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Some results on finite dimensional fuzzy cone normed linear space

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Abstract. In this paper, a new concept of fuzzy cone normed linear space is introduced and some basic results on finite dimensional fuzzy cone normed linear space are established.

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1. Introduction

The theory of fuzzy set was introduced by Zadeh[21]. Since then many authors have developed the theory of fuzzy sets and its applications. Especially, many mathematicians tried to extended classical mathematical results in fuzzy context. In particular, while studying fuzzy topological vector spaces, in 1984 for the first time, the notion of fuzzy norm on a linear space was introduced by Katsaras[12]. After that many researchers started to develop the notion of fuzzy norm in different ways. In 1992, a different approach towards the notion of fuzzy norm was introduced by Felbin[8] whose associated metric is Kaleva[11] type. She defined fuzzy norm by a mapping which assigns a non-negative fuzzy real number corresponding to each element of a linear space. Later in 1994, another idea of fuzzy norm on a linear space was introduced by Cheng and Mordeson[4] whose corresponding induced metric is Kramosil and Michalek[13] type. With a view to formulate decomposition theorem for the fuzzy norm, the definition of fuzzy norm given by Cheng and Moderson[4] was redefined by Bag and Samanta[2].

On the other hand, a number of generalization in metric space, normed linear space and in inner product space have been done.(please see [5, 7, 9, 10, 15, 17]). In 2007, the idea of cone metric space which is a generalization of metric space was introduced by Huang and Zhang[15] by replacing the range of metric with an ordered real Banach space and proved some fixed point theorems on contractive mappings on such spaces. After that, series of article on cone metric space started to appear.

With the idea of cone metric space introduced by Huang and Zhang[15], a new notion of fuzzy cone normed linear space was introduced by Bag[3] which generalizes the corresponding notion of Felbin[8] type fuzzy norm. In this context, it is worth mentioning the work of Somasundaram and Beaula[20], Park and Alaca[18],Choudhury and Das[6], Mohinta and Samanta[16], Saheli[19] .

The purpose of this paper is to introduce a new concept of fuzzy cone normed linear space that generalizes the corresponding notion of fuzzy normed linear space by Bag and Samanta[1]. Some basic definitions on fuzzy cone normed linear space are given and using these some important results on finite dimensional fuzzy cone normed linear space are established.

The organization of the paper is as follows:
In section 2, some preliminary results are given which are used in this paper.
In section 3, a definition of fuzzy cone normed linear space is introduced and some basic results are proved.
In section 4, one fundamental lemma and some basic theorems are established on finite dimensional fuzzy cone normed linear space.

2. Preliminaries

Throughout the paper, we denote a real Banach space by E and the zero element of E by θ_E.

Definition 2.1 ([15]). Let E be a real Banach space and P be a subset of E. Then P is called a cone, if

(i) P is closed, non-empty and $P \neq \{\theta_E\}$,
(ii) $a, b \in R,\ a, b \geq 0,\ x, y \in P \Rightarrow ax + by \in P$,
(iii) $x \in P$ and $-x \in P \quad \Rightarrow x = \theta_E$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ iff $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$ while $x << y$ will stand for $y - x \in \mathrm{Int}P$, where $\mathrm{Int}P$ denotes the interior of P.

The cone P is called normal, if there is a number $K > 0$ such that for all $x, y \in E$ with $\theta_E \preceq x \preceq y$, $\|x\| \leq K\|y\|$.
The least positive number satisfying above is called the normal constant of P.

The cone P is called regular, if every increasing sequence which is bounded from above is convergent. That is if $\{x_n\}$ is a sequence in E such that

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots \preceq y,$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$.
Equivalently, the cone P is regular if every decreasing sequence which is bounded below is convergent. It is clear that a regular cone is a normal cone.

In the following we always assume that P is a cone in E with $\mathrm{Int}P \neq \phi$ and $\preceq$ is a partial ordering with respect to P.

Definition 2.2 ([14]). A binary operation $* : [0\ ,\ 1] \times [0\ ,\ 1] \to [0\ ,\ 1]$ is a t-norm, if it satisfies the following conditions:

(i) $*$ is associative and commutative,
(ii) $a * 1 = a \quad \forall a \in [0\ ,\ 1]$,
(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0\ ,\ 1]$.

If $*$ is continuous, then it is called continuous t-norm.

The following are examples of some t-norms that are frequently used and defined for all $a,\ b \in\ [0,\ 1]$:
(1) Standard intersection: $a * b = min(a,\ b)$.
(2) Algebraic product: $a * b = ab$.
(3) Bounded difference: $a * b = max(0,\ a + b - 1)$. (4) Drastic intersection:

$$a * b = \begin{cases} a & \text{for } b = 1 \\ b & \text{for } a = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.3 ([1])**.** Let U be a linear space over the field F (C or R). A fuzzy subset N of $U \times R$ (R- set of real numbers) is called a fuzzy norm on U, if
(N1) $\forall t \in R$ with $t \leq 0,\ N(x\ ,\ t) = 0$,
(N2) $\forall t \in R, t > 0, N(x\ ,\ t) = 1$ iff $x = \theta_U$ (θ_U denotes the zero element of U),
(N3) $\forall t \in R,\ t > 0, c \in F,\ N(cx\ ,\ t) = N(x\ ,\ \frac{t}{|c|})$ if $c \neq 0$,
(N4) $\forall s, t \in R;\ x, u \in U,\ N(x + u\ ,\ s + t) \geq\ N(x\ ,\ s) * N(u\ ,\ t)$,
(N5) $N(x\ ,\ .)$ is a non-decreasing function of R and $\lim\limits_{t\to\infty} N(x\ ,\ t) = 1$.
The pair $(U\ ,\ N)$ will be referred to as a fuzzy normed linear space.

