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$(\alpha, \beta)_T$-fuzzy H_vMV-ideals

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Abstract. In this paper, based on the concepts of belongingness and quasi-coincidence, new types of fuzzy H_vMV-ideals with respect to a t-norm are introduced. Some properties, characterizations and equivalent conditions are given. The connections among them are obtained, as well.

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1. Introduction

Chang [5] introduced the concept of MV-algebra as an algebraic proof of the completeness theorem for $\aleph_0$-valued Łukasiewicz propositional calculus. After that many mathematicians have worked on MV-algebras and obtained significant results. Tianbang [20] proved that MV-algebras and lattice implication algebras are categorically equivalent. Mundici [18] proved that MV-algebras and bounded commutative BCK-algebras are also categorically equivalent.

The hyperstructure theory (called also multialgebras) was introduced by Marty [16]. Around the 40's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia and Japan. Hyperstructures have many applications to several sectors of both pure and applied sciences. There are applications to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, combinatorics, codes, artificial intelligence and probabilities. In [6] a wealth of applications can be found.

Recently, Ghorbani et al. [8] applied the hyperstructures to MV-algebras and introduced the concept of hyper MV-algebra and investigated some related results. Torkzadeh et al. [21] introduced the concept of hyper MV-ideal and Jun et al. [13] introduced the concept of hyper MV-deductive system and gave some related results.

H_v-structures were introduced by Vougiouklis [22, 23]. The concept of H_v-structure constitute a generalization of the well-known algebraic hyperstructures (hypergroup, hyperring, hypermodule and so on). Actually some axioms concerning the above hyperstructures such as the associative law, the distributive law and so on are replaced by their corresponding weak axioms. The reader will find in [22] some basic definitions and theorems about H_v-structures. Since then the study of H_v-structure theory has been pursued in many directions by Vougiouklis, Davvaz, Spartalis and others.

Recently, H_v-structures have applied to MV-algebras and the concept of $\mathsf{H}_v\mathsf{MV}$-algebra has introduced [1, 2]. The concepts of $\mathsf{H}_v\mathsf{MV}$-subalgebra, $\mathsf{H}_v\mathsf{MV}$-ideal and weak $\mathsf{H}_v\mathsf{MV}$-ideal were defined as well, and some properties and the connections between them were given. Also, quotient structure of an $\mathsf{H}_v\mathsf{MV}$-algebra have studied and some homomorphism theorems have given.

After Zadeh [24] introduced the concept of a fuzzy set, many authors applied it to algebraic structures such as groups, rings and so on. Hoo et al. [9, 10, 11, 12] applied fuzzy sets to MV-algebras and introduced some types of fuzzy ideals and obtain some related results. Jun et al. [14] introduced fuzzy hyper MV-deductive systems and investigated their properties. The present author introduced fuzzy $\mathsf{H}_v\mathsf{MV}$-ideals and fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideals of an $\mathsf{H}_v\mathsf{MV}$-algebra [3] and investigated their properties and the connection between them. He also obtained a characterization for fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal generated by a fuzzy set. Many authors have worked on fuzzy algebraic structures based on the concepts of 'belongingness' and 'quasi-coincidence' (see, for example [4, 7, 17, 19]).

In this paper, the notions of $(\alpha,\beta)_T$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideals and $(\alpha,\beta)_T$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideals are introduced. Their properties and connections between them are investigated, as well. Many theorems to characterize these fuzzy ideals are given. Also, some equivalent conditions together with suitable examples are given.

2. Preliminaries

This section is devoted to give some preliminaries from the literature. For more details we refer to the references [1, 2, 15].

Definition 2.1. An $\mathsf{H}_v\mathsf{MV}$-*algebra* is a nonempty set H endowed with a binary hyper operation '$\oplus$', a unary operation '*' and a constant '0' satisfying the following conditions:

($\mathsf{H}_v\mathsf{MV}1$) $x \oplus (y \oplus z) \cap (x \oplus y) \oplus z \neq \varnothing$ (weak associativity),
($\mathsf{H}_v\mathsf{MV}2$) $(x \oplus y) \cap (y \oplus x) \neq \varnothing$ (weak commutativity),
($\mathsf{H}_v\mathsf{MV}3$) $(x^*)^* = x$,
($\mathsf{H}_v\mathsf{MV}4$) $(x^* \oplus y)^* \oplus y \cap (y^* \oplus x)^* \oplus x \neq \varnothing$,
($\mathsf{H}_v\mathsf{MV}5$) $0^* \in (x \oplus 0^*) \cap (0^* \oplus x)$,
($\mathsf{H}_v\mathsf{MV}6$) $0^* \in (x \oplus x^*) \cap (x^* \oplus x)$,
($\mathsf{H}_v\mathsf{MV}7$) $x \in (x \oplus 0) \cap (0 \oplus x)$,
($\mathsf{H}_v\mathsf{MV}8$) $0^* \in (x^* \oplus y) \cap (y \oplus x^*)$ and $0^* \in (y^* \oplus x) \cap (x \oplus y^*)$ imply $x = y$.

Remark 2.2. On any $\mathsf{H}_v\mathsf{MV}$-algebra H, a binary relation '$\preceq$' is defined by

$$x \preceq y \;\Leftrightarrow\; 0^* \in x^* \oplus y \cap y \oplus x^*.$$

For nonempty subsets A and B of H, $A \preceq B$ means that there exist $a \in A$ and $b \in B$ such that $a \preceq b$. For $A \subseteq H$, we denote the set $\{a^* : a \in A\}$ by A^* and 0^* by 1.

Every hyper MV-algebra is an $\mathsf{H}_v\mathsf{MV}$-algebra. An $\mathsf{H}_v\mathsf{MV}$-algebra which is not a hyper MV-algebra is said to be proper.

Proposition 2.3. *In any $\mathsf{H}_v\mathsf{MV}$-algebra H, the following hold: for all $x, y \in H$ and nonempty subsets A and B of H,*

(1) $A \preceq A$,
(2) $0 \preceq A \preceq 1$,
(3) $A \preceq B$ *implies* $B^* \preceq A^*$,
(4) $(A^*)^* = A$,
(5) $A \cap B \neq \varnothing$ *implies that* $A \preceq B$,
(6) $x \preceq y$ *and* $y \preceq x$ *imply that* $x = y$.

Definition 2.4. Let I be a nonempty subset of H satisfying (I_0), where
(I_0) $x \preceq y$ and $y \in I$ imply $x \in I$. Then I is called:
(i) an $\mathsf{H}_v\mathsf{MV}$-ideal, if $x \oplus y \subseteq I$, for all $x, y \in I$,
(ii) a weak $\mathsf{H}_v\mathsf{MV}$-ideal, if $x \oplus y \preceq I$, for all $x, y \in I$.

A function $T : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ satisfying the following properties is called a t-norm: for all $x, y, z \in [0, 1]$,
(T1) $T(x, 1) = x$,
(T2) $T(x, y) \leq T(x, z)$, if $y \leq z$,
(T3) $T(x, y) = T(y, x)$,
(T4) $T(x, T(y, z)) = T(T(x, y), z)$.

Due to (T4), each t-norm T can be extended in a unique way to an n-ary operation, for $n \in \mathbb{N} \cup \{0\}$:

$$\mathsf{T}_{i=1}^{n} x_i = \begin{cases} 1 & n = 0 \\ T(x_n, \mathsf{T}_{i=1}^{n-1} x_i) & \text{otherwise.} \end{cases}$$

Hence the notatin $T(x_1, x_2, \ldots, x_n)$ is used for $\mathsf{T}_{i=1}^{n} x_i$.

Basic familiar t-norms are:

(1) $T_D(x, y) = \begin{cases} x \wedge y & , \quad x \vee y = 1; \\ 0 & , \quad \text{otherwise.} \end{cases}$ (Drastic product)
(2) $T_L(x, y) = 0 \vee (x + y - 1)$. (Łukasiewicz)
(3) $T_P(x, y) = xy$. (Product)
(4) $T_M(x, y) = x \wedge y$. (Minimum)

where $\wedge = \min$ and $\vee = \max$.

For two t-norms T_1 and T_2, T_1 is said to be weaker than T_2 denoted by $T_1 \leq T_2$, if $T_1(x, y) \leq T_2(x, y)$, for all $(x, y) \in [0, 1]^2$. $T_1 < T_2$ means that $T_1 \leq T_2$ and $T_1 \neq T_2$.

So the drastic t-norm is the weakest and the minimum t-norm is the strongest t-norm, i.e. for every t-norm T,

$$T_D \leq T \leq T_M.$$

Moreover the connection among four the basic t-norms are as follows:

$$T_D < T_L < T_P < T_M.$$

Definition 2.5. Let X be a nonempty set.

(i) A function $\mu : X \longrightarrow [0, 1]$ is called a fuzzy subset of X.

