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Finite dimensional intuitionistic fuzzy n-normed linear spaces

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Abstract. In this paper, definition of intuitionistic fuzzy n-normed linear space is given in general t-norm setting. Some basic results on completeness and compactness of finite dimensional intuitionistic n-fuzzy normed linear spaces have been studied.

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1. Introduction

Theory of fuzzy set was introduced by Zadeh [22] in 1965. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. Among other fields, progressive developments are made in the field of fuzzy metric spaces and fuzzy normed linear spaces [2, 3, 4, 5, 6, 8, 9, 10, 12]. The notion of intuitionistic fuzzy set has been introduced by Atanassov [1] as a generalized fuzzy set. Park [17], who first introduced the idea of intuitionistic fuzzy metric space and studied some basic properties. On the other hand, Saadati and Park [18] made an important contribution on the intuitionistic fuzzy topological spaces.

They have also introduced the notion of intuitionistic fuzzy normed linear space and studied some basic properties in such spaces. There has been a good amount of work done in intuitionistic fuzzy set such as Mandal and Samanta [13, 14]. Recently Vijayabalaji et al.[21] introduced concept of intuitionistic fuzzy n-normed linear space and developed some results. Samanta et al. [19] considered a fuzzy normed linear space which was introduced by Bag and Samanta [2] and defined an intuitionistic fuzzy normed linear space in general setting (taking $*$ and $\diamondsuit$ as t-norm and t-co-norm respectively). They mainly studied different results on finite dimensional intuitionistic fuzzy normed linear space. Bag and Samanta [5] modified the definition of intuitionistic fuzzy normed linear space introduced by Saadati et al. [18] and studied finite dimensional intuitionistic fuzzy normed linear space. In their approach they have avoided the decomposition technique which is very much dependent on the restricted t-norm and t-conorm.

On the other hand, Narayanan and Vijayabalaji [16] extended Bag and Samanta type [2] fuzzy normed linear space to fuzzy n-normed linear space and established some basic results. In 2014, Samanta and Bag [20] considered the fuzzy n-normed linear space introduced by Narayanan and Vijayabalaji [16] and studied some results on completeness and compactness of finite dimensional fuzzy n-normed linear spaces. In this context, it is worth mentioning the work of Dapke and Aage [7], Murugadas and Lalitha [15].

Following the definition of intuitionistic fuzzy n-normed linear space introduced by Vijayabalaji and Thillaigovindan [21], in this paper, a modified definition of intuitionistic fuzzy n-normed linear space is given and some results on finite dimensional intuitionistic fuzzy n-normed linear space are obtained.

The organization of the paper is as follows:
Section 2 comprise of some preliminary results. In Section 3, definition of intuitionistic fuzzy n-normed linear space is given. Some fundamental results on completeness and compactness are established in finite dimensional intuitionistic fuzzy n-normed linear spaces in Section 4.

2. Preliminaries

Definition 2.1 ([11])**.** A binary operation $* : [0\ ,\ 1] \times [0\ ,\ 1] \to [0\ ,\ 1]$ is a t-norm, if it satisfies the following conditions:
(1) $*$ is associative and commutative,
(2) $a * 1 = a \ \ \forall a \in [0\ ,\ 1]$,
(3) $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0\ ,\ 1]$.
If $*$ is continuous, then it is called continuous t-norm.

Following are the examples of some t-norms that are frequently used as fuzzy intersections defined for all $a,\ b \in\ [0,\ 1]$.
(1) Standard intersection: $a * b = min(a,\ b)$.
(2) Algebraic product: $a * b = ab$.
(3) Bounded difference: $a * b = max(0,\ a + b - 1)$.
(4) Drastic intersection:

$$a * b = \begin{cases} a & \text{for } b = 1 \\ b & \text{for } a = 1 \\ 0 & \text{for otherwise.} \end{cases}$$

The relations among these t-norms are

$$a * b(\text{Drastic}) \leq\ max(0,\ a + b - 1) \leq\ ab \leq\ min(a,\ b).$$

Definition 2.2 ([11])**.** A binary operation $\diamondsuit : [0\ ,\ 1] \times [0\ ,\ 1] \to [0\ ,\ 1]$ is a t-co-norm, if it satisfies the following conditions:
(i) $\diamondsuit$ is associative and commutative,
(ii) $a \diamondsuit 0 = a \ \ \forall a \in [0\ ,\ 1]$,
(iii) $a \diamondsuit b \leq c \diamondsuit d$, whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0\ ,\ 1]$.
If $\diamondsuit$ is continuous then it is called continuous t-co-norm.

Following are the examples of some t-co-norms.
(1) Standard union: $a\diamondsuit b = max(a,\ b)$.
(2) Algebraic sum: $a\diamondsuit b = a + b - ab$.
(3) Bounded sum : $a\diamondsuit b = min(1\ ,\ a + b)$.
(4) Drastic union:

$$a\diamondsuit b = \begin{cases} a & \text{for } b = 0 \\ b & \text{for } a = 0 \\ 1 & \text{for otherwise.} \end{cases}$$

Relations among these t-co-norms are

$$a\diamondsuit b \text{ (Drastic)} \geq\ min(1\ ,\ a + b) \geq\ a + b - ab \geq\ max(a,\ b).$$

Definition 2.3 ([16])**.** Let $n \in N$ (The set of all Natural numbers) and X be a real linear space of dimension $d \geq n(d$ can be infinite). A real valued function $||.,.,...,.||$ on X^n satisfying the following four properties is called an n-norm on the linear space X:
(i) $||x_1, x_2, ..., x_n|| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent.
(ii) $||x_1, x_2, ..., x_n||$ invariant under any permutation of $x_1, x_2, ..., x_n$..
(iii) $||x_1, x_2, ..., cx_n|| = |c|||x_1, x_2, ..., x_n||$ for any real c.
(iv) $||x_1, x_2, ..., x_{n-1}, y+z|| \leq ||x_1, x_2, ..., x_{n-1}, y||+||x_1, x_2, ..., x_{n-1}, z||\ \forall y, z \in X$.
The pair $(X, ||.,.,...,.||)$ is called an n-normed linear space.