3. Fuzzy cone normed linear space

In this section, we introduce a concept of fuzzy cone normed linear space in different approach.

Definition 3.1. Let X be a linear space over the field K and E be a real Banach space with cone P. Let $*$ be a t-norm. Then a fuzzy subset $N_c : X \times E \longrightarrow [0, 1]$ is said to be a fuzzy cone norm, if
(FCN1) $\forall\ t \in E$ with $t \preceq \theta_E,\ N_c(x, t) = 0$,
(FCN2) $\forall\ \theta_E \prec t,\ N_c(x, t) = 1$ iff $x = \theta_X$ (θ_X denotes the zero element of X),
(FCN3) $\forall\ \theta_E \prec t$ and $0 \neq c \in K,\ N_c(cx, t) = N_c(x, \frac{t}{|c|})$,
(FCN4) $\forall\ x, y \in X$ and $s, t \in E,\ \ N_c(x + y, s + t) \geq N_c(x, s) * N_c(y, t)$,
(FCN5) $\lim\limits_{\|t\|\to\infty} N_c(x, t) = 1$.
Then $(X, N_c, *)$ is said to be a fuzzy cone normed linear space w.r.t. E.

Remark 3.2. $N_c(x, .)$ is non-decreasing w.r.t. E.

Proof. If $s \preceq t \preceq \theta_E$, then $N_c(x, s) = 0 = N_c(x, t), s, t \in E$
Suppose $\theta_E \prec t \prec s$. Then from (FCN4),

$$\begin{aligned} N_c(x, t) * N_c(\theta_X, s - t) &\leq N_c(x + \theta_X, t + s - t) \\ &= N_c(x, s). \end{aligned}$$

Thus we get $N_c(x, t) * 1 \leq N_c(x, s)$, i.e, $N_c(x, t) \leq N_c(x, s)$. So $N_c(x, .)$ is non-decreasing w.r.t. E. □

Remark 3.3. If we choose $E = R$ (the set of real numbers) and $P = [0, \infty)$ and ordering in E as the usual ordering, then $(X, N_c, *)$ will be a Bag and Samanta[1] type fuzzy normed linear space.

Example 3.4. Let $(X, \| \ \|_1)$ be a normed linear space and take $E = R^2$. Then $P = \{(t_1, t_2) : t_1, t_2 \geq 0\} \subset E$ is a normal cone with normal constant 1. Define a function $N_c : X \times E \longrightarrow [0, 1]$ by

$$N_c(x,t) = \frac{1}{e^{\frac{\|x\|_1}{\|t\|_2}}} \quad \text{if } \theta_E \prec t$$
$$= 0 \quad \text{if } t \preceq \theta_E,$$

where $\| \ \|_2$ is the norm defined on E. If we choose $a * b = ab$, Then $(X, N_c, *)$ is a fuzzy cone normed linear space.

Proof. (i) Let $t \in E$ with $t \preceq \theta_E$. Then, by definition, we have $N_c(x,t) = 0$ for all $x \in X$. Thus (FCN1) holds.

(ii) Let $t \in E$ with $\theta_E \prec t$. Then

$$N_c(x,t) = 1 \Leftrightarrow \frac{1}{e^{\frac{\|x\|_1}{\|t\|_2}}} = 1$$
$$\Leftrightarrow e^{\frac{\|x\|_1}{\|t\|_2}} = 1$$
$$\Leftrightarrow \frac{\|x\|_1}{\|t\|_2} = 0$$
$$\Leftrightarrow \|x\|_1 = 0$$
$$\Leftrightarrow x = \theta_X \quad (\theta_X \text{ denotes the zero element of } X).$$

Thus (FCN2) holds.

(iii) Let $t \in E$ with $\theta_E \prec t$ and $0 \neq c \in K$. Then

$$N_c(cx,t) = \frac{1}{e^{\frac{\|cx\|_1}{\|t\|_2}}} = \frac{1}{e^{\frac{|c|\|x\|_1}{\|t\|_2}}} = \frac{1}{e^{\frac{\|x\|_1}{\frac{\|t\|_2}{|c|}}}} = \frac{1}{e^{\frac{\|x\|_1}{\|\frac{t}{|c|}\|_2}}} = N_c(x, \frac{t}{|c|}).$$

Thus (FCN3) holds.

