Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 13, No. 1, (January 2017), pp. 135–143
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

On paranorm type intuitionistic fuzzy Zweier I-convergent sequence spaces

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Received 4 June 2016; Accepted 24 June 2016

Abstract. In this article we introduce the Paranorm type intuitionistic fuzzy Zweier I-convergent sequence spaces $\mathcal{Z}^I_{(\mu,\nu)}(p)$ and $\mathcal{Z}^I_{0(\mu,\nu)}(p)$ and study the fuzzy topology on the said spaces.

2010 AMS Classification: 46S40

Keywords: Ideal, Filter, I-convergence, Intuitionistic fuzzy normed spaces.

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1. Introduction

After the pioneering work of Zadeh [26], a huge number of research papers have appeared on fuzzy theory and its applications as well as fuzzy analogues of the classical theories. Fuzzy set theory is a powerful hand set for modelling uncertainty and vagueness in various problems arising in field of science and engineering. It has a wide range of applications in various fields: population dynamics [3], chaos control [6], computer programming [7], nonlinear dynamical system [8], etc. Fuzzy topology is one of the most important and useful tools and it proves to be very useful for dealing with such situations where the use of classical theories breaks down. The concept of intuitionistic fuzzy normed space [21] and of intuitionistic fuzzy 2-normed space [18] are the latest developments in fuzzy topology. Quite recently, V. A. Khan and Yasmeen ([10, 11, 12]) studied the notion of I- convergence in Intuitionistic Fuzzy Zweier I-convergent Sequence Spaces.

The notion of statistical convergence is a very useful functional tool for studying the convergence problems of numerical problems/matrices(double sequences) through the concept of density. The notion of I-convergence, which is a generalization of statistical convergence [5], was introduced by Kostyrko, Salat and Wilczynski [13] by using the idea of I of subsets of the set of natural numbers $\mathbb{N}$ and further studied in [19]. Recently, the notion of statistical convergence of double sequences $x = (x_{ij})$ has been defined and studied by Mursaleen and Edely [17]; and for fuzzy numbers by Savaş and Mursaleen [22]. Quite recently, Das et al. [4] studied the notion of I and I^*- convergence of double sequences in $\mathbb{R}$.

2. Preliminaries

We recall some notations and basic definitions used in this paper.

Definition 2.1. A binary operation $* : [0,1]\times[0,1] \to [0,1]$ is said to be continuous t-norm, if it satisfies the following conditions:
(i) $*$ is associative and commutative,
(ii) $*$ is continuous,
(iii) $a * 1 = a$ for all $a \in [0,1]$,
(iv) $a * c \leq b * d$ whenever $a \leq b$ and $c \leq d$ for each $a,b,c,d \in [0,1]$.
For example, $a * b = a.b$ is a continuous t-norm.

Definition 2.2. A binary operation $\diamond : [0,1]\times[0,1] \to [0,1]$ is said to be continuous t-conorm, if it satisfies the following conditions:
(i) $\diamond$ is associative and commutative,
(ii) $\diamond$ is continuous,
(iii) $a \diamond 0 = a$ for all $a \in [0,1]$,
(iv) $a \diamond c \leq b \diamond d$ whenever $a \leq b$ and $c \leq d$ for each $a,b,c,d \in [0,1]$.
For example, $a \diamond b = min\{a+b, 1\}$ is a continuous t-conorm.

Definition 2.3. The five-tuple $(X, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy normed space(for short, IFNS), if X is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and μ, ν are fuzzy sets on $X \times (0,\infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$:
(i) $\mu(x,t) + \nu(x,t) \leq 1$,
(ii) $\mu(x,t) > 0$,
(iii) $\mu(x,t) = 1$ if and only if $x = 0$,
(iv) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
(v) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$,
(vi) $\mu(x, .) : (0,\infty) \to [0,1]$ is continuous,
(vii) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$,
(viii) $\nu(x,t) < 1$,
(ix) $\nu(x,t) = 0$ if and only if $x = 0$,
(x) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
(xi) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y, t+s)$,
(xii) $\nu(x, .) : (0,\infty) \to [0,1]$ is continuous,
(xiii) $\lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1$.
In this case, (μ, ν) is called an intuitionistic fuzzy norm.