(ii) For fuzzy subset μ of X, the set $\mathrm{supp}(\mu) = \{x \in X : \mu(x) > 0\}$ is called the support of μ.

(iii) For fuzzy subset μ of X, the sets

$$\mu_t = \{x \in X : \mu(x) \geq t\} \text{ and } \mu_{\hat{t}} = \{x \in X : \mu(x) > t\} \quad (t \in [0, 1])$$

are called respectively, t-level subset and strong t-level subset of μ.

Let X be a nonempty set and $r \in (0, 1]$. A fuzzy subset μ of X having the form $\mu(x) = r$ and $\mu(y) = 0$, for all $y \in X \setminus \{x\}$, is called a fuzzy point with support x.

Now, let $\alpha \in \{\in, q, \in \wedge q, \in \vee q\}$. Then

(i) $x_r \in \mu$ means that $\mu(x) \geq r$.

(ii) $x_r q\mu$ means that $\mu(x) + r > 1$,

(iii) $x_r \in \wedge q\mu$ means that $x_r \in \mu$ and $x_r q\mu$,

(iv) $x_r \in \vee q\mu$ means that $x_r \in \mu$ or $x_r q\mu$.

(v) $x_r \overline{\alpha}\mu$ means that $x_r \alpha\mu$ does not hold.

For fuzzy subset μ of a nonempty set X and for $r \in (0, 1]$, let

$$\begin{aligned} \mu_r^{\in} &= \{x \in X : x_r \in \mu\}, \\ \mu_r^{q} &= \{x \in X : x_r q\mu\}, \\ \mu_r^{\in\vee q} &= \{x \in X : x_r \in \vee q\mu\}. \end{aligned}$$

Obviously, $\mu_r^{\in\vee q} = \mu_r^{\in} \cup \mu_r^{q}$. Also, it is obvious that $\mu_r^{\in} = \mu_r$, the r-level subset of μ.

Throughout this paper, H will denote an $\mathsf{H}_v\mathsf{MV}$-algebra unless otherwise specified.

3. $(\alpha, \beta)_T$-fuzzy (weak) H_vMV-ideals

In this section, we introduce various types of $(\alpha, \beta)_T$-fuzzy (weak) $\mathsf{H}_v\mathsf{MV}$-ideals of H and investigate their properties and the connections between them.

Definition 3.1. Let T be a t-norm and μ be a fuzzy subset of H. μ is called an

(i) $(\alpha, \beta)_T$-*fuzzy* $\mathsf{H}_v\mathsf{MV}$-*ideal* if for all $x, y, z \in H$ and $r, s \in (0, 1]$,

(1) $x_r \alpha\mu$ and $y_s \alpha\mu$ imply $a_{T(r,s)}\beta\mu$, for all $a \in x \oplus y$,

(2) $y_r \alpha\mu$ and $x \preceq y$ imply $x_r \beta\mu$.

(ii) $(\alpha, \beta)_T$-*fuzzy weak* $\mathsf{H}_v\mathsf{MV}$-*ideal* if for all $x, y, z \in H$ and $r, s \in (0, 1]$,

(1) $x_r \alpha\mu$ and $y_s \alpha\mu$ imply $a_{T(r,s)}\beta\mu$, for some $a \in x \oplus y$,

(2) $y_r \alpha\mu$ and $x \preceq y$ imply $x_r \beta\mu$.

When $T = T_M$, we use the notation (α, β) instead of $(\alpha, \beta)_{T_M}$.

Remark 3.2. Let μ be a fuzzy subset of H such that $\mu(x) \leq 1/2$, for all $x \in H$. If $x \in H$ is such that $x_r \in \wedge q\mu$, then $\mu(x) \geq r$ and $\mu(x) + r > 1$ and so

$$1 < \mu(x) + r \leq \mu(x) + \mu(x) = 2\mu(x).$$

Hence $\mu(x) > 1/2$, which is a contradiction. Thus in Definition 3.1, the case $\alpha = \in \wedge q$ will be omitted.

Theorem 3.3 follows immediately from the definition.

Theorem 3.3. *In any* $\mathsf{H}_v\mathsf{MV}$*-algebra* H*, the following hold:*

(1) *Every* $(\alpha,\beta)_T$*-fuzzy (weak)* $\mathsf{H}_v\mathsf{MV}$*-ideal is an* $(\in\wedge q,\in\vee q)_T$*-fuzzy (weak)* $\mathsf{H}_v\mathsf{MV}$*-ideal.*

(2) *Every* $(\in\vee q,\in\wedge q)_T$*-fuzzy (weak)* $\mathsf{H}_v\mathsf{MV}$*-ideal is an* $(\alpha,\beta)_T$*-fuzzy (weak)* $\mathsf{H}_v\mathsf{MV}$*-ideal.*

(3) *Every* $(\alpha,\in\wedge q)_T$*-fuzzy (weak)* $\mathsf{H}_v\mathsf{MV}$*-ideal is an* $(\alpha,\in)_T$*-fuzzy (weak)* $\mathsf{H}_v\mathsf{MV}$*-ideal and an* $(\alpha,q)_T$*-fuzzy (weak)* $\mathsf{H}_v\mathsf{MV}$*-ideal.*

(4) *Every* $(\alpha,\beta)_T$*-fuzzy (weak)* $\mathsf{H}_v\mathsf{MV}$*-ideal is an* $(\alpha,\in\vee q)_T$*-fuzzy (weak)* $\mathsf{H}_v\mathsf{MV}$*-ideal.*

Theorems 3.4 and 3.5 show the relation between $(\alpha,\beta)_T$-fuzzy (weak) $\mathsf{H}_v\mathsf{MV}$-ideals and (weak) $\mathsf{H}_v\mathsf{MV}$-ideals.

Theorem 3.4. *A non-empty subset* I *of* H *is a (weak)* $\mathsf{H}_v\mathsf{MV}$*-ideal if and only if* χ_I *is an* $(\alpha,\beta)_T$*-fuzzy (weak)* $\mathsf{H}_v\mathsf{MV}$*-ideal of* H*.*

Theorem 3.5. *For every* $(\alpha,\beta)_T$*-fuzzy* $\mathsf{H}_v\mathsf{MV}$*-ideal* μ *of* H*,* $\mathrm{supp}(\mu)$ *is an* $\mathsf{H}_v\mathsf{MV}$*-ideal of* H*, where* $\alpha\neq\in\wedge q$*.*

Corollary 3.6. *For every* $(\alpha,\beta)_T$*-fuzzy* $\mathsf{H}_v\mathsf{MV}$*-ideal* μ *of* H*,* $\mathrm{supp}(\mu)$ *is a weak* $\mathsf{H}_v\mathsf{MV}$*-ideal of* H*, where* $\alpha\neq\in\wedge q$*.*

Theorem 3.7. *Let* μ *be a fuzzy subset of* H *such that* $\mu(x)=1$*, for all* $x\in\mathrm{supp}(\mu)$*. Then* μ *is an* $(\alpha,\beta)_T$*-fuzzy* $\mathsf{H}_v\mathsf{MV}$*-ideal of* H*, where* $\alpha\neq\in\wedge q$*.*

Proof. We shall prove that μ is an $(\in,\in\wedge q)_T$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal of H. Let $x_r\in\mu$ and $y_s\in\mu$, for $r,s\in(0,1]$ and $x,y\in H$. Then $\mu(x)\geq r>0$ and $\mu(y)\geq s>0$ and so $x,y\in\mathrm{supp}(\mu)$, which by Theorem 3.5 implies that $x\oplus y\subseteq\mathrm{supp}(\mu)$. Hence for all $a\in x\oplus y$ we have $\mu(a)=1\geq T(r,s)$ and $\mu(a)+T(r,s)>1$, whence $a_{T(r,s)}\in\wedge q\mu$.

Now, let $x\preceq y$ and $y_r\in\mu$. Then $y\in\mathrm{supp}(\mu)$. Thus we get $x\in\mathrm{supp}(\mu)$. So $\mu(x)=1\geq r$ and $\mu(x)+r>1$, i.e., $x_r\in\wedge q\mu$. Hence μ is an $(\in,\in\wedge q)_T$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal. Therefore μ is an $(\in,\beta)_T$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal, where $\beta\in\{\in,q,\in\vee q\}$.

The proofs of the other cases are similar. □

Example 3.8 shows that the converse of Theorem 3.7 may not be true in general.

Example 3.8. Let $H=\{0,a,b,c,1\}$. Table 1 shows an $\mathsf{H}_v\mathsf{MV}$-algebra structure on H. Define fuzzy subset μ of H by

$$\mu(0)=1,\ \mu(a)=2/3,\ \mu(b)=\mu(c)=\mu(1)=1/2.$$

It is not difficult to check that μ is an $(\in,\in)_{T_M}$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal of H and thus is an $(\in,\in)_T$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal, for every t-norm T. Obviously $a\in\mathrm{supp}(\mu)$ while $\mu(a)\neq 1$.

