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I_λ-convergence in intuitionistic fuzzy n-normed linear space

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Abstract. The notion of lacunary ideal convergence in intuitionistic fuzzy normed linear space (IFNLS) was introduced by the present corresponding author [P. Debnath, Lacunary ideal convergence in intuitionistic fuzzy normed linear spaces, Comput. Math. Appl., 63 (2012), 708-715] and an open problem in that paper was whether every lacunary I-convergent sequence is lacunary I-Cauchy. Further, a new concept of convergence of sequences in an intuitionistic fuzzy n-normed linear space (IFnNLS) was given in [M. Sen, P. Debnath, Lacunary statistical convergence in intuitionistic fuzzy n-normed linear spaces, Math. Comput. Modelling, 54 (2011), 2978-2985]. With the help of this new definition of convergence, the main aim of this paper is to introduce the concept of I_λ-convergence in an IFnNLS, where I is an ideal of a family of subsets of positive integers $\mathbb{N}$. We also define I_λ-limit points and I_λ-cluster points and establish relations between them. Finally we introduce the notion of I_λ-Cauchy sequence in IFnNLS. We improve and extend some existing results and give a positive answer to the open problem mentioned above in the setting of an IFnNLS.

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1. Introduction

Many problems that we study in analysis are concerned with large classes of objects most of which turn out to be vector spaces or linear spaces. Since limit process is indispensable in such problems, a metric or topology may be induced in those classes. If the induced metric satisfies the translation invariance property, a norm can be defined in that linear space and we get a structure of the space which is compatible with that metric or topology. The resulting structure is a normed linear space. There are situations where crisp norm can not measure the length of a vector accurately and in such cases the notion of fuzzy norm happens to be useful. There has been a systematic development of fuzzy normed linear spaces (FNLSs) and one of the important development over FNLS is the notion of intuitionistic fuzzy normed linear space (IFNLS). The study of analytic propertis of IFNLSs, their topological structure and generalizations, therefore, remain well motivated areas of research.

The idea of a fuzzy norm on a linear space was introduced by Katsaras [23] in 1984. In 1992, Felbin [16] introduced the idea of a fuzzy norm whose associated metric is of Kaleva and Seikkala [20] type. In 1994, Cheng and Mordeson [6] introduced another notion of fuzzy norm on a linear space whose associated metric is Kramosil and Michalek [26] type. Again in 2003, following Cheng and Mordeson, one more notion of fuzzy normed linear space was given by Bag and Samanta [3].

The notion of intuitionistic fuzzy set (IFS) introduced by Atanassov [2] has triggered some debate (for details, see [4, 13, 17]) regarding the use of the terminology "intuitionistic" and the term is considered to be a misnomer on the following account:

- The algebraic structure of IFSs is not intuitionistic, since negation is involutive in IFS theory.
- Intuitionistic logic obeys the law of contradiction, IFSs do not.

Also IFSs are considered to be equivalent to interval-valued fuzzy sets and they are particular cases of L-fuzzy sets. In response to this debate, Atanassov justified the terminology in [1]. Apart from the terminological issues, research in intuitionistic fuzzy setting remains well motivated as IFSs give us a very natural tool for modeling imprecision in real life situations which can not be handled with fuzzy set theory alone and also IFS found its application in various areas of science and engineering.

In 2006, with the help of arbitrary continuous t-norm and continuous t-conorm, Saadati and Park [31] introduced the concept of IFNLS. There has been further development over IFNLS, e.g., the topological structure of an intuitionistic fuzzy 2-normed space has been studied by Mursaleen and Lohani in [29]. Further, generalizing the idea of Saadati and Park, an intuitionistic fuzzy n-normed linear space (IFnNLS) has been defined by Vijayabalaji et al. [38] in 2007. Some more recent work in similar context can be found in [5, 7, 9, 10, 11, 12, 14, 30, 32, 37].

The idea of statistical convergence was first introduced by Steinhaus [36] and Fast [15]. Karakus [21] studied statistical convergence on probabilistic normed spaces. Then Karakus et al. [22] generalized it on IFNLSs. The notion of I-convergence was initially introduced by Kostyrko et.al [24] as a generalization of statistical convergence which is based on the structure of the ideal I of subsets of natural numbers $\mathbb{N}$. Further, Kostyrko et.al [25] gave some of basic properties of I-convergence and deal with external I-limit points. Later on, Esi and Hazarika [14] introduced the concept of λ-ideal convergence in intuitionistic fuzzy 2-normed space. Recently Hazarika, Kumar and Guillén [19] studied another generalized ideal convergence in IFNLS.

An unambiguous new definition of convergence of sequences in IFnNLS was given in [34, 35] which is different from [38], and the notion of lacunary ideal convergence in IFNS was introduced in [8]. Combining these, we extend our work to introduce and study the notion of I_λ-convergence in an IFnNLS.

2. Preliminaries

First we collect some preliminaries to be used in this paper. Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and real numbers respectively.

Definition 2.1 ([18]). Let $n \in \mathbb{N}$ and X be a real linear space of dimension $d \geq n$ (d may be infinite). A real valued function $\|.\|$ on $\underbrace{X \times X \times \cdots \times X}_{n} = X^n$ is called an n-norm on X, if it satisfies the following properties:

(i) $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,
(ii) $\|x_1, x_2, \ldots, x_n\|$ is invariant under any permutation,
(iii) $\|x_1, x_2, \ldots, \alpha x_n\| = |\alpha| \|x_1, x_2, \ldots, x_n\|$ for any $\alpha \in \mathbb{R}$,
(iv) $\|x_1, x_2, \ldots, x_{n-1}, y + z\| \leq \|x_1, x_2, \ldots, x_{n-1}, y\| + \|x_1, x_2, \ldots, x_{n-1}, z\|$,

and the pair $(X, \|.\|)$ is called an n-normed linear space.