Definition 2.4. Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then a sequence $x = (x_k)$ is said to be convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\epsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - L, t) > 1 - \epsilon$ and $\nu(x_k - L, t) < \epsilon$ for all $k \geq k_0$. In this case, we write $(\mu, \nu) - \lim x = L$.

Definition 2.5. Let X be a non empty set. Then a family of sets $I \subseteq 2^X$(power set of X) is said to be an ideal, if

(i) I is additive, i.e.. $A, B \in I \Rightarrow A \cup B \in I$,
(ii) I is hereditary, i.e., $A \in I, B \subseteq A \Rightarrow B \in I$.

Definition 2.6. Let X be a non empty set. Then $\mathcal{F} \subset 2^X$ is said to be a filter on X, if
(i) $\phi \notin \mathcal{F}$,
(ii) for $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$,
(iii) for each $A \in \mathcal{F}$ and $B \supset A$ implies $B \in \mathcal{F}$.

Definition 2.7. Let $I \subset 2^{\mathbb{N}}$ be a non trivial ideal and $(X, \mu, \nu, *, \diamond)$ be an IFNS. A sequence $x = (x_k)$ of elements of X is said to be I-convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν), if for every $\epsilon > 0$ and $t > 0$, the set

$$\{k \in \mathbb{N} : \mu(x_k - L, t) \geq 1 - \epsilon \text{ or } \nu(x_k - L, t) \leq \epsilon\} \in I.$$

In this case, L is called the I-limit of the sequence (x_k) with respect to the intuitionistic fuzzy norm (μ, ν) and we write $I_{(\mu,\nu)} - \lim x_k = L$.

Definition 2.8. Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Let $r \in (0, 1), t > 0$ and $x \in X$. Then the set $B_{x^/}(r, t)(p)$
$= \{y \in X : \{k \in \mathbb{N} : [\mu(x_k - y_k, t)]^{p_k} \leq 1 - r \text{ or } [\nu(x_k - y_k, t)]^{p_k} \geq r\} \in I\}$
is called an open ball with centre $x^/$ and radius r with respect to t.

The approach of constructing new sequence spaces by means of the matrix domain of a particular limitation method have been recently employed by Altay, Başar, Mursaleen [1], Malkowsky [16] Ng and Lee [20], and Wang [24]. Şeng*önül* [23] defined the sequence $y = (y_i)$ which is frequently used as the Z^p transformation of the sequence $x = (x_i)$, i.e,

$$y_i = px_i + (1 - p)x_{i-1},$$

where $x_{-1} = 0, p \neq 1, 1 < p < \infty$ and Z^p denotes the matrix $Z^p = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} p, & \text{if } (i = k), \\ 1 - p, & (i - 1 = k); (i, k \in \mathbb{N}) \\ 0, & \text{otherwise} . \end{cases}$$

Analogous to Başar and Altay [2], Şeng*ön*ül [23] introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ as follows:

$$\mathcal{Z} = \{x = (x_k) \in \omega : \mathcal{Z}^p x \in c\},$$
$$\mathcal{Z}_0 = \{x = (x_k) \in \omega : \mathcal{Z}^p x \in c_0\}.$$

Recently Khan, Ebadullah and Yasmeeen [9] introduced the following classes of sequences:
$\mathcal{Z}^I = \{(x_k) \in \omega : \exists L \in \mathbb{C} \text{ such that for a given } \epsilon > 0, \{k \in \mathbb{N} : \mid x^/_k - L \mid \geq \epsilon\} \in I\}$,
$\mathcal{Z}^I_0 = \{(x_k) \in \omega : \text{ for a given } \epsilon > 0, \{k \in \mathbb{N} : \mid x^/_k \mid \geq \epsilon\} \in I\}$,
where $(x^/_k) = (Z^p x)$.