Theorem 3.9. *Let* μ *be a fuzzy subset of* H *and* $t\in(1/2,1]$*. If* μ_t $(\neq\varnothing)$ *is an* $\mathsf{H}_v\mathsf{MV}$*-ideal of* H*, then*

(1) $T(\mu(x),\mu(y))\leq\inf_{a\in x\oplus y}\mu(a)\vee 1/2$,

(2) $\mu(x)\leq\mu(y)\vee 1/2$*, for all* $y\preceq x$*.*

The converse holds when $T=T_M$*.*

$\oplus$	0	a	b	c	1
0	{0}	{a}	{b}	{a,c}	{0,b,1}
a	{a}	{0,a}	{b,1}	{0,a,b,c,1}	{1}
b	{b,1}	{b,1}	{0,a,b,c,1}	{1}	{0,1}
c	{c}	{0,a}	{0,c}	{1}	{1}
1	{0,1}	{a,1}	{1}	{1}	{1}
$*$	1	b	a	c	0

Table 1. The Cayley table of $\oplus$ and *

Proof. (1) Assume that there exist $x, y \in H$ such that

$$\inf_{a\in x\oplus y} \mu(a) \vee 1/2 < T(\mu(x), \mu(y)).$$

Let $T(\mu(x), \mu(y)) = t$. Then $t > 1/2$ and $x, y \in \mu_t$, whence by hypothesis we get $x \oplus y \subseteq \mu_t$. Hence $\inf_{a\in x\oplus y} \mu(a) \vee 1/2 \geq t$, which is a contradiction. Thus (1) holds.

(2) Assume that there exists $y \in H$ such that $y \preceq x$ and $\mu(x) > \mu(y) \vee 1/2$. Let $\mu(x) = t$. Then $x \in \mu_t$ and $t > 1/2$, whence $y \in \mu_t$. Thus $\mu(y) \vee 1/2 > t \vee 1/2 = t$, which is a contradiction.

Conversely, let $T = T_M$ and the conditions (1) and (2) hold. Let $t \in (1/2, 1]$ and $x, y \in \mu_t$, for $x, y \in H$. Then

$$\inf_{a\in x\oplus y} \mu(a) \vee 1/2 \geq \mu(x) \wedge \mu(y) \geq t,$$

whence $\inf_{a\in x\oplus y} \mu(a) \geq t$, proving $x \oplus y \subseteq \mu_t$. By a similar way, we can prove that μ_t satisfies (I_0). Thus μ_t is an $\mathsf{H}_v\mathsf{MV}$-ideal of H. □

An analogous of Theorem 3.9 holds for weak $\mathsf{H}_v\mathsf{MV}$-ideals.

Theorem 3.10. *Let μ be a fuzzy subset of H and $t \in (1/2, 1]$. If μ_t $(\neq \varnothing)$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H, then*

(1) $T(\mu(x), \mu(y)) \leq \sup_{a\in x\oplus y} \mu(a) \vee 1/2$,
(2) $\mu(x) \leq \mu(y) \vee 1/2$, *for all* $y \preceq x$.

The converse holds when $T = T_M$.

In the sequel, we study various types of $(\alpha, \beta)_T$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideals, more closely.

4. $(\in, \in)_T$-fuzzy H_vMV-ideals

We start this section by giving some examples.

Example 4.1. Let $H = \{0, a, b, 1\}$. Table 2 shows an $\mathsf{H}_v\mathsf{MV}$-algebra structure on H. We define fuzzy subset μ of H by

$$\mu(0) = 1,\ \mu(a) = \mu(b) = \mu(1) = 1/2.$$

Then μ is an $(\in, \in)$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal while it is not an $(\in, \in)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal because $0_1 \in \mu$ and $a \in 0 \oplus 0$ while $a_1 \notin \mu$.

⊕	0	a	b	1
0	{0,a}	{0,a,b}	{0,a,b}	{0,a,b,1}
a	{0,a,b,1}	{0,b}	{0,1}	{a,b,1}
b	{a,b}	{0,a,b,1}	{0}	{0,a,b,1}
1	{0,a,1}	{0,a,b,1}	{1}	{0,a,b,1}
∗	1	b	a	0

Table 2. The Cayley table of $\oplus$ and *

Example 4.2. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra H given in Example 3.8 and define fuzzy subset μ of H by

$$\mu(0) = 1,\ \mu(a) = 2/3,\ \mu(b) = \mu(c) = 1/2,\ \mu(1) = 1/3.$$

Then μ is an $(\in, \in)_{T_P}$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal (and so is an $(\in, \in)_T$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal for $T \in \{T_L, T_D\}$) while it is not an $(\in, \in)_{T_P}$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal because $0_{3/4} \in \mu$, $b_{1/2} \in \mu$ and $1 \in b \oplus 0$ while $1_{3/8} \notin \mu$. Furthermore μ is not an $(\in, \in)$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal because $c_{1/2} \in \mu$ and $c \oplus c = \{1\}$ while $1_{1/2} \notin \mu$.

Example 4.3. Let $H = \{0, a, 1\}$ and consider Table 3. Then $(H; \oplus, ^*, 0)$ is a proper $\mathsf{H}_v\mathsf{MV}$-algebra. We define fuzzy subset μ of H by

$$\mu(0) = \mu(a) = 2/3,\ \mu(1) = 1/2.$$

Then μ is an $(\in, \in)_{T_P}$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal while it is not an $(\in, \in)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal because $a_{2/3} \in \mu$ and $1 \in a \oplus a$ while $1_{2/3} \notin \mu$.

⊕	0	a	1
0	{0,a}	{0,a}	{1}
a	{0,a}	{0,a,1}	{1}
1	{1}	{a,1}	{0,1}
∗	1	a	0

Table 3. The Cayley table of $\oplus$ and *

Example 4.4. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra H given in Example 4.3.

(1) We define a fuzzy subset μ of H by $\mu(0) = 3/4,\ \mu(a) = 2/3,\ \mu(1) = 1/3$. It is not difficult to check that μ is an $(\in, \in)_{T_L}$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal while it is not an $(\in, \in)_{T_P}$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal because $a_{2/3} \in \mu$ and $1 \in a \oplus a$ while $1_{4/9} \notin \mu$.

(2) We define a fuzzy subset μ of H by $\mu(0) = 3/4,\ \mu(a) = 2/3,\ \mu(1) = 1/4$. It is not difficult to check that μ is an $(\in, \in)_{T_D}$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal while it is not an $(\in, \in)_{T_L}$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal because $a_{2/3} \in \mu$ and $1 \in a \oplus a$ while $1_{1/3} \notin \mu$.

(3) We define a fuzzy subset μ of H by $\mu(0) = \mu(a) = 1,\ \mu(1) = 1/2$. It is not difficult to check that μ is an $(\in, \in)_{T_D}$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal while it is not an $(\in, \in)_{T_D}$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal because $a_1 \in \mu$ and $1 \in a \oplus a$ while $1_1 \notin \mu$.

Example 4.5. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra H given in Example 3.8. We define a fuzzy subset μ of H by

$$\mu(0) = 1,\ \mu(a) = 2/3,\ \mu(b) = \mu(c) = 1/2,\ \mu(1) = 1/5.$$

Then μ is an $(\in, \in)_{T_L}$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal (and so is an $(\in, \in)_{T_D}$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal) while it is not an $(\in, \in)_{T_P}$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal (and so is not an $(\in, \in)$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal) because $0_{3/4} \in \mu$, $b_{1/2} \in \mu$ and $1 \in b \oplus 0$ while $1_{3/8} \notin \mu$. Furthermore μ is not an $(\in, \in)_{T_D}$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal (and so is not an $(\in, \in)_{T_L}$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal) because $0_1 \in \mu$, $b_{1/2} \in \mu$ and $1 \in b \oplus 0$ while $1_{1/2} \notin \mu$.

The proofs of Theorems 4.6, 4.7 and 4.8 are straightforward, so the proofs are omitted.

Theorem 4.6. *Any fuzzy subset μ of H satisfying the following conditions is an $(\in, \in)_T$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal of H.*

(1) $\inf_{a \in x \oplus y} \mu(a) \geq T(\mu(x), \mu(y))$,
(2) $\mu(y) \geq \mu(x)$, *for all* $y \preceq x$.

The converse hold when $T = T_M$.

Theorem 4.7. *Any fuzzy subset μ of H satisfying the following conditions is an $(\in, \in)_T$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal of H.*

(1) $\sup_{a \in x \oplus y} \mu(a) \geq T(\mu(x), \mu(y))$,
(2) $\mu(y) \geq \mu(x)$, *for all* $y \preceq x$.

The converse hold when $T = T_M$.