(iv) We have to show that

$$N_c(x+y, s+t) \geq N_c(x,s) * N_c(y,t) \ \ \forall x, y \in X \text{ and } s, t \in E.$$

We can consider four cases:

Case (i) $s \preceq \theta_E, \ t \prec \theta_E$;

Case (ii) $s \preceq \theta_E, \ \theta_E \prec t$;

Case (iii) $s \preceq \theta_E, \ t = \theta_E$;

Case (iv) $\theta_E \prec s, \ \ \theta_E \prec t$.

In Cases (i), (ii) and (iii), we can easily prove that (FCN4) holds.

Now suppose Case (iv) holds. Then $s \prec s + t$ and $t \prec s + t$. Since P is a normal cone with normal constant 1,

$$\|s\|_2 \leq \|s+t\|_2 \text{ and } \|t\|_2 \leq \|s+t\|_2.$$

On one hand,

$$\|x+y\|_1 \leq \|x\|_1 + \|y\|_1 \leq \frac{\|s+t\|_2}{\|s\|_2}\|x\|_1 + \frac{\|s+t\|_2}{\|t\|_2}\|y\|_1$$
$$\Rightarrow \frac{\|x+y\|_1}{\|s+t\|_2} \leq \frac{\|x\|_1}{\|s\|_2} + \frac{\|y\|_1}{\|t\|_2}$$
$$\Rightarrow e^{\frac{\|x+y\|_1}{\|s+t\|_2}} \leq e^{\frac{\|x\|_1}{\|s\|_2}} \ e^{\frac{\|y\|_1}{\|t\|_2}}$$
$$\Rightarrow \frac{1}{e^{\frac{\|x\|_1}{\|s\|_2}} \ e^{\frac{\|y\|_1}{\|t\|_2}}} \leq \frac{1}{e^{\frac{\|x+y\|_1}{\|s+t\|_2}}}$$
$$\Rightarrow \frac{1}{e^{\frac{\|x\|_1}{\|s\|_2}}} \frac{1}{e^{\frac{\|y\|_1}{\|t\|_2}}} \leq \frac{1}{e^{\frac{\|x+y\|_1}{\|s+t\|_2}}}$$

$\Rightarrow N_c(x, s)\ N_c(y, t) \leq N_c(x + y, s + t)$.
Thus (FCN4) holds.

(v) If $x \neq \theta_X$, then $\lim_{\|t\|_2 \to \infty} N_c(x, t) = \lim_{\|t\|_2 \to \infty} \frac{1}{e^{\frac{\|x\|_1}{\|t\|_2}}} = 1$.

If $x = \theta_X$, then $\lim_{\|t\|_2 \to \infty} N_c(x, t) = \lim_{\|t\|_2 \to \infty} N_c(\theta_X, t) = 1$.

Thus (FCN5) holds. So $(X, N_c, *)$ is a fuzzy cone normed linear space. □

Definition 3.5. Let $(X, N_c, *)$ be a fuzzy cone normed linear space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to converge to x, if for any $t \in E$ with $\theta_E \prec t$ and $r \in (0, 1)$, $\exists$ a natural number n_0 such that

$$N_c(x_n - x, t) > 1 - r \ \forall\ n \geq n_0,\ \theta_E \prec t.$$

We denote this limit by $\lim_{n\to\infty} x_n = x$

Definition 3.6. Let $(X, N_c, *)$ be a fuzzy cone normed linear space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be a Cauchy sequence, if for any $t \in E$ with $\theta_E \prec t$ and $r \in (0, 1)$, $\exists$ a natural number n_0 such that

$$N_c(x_{n+p} - x_n, t) > 1 - r \ \forall\ n \geq n_0,\ p = 1, 2, ...$$

Definition 3.7. Let $(X, N_c, *)$ be a fuzzy cone normed linear space. A subset B of X is said to be closed, if any sequence $\{x_n\}$ in B converges to x implies that $x \in B$.

Definition 3.8. Let $(X, N_c, *)$ be a fuzzy cone normed linear space. A subset B of X is said to be the closure of A, if for any $x \in B$, $\exists$ a sequence $\{x_n\}$ in A such that

$$\lim_{n\to\infty} N_c(x_n - x, t) = 1 \ \forall\ t \in E,\ \theta_E \prec t.$$

We denote the set B by $\bar{A}$.

Definition 3.9. Let $(X, N_c, *)$ be a fuzzy cone normed linear space. A subset A of X is said to be compact, if any sequence $\{x_n\}$ in A has a subsequence converging to an element of A.

Definition 3.10. Let $(X, N_c, *)$ be a fuzzy cone normed linear space and $A \subset X$. Then A is said to be bounded, if for each r, $0 < r < 1$, there exists $t \in E$ with $\theta_E \prec t$ such that $N_c(x, t) > 1 - r \ \forall\ x \in A$.

Theorem 3.11. *Let $(X, N_c, *)$ be a fuzzy cone normed linear space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x iff*

$$\lim_{n\to\infty} N_c(x_n - x, t) = 1 \ \forall\ t \in E\ (\theta_E \prec t).$$

Proof. Let $\{x_n\}$ be a sequence in X converges to x. Then for any $t \in E$ with $\theta_E \prec t$ and $r \in (0, 1)$, $\exists$ a natural number n_0 such that

$$N_c(x_n - x, t) > 1 - r \ \forall\ n \geq n_0.$$

Since r is arbitrary, it follows that $\lim_{n\to\infty} N_c(x_n - x, t) = 1 \ \forall\ t \in E\ (\theta_E \prec t)$.