Definition 2.9. The concept of paranorm is related to the linear metric spaces. It is a generalization of that of absolute value. Let X be linear space. A function $p : X \to \mathbb{R}$, is called paranorm, if([?])
(p_1) $p(0) \geq 0$,

(p_2) $p(x) \geq 0, \forall x \in X$,
(p_3) $p(-x) = p(x), \forall x \in X$,
(p_4) $p(x+y) \leq p(x) + p(y), \forall x, y \in X$ (triangle inequality),
(p_5) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda (n \to \infty)$ and (x_n) is a sequence of vetors with $p(x_n - x) \to 0 (n \to \infty)$, then $p(x_n\lambda_n - x\lambda) \to (n \to \infty)$,(continuity of multiplication of vectors).

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total. It is well known that the metric of any linear metric space is given by some total paranorm[14].

In this article, we introduce the following sequence spaces as follows:

$$\mathcal{Z}^I_{(\mu,\nu)}(p) = \{(x_k) \in \omega : \{k \in \mathbb{N} : [\mu(x^{/}_k - L, t)]^{p_k} \leq 1 - \epsilon \text{ or } [\nu(x^{/}_k - L, t)]^{p_k} \geq \epsilon\} \in I\},$$

$$\mathcal{Z}^I_{0(\mu,\nu)}(p) = \{(x_k) \in \omega : \{k \in \mathbb{N} : [\mu(x^{/}_k, t)]^{p_k} \leq 1 - \epsilon \text{ or } [\nu(x^{/}_k, t)]^{p_k} \geq \epsilon\} \in I\}.$$

3. Main Results

Theorem 3.1. *$\mathcal{Z}^I_{(\mu,\nu)}(p)$ and $\mathcal{Z}^I_{0(\mu,\nu)}(p)$ are linear spaces.*

Proof. Let $(x^{/}_k), (y^{/}_k) \in \mathcal{Z}^I_{(\mu,\nu)}(p)$ and let α, β be scalars. Then for a given $\epsilon > 0$, we have

$$A_1 = \left\{k \in \mathbb{N} : \left[\mu(x^{/}_k - L_1, \tfrac{t}{2|\alpha|})\right]^{p_k} \leq 1 - \epsilon \text{ or } \left[\nu(x^{/}_k - L_1, \tfrac{t}{2|\alpha|})\right]^{p_k} \geq \epsilon\right\} \in I,$$

$$A_2 = \left\{k \in \mathbb{N} : \left[\mu(y^{/}_k - L_2, \tfrac{t}{2|\beta|})\right]^{p_k} \leq 1 - \epsilon \text{ or } \left[\nu(y^{/}_k - L_2, \tfrac{t}{2|\beta|})\right]^{p_k} \geq \epsilon\right\} \in I.$$

$$A^c_1 = \left\{k \in \mathbb{N} : \left[\mu(x^{/}_k - L_1, \tfrac{t}{2|\alpha|})\right]^{p_k} > 1 - \epsilon \text{ or } \left[\nu(x^{/}_k - L_1, \tfrac{t}{2|\alpha|})\right]^{p_k} < \epsilon\right\} \in \mathcal{F}(I),$$

$$A^c_2 = \left\{k \in \mathbb{N} : \left[\mu(y^{/}_k - L_2, \tfrac{t}{2|\beta|})\right]^{p_k} > 1 - \epsilon \text{ or } \left[\nu(y^{/}_k - L_2, \tfrac{t}{2|\beta|})\right]^{p_k} < \epsilon\right\} \in \mathcal{F}(I).$$

Define the set $A_3 = A_1 \cup A_2$, so that $A_3 \in I$. It follows that A^c_3 is a non-empty set in $\mathcal{F}(I)$,

$$A^c_3 \subset \Big\{k \in \mathbb{N} : \left[\mu\big((\alpha x^{/}_k + \beta y^{/}_k) - (\alpha L_1 + \beta L_2), t\big)\right]^{p_k} > 1 - \epsilon \text{ or } \left[\nu\big((\alpha x^{/}_k + \beta y^{/}_k) - (\alpha L_1 + \beta L_2), t\big)\right]^{p_k} < \epsilon\Big\}.$$