Theorem 4.8. *Let μ be a fuzzy subset of H. If every nonempty r-level subset μ_r (with $r \in (0, 1]$) is a (weak) $\mathsf{H}_v\mathsf{MV}$-ideal of H, then μ is an $(\in, \in)_T$-fuzzy (weak) $\mathsf{H}_v\mathsf{MV}$-ideal of H. The converse holds when $T = T_M$.*

Example 4.9 shows that the converse of Theorem 4.8 may not be true in general.

Example 4.9. Consider the $(\in, \in)_{T_P}$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal μ defined in Example 4.2. Then $\mu_t = \{0, a, b, c\}$ (with $t \in (1/3, 1/2]$) is not a weak $\mathsf{H}_v\mathsf{MV}$-ideal because $b, c \in \mu_t$ while $b \oplus c \not\preceq \mu_t$. Also, μ is an $(\in, \in)_{T_L}$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal while μ_t (with $t \in (1/3, 1/2]$) is not an $\mathsf{H}_v\mathsf{MV}$-ideal.

5. $(\alpha, \in \vee q)_T$-fuzzy $\mathrm{H}_v\mathrm{MV}$-ideals

Example 5.1. Consider the fuzzy subset μ given in Example 4.1. It is not difficult to check that μ is an $(\in, \in \vee q)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal of H.

Example 5.2. Let $H = \{0, a, 1\}$ and consider Table 4. Then $(H; \oplus, ^*, 0)$ is a proper $\mathsf{H}_v\mathsf{MV}$-algebra. We define fuzzy subset μ of H by

$$\mu(0) = \mu(a) = 1,\ \mu(1) = 3/4.$$

It is easy to verify that μ is a $(q, \in \vee q)$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal of H.

Example 5.3. Let $H = \{0, a, 1\}$ and consider Table 5. Then $(H; \oplus, ^*, 0)$ is a proper $\mathsf{H}_v\mathsf{MV}$-algebra. We define fuzzy subset μ of H by

$$\mu(0) = \mu(a) = 1,\ \mu(1) = 2/3.$$

$\oplus$	0	a	1
0	{0,a}	{0,a}	{a,1}
a	{0,a}	{0,a,1}	{a,1}
1	{a,1}	{a,1}	{a,1}
$*$	1	a	0

TABLE 4. The Cayley table of $\oplus$ and *

Then μ is an $(\in\wedge q, \in\vee q)$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal of H but it is not an $(\in\wedge q, \in)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal because $a_{3/4} \in \wedge q\mu$ and $1 \in a \oplus a$ while $1_{3/4} \bar{\in} \mu$.

$\oplus$	0	a	1
0	{0,a}	{0,a}	{a,1}
a	{0,a}	{0,a,1}	{a,1}
1	{a,1}	{a,1}	{0,1}
$*$	1	a	0

TABLE 5. The Cayley table of $\oplus$ and *

Theorem 5.4. *Let μ be a fuzzy subset of H. μ is an $(\in, \in\vee q)_T$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal only if*

(1) $\inf_{a\in x\oplus y} \mu(a) \geq T(\mu(x), \mu(y), 1/2)$, *for all $x, y \in H$,*
(2) $\mu(y) \geq T(\mu(x), 1/2)$, *for all $y \preceq x$.*

The sufficiency holds when $T = T_M$.

Proof. Assume that μ is an $(\in, \in\vee q)_T$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal of H.

(1) We consider two following cases.

Case (i): $T(\mu(x), \mu(y)) < 1/2$, for some $x, y \in H$. If

$$\inf_{a\in x\oplus y} \mu(a) < T(\mu(x), \mu(y), 1/2),$$

then there exists $a \in x \oplus y$ such that

$$\mu(a) < T(\mu(x), \mu(y), 1/2) \leq T(\mu(x), \mu(y)).$$

Let $t \in (0, 1]$ be such that $\mu(a) < T(t, t) \leq t < T(\mu(x), \mu(y))$. Then

$$t < 1/2,\ \mu(x) > t \text{ and } \mu(y) > t.$$

Thus $x_t \in \mu$ and $y_t \in \mu$, while $\mu(a) + T(t, t) < t + t < 1$, i.e., $a_{T(t,t)} \overline{\in\vee q} \mu$. This is a contradiction.

Case (ii): $T(\mu(x), \mu(y)) \geq 1/2$, for some $x, y \in H$. If

$$\inf_{a\in x\oplus y} \mu(a) < T(\mu(x), \mu(y), 1/2),$$

then there exists $a \in x \oplus y$ such that $\mu(a) < T(\mu(x), \mu(y), 1/2)$. For $t \in (0, 1/2]$ with $\mu(a) < T(t, t) \leq t < 1/2$, we have $x_t \in \mu$ and $y_t \in \mu$, while $a_{T(t,t)} \overline{\in\vee q} \mu$. This is a contradiction.

So the condition (1) holds.

(2) We consider two following cases.

Case (i): $\mu(x) < 1/2$, for some $x \in H$. If there exists $y \in H$ with $y \preceq x$ such that $\mu(y) < T(\mu(x), 1/2)$, then $\mu(y) < \mu(x)$. Let $t \in (0, 1/2]$ be such that $\mu(y) < t < \mu(x)$. Then $x_t \in \mu$, while $\mu(y) + t < 2t < 1$, i.e., $y_t \overline{\in \vee q} \mu$. This is a contradiction.

Case (i): $\mu(x) \geq 1/2$, for some $x \in H$. If there exists $y \in H$ such that $y \preceq x$ and $\mu(y) < 1/2$, then $y_{1/2} \overline{\in \vee q} \mu$, while $x_{1/2} \in \mu$. This is a contradiction. Thus (2) holds.

Conversely, assume that μ satisfies the conditions (1) and (2), for $T = T_M$. Let $x_r \in \mu$ and $y_t \in \mu$, for $r, t \in (0, 1]$. Then $\inf_{a \in x \oplus y} \mu(a) \geq r \wedge t \wedge 1/2$. If $r \wedge t > 1/2$, then $\inf_{a \in x \oplus y} \mu(a) + r \wedge t > 1$, i.e., $a_{r \wedge t} q \mu$, for all $a \in x \oplus y$. If $r \wedge t \leq 1/2$, then $\inf_{a \in x \oplus y} \mu(a) \geq r \wedge t$ and thus $a_{r \wedge t} \in \mu$, for all $a \in x \oplus y$. In any case, $a_{r \wedge t} \in \vee q \mu$, for all $a \in x \oplus y$.

Now, let $x_r \in \mu$ and $y \preceq x$. Then $\mu(y) \geq r \wedge 1/2$. If $r > 1/2$, then $\mu(y) + r > 1$ and thus $y_r q \mu$. If $r \leq 1/2$, then $\mu(y) \geq r$ and so $y_r \in \mu$. Hence $y_r \in \vee q \mu$. Therefore μ is an $(\in, \in \vee q)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal of H. □

From Theorems 3.3(4) and 5.4 it follows that

Corollary 5.5. *Every $(\in, \beta)_T$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal satisfies the conditions (1) and (2) of Theorem 5.4.*

Similar to Theorem 5.4 we have the next theorem, for $(\alpha, \beta)_T$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideals; the proof is similar and so it is omitted.

Theorem 5.6. *Let μ be a fuzzy subset of H. μ is an $(\in, \in \vee q)_T$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal only if*

(1) $\sup_{a \in x \oplus y} \mu(a) \geq T(\mu(x), \mu(y), 1/2)$, *for all $x, y \in H$,*

(2) $\mu(y) \geq T(\mu(x), 1/2)$, *for all $y \preceq x$.*

The sufficiency holds when $T = T_M$.

From Theorems 3.3(4) and 5.6 it follows that

Corollary 5.7. *Every $(\in, \beta)_T$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal satisfies the conditions (1) and (2) of Theorem 5.6.*

The following example shows that the converse of Theorem 5.4 may not be true in general.

Example 5.8. Consider the fuzzy subset μ defined in Example 4.5. It is not difficult to check that μ satisfy the conditions (1) and (2) of Theorem 5.4, for $T = T_L$, while it is not an $(\in, \in \vee q)_{T_L}$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal because $0_{3/4} \in \mu$, $b_{1/2} \in \mu$ and $1 \in b \oplus 0$, while $1_{L(3/4, 1/2)} = 1_{1/4} \overline{\in \vee q} \mu$.

Theorem 5.9. *Every $(q, \in \vee q)_T$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal satisfies the conditions (1) and (2) of Theorem 5.4.*

Proof. Assume that μ is a $(q, \in \vee q)_T$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal of H and there exists $x, y \in H$ such that $\inf_{a \in x \oplus y} \mu(a) < T(\mu(x), \mu(y), 1/2)$. Then there exists $a \in x \oplus y$ such that $\mu(a) < T(\mu(x), \mu(y), 1/2)$. Now, for $r \in (0, 1]$ such that

$$1 - T(\mu(x), \mu(y), 1/2) < T(r, r) \leq r < 1 - \mu(a),$$

we have $1-\mu(x)\leq 1-T(\mu(x),\mu(y),1/2)<r$ and similarly $1-\mu(y)<r$, whence $\mu(x)+r>1$ and $\mu(y)+r>1$, i.e., $x_r q\mu$ and $y_r q\mu$.