Conversely, suppose $\lim_{n\to\infty} N_c(x_n - x, t) = 1 \ \forall\ t \in E\ (\theta_E \prec t)$. Then for each $r \in (0, 1)$ and $t \in E\ (\theta_E \prec t)$, $\exists$ a natural number n_0 such that

$$N_c(x_n - x, t) > 1 - r \ \forall\ \ n \geq n_0.$$

Thus $\{x_n\}$ converges to x. This completes the proof. □

Theorem 3.12. *Let $(X, N_c, *)$ be a fuzzy cone normed linear space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence iff*

$$\lim_{n\to\infty} N_c(x_{n+p} - x_n, t) = 1 \ \forall\ t \in E\ (\theta_E \prec t), \quad p = 1, 2, ...$$

Proof. Let $(X, N_c, *)$ be a fuzzy cone normed linear space and $\{x_n\}$ be a Cauchy sequence in X. Then for any $t \in E$ with $\theta_E \prec t$ and $r \in (0, 1)$, $\exists$ a natural number n_0 such that

$$N_c(x_{n+p} - x_n, t) > 1 - r \ \forall\ n \geq n_0, \ \ p = 1, 2, ...$$

Thus $1 - N_c(x_{n+p} - x_n, t) < r \ \forall\ n \geq n_0, \ \ p = 1, 2, ...$
Since r is arbitrary, it follows that $\lim_{n\to\infty} N_c(x_{n+p} - x_n, t) = 1 \ \forall\ t \in E\ (\theta_E \prec t)$.

Conversely, Suppose $\lim_{n\to\infty} N_c(x_{n+p} - x_n, t) = 1 \ \forall\ t \in E\ (\theta_E \prec t), \ \ p = 1, 2, ...$
Then for any $t \in E$ with $\theta_E \prec t$ and $r \in (0, 1)$, $\exists$ a natural number n_0 such that

$$N_c(x_{n+p} - x_n, t) > 1 - r \ \forall\ n \geq n_0, \ \ p = 1, 2, ...$$

Thus $\{x_n\}$ is a Cauchy sequence in X. This completes the proof. □

Lemma 3.13. *Limit of a convergent sequence in a fuzzy cone normed linear space $(X, N_c, *)$ is unique, provided $*$ is continuous at (1,1).*

Proof. Let $\{x_n\}$ be a convergent sequence in $(X, N_c, *)$ and $*$ is continuous at $(1, 1)$. If possible, suppose $\{x_n\}$ converges to x and y, where $(x \neq y)$. Then

$$\lim_{n\to\infty} N_c(x_n - x, s) = 1 \ \ \forall\ s \in E\ (\theta_E \prec s)$$

and

$$\lim_{n\to\infty} N_c(x_n - y, t) = 1 \ \forall\ t \in E\ (\theta_E \prec t).$$

$$\begin{aligned} \text{Now, } N_c(x - y, s + t) &= N_c(x - x_n + x_n - y, s + t) \\ &\geq N_c(x - x_n, s) * N_c(x_n - y, t) \\ &= N_c(x_n - x, s) * N_c(x_n - y, t). \end{aligned}$$

Taking limit as $n \to \infty$, we have

$$\begin{aligned} N_c(x - y, s + t)) &\geq \lim_{n\to\infty} N_c(x_n - x, s) * \lim_{n\to\infty} N_c(x_n - y, t) \\ &= 1 * 1 = 1 \end{aligned}$$

Thus $N_c(x - y, s + t) = 1 \ \forall\ s, t \in E\ (\theta_E \prec t, \theta_E \prec s)$. So $x - y = \theta_X$, by (FCN2) (θ_X denotes the zero element of X). Hence $x = y$. This completes the proof. □

Theorem 3.14. *In a fuzzy cone normed linear space $(X, N_c, *)$, every subsequence of a convergent sequence converges to the limit of the sequence.*

Proof. Let $\{x_n\}$ be a sequence in $(X, N_c, *)$ such that $\{x_n\}$ converges to x. Then for any $t \in E\ (\theta_E \prec t)$ and $r \in (0, 1)$, $\exists$ a natural number N such that

$$N_c(x_n - x, t) > 1 - r \ \forall\ n \geq N, \ \theta_E \prec t.$$

Let $\{x_{k(n)}\}$ be a subsequence of $\{x_n\}$, where $k(n) : Z^+ \longrightarrow \{a \ subset \ of \ Z^+\}$ such that $k(n) < k(m)$, for $n < m$ (Z^+ denotes the set of positive integers). Since $\{x_{k(n)}\}$ is a subsequence, $\exists\ M$ such that $k(n) \geq N$ for $n \geq M$. Thus $N_c(x_{k(n)} - x, t) > 1 - r \ \forall\ n \geq M, \ \theta_E \prec t$. So $\{x_{k(n)}\}$ converges to x. □