Let $m \in A^c_3$. Then

$$\left[\mu(x^{/}_m - L_1, \tfrac{t}{2|\alpha|})\right]^{p_k} > 1 - \epsilon \text{ or } \left[\nu(x^{/}_m - L_1, \tfrac{t}{2|\alpha|})\right]^{p_k} < \epsilon$$

and

$$\left[\mu(y^{/}_m - L_2, \tfrac{t}{2|\beta|})\right]^{p_k} > 1 - \epsilon \text{ or } \left[\nu(y^{/}_m - L_2, \tfrac{t}{2|\beta|})\right]^{p_k} < \epsilon.$$

Thus

$$\begin{aligned}&\left[\mu\big((\alpha x^{/}_m + \beta y^{/}_m) - (\alpha L_1 + \beta L_2), t\big)\right]^{p_k} \\ &\geq \left[\mu(\alpha x^{/}_m - \alpha L_1, \tfrac{t}{2})\right]^{p_k} * \left[\mu(\beta y^{/}_m - \beta L_2, \tfrac{t}{2})\right]^{p_k}. \\ &= \left[\mu(x^{/}_m - L_1, \tfrac{t}{2|\alpha|})\right]^{p_k} * \left[\mu(y^{/}_m - L_2, \tfrac{t}{2|\beta|})\right]^{p_k}, \\ &> (1-\epsilon) * (1-\epsilon).\end{aligned}$$

$= (1-\epsilon)$

and

$$\begin{aligned} &\left[\nu\left((\alpha x_m^{/}+\beta y_m^{/})-(\alpha L_1+\beta L_2),t\right)\right]^{p_k} \\ &\leq \left[\nu(\alpha x_m^{/}-\alpha L_1,\tfrac{t}{2})\right]^{p_k}\diamond\left[\nu(\beta y_m^{/}-\beta L_2,\tfrac{t}{2})\right]^{p_k}. \\ &= \left[\nu(x_m^{/}-L_1,\tfrac{t}{2|\alpha|})\right]^{p_k}\diamond\left[\nu(y_m^{/}-L_2,\tfrac{t}{2|\beta|}))\right]^{p_k}, \\ &< \epsilon\diamond\epsilon. \\ &= \epsilon. \end{aligned}$$

So

$$\begin{aligned} A_3^c \subset \Big\{k\in\mathbb{N}:\left[\mu\left((\alpha x_k^{/}+\beta y_k^{/})-(\alpha L_1+\beta L_2),t\right)\right]^{p_k} > 1-\epsilon \\ \text{or } \left[\nu\left((\alpha x_k^{/}+\beta y_k^{/})-(\alpha L_1+\beta L_2),t\right)\right]^{p_k} < \epsilon\Big\}. \end{aligned}$$

Hence $\mathcal{Z}^I_{(\mu,\nu)}(p)$ is a linear space. □

Theorem 3.2. *Every open ball $B_{x^/}(r,t)(p)$ is an open set in $\mathcal{Z}^I_{(\mu,\nu)}(p)$.*

Proof. Let $B_{x^/}(r,t)(p)$ be an open ball with centre $x^/$ and radius r with respect to t, i.e.,

$$\begin{aligned} &B_{x^/}(r,t)(p) \\ &= \{y\in X:\{k\in\mathbb{N}:[\mu(x_k^{/}-y_k^{/},t)]^{p_k}\leq 1-r \text{ or } [\nu(x_k^{/}-y_k^{/},t)]^{p_k}\geq r\}\in I\}. \end{aligned}$$

Let $y\in B^c_{x^/}(r,t)(p)$. Then $[\mu(x^/-y^/,t)]^{p_k}>1-r$ and $[\nu(x^/-y^/,t)]^{p_k}<r$.