On the other hand,

$\mu(a)+T(r,r)<1$

and

$\mu(a)<T(\mu(x),\mu(y),1/2)\leq 1/2<T(r,r)$, i.e., $a_{T(r,r)}\overline{\in\vee q}\mu$.

This is a contradiction. Thus

$$\mu(a)\geq T(\mu(x),\mu(y),1/2),\ \forall a\in x\oplus y.$$

Similarly, we can show that μ satisfies the condition (2). □

Theorem 5.10. *Every $(q,\in\vee q)_T$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal satisfies the conditions (1) and (2) of Theorem 5.6.*

Proof. The proof is similar to the proof of Theorem 5.9. □

From Theorems 5.4, 5.9 and 5.10 it follows that

Corollary 5.11. *Every $(q,\in\vee q)$-fuzzy (weak) $\mathsf{H}_v\mathsf{MV}$-ideal is an $(\in,\in\vee q)$-fuzzy (weak) $\mathsf{H}_v\mathsf{MV}$-ideal.*

Now, we investigate the relation between $(\alpha,\in\vee q)_T$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideals and their level subsets.

Theorem 5.12. *Let μ be a fuzzy subset of H.*

(1) If $\mu_r\neq\varnothing$ is a (weak) $\mathsf{H}_v\mathsf{MV}$-ideal of H, then μ is an $(\in,\in\vee q)_T$-fuzzy (weak) $\mathsf{H}_v\mathsf{MV}$-ideal of H, for all $r\in(0,1/2]$. The converse hold, when $T=T_M$.

(2) $\mu_r^q\neq\varnothing$ is a (weak) $\mathsf{H}_v\mathsf{MV}$-ideal of H if and only if μ is an $(\in,\in\vee q)$-fuzzy (weak) $\mathsf{H}_v\mathsf{MV}$-ideal of H, for all $r\in(1/2,1]$.

(3) If $\mu_r^{\in\vee q}\neq\varnothing$ is a (weak) $\mathsf{H}_v\mathsf{MV}$-ideal of H, then μ is an $(\in,\in\vee q)_T$-fuzzy (weak) $\mathsf{H}_v\mathsf{MV}$-ideal of H, for all $r\in(0,1]$. The converse hold, when $T=T_M$.

Proof. We shall prove the theorem for $\mathsf{H}_v\mathsf{MV}$-ideals. The proofs for weak $\mathsf{H}_v\mathsf{MV}$-ideals are similar.

(1) Let $x_r\in\mu$ and $y_s\in\mu$, for $r,s\in(0,1/2]$ and $x,y\in H$. Then $x\in\mu_r\subseteq\mu_{T(r,s)}$ and $y\in\mu_s\subseteq\mu_{T(r,s)}$. Thus $x\oplus y\subseteq\mu_{T(r,s)}$. So $a_{T(r,s)}\in\mu$, for all $a\in x\oplus y$, whence $a_{T(r,s)}\in\vee q\mu$, for all $a\in x\oplus y$.

Now, let $y\preceq x$ and $x_r\in\mu$, for $r\in(0,1/2]$. Then $x\in\mu_r$. Thus $y\in\mu_r\subseteq\mu_r^{\in\vee q}$, whence $y_r\in\vee q\mu$. So μ is an $(\in,\in\vee q)_T$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal of H.

Conversely, let $T=T_M$, $t\in(0,1/2]$ and $x,y\in\mu_t$. By Theorem 5.4, we have $\inf_{a\in x\oplus y}\mu(a)\geq\mu(x)\wedge\mu(y)\wedge 1/2\geq t\wedge 1/2=t$, which implies that $x\oplus y\subseteq\mu_t$. It is easily proved that μ_t satisfies (I_0). Thus μ_t is an $\mathsf{H}_v\mathsf{MV}$-ideal of H.

(2) Let $x_r\in\mu$ and $y_s\in\mu$, for $x,y\in H$ and $r,s\in(1/2,1]$. Then $\mu(x)+r\wedge s>1$ and $\mu(y)+r\wedge s>1$, whence $x,y\in\mu^q_{r\wedge s}$. By hypothesis, we get $x\oplus y\subseteq\mu^q_{r\wedge s}\subseteq\mu^{\in\vee q}_{r\wedge s}$, i.e., $a_{r\wedge s}\in\vee q\mu$, for all $a\in x\oplus y$. It is easily proved that μ_r^q satisfies (I_0).

Conversely, let $x,y\in\mu_r^q$, where $r\in(1/2,1]$. From Theorem 5.4, it follows that $\inf_{a\in x\oplus y}\mu(a)\geq\mu(x)\wedge\mu(y)\wedge 1/2>(1-r)\wedge 1/2=1-r$, whence $a_r q\mu$, for all $a\in x\oplus y$, i.e., $x\oplus y\subseteq\mu_r^q$.

Now, let $y\preceq x$ and $x\in\mu_r^q$. Then

$$\mu(y)\geq\mu(x)\wedge 1/2>(1-r)\wedge 1/2=1-r$$

means that $y_r q\mu$. Thus $y \in \mu_r^q$. So μ_r^q is an $\mathsf{H}_v\mathsf{MV}$-ideal of H.

(3) It is proved similar to (1) and (2).

The converse follows from (1) and (2). □

Example 5.13. Consider the $(\in, \in)_{T_D}$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal μ given in Example 4.5. Obviously, μ is an $(\in, \in \vee q)_{T_D}$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal of H. But $\mu^q_{2/3} = \{0, a, b, c\}$ is not a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H because $b, c \in \mu^q_{2/3}$ while $b \oplus c = \{1\}$ and $1 \not\preceq \mu^q_{2/3}$. This example shows that the converse of Theorem 5.12 may not be true in general.

Theorem 5.14. *Let I be a nonempty subset of H and fuzzy subset μ of H be defined as $\mu(x) \geq 1/2$, if $x \in I$, and $\mu(x) = 0$, otherwise. Then μ is an $(\alpha, \in \vee q)_T$-fuzzy (weak) $\mathsf{H}_v\mathsf{MV}$-ideal of H if and only if I is a (weak) $\mathsf{H}_v\mathsf{MV}$-ideal of H, where $\alpha \neq \in \wedge q$.*

Proof. Assume that I is an $\mathsf{H}_v\mathsf{MV}$-ideal of H, and $x_r\alpha\mu$ and $y_s\alpha\mu$, for $x, y \in H$ and $r, s \in (0, 1]$. We consider the following cases.

Case (i): Suppose $\alpha = \in$. Then $\mu(x) \geq r > 0$ and $\mu(y) \geq s > 0$, whence $x, y \in I$. Thus $x \oplus y \subseteq I$. So $\mu(a) \geq 1/2$, for all $a \in x \oplus y$.

Now, if $T(r, s) \leq 1/2$, then $a_{T(r,s)} \in \mu$, for all $a \in x \oplus y$. If $T(r, s) > 1/2$, then $\mu(a) + T(r, s) > 1$. Thus $a_{T(r,s)} q\mu$, for all $a \in x \oplus y$. In any case, $a_{T(r,s)} \in \vee q\mu$, for all $a \in x \oplus y$.

Case (ii): Suppose $\alpha = q$. Then we have $\mu(x) + r > 1$ and $\mu(y) + s > 1$, whence $\mu(x) \geq 1/2$ and $\mu(y) \geq 1/2$. Thus $x, y \in I$ and so $x \oplus y \subseteq I$. Hence $\mu(a) \geq 1/2$, for all $a \in x \oplus y$.

Now, if $T(r, s) \leq 1/2$, then $\mu(a) \geq T(r, s)$. If $T(r, s) > 1/2$, then $\mu(a) + T(r, s) > 1$, for all $a \in x \oplus y$. Thus $a_{T(r,s)} \in \vee q\mu$, for all $a \in x \oplus y$.

The case $\alpha = \in \vee q$ follows from Cases (i) and (ii).

Similarly, we can prove that $y \preceq x$ and $x_r\alpha\mu$ imply $y_r \in \vee q\mu$.

