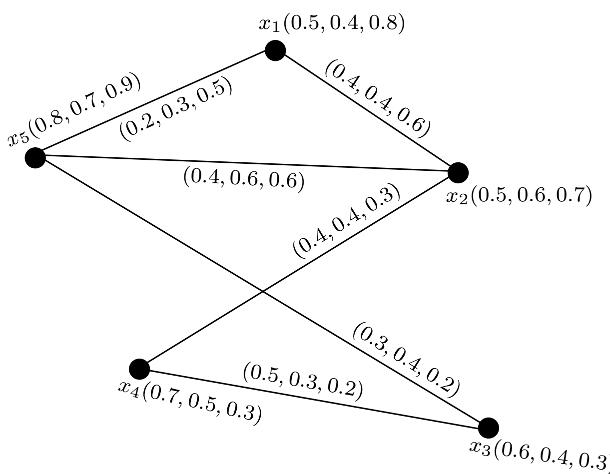


Figure 1. 3-polar fuzzy graph

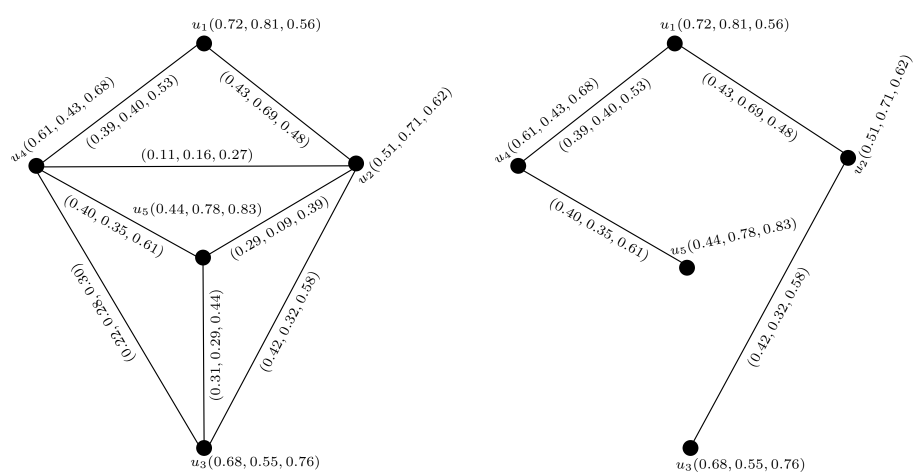
By computations, it is easy to see, x_2x_5, x_1x_2, x_2x_4 are 3-polar fuzzy bridges. x_2 is 3-polar fuzzy cut vertex. x_1, x_5, x_4 are 3-polar fuzzy end vertices of G. x_1x_2, x_2x_5, x_2x_4 are 3-polar fuzzy strong arcs. $x_1 - x_2 - x_5$, $x_1 - x_2 - x_4$ are 3-polar fuzzy strong paths. $x_1 - x_2 - x_5$, $x_1 - x_2 - x_4$, $x_4 - x_2 - x_5$ are strongest 3-polar fuzzy paths.

Definition 2.12. An $m-$*polar fuzzy labeling graph* $G^{\omega}_p = (C^{\omega}_p, D^{\omega}_p)$ is defined as, if the mappings $C^{\omega}_p : X \to [0, 1]^m$ and $D^{\omega}_p : X \times X \to [0, 1]^m$ are bijective, where as all the edges and vertices have distinct membership values and $P_i \circ D^{\omega}_p(xy) < P_i \circ C^{\omega}_p(x) \wedge P_i \circ C^{\omega}_p(y)$ for all $x, y \in X, 1 \leq i \leq m$.

Definition 2.13. A cycle is said to be an $m-$polar fuzzy labeling cycle, if its has an $m-$polar fuzzy labeling.