Since $[\mu(x^/-y^/,t)]^{p_k}>1-r$, there exists $t_0\in(0,1)$ such that

$$[\mu(x^/-y^/,t_0)]^{p_k}>1-r \text{ and } [\nu(x^/-y^/,t_0)]^{p_k}<r.$$

Putting $r_0=[\mu(x^/-y^/,t)]^{p_k}$. Then $r_0>1-r$. Thus there exists $s\in(0,1)$ such that $r_0>1-s>1-r$. So for $r_0>1-s$, we have $r_1,r_2\in(0,1)$ such that

$$r_0 * r_1 > 1-s \text{ and } (1-r_0)\diamond(1-r_2)\leq s.$$

Let $r_3=max\{r_1,r_2\}$ and consider the ball $B^c_{y^/}(1-r_3,t-t_0)(p)$. We prove that

$$B^c_{y^/}(1-r_3,t-t_0)(p)\subset B^c_{x^/}(r,t)(p).$$

Let $z^/\in B^c_{y^/}(1-r_3,t-t_0)(p)$. Then

$$[\mu(y^/-z^/,t-t_0)]^{p_k}>r_3 \text{ and } [\nu(y^/-z^/,t-t_0)]^{p_k}<r_3.$$

Thus

$$\begin{aligned} [\mu(x^/-z^/,t)]^{p_k} &\geq [\mu(x^/-y^/,t_0)]^{p_k} * [\mu(y^/-z^/,t-t_0)]^{p_k} \\ &\geq (r_0*r_3)\geq(r_0*r_1)\geq(1-s)>(1-r) \end{aligned}$$

and

$$\begin{aligned} [\nu(x^/-z^/,t)]^{p_k} &\leq [\nu(x^/-y^/,t_0)]^{p_k}\diamond[\nu(y^/-z^/,t-t_0)]^{p_k} \\ &\leq (1-r_0)\diamond(1-r_3)\leq(1-r_0)\diamond(1-r_2)\leq s<r. \end{aligned}$$

So $z^/\in B^c_{x^/}(r,t)(p)$. Hence $B^c_{y^/}(1-r_3,t-t_0)(p)\subset B^c_{x^/}(r,t)(p)$. □

Remark 3.3. It is clear that $\mathcal{Z}^I_{(\mu,\nu)}(p)$ is an IFNS. Define

$$\begin{aligned} &\tau_{(\mu,\nu)}(p) \\ &= \{A\subset\mathcal{Z}^I_{(\mu,\nu)}(p):\text{ for each } x\in A\ \exists\ t>0 \text{ and } r\in(0,1) \text{ such that } B_{x^/}(r,t)(p)\subset A\}. \end{aligned}$$

Then $\tau_{(\mu,\nu)}(p)$ is a topology on $\mathcal{Z}^I_{(\mu,\nu)}(p)$.

Theorem 3.4. *The topology* $\tau_{(\mu,\nu)}(p)$ *on* $\mathcal{Z}^I_{0(\mu,\nu)}(p)$ *is first countable.*

Proof. It is clear that $\{B_{x^/}(\frac{1}{n},\frac{1}{n})(p) : n = 1, 2, 3, ...\}$ is a local base at $x^/$. Then the topology $\tau_{(\mu,\nu)}(p)$ on $\mathcal{Z}^I_{0(\mu,\nu)}(p)$ is first countable. □

Theorem 3.5. $\mathcal{Z}^I_{(\mu,\nu)}(p)$ *and* $\mathcal{Z}^I_{0(\mu,\nu)}(p)$ *are Housedorff spaces.*

Proof. Let $x^/, y^/ \in \mathcal{Z}^I_{(\mu,\nu)}(p)$ such that $x^/ \neq y^/$. Then

$$0 < [\mu(x^/ - y^/, t)]^{p_k} < 1 \text{ and } 0 < [\nu(x^/ - y^/, t)]^{p_k} < 1.$$

Putting $r_1 = [\mu(x^/ - y^/, t)]^{p_k}$, $r_2 = [\nu(x^/ - y^/, t)]^{p_k}$ and $r = max\{r_1, 1 - r_2\}$. Then for each $r_0 \in (r, 1)$, there exists r_3 and r_4 such that

$$r_3 * r_4 \geq r_0 \text{ and } (1 - r_3) \diamond (1 - r_4) \leq (1 - r_0).$$

Let $r_5 = max\{r_3, 1 - r_4\}$ and consider the open balls

$$B^c_{x^/}(1 - r_5, \tfrac{t}{2})(p) \text{ and } B^c_{y^/}(1 - r_5, \tfrac{t}{2})(p).$$

Then clearly $B^c_{x^/}(1 - r_5, \frac{t}{2})(p) \cap B^c_{y^/}(1 - r_5, \frac{t}{2})(p) = \phi$.