Definition 2.2 ([38]). An IFnNLS is the five-tuple $(X, \mu, \nu, *, \circ)$, where X is a linear space over a field F, $*$ is a continuous t-norm, $\circ$ is a continuous t-conorm, μ, ν are fuzzy sets on $X^n \times (0, \infty)$, μ denotes the degree of membership and ν denotes the degree of non-membership of $(x_1, x_2, \ldots, x_n, t) \in X^n \times (0, 1)$ satisfying the following conditions for every $(x_1, x_2, \ldots, x_n) \in X^n$ and $s, t > 0$:

(i) $\mu(x_1, x_2, \ldots, x_n, t) + \nu(x_1, x_2, \ldots, x_n, t) \leq 1$,
(ii) $\mu(x_1, x_2, \ldots, x_n, t) > 0$,
(iii) $\mu(x_1, x_2, \ldots, x_n, t) = 1$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,
(iv) $\mu(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$,
(v) $\mu(x_1, x_2, \ldots, cx_n, t) = \mu(x_1, x_2, \ldots, x_n, \frac{t}{|c|})$ if $c \neq 0, c \in F$,
(vi) $\mu(x_1, x_2, \ldots, x_n, s) * \mu(x_1, x_2, \ldots, x_n^{'}, t) \leq \mu(x_1, x_2, \ldots, x_n + x_n^{'}, s + t)$,
(vii) $\mu(x_1, x_2, \ldots, x_n, t) : (0, \infty) \to [0, 1]$ is continuous in t,
(viii) $\lim_{t\to\infty} \mu(x_1, x_2, \ldots, x_n, t) = 1$ and $\lim_{t\to 0} \mu(x_1, x_2, \ldots, x_n, t) = 0$,
(ix) $\nu(x_1, x_2, \ldots, x_n, t) < 1$,
(x) $\nu(x_1, x_2, \ldots, x_n, t) = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,
(xi) $\nu(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$,
(xii) $\nu(x_1, x_2, \ldots, cx_n, t) = \nu(x_1, x_2, \ldots, x_n, \frac{t}{|c|})$ if $c \neq 0, c \in F$,
(xiii) $\nu(x_1, x_2, \ldots, x_n, s) \circ \nu(x_1, x_2, \ldots, x_n^{'}, t) \geq \nu(x_1, x_2, \ldots, x_n + x_n^{'}, s + t)$,
(xiv) $\nu(x_1, x_2, \ldots, x_n, t) : (0, \infty) \to [0, 1]$ is continuous in t,
(xv) $\lim_{t\to\infty} \nu(x_1, x_2, \ldots, x_n, t) = 0$ and $\lim_{t\to 0} \nu(x_1, x_2, \ldots, x_n, t) = 1$.

Example 2.3 ([34]). Let $(X, \|.\|)$ be an n-normed linear space. Also let $a * b = ab$ and $a \circ b = \min\{a + b, 1\}$ for all $a, b \in [0, 1]$, $\mu(x_1, x_2, \ldots, x_n, t) = \frac{t}{t+\|x_1,x_2,\ldots,x_n\|}$ and $\nu(x_1, x_2, \ldots, x_n, t) = \frac{\|x_1,x_2,\ldots,x_n\|}{t+\|x_1,x_2,\ldots,x_n\|}$. Then $(X, \mu, \nu, *, \circ)$ is an IFnNLS.

Definition 2.4 ([34]). Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. We say that a sequence $x = \{x_k\}$ in X is convergent to $l \in X$ with respect to the intuitionistic fuzzy n-norm $(\mu, \nu)^n$ if, for every $\epsilon > 0$, $t > 0$ and $y_1, y_2, \ldots, y_{n-1} \in X$, there exists $k_0 \in \mathbb{N}$ such that $\mu(y_1, y_2, \ldots, y_{n-1}, x_k - L, t) > 1 - \epsilon$ and $\nu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) < \epsilon$ for all $k \geq k_0$. It is denoted by $(\mu, \nu)^n - \lim x = l$ or $x_k \overset{(\mu,\nu)^n}{\to} l$ as $k \to \infty$.

Definition 2.5 ([12])**.** Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. For $t > 0$, we define an open ball $B(x, r, t)$ with center at $x \in X$, radius $0 < r < 1$ and $y_1, y_2, \ldots, y_{n-1} \in X$ as

$$B(x, r, t) = \{y \in X : \mu(y_1, y_2, \ldots, y_{n-1}, y - x, t) > 1 - r \text{ and } \nu(y_1, y_2, \ldots, y_{n-1}, y - x, t) < r\}.$$

Definition 2.6 ([34])**.** Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. Then the sequence $x = \{x_k\}$ in X is called a Cauchy sequence with respect to the intuitionistic fuzzy n-norm $(\mu, \nu)^n$ if, for every $\epsilon > 0$, $t > 0$ and $y_1, y_2, \ldots, y_{n-1} \in X$, there exists $k_0 \in \mathbb{N}$ such that $\mu(y_1, y_2, \ldots, y_{n-1}, x_k - x_m, t) > 1 - \epsilon$ and $\nu(y_1, y_2, \ldots, y_{n-1}, x_k - x_m, t) < \epsilon$ for all $k, m \geq k_0$.

Definition 2.7 ([24])**.** Let X be a non-empty set. Then a family of sets $I \subset P(X)$ is called an ideal in X, if
(i) $\varnothing \in I$,
(ii) $A, B \in I$ implies $A \cup B \in I$,
(iii) for each $A \in I$ and $B \subset A$ we have $B \in I$,
where $P(X)$ is the power set of X.

Definition 2.8 ([24])**.** Let X be a non-empty set. Then a non-empty family of sets $F \subset P(X)$ is called a filter on X, if
(i) $\varnothing \notin F$,
(ii) $A, B \in F$ implies $A \cap B \in F$,
(iii) for each $A \in F$ and $B \supset A$, we have $B \in F$.

An ideal I is called non-trivial, if $I \neq \varnothing$ and $X \notin I$. A non-trivial ideal $I \subset P(X)$ is called an admissible ideal in X, if it contains all singletons, i.e., it contains $\{\{x\} : x \in X\}$.

Definition 2.9 ([36])**.** If K is a subset of $\mathbb{N}$, the set of natural numbers, then the natural density of K, denoted by $\delta(K)$, is given by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : k \in K\}\right|,$$

whenever the limit exists, where $|A|$ denotes the cardinality of the set A.

Definition 2.10 ([27])**.** Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to infinity such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. The generalized la Vallée-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in J_n} (x_k),$$

where $J_n = [n - \lambda_n + 1, n]$. A sequence $x = (x_k)$ is said to be (v, λ)-summable to a number l if $t_n(x) \longrightarrow l$ as $n \longrightarrow \infty$. If $\lambda_n = n$, then (V, λ)-summability reduces to $(C, 1)$-summability.