Definition 2.4 ([16])**.** Let U be a linear space over the field F (C or R). A fuzzy subset N of $U \times R$ (R- set of real numbers) is called a fuzzy norm on U, if
(N1) $\forall t \in R$ with $t \leq 0,\ N(x\ ,\ t) = 0$.
(N2) $(\forall t \in R, t > 0, N(x\ ,\ t) = 1)$ iff $x = \underline{0}$.
(N3) $\forall t \in R,\ t > 0,\ N(cx\ ,\ t) = N(x\ ,\ \frac{t}{|c|})$, if $c \neq 0$.
(N4) $\forall s, t \in R;\ x, u \in U,\ N(x + u\ ,\ s + t) \geq\ N(x\ ,\ s) * N(u\ ,\ t)$.
(N5) $N(x,\ .)$ is a non-decreasing function of R and $\lim\limits_{t\to\infty} N(x\ ,\ t) = 1$.
The pair $(U,\ N)$ will be referred to as a fuzzy normed linear space. In [2], particular t-norm "min" is taken for $*$.

Definition 2.5 ([17])**.** Let $*$ be a t-norm, $\diamondsuit$ be a t-conorm and V be a linear space over the field F(R or C). An intuitionistic fuzzy norm (IFN) on V is an object of the form

$$A\ = \{((x\ ,\ t), N(x\ ,\ t),\ M(x\ ,\ t)) :\ (x\ ,\ t) \in V \times R,$$

where N and M are fuzzy sets on $V \times R$, N denotes the degree of membership and M denotes the degree of non-membership $(x\ ,\ t) \in V \times R$ satisfying the following conditions:
(IFN1) $\forall t \in R$ with $t \leq 0,\ N(x\ ,\ t) = 0$.
(IFN2) $(\forall t \in R,\ t > 0, N(x\ ,\ t) = 1)$ iff $x = \underline{0}$.
(IFN3) $\forall t \in R,\ t > 0,\ N(cx\ ,\ t) = N(x\ ,\ \frac{t}{|c|})$, if $c \neq 0$.
(IFN4) $\forall s, t \in R;\ x, u \in U,\ N(x + u\ ,\ s + t) \geq\ N(x\ ,\ s) * N(u\ ,\ t)$.
(IFN5) $\lim\limits_{t\to\infty} N(x\ ,\ t) = 1$.
(IFN6) $\forall t \in R$ with $t \leq 0,\ M(x\ ,\ t) = 1$.
(IFN7) $(\forall t \in R, t > 0, M(x\ ,\ t) = 0)$ iff $x = \underline{0}$.
(IFN8) $\forall t \in R,\ t > 0,\ M(cx\ ,\ t) = M(x\ ,\ \frac{t}{|c|})$, if $c \neq 0$.

(IFN9) $\forall s,t \in R;\ x,u \in V,\ M(x+u\ ,\ s+t) \leq\ M(x\ ,\ s)\diamondsuit M(u\ ,\ t)$.
(IFN10) $\lim\limits_{t\to\infty} M(x\ ,\ t) = 0$.
Then we say $(V,\ A)$ is an intuitionistic fuzzy normed linear space.

The definition of intuitionistic fuzzy n-normed linear space as introduced by Vijayabalaji et al.[21] is given below.

Definition 2.6 ([21])**.** An intuitionistic fuzzy n-normed linear space (or) in short i-f-n-NLS is an object of the form

$$A = \{(X,\ N(x_1,x_2,....,x_n,t),\ M(x_1,x_2,....,x_n,t)) :\ (x_1,x_2,....,x_n) \in X^n\},$$

where X is a linear space over a field F, $*$ is a continuous t-norm, $\diamondsuit$ is a continuous t-conorm and N,M are fuzzy sets on $X^n \times (0,\infty)$, N denotes the the degree of membership and M denotes the degree of non-membership of $(x_1,x_2,....,x_n,t) \in X^n(0,\infty)$ satisfying the following conditions:
(i) $N(x_1,x_2,....,x_n,t) + M(x_1,x_2,....,x_n,t) \leq 1$.
(ii) $N(x_1,x_2,....,x_n,t) > 0$.
(iii) $N(x_1,x_2,....,x_n,t) = 1$ if and only if $x_1,x_2,....,x_n$ are linearly dependent.
(iv) $N(x_1,x_2,....,x_n,t)$ is invariant under any permutation of $x_1,x_2,....,x_n$.
(v) $N(x_1,x_2,....,cx_n,t) = N(x_1,x_2,....,x_n,\frac{t}{|c|})$, if $c \in F,\ c \neq 0$.
(vi) $N(x_1,x_2,....,x_n+x_n',s+t) \geq N(x_1,x_2,....,x_n,s) * N(x_1,x_2,....,x_n',t)$.
(vii) $N(x_1,x_2,....,x_n,t) : (0,\infty) \to [0,1]$ is continuous in t.
(viii) $M(x_1,x_2,....,x_n,t) > 0$.
(xi) $M(x_1,x_2,....,x_n,t) = 0$ if and only if $x_1,x_2,....,x_n$ are linearly dependent.
(x) $M(x_1,x_2,....,x_n,t)$ is invariant under any permutation of $x_1,x_2,....,x_n$.
(xi) $M(x_1,x_2,....,cx_n,t) = M(x_1,x_2,....,x_n,\frac{t}{|c|})$, if $c \in F,\ c \neq 0$.
(xii) $M(x_1,x_2,....,x_n+x_n',s+t) \leq M(x_1,x_2,....,x_n,s)\diamondsuit M(x_1,x_2,....,x_n',t)$.
(xiii) $M(x_1,x_2,....,x_n,t) : (0,\infty) \to [0,1]$ is continuous in t.