For if there exists $z^/ \in B^c_{x^/}(1 - r_5, \frac{t}{2})(p) \cap B^c_{y^/}(1 - r_5, \frac{t}{2})(p)$, then

$$\begin{aligned} r_1 = [\mu(x^/ - y^/, t)]^{p_k} &\geq [\mu(x^/ - z^/, \tfrac{t}{2})]^{p_k} * [\mu(z^/ - y^/, \tfrac{t}{2})]^{p_k} \\ &\geq r_5 * r_5 \geq r_3 * r_3 \geq r_0 > r_1 \end{aligned}$$

and

$$\begin{aligned} r_2 = [\nu(x^/ - y^/, t)]^{p_k} &\leq [\nu(x^/ - z^/, \tfrac{t}{2})]^{p_k} \diamond [\nu(z^/ - y^/, \tfrac{t}{2})]^{p_k} \\ &\leq (1 - r_5) \diamond (1 - r_5) \leq (1 - r_4) \diamond (1 - r_4) \leq (1 - r_0) < r. \end{aligned}$$

This is a contradiction. Thus $\mathcal{Z}^I_{(\mu,\nu)}(p)$ is a Housedorff space.

Similarly, we can prove that $\mathcal{Z}^I_{0(\mu,\nu)}(p)$ is a Housedorff space. □

Theorem 3.6. $\mathcal{Z}^I_{(\mu,\nu)}(p)$ *is an IFNS.* $\tau_{(\mu,\nu)}(p)$ *is a topology on* $\mathcal{Z}^I_{(\mu,\nu)}(p)$. *Then a sequence* $(x^/_x) \in \mathcal{Z}^I_{(\mu,\nu)}(p)$, $x^/_k \to x^/$ *if and only if* $[\mu(x^/_k - x^/, t)]^{p_k} \to 1$ *and* $[\nu(x^/_k - x^/, t)]^{p_k} \to 0$ *as* $k \to \infty$.

Proof. Fix $t_0 > 0$. Suppose $x^/_k \to x^/$. Then for $r \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $x^/_k \in B_{x^/}(r, t)(p)$, for all $k \geq n_0$. On one hand,

$$B_{x^/}(r, t)(p) = \{k \in \mathbb{N} : [\mu(x^/_k - x^/, t)]^{p_k} \leq 1 - r \text{ or } [\nu(x^/_k - x^/, t)]^{p_k} \geq r\} \in I.$$

Thus $B^c_{x^/}(r, t)(p) \in \mathcal{F}(I)$. So

$$1 - [\mu(x^/_k - x^/, t)]^{p_k} < r \text{ and } [\nu(x^/_k - x^/, t)]^{p_k} < r.$$

Hence $[\mu(x^/_k - x^/, t)]^{p_k} \to 1$ and $[\nu(x^/_k - x^/, t)]^{p_k} \to 0$ as $k \to \infty$.

Conversely, suppose for each $t > 0$, $[\mu(x^/_k - x^/, t)]^{p_k} \to 1$ and $[\nu(x^/_k - x^/, t)]^{p_k} \to 0$ as $k \to \infty$. Then for $r \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that

$$1 - [\mu(x^/_k - x^/, t)]^{p_k} < r \text{ and } [\nu(x^/ - x^/, t)]^{p_k} < r, \text{ for all } k \geq n_0.$$

Thus

$$[\mu(x_k^{/} - x^{/}, t)]^{p_k} > 1 - r \text{ and } [\nu(x_k^{/} - x^{/}, t)]^{p_k} < r, \text{ for all } k \geq n_0.$$

So $x_k^{/} \in B_{x^{/}}^c(r,t)(p)$, for all $k \geq n_0$. Hence $x_k^{/} \to x^{/}$. □