Theorem 3.15. *In a fuzzy cone normed linear space $(X, N_c, *)$, with $*$ continuous at $(1, 1)$, every convergent sequence is also a Cauchy sequence.*

Proof. Let $\{x_n\}$ be a convergent sequence in $(X, N_c, *)$ and converges to x. Then $\lim_{n\to\infty} N_c(x_n - x, t) = 1, \ \forall\ t \in E\ (\theta_E \prec t)$.
Now for $\theta_E \prec t,\ \theta_E \prec s$ and $p = 1, 2, ...$, we have

$$\begin{aligned} N_c(x_{n+p} - x_n, s + t) &= N_c(x_{n+p} - x + x - x_n, s + t) \\ &\geq N_c(x_{n+p} - x, s) * N_c(x - x_n, t) \\ &= N_c(x_{n+p} - x, s) * N_c(x_n - x, t). \end{aligned}$$

Taking limit as $n \to \infty$, we have

$$\begin{aligned} \lim_{n\to\infty} N_c(x_{n+p} - x_n, s + t) &\geq \lim_{n\to\infty} N_c(x_{n+p} - x, s) * \lim_{n\to\infty} N_c(x_n - x, t) \\ &= 1 * 1 = 1. \end{aligned}$$

Thus $\lim_{n\to\infty} N_c(x_{n+p} - x_n, s + t) = 1,\ \forall\ s, t \in E\ (\theta_E \prec s, \theta_E \prec t), \quad p = 1, 2, ...$
So $\{x_n\}$ is a Cauchy sequence in $(X, N_c, *)$. □

4. Finite dimensional fuzzy cone normed linear space

In this section, one fundamental lemma is established and by using this lemma some basic theorems on finite dimensional fuzzy cone normed linear space are proved.

Lemma 4.1. *Let $(X, N_c, *)$ be a fuzzy cone normed linear space with the underlying t-norm $*$ continuous at (1,1) and $\{x_1, x_2, ..., x_n\}$ be a linearly independent set of vectors in X. Then $\exists\ c \in E$ with $\theta_E \prec c$ and $\delta \in (0, 1)$ such that for any set of scalars $\{\alpha_1, \alpha_2, ..., \alpha_n\}$,*

$$N_c(\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n, c\sum_{j=1}^{n} |\alpha_j|) < 1 - \delta, \tag{4.1.1}$$

where E is a real Banach space with cone P.

Proof. Let $s = |\alpha_1| + |\alpha_2| + ... + |\alpha_n|$.

If $s = 0$, then $\alpha_j = 0,\ \forall\ j = 1, 2, ..., n$ and the above relation (4.1.1) holds for any $c \in E$ with $\theta_E \prec c$ and $\delta \in (0, 1)$.

Next we suppose that $s > 0$. Then (4.1.1) is equivalent to

$$N_c(\beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n, c) < 1 - \delta, \tag{4.1.2}$$

for some $c \in E$ $(\theta_E \prec c)$ and $\delta \in (0, 1)$, and for all scalars $\beta' s$ with $\sum_{j=1}^{n} |\beta_j| = 1$.

If possible, suppose that (4.1.2) does not hold. Then for each $c \in E$ with $\theta_E \prec c$ and $\delta \in (0, 1)$, $\exists$ a set of scalars $\{\beta_1, \beta_2, ..., \beta_n\}$ with $\sum_{j=1}^{n} |\beta_j| = 1$ such that

$$N_c(\beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n, c) \geq 1 - \delta.$$

Thus for $c_m \in E$ $(\theta_E \prec c_m)$ with $\|c_m\| = \delta = \frac{1}{m}, m = 1, 2, ..., \exists$ a set of scalars $\{\beta_1^m, \beta_2^m, ..., \beta_n^m\}$ with $\sum_{j=1}^{n} |\beta_j^m| = 1$ such that $N_c(y_m, c_m) \geq 1 - \frac{1}{m}$, where

$$y_m = \beta_1^m x_1 + \beta_2^m x_2 + ... + \beta_n^m x_n.$$

Since $\sum_{j=1}^{n} |\beta_j^m| = 1$, we have $0 \leq |\beta_j^m| \leq 1$ for $j = 1, 2, ..., n$. So for each fixed j, the sequence $\{\beta_j^m\}$ is bounded. Hence $\{\beta_1^m\}$ has a convergent subsequence. Let β_1 denote the limit of that subsequence and let $\{y_{1,m}\}$ denote the corresponding

subsequence of $\{y_m\}$. By the same argument, $\{y_{1,m}\}$ has a subsequence $\{y_{2,m}\}$ for which the corresponding subsequence of scalars $\{\beta_2^m\}$ converges to β_2(say).
Continuing in this way, after n steps, we obtain a subsequence $\{y_{n,m}\}$, where
$y_{n,m} = \sum_{j=1}^{n} \gamma_j^m x_j$ with $\sum_{j=1}^{n} |\gamma_j^m| = 1$ and $\gamma_j^m \to \beta_j$ as $m \to \infty$.