Definition 2.11 ([28])**.** A sequence $x = (x_k)$ is said to be λ-statistically convergent to the number l if for every $\epsilon > 0$,

$$\lim_{n\longrightarrow\infty} \frac{1}{\lambda_n} |\{k \in J_n : |x_k - l| \geq \varepsilon\}| = 0.$$

Let S_λ denote the set of all λ-statistically convergent sequences. If $\lambda_n = n$, then S_λ is the same as S.

Definition 2.12 ([33]). Let $I \subset 2^{\mathbb{N}}$ be a non-trivial ideal. A sequence $x = (x_k)$ is said to be $I - [V, \lambda]$-summability to a number l if, for every $\varepsilon > 0$

$$\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} |x_k - l| \geq \varepsilon\} \in I.$$

Through out the paper, we denote by I as admissible ideal of subsets of $\mathbb{N}$ and $\lambda = (\lambda_n)$ is a sequence as defined in Definition 2.9.

3. Main Results

Now we discuss our main results.

Definition 3.1. Let $I \subset 2^{\mathbb{N}}$ and let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. A sequence $x = \{x_k\}$ in X is said to be I_λ- convergent to $l \in X$ with respect to the intuitionistic fuzzy n-norm $(\mu, \nu)^n$ if, for every $\epsilon > 0$, $t > 0$ and $y_1, y_2, \ldots, y_{n-1} \in X$, we have

$$\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) \leq 1 - \epsilon \quad \text{or} \quad \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) \geq \epsilon\} \in I.$$

In this case, l is called the I_λ- limit of the sequence $x = (x_k)$ and we write $I_\lambda^{(\mu,\nu)^n} - \lim x = l$.

Example 3.2. Let $(\mathbb{R}, |\cdot|)$ denote the space of all real numbers with the usual n-norm and let $a * b = ab$ and $a \circ b = min\{a + b, 1\}$ for all $a, b \in [0, 1]$. For all $x_1, x_2, \ldots, x_{n-1} \in \mathbb{R}$ and every $t > 0$ consider $\mu(x_1, x_2, \ldots, x_n, t) = \frac{t}{t+|x_1,x_2,\ldots,x_n|}$ and $\nu(x_1, x_2, \ldots, x_n, t) = \frac{|x_1,x_2,\ldots,x_n|}{t+|x_1,x_2,\ldots,x_n|}$. Then $(\mathbb{R}, \mu, \nu, *, \circ)$ is an IFnNLS. If we take $I = \{A \subset \mathbb{N} : \delta(A) = 0\}$, where $\delta(A)$ denote natural density of the set A, then I is non trivial admissible ideal. Define a sequence $x = (x_k)$ as follows:

$$x_k = \begin{cases} 1, & \text{if } k = i^2, i \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Then for every $\epsilon \in (0, 1)$, for any $t > 0$ and $y_1, y_2, \ldots, y_{n-1} \in X$, the set

$$K(\epsilon, t) = \{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k, t) \leq 1 - \epsilon \quad \text{or} \quad \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_k, t) \geq \epsilon\}$$

will be a finite set. Thus $\delta(K(\epsilon, t)) = 0$. So $K(\epsilon, t) \in I$, i.e.,$I_\lambda^{(\mu,\nu)^n} - \lim x = 0$.

Lemma 3.3. *Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS and let $x = (x_k)$ be a sequence in X. Then for every $\varepsilon \in (0, 1)$, $t > 0$ and $y_1, y_2, \ldots, y_{n-1} \in X$ the following statements are equivalent:*

(1) $I_\lambda^{(\mu,\nu)^n} - \lim x = l$.
(2) $\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) \leq 1 - \varepsilon\} \in I$
and $\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) \geq \varepsilon\} \in I$.
(3) $\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) > 1 - \varepsilon$
and $\frac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) < \varepsilon\} \in F(I)$.
(3) $\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) > 1 - \varepsilon\} \in F(I)$
and $\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) < \varepsilon\} \in F(I)$.
(4) $I_\lambda - \lim \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) = 1$
and $I_\lambda - \lim \nu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) = 0$.

Theorem 3.4. *Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. If a sequence $x = (x_k)$ in X is I_λ-convergent to $l \in X$ with respect to the intuitionistic fuzzy n-norm (μ, ν), then $I_\lambda^{(\mu,\nu)^n} - \lim x$ is unique.*

Proof. If possible suppose, $I_\lambda^{(\mu,\nu)^n} - \lim x = l_1$ and $I_\lambda^{(\mu,\nu)^n} - \lim x = l_2$. For a fixed $\varepsilon \in (0,1)$, choose $\gamma \in (0,1)$ such that $(1-\gamma) * (1-\gamma) > 1 - \varepsilon$ and $\gamma \circ \gamma < \varepsilon$. Then for any $t > 0$ and $y_1, y_2, \ldots, y_{n-1} \in X$ define the following sets:

$$K_{\mu,1}(\gamma, t) = \{n \in \mathbb{N} : \tfrac{1}{\lambda_n} \textstyle\sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l_1, \tfrac{t}{2}) \leq 1 - \gamma\},$$

$$K_{\mu,2}(\gamma, t) = \{n \in \mathbb{N} : \tfrac{1}{\lambda_n} \textstyle\sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l_2, \tfrac{t}{2}) \leq 1 - \gamma\},$$

$$K_{\nu,1}(\gamma, t) = \{n \in \mathbb{N} : \tfrac{1}{\lambda_n} \textstyle\sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_k - l_1, \tfrac{t}{2}) \geq \gamma\},$$

$$K_{\nu,2}(\gamma, t) = \{n \in \mathbb{N} : \tfrac{1}{\lambda_n} \textstyle\sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_k - l_2, \tfrac{t}{2}) \geq \gamma\}.$$

Now $I_\lambda^{(\mu,\nu)^n} - \lim x = l_1$ implies that (using Lemma 3.3) $K_{\mu,1}(\gamma, t) \in I$ and $K_{\nu,1}(\gamma, t) \in I$ for all $t > 0$. Also using $I_\lambda^{(\mu,\nu)^n} - \lim x = l_2$, we get $K_{\mu,2}(\gamma, t) \in I$ and $K_{\nu,2}(\gamma, t) \in I$ for all $t > 0$.