Theorem 3.7. *A sequence $x = (x_k^{/}) \in \mathcal{Z}_{(\mu,\nu)}^I(p)$ I-converges if and only if for every $\epsilon > 0$ and $t > 0$, there exists a number $N = N(x, \epsilon, t)$ such that*

$$\Big\{k \in \mathbb{N} : \big[\mu\big(x_k^{/} - L, \tfrac{t}{2}\big)\big]^{p_k} > 1 - \epsilon \text{ or } \big[\nu\big(x_k^{/} - L, \tfrac{t}{2}\big)\big]^{p_k} < \epsilon\Big\} \in \mathcal{F}(I).$$

Proof. Suppose that $I_{(\mu,\nu)} - x = L$ and let $\epsilon > 0$ and $t > 0$. For a given $\epsilon > 0$, choose $s > 0$ such that $(1-\epsilon) * (1-\epsilon) > 1 - s$ and $\epsilon \diamond \epsilon < 0$. Then for each $x \in \mathcal{Z}_{(\mu,\nu)}^I(p)$,

$$A = \{k \in \mathbb{N} : [\mu(x_k^{/} - L, \frac{t}{2})]^{p_k} \leq 1 - \epsilon \text{ or } [\nu(x_k^{/} - L, \frac{t}{2})]^{p_k} \geq \epsilon\} \in I.$$

Thus

$$A^c = \{k \in \mathbb{N} : [\mu(x_k^{/} - L, \frac{t}{2})]^{p_k} > 1 - \epsilon \text{ or } [\nu(x_k^{/} - L, \frac{t}{2})]^{p_k} < \epsilon\} \in \mathcal{F}(I).$$

Conversely let us choose $N \in A^c$. Then

$$\big[\mu\big(x_N^{/} - L, \tfrac{t}{2}\big)\big]^{p_k} > 1 - \epsilon \text{ or } \big[\nu\big(x_N^{/} - L, \tfrac{t}{2}\big)\big]^{p_k} < \epsilon.$$

Now we want to show that there exists a number $N = N(x_N^{/}, \epsilon, t)$ such that

$$\{k \in \mathbb{N} : [\mu(x_k^{/} - x_N^{/}, t)]^{p_k} \leq 1 - s \text{ or } [\nu(x_k^{/} - x_N^{/}, t)]^{p_k} \geq s\} \in I.$$

For this, define for each $x \in \mathcal{Z}_{(\mu,\nu)}^I(p)$,

$$B = \{k \in \mathbb{N} : [\mu(x_k^{/} - x_N^{/}, t)]^{p_k} \leq 1 - s \text{ or } [\nu(x_k^{/} - x_N^{/}, t)]^{p_k} \geq s\} \in I.$$

Assume that $B \not\subset A$. Then there exists $n \in B$ and $n \notin A$. Thus we have

$$[\mu(x_n^{/} - x_N^{/}, t)]^{p_k} \leq 1 - s \text{ and } [\mu(x_k^{/} - L, \tfrac{t}{2})]^{p_k} > 1 - \epsilon.$$

In particular, $[\mu(x_N^{/} - L, \frac{t}{2})]^{p_k} > 1 - \epsilon$. Thus we have

$$\begin{aligned} 1 - s &\geq [\mu(x_n^{/} - x_N^{/}, t)]^{p_k} \\ &\geq [\mu(x_n^{/} - L, \tfrac{t}{2})]^{p_k} * [\mu(x_N^{/} - L, \tfrac{t}{2})]^{p_k} \\ &\geq (1-\epsilon) * (1-\epsilon) > 1 - s. \end{aligned}$$

This is not possible.

On the other hand,

$$[\nu(x_n^{/} - x_N^{/}, t)]^{p_k} \geq s \text{ and } [\nu(x_k^{/} - L, \tfrac{t}{2})]^{p_k} > \epsilon.$$

In particular, $[\nu(x_N^{/} - L, \frac{t}{2})]^{p_k} > \epsilon$. Then we have

$$s \leq [\nu(x_n^{/} - x_N^{/}, t)]^{p_k} \leq [\nu(x_n^{/} - L, \frac{t}{2})]^{p_k} \diamond [\nu(x_N^{/} - L, \frac{t}{2})]^{p_k} \leq \epsilon \diamond \epsilon < s.$$

This is not possible. Thus $B \subset A$. So $A \in I$ implies $B \in I$. □

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