3. Intuitionistic fuzzy n-normed linear space

Following the definition of intuitionistic fuzzy n-normed linear space given by Vijayabalaji et al.[21], in this Section a modified definition of intuitionistic fuzzy n-normed linear space is given.

Definition 3.1. Let $*$ be a t-norm, $\diamondsuit$ be a t-conorm and V be a linear space over the field F (R or C) of dimension $d \geq n$. An intuitionistic fuzzy n-norm (IFN) on V is an object of the form

$$A = \{(x_1,x_2,....,x_n,t), N(x_1,x_2,....,x_n,t), M(x_1,x_2,....,x_n,t) :\ (x_1,x_2,....,x_n,t) \in V^n \times R\},$$

where N and M are fuzzy sets on $V^n \times R$, N denotes the degree of membership and M denotes the degree of non-membership of $(x_1,x_2,....,x_n,t) \in V^n \times R$ satisfying the following conditions:
(IFN1) $\forall t \in R$ with $t \leq 0,\ N(x_1,x_2,....,x_n,t) = 0$.
(IFN2) $\forall t \in R$ with $t > 0, N(x_1,x_2,....,x_n,t)\ = 1$ if and only if $x_1,x_2,....,x_n$ are linearly dependent.
(IFN3) $N(x_1,x_2,....,x_n,t)$ is invariant under any permutation of $x_1,x_2,....,x_n$.
(IFN4) $\forall c \in F,\ c \neq 0,\ N(x_1,x_2,....,cx_n,t) =\ N(x_1,x_2,....,x_n,\ \frac{t}{|c|})$.

(IFN5) $\forall s, t \in R$,

$$N(x_1, x_2, ..., x_n + x'_n, s + t) \geq N(x_1, x_2, ..., x_n, s) * N(x_1, x_2, ..., x'_n, t).$$

(IFN6) $\lim_{t\to\infty} N(x_1, x_2, ..., x_n, t) = 1$.

(IFN7) $\forall t \in R$ with $t \leq 0,\ M(x_1, x_2, ..., x_n, t) = 1$.

(IFN8) $\forall t \in R$ with $t > 0, M(x_1, x_2, ..., x_n, t) = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent.

(IFN9) $M(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$.

(IFN10) $\forall c \in F,\ c \neq 0,\ M(x_1, x_2, ..., cx_n, t) = M(x_1, x_2, ..., x_n, \frac{t}{|c|})$.

(IFN11) $\forall s, t \in R$,

$$M(x_1, x_2, ..., x_n + x'_n, s + t) \geq M(x_1, x_2, ..., x_n, s) \diamondsuit M(x_1, x_2, ..., x'_n, t).$$

(IFN12) $\lim_{t\to\infty} M(x_1, x_2, ..., x_n, t) = 0$.

Then $(V,\ A)$ is called an intuitionistic fuzzy n-normed linear space.

Remark 3.2. The non-decreasing property of $N(x_1, x_2, ..., x_n, .)$ follows from (IFN2), and (IFN5) and non-increasing property of $M(x_1, x_2, ..., x_n, .)$ follows from (IFN8) and (IFN11).

Example 3.3. Let $(X\ ,\ ||., ., ..., .||)$ be an n-normed linear space as in Definition 2.3[16]. Define

$$N(x_1, x_2, ..., x_n, t) = \begin{cases} \frac{t}{t+||x_1,x_2,....,x_n||} & \text{for } t \in R^+, (x_1, x_2, ..., x_n) \in X^n \\ 0 & \text{when } t \leq 0, \end{cases}$$

$$M(x_1, x_2, ..., x_n, t) = \begin{cases} \frac{||x_1,x_2,....,x_n||}{||x_1,x_2,....,x_n||+t} & \text{for } t \in R^+, (x_1, x_2, ..., x_n) \in X^n \\ 1 & \text{when } t \leq 0. \end{cases}$$

Solution. In Example 3.3[20], it is shown that N satisfies the conditions (IFN1) to (IFN6). Now we shall show that M satisfies the conditions (IFN11) to (IFN12).

(i) (IFN7IFN7) follows from Definition.

(ii) $\forall t > 0,\ M(x_1, x_2, ..., x_n, t) = 0$

$\Leftrightarrow \frac{||x_1,x_2,....,x_n||}{||x_1,x_2,....,x_n||+t} = 0$

$\Leftrightarrow ||x_1, x_2, ..., x_n|| = 0$

$\Leftrightarrow x_1, x_2, ..., x_n$ are linearly independent.

Then (IFN8) holds.

(iii) Since $||x_1, x_2, ..., x_n||$ is invariant under any permutation of $x_1, x_2, ..., x_n$, $M(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$.

Then (IFN9) holds.

(iv) For $c \neq 0$ and $\forall t > 0$ we have

$$\begin{aligned} M(x_1, x_2, ..., cx_n, t) &= \frac{||x_1,x_2,....,cx_n||}{||x_1,x_2,....,cx_n||+t} \\ &= \frac{|c|||x_1,x_2,....,x_n||}{|c|||x_1,x_2,....,x_n||+t} \\ &= \frac{||x_1,x_2,....,x_n||}{||x_1,x_2,....,x_n||+\frac{t}{|c|}} \\ &= M(x_1, x_2, ..., x_n, \tfrac{t}{|c|}). \end{aligned}$$

Then (IFN10) holds.