Again suppose $K_{\mu,\nu}(\gamma, t) = (K_{\mu,1}(\gamma, t) \cup K_{\mu,2}(\gamma, t)) \cap (K_{\nu,1}(\gamma, t) \cup K_{\nu,2}(\gamma, t))$. Then $K_{\mu,\nu}(\gamma, t) \in I$. This implies that its complement $K_{\mu,\nu}^C(\gamma, t)$ is a non-empty set in $F(I)$. If we take $n \in K_{\mu,\nu}^C(\gamma, t)$, first consider the case $n \in (K_{\mu,1}^C(\gamma, t) \cap K_{\mu,2}^C(\gamma, t))$. Then we have

$$\frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l_1, \frac{t}{2}) > 1 - \gamma$$

and

$$\frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l_2, \frac{t}{2}) > 1 - \gamma.$$

Clearly, we will get a $p \in \mathbb{N}$ such that

$$\begin{aligned} & \mu(y_1, y_2, \ldots, y_{n-1}, x_p - l_1, \frac{t}{2}) \\ > & \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l_1, \frac{t}{2}) \\ > & 1 - \gamma \end{aligned}$$

and

$$\begin{aligned} & \mu(y_1, y_2, \ldots, y_{n-1}, x_p - l_2, \frac{t}{2}) \\ > & \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l_2, \frac{t}{2}) \\ > & 1 - \gamma. \end{aligned}$$

(e.g., consider $\max\{\mu(y_1, y_2, \ldots, y_{n-1}, x_k - l_1, \frac{t}{2}), \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l_2, \frac{t}{2})$ $: k \in J_n\}$ and choose that k as p for which the maximum occurs).
Then we have

$$\begin{aligned} & \mu(y_1, y_2, \ldots, y_{n-1}, l_1 - l_2, t) \\ \geq & \mu(y_1, y_2, \ldots, y_{n-1}, x_p - l_1, \frac{t}{2}) * \mu(y_1, y_2, \ldots, y_{n-1}, x_p - l_2, \frac{t}{2}) \\ > & (1 - \gamma) * (1 - \gamma) \\ > & 1 - \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $\mu(y_1, y_2, \ldots, y_{n-1}, l_1 - l_2, t) = 1$ for all $t > 0$, which implies that $l_1 = l_2$.

Again if $n \in (K^C_{\nu,1}(r, t) \cap K^C_{\nu,2}(r, t))$, then using a similar technique it can be proved that

$$\nu(y_1, y_2, \ldots, y_{n-1}, l_1 - l_2, t) < \varepsilon,$$

for all $t > 0$ and arbitrary $\varepsilon > 0$. Thus $l_1 = l_2$. This proves that $I_\lambda^{(\mu,\nu)^n} - \lim x$ is unique. □

The following result shows that the collection of all ideal convergence sequences in an IFnNLS is closed under addition and scalar multiplication.

Theorem 3.5. *Let X be an IFnNLS. Then*

(1) *If $I_\lambda^{(\mu,\nu)^n} - \lim x = l_1$ and $I_\lambda^{(\mu,\nu)^n} - \lim y = l_2$, then $I_\lambda^{(\mu,\nu)^n} - \lim(x + y) = l_1 + l_2$.*

(2) *If $I_\lambda^{(\mu,\nu)^n} - \lim x = l$ and $\alpha \in \mathbb{R}$, then $I_\lambda^{(\mu,\nu)^n} - \lim \alpha x = \alpha l$.*

(3) *If $I_\lambda^{(\mu,\nu)^n} - \lim x = l_1$ and $I_\lambda^{(\mu,\nu)^n} - \lim y = l_2$, then $I_\lambda^{(\mu,\nu)^n} - \lim(x - y) = l_1 - l_2$.*

Proof. (1) For a given $\varepsilon > 0$, choose $\gamma > 0$ such that $(1 - \gamma) * (1 - \gamma) > 1 - \varepsilon$. Now for any $t > 0$, we define the following sets:

$$K_{\mu,1}(\gamma, t) = \{k \in \mathbb{N} : \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l_1, \frac{t}{2}) > 1 - \gamma\}$$

and

$$K_{\mu,2}(\gamma, t) = \{k \in \mathbb{N} : \mu(y_1, y_2, \ldots, y_{n-1}, y_k - l_2, \frac{t}{2}) > 1 - \gamma\}.$$

Since $I_\lambda^{(\mu,\nu)^n} - \lim x = l$, clearly $K_{\mu,1}(\gamma, t) \in F(I)$. Similarly, $K_{\mu,2}(\gamma, t) \in F(I)$. Let $K_\mu(\gamma, t) = K_{\mu,1}(\gamma, t) \cap K_{\mu,2}(\gamma, t)$. Then $K_\mu(\gamma, t) \in F(I)$. Now if $k \in K_\mu(\gamma, t)$, we have

$$\begin{aligned}&\mu(y_1, y_2, \ldots, y_{n-1}, x_k + y_k - (l_1 + l_2), t)\\ &\geq \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l, \frac{t}{2}) * \mu(y_1, y_2, \ldots, y_{n-1}, y_k - l_2, \frac{t}{2})\\ &> (1-\gamma) * (1-\gamma)\\ &> 1-\varepsilon.\end{aligned}$$

This shows that $\{k \in I_n : \mu(y_1, y_2, \ldots, y_{n-1}, x_k + y_k - (l_1 + l_2), t) \leq 1 - \epsilon\} \in I$. Thus $I_\lambda^{(\mu,\nu)^n} - \lim(x+y) = l_1 + l_2$.

(2) Let $\alpha = 0$. Then for every $r > 0$, $t > 0$, $y_1, y_2, \ldots, y_{n-1} \in X$, there exists $n_0 = 1$ such that

$$\mu(y_1, y_2, \ldots, y_{n-1}, 0x_k - 0l, t) = 1 > 1 - r$$

and

$$\nu(y_1, y_2, \ldots, y_{n-1}, 0x_k - 0l, t) = 0 < r,$$

for all $k \geq n_0$. Thus $(\mu, \nu)^\lambda - \lim 0x = 0l$. This implies that $I_\lambda^{(\mu,\nu)^n} - \lim 0x = 0l$.