Definition 2.14. An $m-$polar fuzzy labeling tree $G_p^\omega = (C_p^\omega, D_p^\omega)$ is defined as if it has an $m-$polar fuzzy labeling and an $m-$polar fuzzy spanning subgraph $S_p^\omega = (C_p^\omega, F_p^\omega)$ which is a tree, where for all arcs (x, y) not in S_p^ω, $P_i \circ D_p^\omega(xy) < (P_i \circ F_p^\omega(xy))^\infty$, where $1 \leq i \leq m$.

Example 2.15. A 3−polar fuzzy labeling tree can be seen in Fig.2.



3−polar fuzzy labeling tree   3−polar fuzzy Spanning subgraph

FIGURE 2. 3−polar fuzzy labeling tree

Theorem 2.16. *If G_p^ω is an $m-$polar fuzzy labeling tree then arcs of $m-$polar fuzzy spanning subgraph S_p^ω are $m-$polar fuzzy bridges of G_p^ω.*

Proof. Given that G_p^ω is an $m-$polar fuzzy labeling tree generated by an $m-$polar fuzzy spanning subgraph S_p^ω. Let (a, b) be an arc in S_p^ω. Then $(P_i \circ D'(ab))^\infty < P_i \circ D(ab) \leq (P_i \circ D(ab))^\infty$, where $1 \leq i \leq m$. Thus arc (a, b) is an $m-$polar fuzzy bridge of G_p^ω. □

Remark 2.17. Every $m-$polar fuzzy labeling graph is not an $m-$polar fuzzy labeling tree. As shown in Example 2.17.

Example 2.18. Consider a 3−polar fuzzy labeling graph.

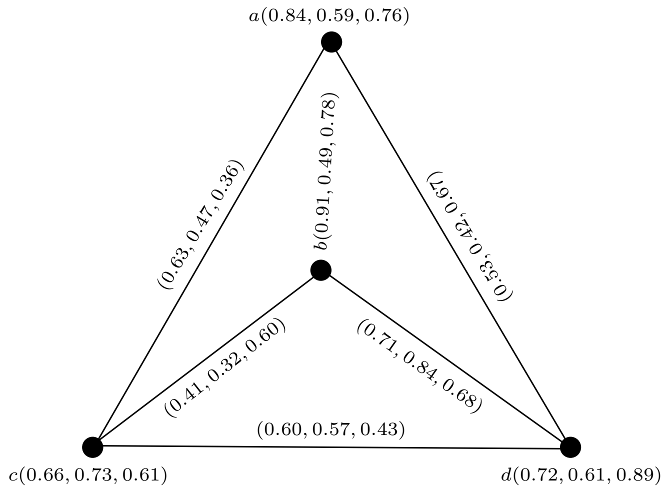


FIGURE 3. 3−polar fuzzy labeling graph

By direct calculations, it can be seen that a 3−polar fuzzy labeling graph is not a 3−polar fuzzy labeling tree because it does not have any 3−polar fuzzy spanning subgraph which fulfill the condition $P_i \circ D_p^\omega(xy) < (P_i \circ F_p^\omega(xy))^\infty$, where $1 \leq i \leq m$.

Proposition 2.19. *If $G_p^\omega = (C_p^\omega, D_p^\omega)$ is an $m-$polar fuzzy labeling tree, then its spanning subgraph $S_p^\omega = (C_p^\omega, F_p^\omega)$ is also an $m-$polar fuzzy labeling graph.*

Proof. Let $G_p^\omega = (C_p^\omega, D_p^\omega)$ be an $m-$polar fuzzy labeling tree. Then, by definition of $m-$polar fuzzy labeling graph, C_p^ω and D_p^ω are bijective in G_p^ω. Since S_p^ω is an$m-$polar fuzzy spanning subgraph of G_p^ω, $D_p^\omega = F_p^\omega$ if $(x, y) \in F_p^*$, which implies that bijection is preserved in S_p^ω. Thus S_p^ω is an $m-$polar fuzzy labeling graph. □

Remark 2.20. Let G^* be complete and G_p^ω is an $m-$polar fuzzy labeling tree. Then $d_{G_p^\omega}(x) \neq d_{S_p^\omega}(x)$ where S_p^ω, is an $m-$polar fuzzy spanning subgraph of G_p^ω.