Let $y = \beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n$. Then we have

$$\lim_{m\to\infty} N_c(y_{n,m} - y, t) = 1 \ \forall \ t \in E \ (\theta_E \prec t). \tag{4.1.3}$$

Now for $\theta_E \prec k$, choose m such that $c_m \prec k$. Then we have

$$\begin{aligned} N_c(y_{n,m}, k) &= N_c(y_{n,m} + \theta_X, c_m + k - c_m) \\ &\geq N_c(y_{n,m}, c_m) * N_c(\theta_X, k - c_m) \\ &\geq (1 - \frac{1}{m}) * N_c(\theta_X, k - c_m), \end{aligned}$$

where θ_X denotes the zero element of X. Thus $N_c(y_{n,m}, k) \geq (1-\frac{1}{m}) * N_c(\theta_X, k-c_m)$. So $\lim\limits_{m\to\infty} N_c(y_{n,m}, k) \geq 1 * 1$. Hence

$$\lim_{m\to\infty} N_c(y_{n,m}, k) = 1. \tag{4.1.4}$$

On one hand,

$$N_c(y, 2k) = N_c(y - y_{n,m} + y_{n,m}, k + k) \geq N_c(y - y_{n,m}, k) * N_c(y_{n,m}, k).$$

Then, by the continuity of t-norm $*$ at (1,1),

$$N_c(y, 2k) \geq \lim_{m\to\infty} N_c(y - y_{n,m}, k) * \lim_{m\to\infty} N_c(y_{n,m}, k).$$

Thus, by (4.1.3) and (4.1.4) $N_c(y, 2k) \geq 1 * 1$. So $N_c(y, 2k) = 1$.
Since $\theta_E \prec k$ is arbitrary, it follows that $y = \theta_X$, by (FCN2). Again since $\sum_{j=1}^{n} |\beta_j| = 1$ and $\{x_1, x_2, ..., x_n\}$ are linearly independent set of vectors,

$$y = \beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n \neq \theta_X.$$

So we arrive at a contradiction. Hence the lemma is proved. □

Theorem 4.2. *Let $(X, N_c, *)$ be a finite dimensional fuzzy cone normed linear space with the continuity of the underlying t-norm $*$ at $(1, 1)$, P be a normal cone with normal constant K. Then $(X, N_c, *)$ is complete.*

Proof. Let $(X, N_c, *)$ be a fuzzy cone normed linear space and dim$X$=k(say). Let $\{e_1, e_2, ..., e_k\}$ be a basis for X and $\{x_n\}$ be a Cauchy sequence in X. Let

$$x_n = \beta_1^n e_1 + \beta_2^n e_2 + ... + \beta_k^n e_k,$$

where $\beta_1^n, \beta_2^n, ..., \beta_k^n$ are suitable scalars. Then

$$\lim_{m,n\to\infty} N_c(x_m - x_n, t) = 1, \ \forall \ t \in E \ (\theta_E \prec t). \tag{4.2.1}$$

From Lemma 4.1, it follows that $\exists \ c \in E$ with $\theta_E \prec c$ and $\delta \in (0, 1)$ such that

$$N_c(\sum_{i=1}^{k} (\beta_i^m - \beta_i^n) e_i, c \sum_{i=1}^{k} |\beta_i^m - \beta_i^n|) < 1 - \delta. \tag{4.2.2}$$

Again for $0 < \delta < 1$, from (4.2.1), it follows that for any $\theta_E \prec \frac{t}{2}$, $\exists$ a positive integer $n_0(\delta, t)$ such that

$$N_c(x_m - x_n, \frac{t}{2}) > 1 - \delta \ \forall m, n \geq n_0(\delta, t).$$

Thus

$$N_c(\sum_{i=1}^{k}(\beta_i^m - \beta_i^n)e_i, \frac{t}{2}) > 1 - \delta, \ \forall m, n \geq n_0(\delta, t). \tag{4.2.3}$$

From (4.2.2) and (4.2.3), $N_c(\sum_{i=1}^{k}(\beta_i^m - \beta_i^n)e_i, \frac{t}{2}) > 1 - \delta$. So

$> N_c(\sum_{i=1}^{k}(\beta_i^m - \beta_i^n)e_i, c\sum_{i=1}^{k}|\beta_i^m - \beta_i^n|), \ \forall m, n \geq n_0(\delta, t)$

$\Rightarrow c\sum_{i=1}^{k}|\beta_i^m - \beta_i^n| \prec \frac{t}{2}, \ \forall m, n \geq n_0(\delta, t)$

(Since $N_c(x, .)$ is non decreasing w.r.t. E)

$\Rightarrow \|c\sum_{i=1}^{k}|\beta_i^m - \beta_i^n|\| \leq K\|\frac{t}{2}\|, \ \ \forall m, n \geq n_0(\delta, t)$

(Since P is a normal cone with normal constant K)

$\Rightarrow \sum_{i=1}^{k}|\beta_i^m - \beta_i^n|\|c\| \leq K\frac{\|t\|}{2}, \ \ \forall m, n \geq n_0(\delta, t)$

$\Rightarrow \sum_{i=1}^{k}|\beta_i^m - \beta_i^n| \leq \frac{K||t||}{2||c||}, \ \ \forall m, n \geq n_0(\delta, t)$

$\Rightarrow |\beta_i^m - \beta_i^n| \leq \frac{K\|t\|}{2\|c\|} < \frac{K\|t\|}{\|c\|}, \ \ \forall m, n \geq n_0(\delta, t)$ and $i = 1, 2, ..., k$.