Now let $\alpha (\neq 0) \in \mathbb{R}$ and let

$$K_n(\varepsilon, t) = \{k \in \mathbb{N} : \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l, \frac{t}{|\alpha|}) > 1 - \varepsilon\}.$$

Since $I_\lambda^{(\mu,\nu)^n} - \lim x = l$, we have $K_n(\varepsilon, t) \in F(I)$. Now if $k \in K_n(\varepsilon, t)$, then

$$\mu(y_1, y_2, \ldots, y_{n-1}, \alpha x_k - \alpha l, t) = \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l, \frac{t}{|\alpha|}) > 1 - \varepsilon.$$

Thus $\{k \in I_n : \mu(y_1, y_2, \ldots, y_{n-1}, \alpha x_k - \alpha l, t) \leq 1 - \varepsilon\} \in I$. So $I_\lambda^{(\mu,\nu)^n} - \lim \alpha x = \alpha l$.

(3) The proof follows from (1) and (2). □

Theorem 3.6. *Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS and let $x = (x_k)$ be a sequence in X. Let I be a non trivial ideal in $\mathbb{N}$. If there is a $I_\lambda^{(\mu,\nu)^n}$-convergent sequence $y = (y_k)$ in X such that $\{n \in \mathbb{N} : y_k \neq x_k, k \in J_n\} \in I$, then x is also $I_\lambda^{(\mu,\nu)^n}$-convergent to the same limit.*

Proof. Suppose that $\{k \in \mathbb{N} : y_k \neq x_k\} \in I$ and $I_\lambda^{(\mu,\nu)^n} - y = l$. Then for every $\varepsilon \in (0, 1)$, $t > 0$ and $y_1, y_2, \ldots, y_{n-1} \in X$, we have

$$\{n \in \mathbb{N} : \tfrac{1}{\lambda_n} \textstyle\sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, y_k - l, t) \leq 1 - \varepsilon \text{ or } \tfrac{1}{\lambda_n} \textstyle\sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, y_k - l, t) \geq \varepsilon\} \in I.$$

For every $0 < \varepsilon < 1$, $t > 0$ and $y_1, y_2, \ldots, y_{n-1} \in X$, we have

$$\begin{aligned}&\{n \in \mathbb{N} : \tfrac{1}{\lambda_n} \textstyle\sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) \leq 1 - \varepsilon \text{ or } \tfrac{1}{\lambda_n} \textstyle\sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) \geq \varepsilon\}\\ &\subseteq \{n \in \mathbb{N} : y_k \neq x_k, \text{ for some } k \in J_n\}\\ &\cup \{n \in \mathbb{N} : \tfrac{1}{\lambda_n} \textstyle\sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, y_k - l, t) \leq 1 - \varepsilon\end{aligned}$$

or $\frac{1}{\lambda_n}\sum_{k\in J_n}\nu(y_1,y_2,\dots,y_{n-1},y_k-l,t)\geq\varepsilon\}$.
As both the right-hand side member of the above equation are in I, we have

$$\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in J_n}\mu(y_1,y_2,\dots,y_{n-1},x_k-l,t)\leq 1-\varepsilon$$
$$\text{or }\frac{1}{\lambda_n}\sum_{k\in J_n}\nu(y_1,y_2,\dots,y_{n-1},x_k-l,t)\geq\varepsilon\}\in I.$$

Thus x is $I_\lambda^{(\mu,\nu)^n}$-convergent. □

4. Lambda-convergence in IFnNLS

In this section, we introduce the concept of λ-convergence in IFnNLS and establish its relation with I_λ-convergence.

Definition 4.1. Let $(X,\mu,\nu,*,\circ)$ be an IFnNLS. A sequence $x=\{x_k\}$ in X is λ-convergent to $l\in X$ with respect to the intuitionistic fuzzy n-norm $(\mu,\nu)^n$, if for every $t>0$, $\varepsilon\in(0,1)$ and $y_1,y_2,\dots,y_{n-1}\in X$, there exists $n_0\in\mathbb{N}$ such that

$$\frac{1}{\lambda_n}\sum_{k\in J_n}\mu(y_1,y_2,\dots,y_{n-1},x_k-l,t)>1-\varepsilon \text{ and } \frac{1}{\lambda_n}\sum_{k\in J_n}\nu(y_1,y_2,\dots,y_{n-1},x_k-l,t)<\varepsilon$$

for all $n\geq n_0$. In this case, we write $(\mu,\nu)^n_\lambda-\lim x=l$.

Theorem 4.2. *Let $(X,\mu,\nu,*,\circ)$ be an IFnNLS and $x=\{x_k\}$ be a sequence in X. If $x=\{x_k\}$ is λ-convergent with respect to the intuitionistic fuzzy n-norm $(\mu,\nu)^n$, then $(\mu,\nu)^n_\lambda-\lim x$ is unique.*

Proof. Let $(\mu,\nu)^n_\lambda-\lim x=l_1$ and $(\mu,\nu)^n_\lambda-\lim x=l_2$ $(l_1\neq l_2)$. For a fixed $\varepsilon>0$, choose $\gamma\in(0,1)$ such that $(1-\gamma)*(1-\gamma)>1-\varepsilon$ and $\gamma\circ\gamma<\varepsilon$. Now, for every $t>0$, $y_1,y_2,\dots,y_{n-1}\in X$ there exists $n_1\in\mathbb{N}$ such that

$$\frac{1}{\lambda_n}\sum_{k\in J_n}\mu(y_1,y_2,\dots,y_{n-1},x_k-l_1,t)>1-\varepsilon$$

and

$$\frac{1}{\lambda_n}\sum_{k\in J_n}\nu(y_1,y_2,\dots,y_{n-1},x_k-l_1,t)<\varepsilon,$$

for all $n\geq n_1$. Also there exists $n_2\in\mathbb{N}$ such that

$$\frac{1}{\lambda_n}\sum_{k\in J_n}\mu(y_1,y_2,\dots,y_{n-1},x_k-l_2,t)>1-\varepsilon$$

and

$$\frac{1}{\lambda_n}\sum_{k\in J_n}\nu(y_1,y_2,\dots,y_{n-1},x_k-l_2,t)<\varepsilon,$$

for all $n\geq n_2$.