Example 2.21. Consider a 4−polar fuzzy labeling tree as shown in Fig.4.

In 4−polar fuzzy labeling tree,

$$d_{G_p^\omega}(x) = (1.16, 1.73, 0.92, 1.5), d_{G_p^\omega}(y) = (1.34, 1.79, 1.3, 1.68),$$

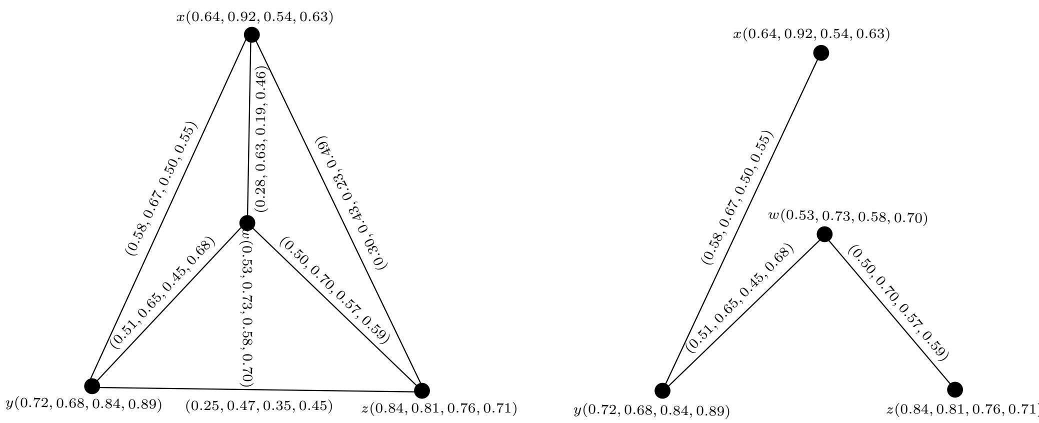
$$d_{G_p^\omega}(z) = (1.05, 1.6, 1.15, 1.53), d_{G_p^\omega}(w) = (1.29, 1.98, 1.21, 1.73).$$

In 4−polar fuzzy spanning subgraph,

$$d_{S_p^\omega}(x) = (0.58, 0.67, 0.50, 0.55), d_{S_p^\omega}(y) = (1.09, 1.32, 0.95, 1.23),$$

$$d_{S_p^\omega}(z) = (0.50, 0.70, 0.57, 0.59), d_{S_p^\omega}(w) = (1.01, 1.35, 1.02, 1.27).$$

Routine calculations show that $d_{G_p^\omega}(x) \neq d_{S_p^\omega}(x)$ for all $x, y \in X$.



4−polar fuzzy labeling tree

4−polar fuzzy spanning subgraph

FIGURE 4. 4−polar fuzzy labeling tree

Definition 2.22. Let $G = (C, D)$ be an $m-$polar fuzzy graph. The height of an $m-$polar fuzzy graph G denoted by $H(G)$ is defined as

$$(\sup_{1\leq j\leq n}(P_1 \circ D(xy))), (\sup_{1\leq j\leq n}(P_2 \circ D(xy))), \cdots, (\sup_{1\leq j\leq n}(P_m \circ D(xy)))$$

Proposition 2.23. *If $S_p^\omega = (C_p^\omega, F_p^\omega)$ is an $m-$polar fuzzy spanning subgraph of an $m-$polar fuzzy labeling tree $G_p^\omega = (C_p^\omega, D_p^\omega)$, then for all (x, y) not in S_p^ω, $(P_i \circ F(xy))^\infty \neq$ height of G_p^ω.*

