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Intra-regular and weakly regular ordered ternary semigroups

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Abstract. In this paper we characterize regular and weakly regular, intra-regular and weakly regular, regular and intra-regular ordered ternary semigroups in terms of fuzzy ideals, respectively.

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1. Introduction

The eminent concept of a fuzzy subset was introduced by Zadeh [36] in 1965. Since then enormous studies have been published on fuzzy sets proving its importance in fields of set theory, logic, measure theory, group theory, real analysis, semigroups and topology etc. Kuroki was another main contributor to the study of fuzzy sets on semigroups. The fuzzy ideals in semigroups were first introduced by Kuroki, and he also studied fuzzy ideals, fuzzy bi-ideals, fuzzy quasi-ideals and fuzzy semiprime ideals of semigroups ([20, 21, 22, 23, 24]). Ordered semigroups and computer science are closely related fields, especially in case of theory of automata and formal language, and it has been widely studied by many researchers (see, [9, 12, 13, 14]).

Ternary semigroups have been studied by many authors in [6, 19, 28, 33]. Sioson [33] investigated the ideal theory in ternary semigroups and regular ternary semigroups and developed several results by using the properties of quasi-ideals. Ordered ternary semigroups were studied in [2, 4, 7, 8]. Among others, several regularities such as von Neumann regularity, weak regularity and intra-regularity have been characterized in semigroups, ordered semigroups, ternary semigroups and ordered ternary semigroups in terms of various kinds of ideals and fuzzy ideals in [4, 10, 16, 17, 18, 26, 27, 29, 30, 31, 32, 34, 35].

The main purpose of this paper is to study regular and weakly regular, intra-regular and weakly regular, regular and intra-regular ordered ternary semigroups. We characterize these kinds of ordered ternary semigroups in terms of fuzzy ideals, respectively. Several results in [16, 17, 18, 29, 31, 32, 34] are generalized.

2. Preliminaries

A ternary semigroup is a non-empty set S with a ternary operation denoted by juxtaposition such that $(uvw)xy = u(vwx)y = uv(wxy)$ for all $u, v, w, x, y \in S$. An ordered ternary semigroup is a ternary semigroup S with a partial ordering $\leq$ such that $x_1x_2x_3 \leq y_1y_2y_3$ for any x_i, $y_i \in S$ with $x_i \leq y_i$, $i = 1, 2, 3$.

An ordered semigroup is an ordered ternary semigroup with the same ordering and with the ternary operation xyz induced by the binary operation of the semigroup. A ternary semigroup is an ordered ternary semigroup with the trivial ordering.

We write A^3 for AAA for any element or non-empty subset A of an ordered ternary semigroup.

In what follows a (fuzzy) subset always means a non-empty one.

Let S be an ordered ternary semigroup. For $A \subseteq S$ write

$$(A] = \{t \in S \mid t \leq a \text{ for some } a \in A\}.$$

We have $((A](B](C]] = (ABC]$ for any subsets A, B, C of S.

A subset A of an ordered ternary semigroup S is called a ternary subsemigroup of S if $(A] \subseteq A$ and $AAA \subseteq A$.

A subset A of an ordered ternary semigroup S is called left (resp. lateral, right) ideal of S, if $(A] = A$ and $SSA \subseteq A$ (resp. $SAS \subseteq A$, $ASS \subseteq A$). A left and right ideal is called a two-sided ideal. A left, lateral and right ideal is called an ideal.

A subset Q of an ordered ternary semigroup S is called a quasi-ideal if $(Q] = Q$, $(QSS] \cap (SQS] \cap (SSQ] \subseteq Q$ and $(QSS] \cap (SSQSS] \cap (SSQ] \subseteq Q$.

A ternary subsemigroup B of an ordered ternary semigroup S is called a bi-ideal if $(B] = B$ and $BSBSB \subseteq B$.