Consider $n_0=\max\{n_1,n_2\}$. Then for $n\geq n_0$, we will get a $p\in\mathbb{N}$ such that

$$\mu(y_1,y_2,\dots,y_{n-1},x_p-l_1,\tfrac{t}{2})>\tfrac{1}{\lambda_n}\textstyle\sum_{k\in J_n}\mu(y_1,y_2,\dots,y_{n-1},x_k-l_1,\tfrac{t}{2})$$
$$>1-\gamma$$

and

$$\mu(y_1, y_2, \ldots, y_{n-1}, x_p - l_2, \tfrac{t}{2}) > \tfrac{1}{\lambda_n}\sum_{k\in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l_2, \tfrac{t}{2}) > 1-\gamma.$$

Thus we have

$$\begin{aligned}
&\mu(y_1, y_2, \ldots, y_{n-1}, l_1 - l_2, t)\\
&\geq \mu(y_1, y_2, \ldots, y_{n-1}, x_p - l_1, \frac{t}{2}) * \mu(y_1, y_2, \ldots, y_{n-1}, x_p - l_2, \frac{t}{2})\\
&> (1-\gamma) * (1-\gamma)\\
&> 1-\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $\mu(y_1, y_2, \ldots, y_{n-1}, l_1 - l_2, t) = 1$ for all $t > 0$, which implies that $l_1 = l_2$. By using similar technique, it can be proved that

$$\nu(y_1, y_2, \ldots, y_{n-1}, l_1 - l_2, t) < \varepsilon,$$

for all $t > 0$ and arbitrary $\varepsilon > 0$. So $l_1 = l_2$. Hence $(\mu,\nu)^n_\lambda - \lim x$ is unique. □

The following theorem shows that λ-convergence is stronger than I_λ-convergence.

Theorem 4.3. *Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS and let $x = (x_k)$ in X. If $(\mu,\nu)^n_\lambda - \lim x = l$, then $I^{(\mu,\nu)^n}_\lambda - \lim x = l$.*

Proof. Suppose that $(\mu,\nu)^n_\lambda - \lim x = l$. Then for every $t > 0$, $\varepsilon \in (0,1)$ and $y_1, y_2, \ldots, y_{n-1} \in X$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\lambda_n}\sum_{k\in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) > 1-\varepsilon$$

and

$$\frac{1}{\lambda_n}\sum_{k\in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) < \varepsilon,$$

for all $n \geq n_0$. Thus, we have

$$\begin{aligned}
A = \{n\in\mathbb{N} : &\tfrac{1}{\lambda_n}\textstyle\sum_{k\in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) \leq 1-\varepsilon\\
&\text{or } \tfrac{1}{\lambda_n}\textstyle\sum_{k\in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) \geq \varepsilon\}\\
\subseteq \{1, 2, &\ldots, n_0 - 1\}.
\end{aligned}$$

But I being admissible, we have $A \in I$. sO $I^{(\mu,\nu)^n}_\lambda - \lim x = l$. □

Theorem 4.4. *Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS and $x = \{x_k\}$ a sequence in X. If $(\mu,\nu)^n_\lambda - \lim x = l$, then there exists a subsequence $\{x_{m_k}\}$ of $x = (x_k)$ such that $(\mu,\nu)^n - \lim x_{m_k} = l$.*

Proof. Let $(\mu,\nu)^n_\lambda - \lim x = l$. Then for every $t > 0$ and $\varepsilon \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\lambda_n}\sum_{k\in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) > 1-\epsilon$$

and

$$\frac{1}{\lambda_n}\sum_{k\in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) < \varepsilon,$$

for all $n \geq n_0$. Clearly, for each $n \geq n_0$, we can select an $m_k \in J_n$ such that

$$\mu(y_1, y_2, \ldots, y_{n-1}, x_{m_k} - l, t) > \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) > 1 - \varepsilon$$

and

$$\nu(y_1, y_2, \ldots, y_{n-1}, x_{m_k} - l, t) < \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) < \varepsilon.$$

It follows that $(\mu, \nu)^n - \lim x_{m_k} = l$. □

Theorem 4.5. *Let X be an IFnNLS and I be a nontrivial ideal of $\mathbb{N}$. If a sequence $x = \{x_k\}$ is λ-convergent in X and $y = \{y_k\}$ is a sequence in X such that $\{n \in \mathbb{N} : x_k \neq y_k$ for some $k \in J_n\} \in I$, then y is λ-convergent to the same limit.*

Proof. Let $\varepsilon \in (0, 1)$, $t > 0$, $y_1, y_2, \ldots, y_{n-1} \in X$,$\{n \in \mathbb{N} : x_k \neq y_k$ for some $k \in J_n\} \in I$ and $(\mu, \nu)^n_\lambda - \lim x = l$. Then

$$\frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) > 1 - \varepsilon$$

and

$$\frac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) < \varepsilon\},$$

for all $n \geq n_0$. Thus

$$\{n \in \mathbb{N} : \tfrac{1}{\lambda_n} \textstyle\sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) > 1 - \varepsilon \text{ and } \tfrac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) < \varepsilon\} \notin I.$$

On one hand,

$$\{n \in \mathbb{N} : \tfrac{1}{\lambda_n} \textstyle\sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) \leq 1 - \varepsilon \text{ or } \tfrac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) \geq \varepsilon\} \in I.$$

Now, for every $0 < \varepsilon < 1$, $t > 0$ and $y_1, y_2, \ldots, y_{n-1} \in X$, we have

$$\begin{aligned}&\{n \in \mathbb{N} : \tfrac{1}{\lambda_n} \textstyle\sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, y_k - l, t) \leq 1 - \varepsilon \text{ or } \tfrac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, y_k - l, t) \geq \varepsilon\} \\ &\subseteq \{n \in \mathbb{N} : x_k \neq y_k, \text{ for some } k \in J_n\} \\ &\quad \cup \{n \in \mathbb{N} : \tfrac{1}{\lambda_n} \textstyle\sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) \leq 1 - \varepsilon \text{ or } \tfrac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) \geq \varepsilon\}.\end{aligned}$$

As both the right-hand side member of the above equation are in I, we have

$$\{n \in \mathbb{N} : \tfrac{1}{\lambda_n} \textstyle\sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, y_k - l, t) \leq 1 - \varepsilon \text{ or } \tfrac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, y_k - l, t) \geq \varepsilon\} \in I.$$

So y is $(\mu, \nu)^n_\lambda$-convergent. □

5. Limit point and cluster point in IFnNLS

Limit points and cluster points are essential concepts in the study of closedness of a set. Here we study the analogous concepts in an IFnNLS.