Conversely, if μ is an $(\alpha, \in \vee q)_T$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal, where $\alpha \in \{\in, q, \in \vee q\}$, then $I = \mathrm{supp}(\mu)$. Thus by Theorem 3.5, I is an $\mathsf{H}_v\mathsf{MV}$-ideal of H. □

6. $(\in \wedge q, \beta)_T$ and $(q, \beta)_T$-fuzzy $\mathrm{H}_v\mathrm{MV}$-ideals

We first investigate the properties of $(\in \wedge q, \beta)_T$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideals. We first give some examples

Example 6.1. Consider the fuzzy subset μ given in Example 5.3. It is not difficult to check that μ is an $(\in \wedge q, \in)$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal but it is not an $(\in \wedge q, \in)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal because $a_{3/4} \in \wedge q\mu$ and $1 \in a \oplus a$ while $1_{3/4}\bar{\in}\mu$. Also, μ is an $(\in \wedge q, q)$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal of H. Hence μ is an $(\in \wedge q, \in \wedge q)$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal, too.

Theorem 6.2. *If μ is an $(\in \wedge q, \beta)$-fuzzy (weak) $\mathsf{H}_v\mathsf{MV}$-ideal of H, $\mu_{\widehat{1/2}} \neq \varnothing$ is a (weak) $\mathsf{H}_v\mathsf{MV}$-ideal of H. The converse hold for $\beta = \in \vee q$.*

Proof. Assume that $\mu_{\widehat{1/2}} \neq \varnothing$ and $x, y \in \mu_{\widehat{1/2}}$, for $x, y \in H$. We consider the following cases.

Case (i): Suppose $\beta \in \{\in, \in \wedge q\}$. Then for $r \in (0,1]$ such that $\mu(x) \wedge \mu(y) > r > 1/2$, we have $x_r \in \mu$, $y_r \in \mu$, $\mu(x) + r > 1$ and $\mu(y) + r > 1$, whence $x_r \in \wedge q\mu$ and $y_r \in \wedge q\mu$. Thus $a_r \in \mu$, for all $a \in x \oplus y$, i.e., $x \oplus y \subseteq \mu_r \subseteq \mu_{\widehat{1/2}}$.

Case (ii): Suppose $\beta = q$. From $\mu(x) > 1/2$ and $\mu(y) > 1/2$, it follows that $x_{1/2} \in \wedge q\mu$ and $y_{1/2} \in \wedge q\mu$, whence $a_{1/2} q\mu$, for all $a \in x \oplus y$, i.e., $x \oplus y \subseteq \mu_{\widehat{1/2}}$.

Case (iii): Suppose $\beta = \in \vee q$. Then it is similar to the proof of Cases (i) and (ii).

Now, let $y \preceq x$ and $x \in \mu_{\widehat{1/2}}$, for $x, y \in H$. We consider the following cases.

Case (i): Suppose $\beta \in \{\in, \in \wedge q\}$. Then for $r \in (0,1]$ with $\mu(x) > r > 1/2$, we have $x_r \in \wedge q\mu$, whence $y_r \in \mu$. Thus $\mu(y) \geq r > 1/2$. So $y \in \mu_{\widehat{1/2}}$.

Case (ii): Suppose $\beta = q$. Then from $\mu(x) > 1/2$, it follows that $x_{1/2} \in \wedge q\mu$. Thus $y_{1/2} q\mu$, whence $y \in \mu_{\widehat{1/2}}$.

Case (iii): Suppose $\beta = \in \vee q$. Then it is similar to Cases (i) and (ii).

So $\mu_{\widehat{1/2}}$ is an $\mathsf{H}_v\mathsf{MV}$-ideal of H.

The proofs for $(\in \wedge q, \beta)$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideals are similar.

Conversely, let $x_r \in \wedge q\mu$ and $y_s \in \wedge q\mu$, for $x, y \in H$ and $r, s \in (0,1]$. Then $\mu(x) \geq r$ and $\mu(x) > 1 - r$, $\mu(y) \geq s$ and $\mu(y) > 1 - s$, whence $\mu(x) > 1/2$ and $\mu(y) > 1/2$, i.e., $x, y \in \mu_{\widehat{1/2}}$. By hypothesis, we get $x \oplus y \subseteq \mu_{\widehat{1/2}}$. Thus $\mu(a) > 1/2$, for all $a \in x \oplus y$. If $r \wedge s \leq 1/2$, then $a_{r\wedge s} \in \mu$ and thus $a_{r\wedge s} \in \vee q\mu$. If $r \wedge s > 1/2$, then $\mu(a) + r \wedge s > 1$, i.e., $a_{r\wedge s} q\mu$ and so $a_{r\wedge s} \in \vee q\mu$. In any case, $a_{r\wedge s} \in \vee q\mu$, for all $a \in x \oplus y$.

Similarly, it is proved that if $y \preceq x$ and $x_r \in \wedge q\mu$, then $y_r \in \vee q$. □

Examples 6.3 and 6.4 show that the converse of Theorem 6.2 may not be true in general.

Example 6.3. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra H given in Example 3.8 and define fuzzy subset μ of H by $\mu(0) = 1$, $\mu(a) = 4/5$, $\mu(b) = \mu(c) = \mu(1) = 1/2$. It is easy to check that $\mu_{\widehat{1/2}} = \{0, a\}$ is an $\mathsf{H}_v\mathsf{MV}$-ideal, while μ is not an $(\in \wedge q, q)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal because $0_{1/5} \in \wedge q\mu$, $a_{1/3} \in \wedge q\mu$ and $a \in a \oplus 0$ but $a_{1/5}\overline{q}\mu$.

Example 6.4. Consider the fuzzy subset μ given in Example 5.3. By Example 6.1, μ is not an $(\in \wedge q, \in)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal of H. Obviously, $\mu_{\widehat{1/2}} = H$ is an $\mathsf{H}_v\mathsf{MV}$-ideal of H.

Theorem 6.5. *Every $(\in \wedge q, q)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal satisfies the following conditions:*

(1) $\mu(x) \wedge \mu(y) \leq \inf_{a\in x\oplus y} \mu(a) \vee 1/2$,
(2) $\mu(x) \leq \mu(y) \vee 1/2$, *for all* $y \preceq x$.

Proof. If $\mu(x) \wedge \mu(y) > \inf_{a\in x\oplus y} \mu(a) \vee 1/2$, for some $x, y \in H$, then $\mu(x) \wedge \mu(y) > \mu(a) \vee 1/2$, for some $a \in x \oplus y$. For $r \in (0,1]$ with $1 - \mu(a) \vee 1/2 > r > 1 - \mu(x) \wedge \mu(y)$ we have $r > 1 - \mu(x)$, $r > 1 - \mu(y)$, $r < 1/2$ and $1 - \mu(a) > r$. Thus $\mu(x) > 1 - r > 1/2 > r$ and $\mu(y) > 1 - r > 1/2 > r$, whence $x_r \in \wedge q\mu$ and $y_r \in \wedge q\mu$. But $\mu(a) + r < 1$, i.e. $a_r \bar{q}\mu$, which is a contradiction.

The second condition is proved similarly. □

Example 6.6. Consider the fuzzy subset μ given in Example 6.3. Although μ in not an $(\in \wedge q, q)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal, but it satisfies the conditions (1) and (2) of Theorem 6.5, means that the converse of Theorem 6.5 may not be true in general.

Theorem 6.7. *Every* $(\in \wedge q, q)$*-fuzzy weak* $\mathsf{H}_v\mathsf{MV}$*-ideal satisfies the following conditions:*

(1) $\mu(x) \wedge \mu(y) \leq \sup_{a\in x\oplus y} \mu(a) \vee 1/2$,
(2) $\mu(x) \leq \mu(y) \vee 1/2$, *for all* $y \preceq x$.

Proof. The proof is similar to the proof of Theorem 6.5. □

Example 6.8. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra H given in Example 4.1 and define fuzzy subset μ by $\mu(0) = 1,\ \mu(a) = 4/5,\ \mu(b) = 3/4,\ \mu(1) = 1/2$. Routine calculations show that μ satisfies the conditions (1) and (2) of Theorem 6.7 while μ is not an $(\in \wedge q, q)$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal because $0_{1/5} \in \wedge q\mu$, $b_{2/3} \in \wedge q\mu$ and $b \oplus 0 = \{a, b\}$, while $a_{1/5}\overline{q}\mu$ and $b_{1/5}\overline{q}\mu$. Thus the converse of Theorem 6.7 may not be true in general.

Theorem 6.9. *Every* $(\in \wedge q, q)$*-fuzzy (weak)* $\mathsf{H}_v\mathsf{MV}$*-ideal* μ *is constant on* $\mu_{\widehat{1/2}}$.

Proof. Assume that μ is an $(\in \wedge q, q)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal which is not constant on $\mu_{\widehat{1/2}}$. Then there exist $x, y \in \mu_{\widehat{1/2}}$ such that $\mu(x) \neq \mu(y)$. We consider two following cases.