Proof. Let (x, y) be an arc not in S_p^ω. Then $(x, y) \in G_p^\omega$ and (x, y) is not $m-$polar fuzzy bridge of G_p^ω, because the arcs of S_p^ω are $m-$polar fuzzy bridges of G_p^ω. By definition of $m-$polar fuzzy labeling tree, if (x, y) is not in S_p^ω, then $(x, y) < (P_i \circ F(xy))^\infty$. We know S_p^ω is a tree. Thus there will be only one path between x and y. So strength of connectedness between x and y is equal to strength of $m-$polar fuzzy path, i.e.,

$$(\inf_{1\leq j\leq n}(P_1 \circ D(xy))), (\inf_{1\leq j\leq n}(P_2 \circ D(xy))), \cdots, (\inf_{1\leq j\leq n}(P_m \circ D(xy))).$$

This shows that $(P_i \circ F(xy))^\infty$ is not equal to maximum of $P_i \circ C's$. Hence $(P_i \circ F(xy))^\infty \neq$ height of G. □

Proposition 2.24. *If G_p^ω is an $m-$polar fuzzy labeling tree then there exists exactly one strong path between any two vertices of G_p^ω.*

Proof. Proof is obvious, if G^* is a tree.

Now choose a path (x, y) from an $m-$polar fuzzy labeling tree G_p^ω s.t. $P_i \circ D(x_j y_j) > 0$ for all $1 \leq j \leq n$. As G_p^ω is an $m-$polar fuzzy labeling tree and in its spanning subgraph the path connecting all the vertices is strong, all the arcs are strong. Thus between any two vertices arcs are strong. Similarly, choose another path between x and y, because G_p^ω is connected. But $P_i \circ D$ is bijective. So, getting another strong path is impossible. Hence there exists exactly one strong path between any two vertices. □

3. Bipartite $m-$polar fuzzy labeling tree

Definition 3.1. A bipartite $m-$polar fuzzy labeling graph $G_p^\omega = (C_p^\omega, D_p^\omega)$ is defined as, if set of vertices X can be distributed into two nonempty $m-$polar fuzzy independent sets X_1 and X_2. Where as, two vertices of an $m-$polar fuzzy graph are called $m-$polar fuzzy independent. If there does not exist any strong arc between them.

Example 3.2. Consider a 3−polar fuzzy labeling graph as shown in Fig. 5.

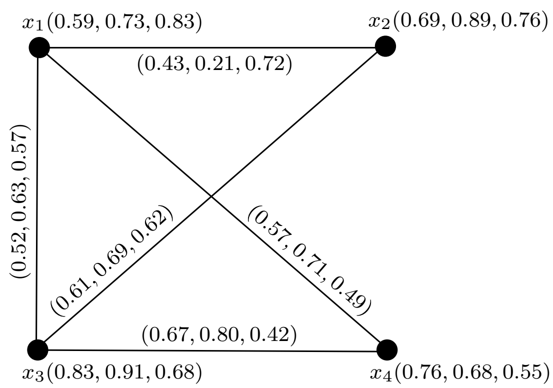


Figure 5. bipartite 3−polar fuzzy labeling graph

It is easy to compute that given graph is bipartite 3−polar fuzzy labeling graph, because set of vertices X can be distributed into two nonempty 3−polar fuzzy independent sets X_1 and X_2. Here, $X_1 = \{x_1, x_2\}$ and $X_2 = \{x_3, x_4\}$.

Proposition 3.3. *In any pair of vertices there will be a strong $m-$polar fuzzy path if G_p^ω is connected $m-$polar fuzzy labeling graph.*

Proposition 3.4. *Every $m-$polar fuzzy labeling tree is a bipartite $m-$polar fuzzy graph.*

Proof. Suppose G_p^ω is an $m-$polar fuzzy labeling tree and it is connected. Then, by Proposition 3.3, there exists a strong $m-$polar fuzzy path between any two vertices of G_p^ω. Thus, there exists $m-$polar fuzzy independent sets X_1 and X_2, such that the strong arc of the path have one vertex in X_1 and other in X_2. □