Definition 5.1. Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. $l \in X$ is called a limit point of of the sequence $x = \{x_k\}$ with respect to the intuitionistic fuzzy n-norm $(\mu, \nu)^n$ provided that there is a subsequence of x that converges to l with respect to the intuitionistic fuzzy n-norm $(\mu, \nu)^n$.

Definition 5.2. Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS and let $x = (x_k)$ be a sequence in X. Then

(i) An element $l \in X$ is said to be an I_λ-limit point of $x = (x_k)$, if there is a set $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \subset \mathbb{N}$ such that $M' = \{n \in \mathbb{N} : m_k \in J_n\} \notin I$ and $(\mu, \nu)^n_\lambda - \lim x_{m_k} = l$.

(ii) An element $l \in X$ is said to be an I_λ-cluster point of $x = (x_k)$, if for every $t > 0$, $\varepsilon \in (0, 1)$ and $y_1, y_2, \ldots, y_{n-1} \in X$, we have

$\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) > 1 - \epsilon$
or $\frac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) < \epsilon\} \notin I$.

Let $\Lambda^I_{(\mu,\nu)^\lambda}(x)$ denote the set of all I^λ-limit points and $\Gamma^I_{(\mu,\nu)^\lambda}(x)$ denote the set of all I^λ-cluster points in X, respectively.

Theorem 5.3. *Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. Then for each sequence $x \in X$, $\Lambda^I_{(\mu,\nu)^\lambda}(x) \subset \Gamma^I_{(\mu,\nu)^\lambda}(x)$.*

Proof. Let $l \in \Lambda^I_{(\mu,\nu)^\lambda}(x)$. Then there exists a set $M \subset \mathbb{N}$ such that $M' \notin I$, where M and M' are as in the Definition 5.2, satisfies $(\mu, \nu)^n_\lambda - \lim x_{m_k} = l$. Thus for every $t > 0$, $\epsilon \in (0, 1)$ and $y_1, y_2, \ldots, y_{n-1} \in X$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_{m_k} - l, t) > 1 - \varepsilon$$

and

$$\frac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_{m_k} - l, t) < \varepsilon,$$

for all $n \geq n_0$. So

$B = \{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) > 1 - \varepsilon$
and $\frac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_k - l, t) < \varepsilon\}$
$\supseteq M' \setminus \{m_1, m_2, \ldots m_{k_0}\}$.

Now, with I being admissible, we must have $M' \setminus \{m_1, m_2, \ldots m_{k_0}\} \notin I$ and as such $B \notin I$. Hence $l \in \Gamma^I_{(\mu,\nu)^\lambda}(x)$, i.e., $\Lambda^I_{(\mu,\nu)^\lambda}(x) \subset \Gamma^I_{(\mu,\nu)^\lambda}(x)$. □

Theorem 5.4. *Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. For each sequence $x = (x_k)$ in X, the set $\Gamma^I_{(\mu,\nu)^\lambda}(x)$ is closed set in X with respect to the usual topology induced by the intuitionistic fuzzy norm $(\mu, \nu)^n$.*

Proof. Let $y \in \overline{\Gamma^I_{(\mu,\nu)^\lambda}(x)}$. Take $t > 0$ and $\varepsilon \in (0, 1)$. Then there exists $l_0 \in \Gamma^I_{(\mu,\nu)^\lambda}(x) \cap B(y, \varepsilon, t)$. Choose $\delta > 0$ such that $B(l_0, \delta, t) \subset B(y, \varepsilon, t)$. We have

$G = \{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - y, t) > 1 - \varepsilon$
and $\frac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_k - y, t) < \varepsilon\}$
$\supseteq \{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l_0, t) > 1 - \delta$
and $\frac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_k - l_0, t) < \delta\}$
$= H$.

Thus $H \notin I$. So $G \notin I$. Hence $y \in \Gamma^I_{(\mu,\nu)^\lambda}(x)$. Therefore $\Gamma^I_{(\mu,\nu)^\lambda}(x)$ is closed set in X. □

Theorem 5.5. *Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS and let $x = (x_k)$ in X. Then the following statements are equivalent:*

(1) *l is a I_λ-limit point of x.*

(2) *There exists two sequences y_k and z_k in X such that $x = y_k + z_k$ and $(\mu, \nu)^n_\lambda - \lim y_k = l$ and $\{n \in \mathbb{N} : k \in J_n, z_k \neq \theta\} \in I$, where θ is the zero element of X.*

Proof. Suppose that (1) holds. Then there exist sets M and M' as in Definition 5.2 such that $M' \notin I$ and $(\mu, \nu)^n_\lambda - \lim x_{m_k} = l$. Define the sequences y and z as follows:

$$y_k = \begin{cases} x_k, & \text{if } k \in J_n; n \in M' \\ l, & \text{otherwise.} \end{cases}$$

and

$$z_k = \begin{cases} \theta, & \text{if } k \in J_n; n \in M' \\ x_k - l, & \text{otherwise.} \end{cases}$$

We consider the case $k \in J_n$ such that $n \in \mathbb{N} - M'$. Then for each $\epsilon \in (0, 1)$, $t > 0$ and $y_1, y_2, \ldots, y_{n-1} \in X$, we have

$$\mu(y_1, y_2, \ldots, y_{n-1}, y_k - l, t) = 1 > 1 - \epsilon$$

and

$$\nu(y_1, y_2, \ldots, y_{n-1}, y_k - l, t) = 0 < \epsilon.$$

Thus

$$\frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, y_k - l, t) = 1 > 1 - \varepsilon$$

and

$$\frac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, y_k - l, t) = 0 < \varepsilon.$$

So $(\mu, \nu)^n_\lambda - \lim y = l$ and $\{n \in \mathbb{N} : k \in J_n, z_k \neq \theta\} \in I$.

Suppose that (2) holds. Let $M' = \{n \in \mathbb{N} : k \in J_n, z_k = \theta\}$. Then, clearly $M' \in F(I)$ and so it is an infinite set. Construct the set $M = \{m_1 < m_2 < \ldots m_k < \ldots\} \subset \mathbb{N}$ such that $m_k \in J_n$ and $z_{m_k} = \theta$. Since $x_{m_k} = y_{m_k}$ and $(\mu, \nu)^n_\lambda - \lim y = l$, we obtain $(\mu, \nu)^n_\lambda - \lim x_{m_k} = l$. □

6. Cauchy sequences in IFnNLS

Here we define some variants of Cauchy sequences and give a positive answer to the open problem mentioned in [8] that every I_λ-convergent sequence is I_λ-Cauchy in the new setting of IFnNLS.