Case (i): Suppose $\mu(x) > \mu(y)$. Then from Theorem 6.5, it follows that

$$\mu(a) \vee 1/2 \geq \inf_{a\in x\oplus y} \mu(a) \vee 1/2 \geq \mu(y) > 1/2, \quad \forall a \in x \oplus y,$$

whence $\mu(a) > 1/2$, for all $a \in x \oplus y$. Now, for $r \in (0, 1]$ such that

$$1/2 > 1 - \mu(y) > r > 1 - \mu(x)$$

we have $x_r \in \wedge q\mu$ and $y_{1/2} \in \wedge q\mu$ while $a_{r\wedge 1/2} = a_r\overline{q}\mu$, which is a contradiction.

Case (ii): Suppose $\mu(y) > \mu(x)$. Then it is similar to Case (i).

Similarly, it is proved that every $(\in \wedge q, q)$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal μ is constant on $\mu_{\widehat{1/2}}$. □

Theorem 6.10. *If* μ *is an* $(\in \wedge q, q)$*-fuzzy (weak)* $\mathsf{H}_v\mathsf{MV}$*-ideal, then* $\mu_r^q \neq \varnothing$ *is a (weak)* $\mathsf{H}_v\mathsf{MV}$*-ideal of* H*, for all* $r \in (0, 1/2]$.

Proof. Let $x, y \in \mu_r^q$, where $x, y \in H$ and $r \in (0, 1/2]$. Then $\mu(x) + r > 1$ and $\mu(y) + r > 1$, whence $\mu(x) > 1/2 \geq r$ and $\mu(y) > 1/2 \geq r$. Thus $x_r \in \wedge q\mu$ and $y_r \in \wedge q\mu$ and so $a_r q\mu$, for all $a \in x \oplus y$, proving $x \oplus y \subseteq \mu_r^q$.

Now, let $x \in \mu_r^q$ and $y \preceq x$. Since $r \leq 1/2$, $\mu(x) > 1 - r \geq 1/2 \geq r$. Thus $x_r \in \wedge q\mu$. So $y_r q\mu$, i.e., $y \in \mu_r^q$. Hence μ_r^q is an $\mathsf{H}_v\mathsf{MV}$-ideal of H.

The proof for $(\in \wedge q, q)$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideals is similar. □

Example 6.11. Consider the fuzzy subset μ defined in Example 6.8. We see that μ is not an $(\in \wedge q, q)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal of H while routine calculations show that for all $r \in (0, 1/2]$, $\mu_r^q = \{0, a, b\}$ is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H. Also, the fuzzy subset μ defined in Example 3.8 is not an $(\in \wedge q, q)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal because $0_{1/4} \in \wedge q\mu$, $a_{2/3} \in \wedge q\mu$ while $a_{1/4}\overline{q}\mu$, where $a \in a \oplus 0$. We can see that $\mu_r^q = \{0, a\}$ (with $r \in (1/3, 1/2]$) and $\mu_r^q = \{0\}$ (with $r \in (0, 1/3]$) are $\mathsf{H}_v\mathsf{MV}$-ideals of H. Hence the converse of Theorem 6.10 may not be true in general.

Theorem 6.12. *Every* $(\in \wedge q, \in)$*-fuzzy* $\mathsf{H}_v\mathsf{MV}$*-ideal satisfies the conditions (1) and (2) of Theorem 6.5.*

Proof. Assume that μ is an $(\in \wedge q, \in)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal of H. If there exist $x, y \in H$ such that

$$\mu(x) \wedge \mu(y) > \inf_{a \in x \oplus y} \mu(a) \vee 1/2,$$

then $\mu(x) \wedge \mu(y) > \mu(a) \vee 1/2$, for some $a \in x \oplus y$. Let $\mu(x) \wedge \mu(y) = r$. Then $r > 1/2$, $\mu(x) \geq r$, $\mu(y) \geq r$, $\mu(x) + r \geq 2r > 1$, $\mu(y) + r > 1$ and $\mu(a) < r$. Thus $x_r \in \wedge q\mu$ and $y_r \in \wedge q\mu$, while $a_r \overline{\in} \mu$. This is a contradiction. Also if there exist $x, y \in H$ (with $y \preceq x$) such that $\mu(x) = r > \mu(y) \vee 1/2$, then $r > 1/2$, $\mu(x) \geq r$ and $\mu(x) + r > 1$, while $\mu(y) < r$, i.e., $x_r \in \wedge q\mu$, while $y_r \overline{\in} \mu$. This is a contradiction. So the conditions (1) and (2) hold. □

Theorem 6.13. *Every $(\in \wedge q, \in)$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal satisfies the conditions (1) and (2) of Theorem 6.7.*

Proof. The proof is similar to the proof of Theorem 6.12. □

Theorem 6.14. *Let μ be a fuzzy subset of H. If μ is an $(\in \wedge q, \in)$-fuzzy (weak) $\mathsf{H}_v\mathsf{MV}$-ideal, then $\mu_r \neq \varnothing$ is a (weak) $\mathsf{H}_v\mathsf{MV}$-ideal of H, for all $r \in (1/2, 1]$. Conversely, if $\mu_r \neq \varnothing$ is a (weak) $\mathsf{H}_v\mathsf{MV}$-ideal of H, for $r \in (1/2, 1]$, then μ is an $(\in \wedge q, \in)$-fuzzy (weak) $\mathsf{H}_v\mathsf{MV}$-ideal of H.*

Proof. Assume that μ is an $(\in \wedge q, \in)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal of H and $x, y \in \mu_r$, for $r \in (1/2, 1]$. Then by Theorem 6.12, we get

$$\inf_{a \in x \oplus y} \mu(a) \vee 1/2 \geq \mu(x) \wedge \mu(y) \geq r.$$

Thus $\mu(a) \geq r$, for all $a \in x \oplus y$. So $x \oplus y \subseteq \mu_r$.

Now, let $y \preceq x$ and $x \in \mu_r$. Then $\mu(y) \vee 1/2 \geq \mu(x) \geq r > 1/2$. Thus $\mu(y) \geq r$, i.e., $y \in \mu_r$. So μ_r is an $\mathsf{H}_v\mathsf{MV}$-ideal of H.

Conversely, let $x_r \in \wedge q\mu$ and $y_s \in \wedge q\mu$. Then

$$\mu(x) \geq r,\ \mu(x) + r > 1,\ \mu(y) \geq s \text{ and } \mu(y) + s > 1.$$

Thus $\mu(x) > 1/2$ and $\mu(y) > 1/2$.

Suppose $r \wedge s > 1/2$. From $x \in \mu_r \subseteq \mu_{r \wedge s}$ and $y \in \mu_s \subseteq \mu_{r \wedge s}$ and that $\mu_{r \wedge s}$ is an $\mathsf{H}_v\mathsf{MV}$-ideal, it follows that $x \oplus y \subseteq \mu_{r \wedge s}$, i.e., $a_{r \wedge s} \in \mu$, for all $a \in x \oplus y$.

Suppose $r \wedge s < 1/2$. Since $\mu_{r \wedge s}$ is an $\mathsf{H}_v\mathsf{MV}$-ideal of H, from $x \in \mu_r$ and $y \in \mu_s$, it follows that $x, y \in \mu_{r \wedge s}$. Thus $x \oplus y \subseteq \mu_{r \wedge s}$, i.e., $a_{r \wedge s} \in \mu$, for all $a \in x \oplus y$.

Similarly, we can prove that $x_r \in \wedge q\mu$ and $y \preceq x$ imply $y_r \in \mu$. So μ is an $(\in \wedge q, \in)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal of H.

The proof for the case 'weak $\mathsf{H}_v\mathsf{MV}$-ideals' is similar. □

In the sequel we study (q, β)-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideals and their properties. We first give some examples.

Example 6.15. Consider the fuzzy subset μ given in Example 5.2. It is easily verified that μ is a (q, q)-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal of H but it is not a (q, q)-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal because $a_{1/5} q\mu$ and $1 \in a \oplus a$ while $1_{1/5} \bar{q}\mu$.

Theorem 6.16. *A fuzzy subset μ of H is a $(q, \in)$-fuzzy (weak) $\mathsf{H}_v\mathsf{MV}$-ideal if and only if $\mu(x) = 1$, for all $x \in \operatorname{supp}(\mu)$.*

Proof. Assume that μ is a $(q, \in)$-fuzzy (weak) $\mathsf{H}_v\mathsf{MV}$-ideal of H. If there exists $x \in \text{supp}\ (\mu)$ such that $\mu(x) < 1$, then for all $y \in \text{supp}\ (\mu)$ with $x \preceq y$, we have $y_1 q\mu$ while $x_1 \overline{\in} \mu$, a contradiction.