Since $\theta_E \prec t$ is arbitrary, $\lim_{m,n\to\infty} |\beta_i^m - \beta_i^n| = 0, \ \ i = 1, 2, ..., k$.

This implies that $\{\beta_i^n\}$ is a Cauchy sequence of scalars for each $i = 1, 2, ..., k$. Hence each $\{\beta_i^n\}$ converges.

Let $\lim_{n\to\infty} \beta_i^n = \beta_i$ for $i = 1, 2, ..., k$ and $x = \sum_{i=1}^{k}\beta_i e_i$. Then clearly, $x \in X$. Now for all $t \in E$ $(\theta_E \prec t)$,

$$N_c(x_n - x, t) = N_c(\sum_{i=1}^{k}\beta_i^n e_i - \sum_{i=1}^{k}\beta_i e_i, t) = N_c(\sum_{i=1}^{k}(\beta_i^n - \beta_i)e_i, t).$$

Thus

$$N_c(x_n - x, t) \geq N_c(e_1, \frac{t}{k|\beta_1^n - \beta_1|}) * N_c(e_2, \frac{t}{k|\beta_2^n - \beta_2|}) * ... * N_c(e_k, \frac{t}{k|\beta_k^n - \beta_k|}). \tag{4.2.4}$$

Since $\beta_i^n \to \beta_i$ as $n \to \infty$ for $i = 1, 2, ..., k$, when $n \to \infty$,

$$\|\tfrac{t}{k|\beta_i^n - \beta_i|}\| \to \infty, \ \ \forall \ t \in E \ (\theta_E \prec t) \text{ and for } i = 1, 2, ..., k.$$

From (4.2.4), using the continuity of t-norm $*$ at $(1, 1)$, we get

$$\lim_{n\to\infty} N_c(x_n - x, t) \geq 1 * 1 * ... * 1, \ \ \forall \ t \in E \ (\theta_E \prec t).$$

So $\lim_{n\to\infty} N_c(x_n - x, t) = 1, \ \ \forall \ t \in E \ (\theta_E \prec t)$. Hence $x_n \to x$. Therefore $(X, N_c, *)$ is complete. □

Theorem 4.3. *Let $(X, N_c, *)$ be a finite dimensional fuzzy cone normed linear space with the underlying t-norm $*$ continuous at $(1, 1)$, P be a normal cone with normal constant K. Then a subset $A \subset X$ is compact iff A is closed and bounded.*

Proof. First we suppose that A is compact. We will now show that A is closed and bounded.

Let $x \in \bar{A}$. Then $\exists$ a sequence $\{x_n\}$ in A such that $\lim_{n\to\infty} x_n = x$. Since A is compact, $\exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to a point in A. Since $\{x_n\} \to x$, $\{x_{n_k}\} \to x$. Then $x \in A$. Thus A is closed.

If possible, suppose that A is not bounded. Then $\exists\ r = r_0$, $0 < r_0 < 1$ such that for each $t \in E$ with $\theta_E \prec t$, there is an element $x \in A$ such that $N_c(x,t) \leq 1 - r_0$. Thus for each $c_n \in Eand\ \theta_E \prec c_n$ with $\|c_n\| < \|c_{n+1}\|$,

$$\exists\ \ x_n \in A \text{ such that } N_c(x_n, c_n) \leq 1 - r_0 \ (n \in N, \text{the set of natural numbers}).$$

Since A is compact, $\exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to some element $x \in A$. So $\lim_{k\to\infty} N_c(x_{n_k} - x, t) = 1,\ \forall\, t \in E,\ \ \theta_E \prec t$. Since $N_c(x_{n_k}, c_{n_k}) \leq 1 - r_0$,

$1 - r_0 \geq N_c(x_{n_k}, c_{n_k}) = N_c(x_{n_k} - x + x, c_{n_k} - t + t),\ \ \theta_E \prec t$

$\Rightarrow 1 - r_0 \geq N_c(x_{n_k} - x, t) * N_c(x, c_{n_k} - t)$

$\Rightarrow 1 - r_0 \geq \lim_{k\to\infty} N_c(x_{n_k} - x, t) * \lim_{k\to\infty} N_c(x, c_{n_k} - t)$

(Using the continuity of t-norm $*$ at $(1,1)$)

$\Rightarrow 1 - r_0 \geq 1 * 1$

(Since $\|c_{n_k}\| \to \infty$ as $k \to \infty$, $\|c_{n_k} - t\| \to \infty$ as $k \to \infty$, thus $\|c_{n_k} - t\| \geq |\|c_{n_k}\| - \|t\||$)

$\Rightarrow r_0 \leq 0$.

This is a contradiction. Hence A is bounded.