Definition 6.1. Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. A sequence $x = (x_k)$ in X is said to be λ-Cauchy sequence with respect to the intuitionistic fuzzy norm $(\mu, \nu)^n$, if for every $t > 0$, $\varepsilon \in (0, 1)$ and $y_1, y_2, \ldots, y_{n-1} \in X$, there exist $n_0, m \in \mathbb{N}$ satisfying

$$\frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - x_m, t) > 1 - \varepsilon$$

and

$$\frac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_k - x_m, t) < \varepsilon,$$

for all $n \geq n_0$.

Definition 6.2. Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. A sequence $x = (x_k)$ in X is said to be I_λ-Cauchy sequence with respect to the intuitionistic fuzzy norm $(\mu, \nu)^n$, if for every $t > 0$, $y_1, y_2, \ldots, y_{n-1} \in X$ and $\varepsilon \in (0, 1)$ there exist $m \in \mathbb{N}$ satisfying
$\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - x_m, t) > 1 - \varepsilon$
and $\frac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_k - x_m, t) < \varepsilon\} \in F(I)$.

Definition 6.3. Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. A sequence $x = (x_k)$ in X is said to be I^*_λ-Cauchy sequence with respect to the intuitionistic fuzzy norm $(\mu, \nu)^n$, if there exists a set $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \subset \mathbb{N}$ such that the set $M^{'} = \{n \in \mathbb{N} : m_k \in J_n\} \in F(I)$ and the subsequence (x_{m_k}) of $x = (x_k)$ is a Cauchy sequence with respect to the intuitionistic fuzzy norm $(\mu, \nu)^n$.

Theorem 6.4. *Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. If a sequence $x = (x_k)$ is I_λ-convergent with respect to $(\mu, \nu)^n$, then it is I_λ-Cauchy.*

Proof. Suppose that $x = (x_k)$ be a I_λ-convergent sequence converging to l. For a given $\varepsilon > 0$, choose $\gamma > 0$ such that $(1 - \gamma) * (1 - \gamma) > 1 - \varepsilon$ and $\gamma \circ \gamma < \varepsilon$. Then for any $t > 0$ and $y_1, y_2, \ldots, y_{n-1} \in X$, define

$$K_\mu(\gamma, t) = \{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l, \frac{t}{2}) > 1 - \gamma\}$$

and

$$K_\nu(\gamma, t) = \{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_1, y_2, \ldots, y_{n-1}, x_k - l, \frac{t}{2}) < \gamma\}.$$

Thus $K_\mu(\gamma, t) \in F(I)$ and $K_\nu(\gamma, t) \in F(I)$.

Let $K(\gamma, t) = K_\mu(\gamma, t) \cap K_\nu(\gamma, t)$. Then $K(\gamma, t) \in F(I)$. If $n \in K(\gamma, t)$ and we choose a fixed $m \in K(\gamma, t)$, then

$$\begin{aligned} &\mu(y_1, y_2, \ldots, y_{n-1}, x_k - x_m, t) \\ &\geq \mu(y_1, y_2, \ldots, y_{n-1}, x_k - l, \frac{t}{2}) * \mu(y_1, y_2, \ldots, y_{n-1}, x_m - l, \frac{t}{2}) \\ &> (1 - \gamma) * (1 - \gamma) \\ &> 1 - \varepsilon. \end{aligned}$$

This clearly implies that

$$\frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_1, y_2, \ldots, y_{n-1}, x_k - x_m, t) > 1 - \varepsilon.$$

Also,

$$\begin{aligned} &\nu(y_1, y_2, \ldots, y_{n-1}, x_k - x_m, t) \\ &\leq \nu(y_1, y_2, \ldots, y_{n-1}, x_k - l, \frac{t}{2}) \circ \nu(y_1, y_2, \ldots, y_{n-1}, x_m - l, \frac{t}{2}) \\ &< \gamma \circ \gamma \\ &< \varepsilon. \end{aligned}$$

So,

$$\frac{1}{\lambda_n}\sum_{k\in J_n}\nu(y_1,y_2,\dots,y_{n-1},x_k-x_m,t)<\varepsilon.$$

Hence

$$\{n\in\mathbb{N}:\tfrac{1}{\lambda_n}\textstyle\sum_{k\in J_n}\mu(y_1,y_2,\dots,y_{n-1},x_k-x_m,t)>1-\varepsilon$$
$$\text{and } n\in\mathbb{N}:\tfrac{1}{\lambda_n}\textstyle\sum_{k\in J_n}\nu(y_1,y_2,\dots,y_{n-1},x_k-x_m,t)<\varepsilon\}\in F(I).$$

Therefore $x=(x_k)$ is I_λ-Cauchy. □

The following is an analogue of Theorem 4.3 in the sense that λ-Cauchyness is stronger than I_λ-Cauchyness of sequences in IFnNLS whose proof follows likewise.

Theorem 6.5. *Let $(X,\mu,\nu,*,\circ)$ be an IFnNLS. If a sequence $x=(x_k)$ in X is λ-Cauchy with respect to $(\mu,\nu)^n$, then it is I_λ-Cauchy with respect to same.*

The proof of the following theorems can be easily established using definitions.

Theorem 6.6. *Let $(X,\mu,\nu,*,\circ)$ be an IFnNLS. If a sequence $x=(x_k)$ in X is λ-Cauchy with respect to $(\mu,\nu)^n$, then there is a subsequence of $x=(x_k)$ which is ordinary Cauchy sequence with respect to the same norm.*

Theorem 6.7. *Let $(X,\mu,\nu,*,\circ)$ be an IFnNLS. If a sequence $x=(x_k)$ is I^*_λ-Cauchy, then it is I_λ-Cauchy as well.*

7. Conclusions

In this paper, the concept of I_λ-convergence has been introduced in IFnNLS and based on this concept some existing results are extended and some new results are established. We have also introduced the concept of I_λ-Cauchy sequences and established the relation between I_λ-convergence and I_λ-Cauchy. In establishing most of the proofs, different approach than their classical analogues has been adopted. The results obtained in this paper are more general than the corresponding results for normed spaces.

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