(v) We consider only the case when $s, t > 0$. Since other cases are obvious, suppose $\frac{||x_1,x_2,....,x_{n-1},y||}{||x_1,x_2,....,x_{n-1},y||+s} \leq \frac{||x_1,x_2,....,x_{n-1},z||}{||x_1,x_2,....,x_{n-1},z||+t}$. Then we have,

(A) $$||x_1, x_2, ..., x_{n-1}, z|| - t||x_1, x_2, ..., x_{n-1}, y|| \geq 0.$$

On the other hand,

$$\begin{aligned}
&\frac{||x_1,x_2,....,x_{n-1},z||}{||x_1,x_2,....,x_{n-1},z||+t} - \frac{||x_1,x_2,....,x_{n-1},y+z||}{||x_1,x_2,....,x_{n-1},y+z||+s+t} \\
&\geq \frac{||x_1,x_2,....,x_{n-1},z||}{||x_1,x_2,....,x_{n-1},z||+t} - \frac{||x_1,x_2,....,x_{n-1},y+z||}{||x_1,x_2,....,x_{n-1},y||+s+t+||x_1,x_2,....,x_{n-1},z||} \\
&\geq \frac{||x_1,x_2,....,x_{n-1},z||}{||x_1,x_2,....,x_{n-1},z||+t} - \frac{||x_1,x_2,....,x_{n-1},y||+||x_1,x_2,....,x_{n-1},z||}{||x_1,x_2,....,x_{n-1},y||+||x_1,x_2,....,x_{n-1},z||+s+t}.
\end{aligned}$$

By using (A), we get

$$\frac{||x_1, x_2, ..., x_{n-1}, z||}{||x_1, x_2, ..., x_{n-1}, z|| + t} - \frac{||x_1, x_2, ..., x_{n-1}, y + z||}{||x_1, x_2, ..., x_{n-1}, y + z|| + s + t} \geq 0.$$

Thus, $\frac{||x_1,x_2,....,x_{n-1},z||}{||x_1,x_2,....,x_{n-1},z||+t} \geq \frac{||x_1,x_2,....,x_{n-1},y+z||}{||x_1,x_2,....,x_{n-1},y+z||+s+t}$. So

$$\begin{aligned}
&max\ \{M(x_1, x_2, ..., x_{n-1}, y, s),\ M(x_1, x_2, ..., x_{n-1}, z, t)\} \\
&\geq M(x_1, x_2, ..., x_{n-1}, y + z, s + t).
\end{aligned}$$

Hence $M(x_1, x_2, ..., x_{n-1}, y, s) \diamondsuit M(x_1, x_2, ..., x_{n-1}, z, t)\}$
$\geq M(x_1, x_2, ..., x_{n-1}, y + z, s + t)$.

Therefore (IFN11) holds.

(vi) It is clear that $\lim_{t\to\infty} M(x_1, x_2, ..., x_n, t) = 0$.

Hence $(X, N, M, *, \diamondsuit)$ is an intuitionistic fuzzy n-normed linear space.

Definition 3.4. Let $(X, N, M, *, \diamondsuit)$ be an intuitionistic fuzzy n-normed linear space. A sequence $\{x_n\}$ in X is said to be convergent and converges to x, if for each $y_1, y_2, ..., y_{n-1} \in X$,

$$\lim_{n\to\infty} N(y_1, y_2, ..., y_{n-1}, x_n - x, t) = 1$$

and

$$\lim_{n\to\infty} M(y_1, y_2, ..., y_{n-1}, x_n - x, t) = 0 \quad \forall t > 0.$$

Definition 3.5. Let $(X, N, M, *, \diamondsuit)$ be an intuitionistic fuzzy n-normed linear space. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence, if for each $y_1, y_2, ..., y_{n-1} \in X$,

$$\lim_{n\to\infty} N(y_1, y_2, ..., y_{n-1}, x_{n+p} - x_n, t) = 1$$

and

$$\lim_{n\to\infty} M(y_1, y_2, ..., y_{n-1}, x_{n+p} - x_n, t) = 0,\ \forall t > 0$$

and it is uniformly on $p = 1, 2, 3, ...$

Theorem 3.6. *If a sequence in an intuitionistic fuzzy n-normed linear space $(X, N, M, *, \diamondsuit)$ is convergent then its limit is unique provided $*$ is continuous at $(1, 1)$ and $\diamondsuit$ is continuous at $(0, 0)$.*

Proof. Let $\{x_p\}$ be a convergent sequence in X. If possible suppose that $\exists x, y (x \neq y)$ such that $\{x_p\} \to x$ and $\{x_p\} \to y$ as $p \to \infty$. As $x \neq y$ and $dimX \geq n$, $\exists$ a linearly independent set of vectors $\{u_1, u_2, ..., u_{n-1}, x - y\}$ in X.

Now,

$$N(u_1, u_2, ..., u_{n-1}, x - y, 2s)$$

$= N(u_1, u_2, ..., u_{n-1}, x - x_p + x_p - y, s + s)$
$\geq N(u_1, u_2, ..., u_{n-1}, x - x_p, s) * N(u_1, u_2, ..., u_{n-1}, x_p - y, s)$.
Since $*$ is continuous at (1,1), $\lim\limits_{n\to\infty} N(u_1, u_2, ..., u_{n-1}, x - y, 2s) \geq 1 * 1 = 1 \ \forall s > 0$.
Then $\{u_1, u_2, ..., u_{n-1}, x - y\}$ are linearly dependent. This is a contradiction.
Again,
$M(u_1, u_2, ..., u_{n-1}, x - y, 2s)$
$= M(u_1, u_2, ..., u_{n-1}, x - x_p + x_p - y, s + s)$
$\leq M(u_1, u_2, ..., u_{n-1}, x - x_p, s) \diamondsuit M(u_1, u_2, ..., u_{n-1}, x_p - y, s)$.
Since $\diamondsuit$ is continuous at (0,0), $\lim\limits_{n\to\infty} M(u_1, u_2, ..., u_{n-1}, x - y, 2s) \leq 0 \diamondsuit 0 = 0 \ \forall s > 0$.
Thus $\{u_1, u_2, ..., u_{n-1}, x - y\}$ are linearly dependent, which is a contradiction.
So $x - y = \theta$, i.e., $x = y$. □

Definition 3.7. Let $(X, N, M, *, \diamondsuit)$ be an intuitionistic fuzzy n-normed linear space. A subset F of X is said to be closed if for any sequence $\{x_n\}$ in F such that $x_n \to x$ implies $x \in F$.