Conversely, suppose that A is closed and bounded. Let $\dim X = n$ and $\{e_1, e_2, ..., e_n\}$ be a basis for X. Choose a sequence $\{x_k\}$ in A and suppose $x_k = \beta_1^k e_1 + \beta_2^k e_2 + ... + \beta_n^k e_n$, where $\beta_1^k, \beta_2^k, ..., \beta_n^k$ are scalars. Then, from Lemma 4.1, $\exists\ c \in E$, $(\theta_E \prec c)$ and $\delta \in (0,1)$ such that

$$N_c\Big(\sum_{i=1}^{n} \beta_i^k e_i, c\sum_{i=1}^{n} |\beta_i^k|\Big) < 1 - \delta. \tag{4.3.1}$$

Since A is bounded, for $\delta \in (0,1)$, $\exists\ \ t \in E\ (\theta_E \prec t)$ such that $N_c(x_k, t) > 1 - \delta$,i.e.,

$$N_c\Big(\sum_{i=1}^{n} \beta_i^k e_i, t\Big) > 1 - \delta. \tag{4.3.2}$$

From (4.3.1) and (4.3.2), we get

$N_c(\sum_{i=1}^n \beta_i^k e_i, c\sum_{i=1}^n |\beta_i^k|) < 1 - \delta < N_c(\sum_{i=1}^n \beta_i^k e_i, t)$

$\Rightarrow N_c(\sum_{i=1}^n \beta_i^k e_i, c\sum_{i=1}^n |\beta_i^k|) < N_c(\sum_{i=1}^n \beta_i^k e_i, t)$

$\Rightarrow c\sum_{i=1}^n |\beta_i^k| \prec t$ ($N_c(x,.)$ is non decreasing w.r.t. E)

$\Rightarrow \|c\sum_{i=1}^n |\beta_i^k|\| \leq K\|t\|$

(Since P is a normal cone with normal constant K)

$\Rightarrow \sum_{i=1}^n |\beta_i^k| \|c\| \leq K\|t\|$

$\Rightarrow |\beta_i^k| \leq \frac{K\|t\|}{\|c\|}$ for $k = 1, 2, ...$ and $i = 1, 2, ..., n$.

Thus each $\{\beta_i^k\}$ $(i = 1, 2, ..., n)$ is bounded. By repeated application of Bolzano-Weierstrass theorem, it follows that each of the sequences $\{\beta_i^k\}$ has a convergent subsequence say $\{\beta_i^{k_l}\}$, $\forall$ $i = 1, 2, ..., n$. Let

$$x_{k_l} = \beta_1^{k_l} e_1 + \beta_2^{k_l} e_2 + ... + \beta_n^{k_l} e_n,$$

where $\{\beta_1^{k_l}\}, \{\beta_2^{k_l}\}, ..., \{\beta_n^{k_l}\}$ are all convergent. Let $\beta_i = \lim_{l\to\infty} \beta_i^{k_l}$, $i = 1, 2, ..., n$ and let $x = \beta_1 e_1 + \beta_2 e_2 + ... + \beta_n e_n$. Then for $t \in E$ $(\theta_E \prec t)$, we have

$$N_c(x_{k_l} - x, t) = N_c(\sum_{i=1}^{n} (\beta_i^{k_l} - \beta_i) e_i, t).$$

Thus

$$N_c(x_{k_l} - x, t) \geq N_c(e_1, \frac{t}{n|\beta_1^{k_l} - \beta_1|}) * N_c(e_2, \frac{t}{n|\beta_2^{k_l} - \beta_2|}) * ... * N_c(e_n, \frac{t}{n|\beta_n^{k_l} - \beta_n|}). \tag{4.3.3}$$

Since $\beta_i^{k_l} \to \beta_i$ as $l \to \infty$ for $i = 1, 2, ..., n$, when $l \to \infty$,

$$\|\frac{t}{n|\beta_i^{k_l} - \beta_i|}\| \to \infty, \quad \forall \quad \theta_E \prec t \text{ and for } i = 1, 2, ..., n.$$

From (4.3.3), using the continuity of t-norm $*$ at $(1, 1)$, we get

$$\lim_{l\to\infty} N_c(x_{k_l} - x, t) \geq 1 * 1 * ... * 1.$$

So $\lim_{l\to\infty} N_c(x_{k_l} - x, t) = 1$. Since $\theta_E \prec t$ is arbitrary, it follows that $\lim_{l\to\infty} x_{k_l} = x$, i.e, $\{x_{k_l}\}$ is a convergent subsequence of $\{x_k\}$ and converges to x. Since A is closed and $\{x_k\}$ is a sequence in A, it follows that $x \in A$. Hence every sequence in A has a convergent subsequence that converges to an element of A. Therefore A is compact. □

5. Conclusion

In this paper, a notion of fuzzy cone normed linear space is introduced in a different approach which is a generalization of fuzzy normed linear space. Here a real Banach space is considered instead of R(the set of real numbers) in fuzzy normed linear space. It is seen that Bag and Samanta type[1] fuzzy normed linear space is a particular case of fuzzy cone normed linear space. By using this concept, some basic results on finite dimensional fuzzy cone normed linear space are established. Since fuzzy mathematics along with the classical ones are constantly developing, so the concept of fuzzy cone normed linear can also play an important part in the new fuzzy area and fuzzy funtional analysis.

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