Proposition 3.5. *If G^* is $K_{1,n}^*$ and G_p^ω is an $m-$polar fuzzy labeling tree, then G_p^ω is a complete bipartite $m-$polar fuzzy graph.*

Proof. It is trivial that G_p^ω is an $m-$polar fuzzy labeling tree, if G^* is a tree. Then, $K_{1,n}^*$ is an $m-$polar fuzzy labeling tree, which is also a complete bipartite graph. Since $K_{1,n}^*$ graph can be distributed into two non empty independent sets X_1 and X_2, $X_1 = \{x\}$ and $X_2 = \{x_1, x_2, \cdots, x_n\}$. All the arcs of G_p^ω are strong arcs. Thus the vertices $x \in X$ is a strong neighbor of $\{x_1, x_2, \cdots, x_n\} \in X_2$. □

Remark 3.6. Every $m-$polar fuzzy labeling graph is not a complete bipartite $m-$polar fuzzy graph. For example $k^*_{2,n}$ is not complete bipartite $m-$polar fuzzy graph.

Algorithm for finding $m-$polar fuzzy spanning subgraph S^ω_p of an $m-$polar fuzzy labeling tree G^ω_p, when degree of membership of edges are in increasing order s.t $e_1 < e_2 < \cdots < e_n$ and $e_i = (r_1, r_2, \cdots, r_m)$, where G^* is complete.

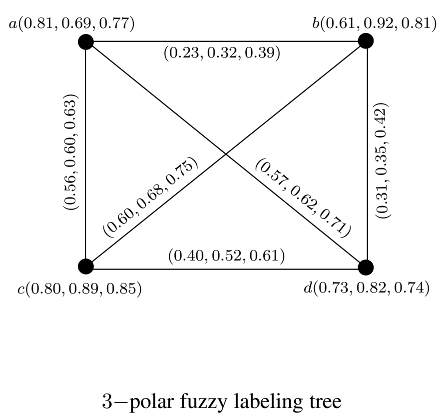
Step 1. Consider an $m-$polar fuzzy labeling tree such that G^* is complete with $|X| = n$.

Step 2. Choose an arbitrary cycle and remove an $m-$polar fuzzy weakest arc (there exist only one $m-$polar fuzzy weakest arc because degree of membership of all the edges are in increasing order as well as $P_i \circ D^\omega_p$ is bijective).

Step 3. Repeat step 2 until no cycle remains.

Step 4. The remaining graph is the $m-$polar fuzzy spanning subgraph S^ω_p of an $m-$polar fuzzy labeling graph G^ω_p, where all arcs of S^ω_p are $m-$polar fuzzy bridges of G^ω_p.

Example 3.7. The above algorithm is explained with the following 3$-$polar fuzzy labeling tree as shown in Fig 6.



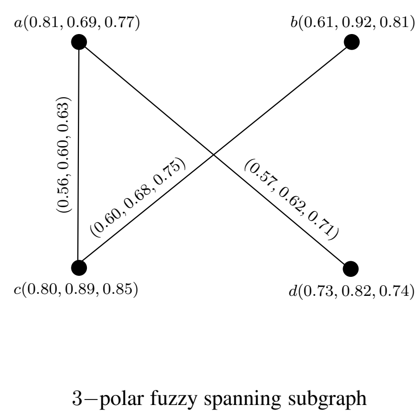


Figure 6. 3$-$polar fuzzy labeling tree when G^* is complete

4. Conclusion

Fuzzy graph theory plays an important role in many fields including decision makings, computer networking and management sciences. An m-polar fuzzy graph, generalization of a fuzzy graph, is useful for handling multi attribute, multi agents and multipolar information models. In this research article, we have introduced the concept of an m-polar fuzzy labeling tree G^ω_p generated by $m-$polar fuzzy spanning subgraph S^ω_p. We also precede the concept of bipartite $m-$polar fuzzy labeling graphs. We are extending our research work to (1) m-polar fuzzy magic graphs, (2) m-polar fuzzy hypergraphs, (3) m-polar fuzzy soft graphs.

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