Definition 3.8. Let $(X, N, M, *, \diamondsuit)$ be an intuitionistic fuzzy n-normed linear space and $F \subset X$. Then the closure of F denoted by $\bar{F}$ and is defined by

$$\bar{F} = \{x \in X : \ \exists \text{ a sequence } \{x_n\} \text{ in F converging to } x\}.$$

Definition 3.9. Let $(X, N, M, *, \diamondsuit)$ be an intuitionistic fuzzy n-normed linear space. A subset V of X is said to be compact if any sequence $\{x_n\}$ in V has a convergent subsequence which converges to some element in V.

4. Finite dimensional intuitionistic fuzzy normed linear space

In this Section some results on finite dimensional intuitionistic fuzzy normed linear spaces are established.

Lemma 4.1. *Let $(X, N, M, *, \diamondsuit)$ be an intuitionistic fuzzy n-normed linear space with the underlying t-norm $*$ is continuous $(1, 1)$ and the underlying t-conorm $\diamondsuit$ is continuous at $(0, 0)$ and $\{x_1, x_2, ..., x_k\}$ be a linearly independent set of vectors in X, then $\exists c_1, c_2 > 0$ and $\exists \delta_1, \delta_2 \in (0, 1)$ such that for any set of scalars $\{\alpha_1, \ \alpha_2, ..., \alpha_k\}, \exists y_1, y_2, ..., y_k \in X$ such that ;*

$$N(y_1, y_2, ..., y_{n-1}, \sum_{i=1}^{n} \alpha_i x_i, c_1 \sum_{i=1}^{n} |\alpha_i|) \ < \ 1 - \delta_1 \tag{4.1.1}$$

and

$$M(y_1, y_2, ..., y_{n-1}, \sum_{i=1}^{n} \alpha_i x_i, c_2 \sum_{i=1}^{n} |\alpha_i|) \ > \ \delta_2. \tag{4.1.2}$$

Proof. We prove the second part (4.1.2). First part follows from the Lemma 3.7[20]. Let $s = |\alpha_1| + |\alpha_2| + ... + |\alpha_k|$.

If $s = 0$, then $\alpha_j = 0 \ \forall j = 1, 2, ..., k$. Thus the relation (4.1.2) holds for any $c > 0$ and $\delta \in (0 \ , \ 1)$.

Next we suppose that $s > 0$. Then (4.1.2) is equivalent to

$$(4.1.3) \qquad M(y_1, y_2, ..., y_{n-1}, \sum_{i=1}^{k} \beta_i x_i,\ c_2)\ >\ \delta_2,$$

for all scalars β_j's with $\sum_{j=1}^{k} |\beta_j| = 1$.

If possible, suppose that (4.1.3) does not hold. Then for any $c > 0$ and $\delta \in (0\ ,\ 1)$, $\exists$ a set of scalars $\{\beta_1,\ \beta_2, ..., \beta_n\}$ with $\sum_{j=1}^{k} |\beta_j| = 1$ such that

for any $y_1, y_2, ..., y_{n-1} \in X$, $M(y_1, y_2, ..., y_{n-1}, \sum_{i=1}^{k} \beta_i x_i,\ c)\ \leq \delta$.

In particular, for $c = \delta = \frac{1}{m}$, $m = 1, 2, ...$, $\exists$ a set of scalars $\{\gamma_1^{(m)},\ \gamma_2^{(m)}, ..., \gamma_k^{(m)}\}$ with $\sum_{j=1}^{k} |\gamma_j^{(m)}| = 1$ such that $M(y_1, y_2, ..., y_{n-1}, z_m\ ,\ \frac{1}{m})\ \leq \frac{1}{m}$,

where $z_m =\ \gamma_1^{(m)} x_1 + \gamma_2^{(m)} x_2 + ... + \gamma_k^{(m)} x_k$. Since $\sum_{j=1}^{k} |\gamma_j^{(m)}| = 1$, we have

$0 \leq |\gamma_j^{(m)}|\ \leq 1$ for $j = 1, 2, ..., k$ and $m = 1, 2, 3, ...$ Thus for each fixed j, $\{\gamma_j^{(m)}\}$ is bounded. So in particular, $\{\gamma_1^{(m)}\}$ is also so. By Bolzano-Weierstrass Theorem, it follows that $\{\gamma_1^{(m)}\}$ has a convergent subsequence which converges to γ_1 (say).

Let $\{z_{1,m}\}$ denotes the corresponding subsequence of $\{z_m\}$. By the same argument as above, $\{z_{1,m}\}$ has a subsequence say $\{z_{2,m}\}$ for which the corresponding subsequence of scalars $\{\gamma_2^{(m)}\}$ converges to γ_2. Continuing in this way, after k-steps, we obtain a subsequence $\{z_{k,m}\}$, where

$z_{k,m} = \sum_{j=1}^{k} \eta_j^{(m)} x_j$ with $\sum_{j=1}^{k} |\eta_j^{(m)}| = 1$ and $\eta_j^{(m)} \to \eta_j$ as $m \to \infty$ for $j = 1, 2, 3, ..., k$.