The converse follows from Theorem 3.7. □

Theorem 6.17. *A fuzzy subset μ of H is a $(q, \in)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal if and only if it is a $(q, \in \wedge q)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal.*

Proof. Obviously, every $(q, \in \wedge q)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal is a $(q, \in)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal. Now, assume that μ is a $(q, \in)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal of H, and $x_r q\mu$ and $y_s q\mu$, for $r, s \in (0, 1]$ and $x, y \in H$. Then $\mu(x) > 1 - r \geq 0$ and $\mu(y) > 1 - s \geq 0$. Thus $x, y \in \text{supp}\ (\mu)$. By Theorems 3.5 and 6.16, we get $\mu(a) = 1$, for all $a \in x \oplus y$. So $\mu(a) \geq r \wedge s$ and $\mu(a) + r \wedge s > 1$, i.e., $a_{r\wedge s} \in \wedge q\mu$, for all $a \in x \oplus y$. Let $y \preceq x$ and $x_r q\mu$. Then $x \in \text{supp}\ (\mu)$. Thus $y \in \text{supp}\ (\mu)$. So $\mu(y) = 1 \geq r$ and $\mu(y) + r > 1$, i.e., $y \in \wedge q\mu$. Hence μ is an $(q, \in \wedge q)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal of H. □

Theorem 6.18. *A fuzzy subset μ of H is a $(q, \in)$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal if and only if it is a $(q, \in \wedge q)$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal.*

Proof. The proof is similar to the proof of Theorem 6.17. □

From Theorems 3.3, 6.17 and 6.18, we get

Corollary 6.19. *Every $(q, \in)$-fuzzy (weak) $\mathsf{H}_v\mathsf{MV}$-ideal is a (q, q)-fuzzy (weak) $\mathsf{H}_v\mathsf{MV}$-ideal.*

Theorem 6.20. *Let μ be a $(q, \in)$-fuzzy (weak) $\mathsf{H}_v\mathsf{MV}$-ideal of H. Then $\mu_r^\alpha = \varnothing$ or μ_r^α is a (weak) $\mathsf{H}_v\mathsf{MV}$-ideal of H, for all $r \in (1/2, 1]$, where $\alpha \in \{\in, q\}$.*

Proof. Assume that μ is a $(q, \in)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal of H and $\mu_r^\in \neq \varnothing$, for $r \in (1/2, 1]$. Let $x, y \in \mu_r^\in$. Then $\mu(x) + r \geq 2r > 1$ and $\mu(y) + r > 1$. Thus $x_r q\mu$ and $y_r q\mu$. So $a_r \in \mu$, for all $a \in x \oplus y$. Hence $x \oplus y \subseteq \mu_r$. Similarly, we can prove that $x \in \mu_r$ and $y \preceq x$ imply $y \in \mu_r$. Then μ_r is an $\mathsf{H}_v\mathsf{MV}$-ideal of H.

By a similar way, it is proved that $\mu_r^q \neq \varnothing$ is an $\mathsf{H}_v\mathsf{MV}$-ideal of H, for all $r \in (1/2, 1]$.

Analogously, it is proved that if μ is a $(q, \in)$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal, then $\mu_r^\alpha \neq \varnothing$ (with $\alpha \in \{\in, q\}$) is a weak $\mathsf{H}_v\mathsf{MV}$-ideal of H. □

The following example shows that the converse of Theorem 6.20 may not be true in general.

Example 6.21. Consider the $\mathsf{H}_v\mathsf{MV}$-algebra H defined in Example 4.1 and define fuzzy subset μ of H by $\mu(0) = 1$, $\mu(a) = 2/3$, $\mu(b) = \mu(1) = 1/3$. Then $\mu_r^\in = \{0\}$, for all $r \in (2/3, 1]$ and $\mu_r^q = \{0, a\}$, for all $r \in (1/2, 1]$, which are weak $\mathsf{H}_v\mathsf{MV}$-ideals of H, while μ is not a $(q, \in)$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideal of H, because $1_{3/4} q\mu$, $b_{3/4} q\mu$ but $1_{3/4} \overline{\in} \mu$, where $1 \in 1 \oplus b$.

Now, consider the fuzzy subset μ defined in Example 3.8. Then $\mu_r^\in = \{0\}$ (with $r \in (2/3, 1]$) and $\mu_r^q = \{0, a\}$ (with $r \in (1/2, 1]$) are $\mathsf{H}_v\mathsf{MV}$-ideals of H, while μ is not a $(q, \in)$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideal, because $0_{2/3} q\mu$ and $b_{2/3} q\mu$, while $b_{2/3} \overline{\in} \mu$, where $b \in 0 \oplus b$.

7. Conclusions and the future works

In this paper, another types of fuzzy $\mathsf{H}_v\mathsf{MV}$-ideals was introduced. Based on the concepts of belongingness and quasi-coincidence, $(\alpha,\beta)_T$-fuzzy $\mathsf{H}_v\mathsf{MV}$-ideals and $(\alpha,\beta)_T$-fuzzy weak $\mathsf{H}_v\mathsf{MV}$-ideals with respect to a t-norm were introduced. It was proved that some of them coincide, with respect to special t-norms. Some characterizations and equivalent conditions were investigated. Giving suitable examples it was shown that some theorems are not true in general. Furthermore, some characterizations based on level subsets are obtained.

There are still many subjects which are interesting to study such as $(\alpha,\beta)_T$-fuzzy implicative $\mathsf{H}_v\mathsf{MV}$-ideals, $(\alpha,\beta)_T$-fuzzy prime $\mathsf{H}_v\mathsf{MV}$-ideals, interval-valued fuzzy (weak) $\mathsf{H}_v\mathsf{MV}$-ideals and many other concepts.

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References

[1] M. Bakhshi, $\mathsf{H}_v\mathsf{MV}$-algebras I, Quasigroups Related Systems 22 (2014) 9–18.
[2] M. Bakhshi, $\mathsf{H}_v\mathsf{MV}$-algebras II, J. Algebraic Systems 3 (1) (2015) 49–64.
[3] M. Bakhshi, Fuzzy $\mathsf{H}_v\mathsf{MV}$-algebras, Afr. Mat. 27 (2016) 379–392.
[4] M. Bakhshi, Generalized fuzzy filters in non-commutative residuated lattices, Afr. Mat. 25 (2014) 289-305.
[5] C. C. Chang, Algebraic analysis of many valued logics, Trans. Amer. Math. Soc. 88 (1958) 467–490.
[6] P. Corsini and V. Leoreanu, Applications of Hyperstructure Theory, Kluwer Academic Publishers, Dordrecht 2003.
[7] B. Davvaz and P. Corsini, On (α,β)-fuzzy H_v-ideals of H_v-rings, Iran. J. Fuzzy Syst. 5 (2008) 35–47.
[8] Sh. Ghorbani, A. Hassankhani and E. Eslami, Hyper MV-algebras, Set-Valued Mathematics and Applications 1 (2008) 205–222.
[9] C. S. Hoo, Fuzzy implicative and Boolean ideals of MV-algebras, Fuzzy Sets and Systems 66 (1994) 315–327.
[10] C. S. Hoo, Fuzzy ideals of BCI and MV-algebras, Fuzzy Sets and Systems 62 (1994) 111-114.
[11] C. S. Hoo, Some fuzzy concepts of BCI, BCK and MV-algebras, Internat. J. Approx. Reason. 18 (1998) 177–189.
[12] C. S. Hoo and S. Sessa, Fuzzy maximal ideals of BCI and MV-algebras, Inform. Sci. 80 (1994) 299–309.
[13] Y. B. Jun, M. S. Kang and H. S. Kim, Hyper MV-deductive systems of hyper MV-algebras, Commun. Korean Math. Soc. 25 (2010) 537–545.
[14] Y. B. Jun, M. S. Kang and H. S. Kim, Fuzzy structures of hyper MV-deductive systems in hyper MV-algebras, Comput. Math. Appl. 59 (2010) 2982–2989.
[15] E. P. Klement and R. Mesiar, Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms, Netherlands 2005.
[16] F. Marty, Sur une generalization de la notion de groups, 8th congress Math. Scandinaves, Stockhholm 1934.
[17] B. L. Meng, CI-algebras, Sci. Math. Jpn. 71 (2009) 695–701.
[18] D. Mundici, MV-algebras are categorically equivalent to bounded commutative BCK-algebras, Math. Jpn 31 (1986) 889–894.
[19] A. Namdar, A. Borumand Saeid and G. Jabbari, On $(\alpha, \in \vee q_k)$-fuzzy filters of CI-algebras, Ann. Fuzzy Math. Inform. 7 (2014) 851–858.

[20] G. Tianbang, Lattice implication algebras and MV-algebras, Chinese Quart. J. Math. 14 (3) (1999) 17–23.
[21] L. Torkzadeh and A. Ahadpanah, Hyper MV-ideals in hyper MV-algebras, Math. Log. Q. 56 (2010) 51–62.
[22] T. Vougiouklis, Hyperstructures and their representations, Hadronic, Florida 1994.
[23] T. Vougiouklis, A new class of hyperstructures, J. Combin. inform. Syst. Sci. 20 (1995) 229–235.
[24] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

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