Let $z = \eta_1 x_1 + \eta_2 x_2 + ... + \eta_k x_k$. Then $\forall t > 0$ and $\forall y_1, y_2, ..., y_{n-1} \in X$, we have,

$$\begin{aligned}
&\lim_{m\to\infty} M(y_1, y_2, ..., y_{n-1}, z_{k,m} - z, t) \\
&= \lim_{m\to\infty} M(y_1, y_2, ..., y_{n-1}, \sum_{j=1}^{k} (\gamma_j^{(m)} - \eta_j) x_j, \frac{kt}{k}) \\
&\leq \lim_{m\to\infty} M(y_1, y_2, ..., y_{n-1}, x_1, \frac{t}{n|\gamma_1^{(m)} - \eta_1|}) \\
&\quad \diamondsuit \lim_{m\to\infty} M(y_1, y_2, ..., y_{n-1}, x_2, \frac{t}{n|\gamma_2^{(m)} - \eta_2|}) \\
&\quad \diamondsuit .....\diamondsuit \lim_{m\to\infty} M(y_1, y_2, ..., y_{n-1}, x_k, \frac{t}{n|\gamma_k^{(m)} - \eta_k|}) \\
&= 0 \diamondsuit 0 \diamondsuit .....\diamondsuit 0 = 0 \text{ (by the continuity of } t\text{-conorms } \diamondsuit \text{ at (0,0)).}
\end{aligned}$$

Thus $\lim_{m\to\infty} M(y_1, y_2, ..., y_{n-1}, z_{k,m} - z, t) = 0,\ \forall t > 0.$

In Lemma 3.7[20], it is proved that $\lim_{n\to\infty} N(y_1, y_2, ..., y_{n-1}, z_{k,m} - z, t) = 0, \forall t > 0$.
So $\{z_{k,m}\} \to z$ as $m \to \infty$.

Now for $s > 0$, choose m such that $\frac{1}{m} < s$. Then we have,

$$\begin{aligned}
&M(y_1, y_2, ..., y_{n-1}, z_{k,m}, s)\\
&= M(y_1, y_2, ..., y_{n-1}, z_{k,m} - \theta, s + \tfrac{1}{m} - \tfrac{1}{m})\\
&\leq M(y_1, y_2, ..., y_{n-1}, z_{k,m}, \tfrac{1}{m}) \diamondsuit M(y_1, y_2, ..., y_{n-1}, \theta, s - \tfrac{1}{m}).
\end{aligned}$$

Thus $M(y_1, y_2, ..., y_{n-1}, z_{k,m}, s) \leq \frac{1}{m} \diamondsuit 0 = 0$.
So $\lim_{m\to\infty} M(y_1, y_2, ..., y_{n-1}, z_{k,m}, s) = 0, \ \forall s > 0$.
In Lemma 3.7[20], it is proved that $\lim_{m\to\infty} N(y_1, y_2, ..., y_{n-1}, z_{k,m}, s) = 1, \forall s > 0$. Hence $\{z_{k,m}\} \to \theta$ as $m \to \infty$. Since the limit of a convergence sequence in $(X, N, M, *, \diamondsuit)$ is unique, $z = \theta$. This implies that $\eta_1 = \eta_2 = ... = \eta_k = 0$.

Now $\gamma_i^{(m)} \to \eta_i$ as $m \to \infty$ for each $i = 1, 2, 3, ..., k$ where $\sum_{j=1}^{k} |\eta_j| = 1$ which contradicts the fact that $\eta_1 = \eta_2 = ... = \eta_k = 0$. Therefore (4.1.3) holds. □

Theorem 4.2. *If $(X, N, M, *, \diamondsuit)$ is a finite dimensional intuitionistic fuzzy n-normed linear space with the underlying t-norm $*$ is continuous at $(1, 1)$ and the underlying t-conorm $\diamondsuit$ is continuous at $(0, 0)$, then $(X, N, M, *, \diamondsuit)$ is complete.*

Proof. Let dim X = k and $\{e_1, e_2, ..., e_k\}$ be a basis for X. Let $\{x_r\}$ be a Cauchy sequence in X. Then ∃ scalars $\beta_i^{(r)}, \ i = 1, 2, ..., k; \ r = 1, 2, ...$ such that

$$x_r = \sum_{i=1}^{k} e_i \beta_i^{(r)}, \ r = 1, 2, 3, ...$$

Since $\{x_r\}$ is a Cauchy sequence, for each $y_1, y_2, ..., y_{n-1} \in X$,

$$\text{(4.2.1)} \qquad \lim_{r,s\to\infty} N(y_1, y_2, ..., y_{n-1}, x_r - x_s, t) = 1, \quad \forall t > 0$$

and

$$\text{(4.2.2)} \qquad \lim_{r,s\to\infty} M(y_1, y_2, ..., y_{n-1}, x_r - x_s, t) = 0, \quad \forall t > 0.$$

Now by Lemma 4.1, $\exists c_1, c_2 > 0$ and $\delta_1, \delta_2 \in (0, 1)$ such that

$$\text{(4.2.3)} \qquad N(y_1, y_2, ..., y_{n-1}, \ \sum_{i=1}^{k} e_i(\beta_i^{(r)} - \beta_i^{(s)}), \ c_1 \sum_{i=1}^{k} |\beta_i^{(r)} - \beta_i^{(s)}|) \ < \ 1 - \delta_1$$

and

$$\text{(4.2.4)} \qquad M(y_1, y_2, ..., y_{n-1}, \ \sum_{i=1}^{k} e_i(\beta_i^{(r)} - \beta_i^{(s)}), \ c_2 \sum_{i=1}^{k} |\beta_i^{(r)} - \beta_i^{(s)}|) \ > \ \delta_2.$$

From (4.2.2), it follows that for $\delta_2 > 0$, $\exists p \in N$ such that

$$M(y_1, y_2, ..., y_{n-1}, x_r - x_s, t) \ < \delta_2, \quad \forall r, s \geq p \text{ and for } t > 0.$$

Then,

$$M(y_1, y_2, ..., y_{n-1}, \sum_{i=1}^{k} e_i(\beta_i^{(r)} - \beta_i^{(s)}), \ c_2 \sum_{i=1}^{k} |\beta_i^{(r)} - \beta_i^{(s)}|) \ > \ \delta_2$$
$$> M(y_1, y_2, ..., y_{n-1}, x_r - x_s, t), \quad \forall r, s \geq p \text{ and for } t > 0$$
$$\Rightarrow c_2 \sum_{i=1}^{k} |\beta_i^{(r)} - \beta_i^{(s)}| < t \ \forall r, s \geq p \text{ and for } t > 0$$
$$\Rightarrow \sum_{i=1}^{k} |\beta_i^{(r)} - \beta_i^{(s)}| < \frac{t}{c_2} \ \forall r, s \geq p \text{ and for } t > 0$$
$$\Rightarrow |\beta_i^{(r)} - \beta_i^{(s)}| < \tfrac{t}{c_2} \ \forall r, s \geq p \text{ for } t > 0 \text{ and } i = 1, 2, ..., k.$$

From above, it follows that, for each $i = 1, 2, ..., k$, $\{\beta_i^{(r)}\}$ is a Cauchy sequence in C. Since C is complete, $\{\beta_i^{(r)}\}$ is convergent for each $i = 1, 2, ..., k$.

Let $\beta_i^{(r)} \to \beta_i$ as $r \to \infty$ and for $i = 1, 2, ..., k$. Let $y = \sum_{i=1}^{k} \beta_i e_i$. Then $y \in X$. Thus for $y_1, y_2, ..., y_{n-1} \in X$,

$$M(y_1, y_2, ..., y_{n-1}, x_r - x, t)$$
$$= M(y_1, y_2, ..., y_{n-1}, \sum_{i=1}^{k} e_i(\beta_i^{(r)} - \beta_i), \frac{kt}{k})$$
$$\leq M(y_1, y_2, ..., y_{n-1}, e_1, \tfrac{t}{k|\beta_1^{(r)} - \beta_1|}) \diamondsuit M(y_1, y_2, ..., y_{n-1}, e_2, \tfrac{t}{k|\beta_2^{(r)} - \beta_2|})$$
$$\diamondsuit ......\diamondsuit M(y_1, y_2, ..., y_{n-1}, e_k, \tfrac{t}{k|\beta_k^{(r)} - \beta_k|}).$$

Now as $p \to \infty$, $\frac{t}{k|\beta_i^{(p)} - \beta_i|} \to \infty$ for $i = 1, 2, ..., k$. So

$$\lim_{p \to \infty} M(y_1, y_2, ..., y_{n-1}, e_i, \frac{t}{k|\beta_i^{(p)} - \beta_i|}) = 0 \text{ for } i = 1, 2, ..., k.$$

Using continuity of t-co-norm at $(0, 0)$, we get

$$\lim_{p \to \infty} M(y_1, y_2, ..., y_{n-1}, x_r - x, t) = 0, \forall t > 0. \tag{4.2.5}$$

Using continuity of t-norm at $(1, 1)$, we get

$$\lim_{p \to \infty} N(y_1, y_2, ..., y_{n-1}, x_r - x, t) = 1, \forall t > 0, \tag{4.2.6}$$

(See Theorem 3.8 [20]). From (4.2.5) and (4.2.6), it follows that $x_r \to x$. Since $\{x_r\}$ is an arbitrary sequence in X, it follows that X is complete.

□

Theorem 4.3. *Let $(X, N, M, *, \diamondsuit)$ be a finite dimensional intuitionistic fuzzy n-normed linear space with the underlying t-norm $*$ is continuous at $(1, 1)$ and the underlying t-conorm $\diamondsuit$ is continuous at $(0, 0)$ and $A \subset X$. Then A is compact iff A is closed and bounded.*

Proof. First we suppose that A is compact.

If possible suppose that A is not bounded. Then $\exists y_1, y_2, ..., y_{n-1} \in X, r \in (0,1)$ such that for each positive integer $m, \exists x_m \in A$ such that

$$M(y_1, y_2, ..., y_{n-1}, x_m, m) > r. \tag{4.3.1}$$

Now $\{x_m\}$ is a sequence in A. As A is compact, there exists a subsequence $\{x_{m_i}\}$ of $\{x_m\}$ converging to a point $x \in A$. Thus for each $y_1, y_2, ..., y_{n-1} \in X$,

$$\lim_{i\to\infty} M(y_1, y_2, ..., y_{n-1}, x_{m_i} - x, t) = 0,\ \forall t > 0. \tag{4.3.2}$$

From (4.3.1), we have $M(y_1, y_2, ..., y_{n-1}, x_{m_i}, m_i) > r$. So

$$\begin{aligned}
& r < M(y_1, y_2, ..., y_{n-1}, x_{m_i} - x + x, m_i - t + t) \\
& \quad \le M(y_1, y_2, ..., y_{n-1}, x_{m_i} - x, t) \diamondsuit M(y_1, y_2, ..., y_{n-1}, x, m_i - t) \\
& \Rightarrow r \le \lim_{i\to\infty} M(y_1, y_2, ..., y_{n-1}, x_{m_i} - x, t) \diamondsuit \lim_{i\to\infty} M(y_1, y_2, ..., y_{n-1}, x, m_i - t) \\
& \Rightarrow r \le 0 \diamondsuit 0 = 0 \text{ (using the continuity of } t\text{-co-norm at } (0,0)) \\
& \Rightarrow r = 0.
\end{aligned}$$

This is a contradiction. If we consider N, then we also arrive at the same contradiction (Please see Theorem 3.10[20]). Hence A is bounded.

Next consider a sequence $\{x_n\}$ in A converging to a point x in X. Since A is compact, the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ converging to a point in A. As every subsequence of a convergent sequence converges to the same limit, it follows that $\{x_{n_k}\}$ converges to x. Then $x \in A$ and Thus A is closed.

Conversely, suppose that A is closed and bounded. We have to show that A is compact. Let dimX $= k(\ge n)$. and let $B = \{e_1, e_2, ..., e_k\}$ be a basis for X and $\{x_m\}$ be a sequence in A. Then $\exists$ scalars $\beta_i^{(m)}, i = 1, 2, ..., k$ such that $x_m = \sum_{i=1}^{k} \beta_i^{(m)} e_i; m = 1, 2, ...$ By Lemma 4.1, $\exists c_1, c_2 > 0$ and $\delta_1, \delta_2 \in (0,1)$ such that for scalars $\beta_i^{(m)}, i = 1, 2, ..., k$, $\exists y_1, y_2, ..., y_{n-1} \in X$ such that

$$N(y_1, y_2, ..., y_{n-1}, \sum_{i=1}^{k} \beta_i^{(m)} e_i, c_1 \sum_{i=1}^{k} |\beta_i^{(m)}|) \ < \ 1 - \delta_1. \tag{4.3.3}$$

and

$$M(y_1, y_2, ..., y_{n-1}, \sum_{i=1}^{k} \beta_i^{(m)} e_i, c_2 \sum_{i=1}^{k} |\beta_i^{(m)}|) \ > \ \delta_2. \tag{4.3.4}$$

Since $\{x_m\}$ is bounded, for $\delta_2 \in (0,1)$ and $y_1, y_2, ..., y_{n-1} \in X, \exists t > 0$ such that

$$M(y_1, y_2, ..., y_{n-1}, \sum_{i=1}^{k} \beta_i^{(m)} e_i, t) \ < \ \delta_2 \ m = 1, 2, 3, ...$$

From (4.3.4) we have,

$$\begin{aligned}
& M(y_1, y_2, ..., y_{n-1}, \sum_{i=1}^{k} \beta_i^{(m)} e_i, c_2 \sum_{i=1}^{k} |\beta_i^{(m)}|) \\
& > \delta_2
\end{aligned}$$

$$> M(y_1, y_2, ..., y_{n-1}, \sum_{i=1}^{k} \beta_i^{(m)} e_i, t).$$

Thus $c_2 \sum_{i=1}^{k} |\beta_i^{(m)}| \ < t, \ m = 1, 2, 3, ...$ So $\sum_{i=1}^{k} |\beta_i^{(m)}| \leq \frac{t}{c_2}, \ m = 1, 2, 3, ...$

Hence for each $i = 1, 2, ..., k$; $\{\beta_i^{(m)}\}$ is bounded in R or C. In particular, $\{\beta_1^{(m)}\}$ is bounded and thus it has a convergent subsequence $\{\beta_1^{(m,1)}\}$ converging to a point β_1 (say) in R or C.

Let $\{x_{m,1}\}$ be the corresponding subsequence of $\{x_m\}$. Again consider the bounded sequence $\{\beta_2^{(m)}\}$. Then it has a converging subsequence $\{\beta_2^{(m,2)}\}$ and corresponding subsequence of $\{x_{m,1}\}$ is $\{x_{m,2}\}$.

Continuing in this way, after k-steps, suppose $\{x_{m,k}\}$ be the corresponding subsequence of $\{x_m\}$,

where $x_{m,k} \ = \sum_{i=1}^{k} \beta_i^{(m,k)} e_i; \ m = 1, 2, 3, ...$ and $\lim_{m\to\infty} \beta_i^{(m,k)} = \beta_i, \ 1 \leq i \leq k.$

Let $y = \sum_{i=1}^{k} \beta_i e_i$. Now for $y_1, y_2, ..., y_{n-1} \in X$, we get

$$\begin{aligned}
& M(y_1, y_2, ..., y_{n-1}, x_{k,m} - y, t) \\
&= M(y_1, y_2, ..., y_{n-1}, \sum_{i=1}^{k} (\beta_i^{(m,k)} - \beta_i) e_i, \frac{kt}{k}) \\
&\leq M(y_1, y_2, ..., y_{n-1}, (\beta_1^{(m,k)} - \beta_1) e_1, \tfrac{t}{k}) \diamondsuit M(y_1, y_2, ..., y_{n-1}, (\beta_2^{(m,k)} - \beta_2) e_2, \tfrac{t}{k}) \\
&\qquad \diamondsuit .....\diamondsuit M(y_1, y_2, ..., y_{n-1}, (\beta_k^{(m,k)} - \beta_k) e_1, \tfrac{t}{k}) \\
&= M(y_1, y_2, ..., y_{n-1}, e_1, \tfrac{t}{k|\beta_1^{(m,k)} - \beta_1|}) \diamondsuit .....\diamondsuit M(y_1, y_2, ..., y_{n-1}, e_1, \tfrac{t}{k|\beta_k^{(m,k)} - \beta_k|}).
\end{aligned}$$

Thus $\lim_{m\to\infty} M(y_1, y_2, ..., y_{n-1}, x_{k,m} - y, t) \leq 0 \diamondsuit 0 \diamondsuit ...... \diamondsuit 0 = 0.$

(Since $\frac{t}{k|\beta_k^{(m,k)} - \beta_k|} \to \infty$ as $m \to \infty$ and using the continuity of $\diamondsuit$ at (0 , 0)).

On the other hand, in similar way, it can be proved that

$\lim_{m\to\infty} N(y_1, y_2, ..., y_{n-1}, x_{k,m} - y, t) = 1$ (Please see Theorem 3.10[20]). So $x_{k,m} \to y$ as $m \to \infty$. Hence the sequence $\{x_m\}$ has a convergent subsequence converging to a point in X. Since A is closed, $x \in A$. As $\{x_m\}$ is an arbitrary sequence in A, it follows that A is compact. □

5. Conclusion

Extending a recent approach of Bag and Samanta [4] towards the study of fuzzy normed linear spaces with general t-norm, it has been possible to develop finite dimensional intuitionistic fuzzy n-normed linear spaces and have studied compactness and completeness in such spaces. Since in general t-norm setting, decomposition theorem is not applicable, so that a different technique is required to handle such situations. There is a wide scope of research in studying intuitionistic fuzzy normed linear spaces with underlying general t-norm setting in the triangle inequality of the fuzzy norm.

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