We denote by $L(a)$ (resp. $R(a)$, $M(a)$, $T(a)$, $I(a)$, $Q(a)$, $B(a)$) the left (resp. right, lateral, two-sided, ideal, quasi- and bi-) ideal of S generated by a. Then we have for example

$$\begin{aligned}
L(a) &= (a \cup SSa], \\
R(a) &= (a \cup aSS], \\
T(a) &= (a \cup aSS \cup SSa \cup SSaSS], \\
Q(a) &= (a \cup aSS] \cap (a \cup SaS \cup SSaSS] \cap (a \cup SSa], \\
B(a) &= (a \cup a^3 \cup aSaSa].
\end{aligned}$$

Let S be an ordered ternary semigroup. By a fuzzy subset of S we mean a mapping $f : S \to [0, 1]$. For $A \subseteq S$ the characteristic function f_A of A is a fuzzy subset of S. We write $f_S = 1$ and $f_\varnothing = 0$. For any $a \in S$, we define

$$A_a = \{(x, y, z) \in S \times S \times S \mid a \leq xyz\}.$$

We denote by $F(S)$ the set of all fuzzy subsets of S. We define an ordered relation $\preceq$ on $F(S)$ as follows:

$$f \preceq g \text{ if and only if } f(x) \leq g(x) \text{ for all } x \in S.$$

It is easy to see that $(F(S), \preceq)$ is a poset. Furthermore for $f, g, h \in F(S)$, we define $f \circ g \circ h : S \to [0,1]$ by

$$a \mapsto \begin{cases} \bigvee_{(x,y,z)\in A_a} \min\{f(x), g(y), h(z)\}, & \text{if } A_a \neq \varnothing; \\ 0, & \text{if } A_a = \varnothing. \end{cases}$$

Then $F(S)$ is an ordered ternary semigroup (see, [1]). Moreover, if S is an ordered semigroup, then $F(S)$ is an ordered semigroup ([15]) and the induced ordered ternary semigroup is the same as that defined above ([1]), which ensures that our discussion on ordered ternary semigroups includes ordered semigroups as a special case.

According to [2], a fuzzy subset f of an ordered ternary semigroup S is called a fuzzy subsemigroup, if for all $x, y, z \in S$, we have

(i) $x \leq y$ implies $f(x) \geq f(y)$,

(ii) $f(xyz) \geq \min\{f(x), f(y), f(z)\}$.

A fuzzy subset f of S is called a fuzzy left (resp. lateral, right) ideal of S, if for all $x, y, z \in S$, we have

(i) $x \leq y$ implies $f(x) \geq f(y)$,

(ii) $f(xyz) \geq f(z)$ (resp. $f(xyz) \geq f(y)$, $f(xyz) \geq f(x)$).

A fuzzy left and fuzzy right ideal is called a fuzzy two-sided ideal. A fuzzy left, fuzzy lateral and fuzzy right ideal is called a fuzzy ideal.

According to [26], a fuzzy subset f of an ordered ternary semigroup S is called a fuzzy quasi-ideal of S, if

(i) $x \leq y$ implies $f(x) \geq f(y)$ for any $x, y \in S$,

(ii) $(f \circ 1 \circ 1) \wedge (1 \circ f \circ 1) \wedge (1 \circ 1 \circ f) \preceq f$,

(iii) $(f \circ 1 \circ 1) \wedge (1 \circ 1 \circ f \circ 1 \circ 1) \wedge (1 \circ 1 \circ f) \preceq f$.

A fuzzy subsemigroup f of S is called a fuzzy bi-ideal of S, if for any $p, q, x, y, z \in S$,

(i) $x \leq y$ implies $f(x) \geq f(y)$,

(ii) $f(xpyqz) \geq f(x) \wedge f(y) \wedge f(z)$.

By [2, 10], a subset A of an ordered ternary semigroup S is a left (resp. right, lateral, two-sided, quasi-, bi-) ideal of S if and only if the characteristic function f_A of A is a fuzzy left (resp. right, lateral, two-sided, quasi-, bi-) ideal of S, and moreover, (fuzzy) one-sided ideals of S are (fuzzy) quasi-ideals, and (fuzzy) quasi-ideals are (fuzzy) bi-ideals.

Let S be an ordered semigroup such that $(S^3] = S$. Then left ideals, right ideals and ideals of the ordered semigroup S are the same as left ideals, right ideals and two-sided ideals of the induced ordered ternary semigroup S, respectively. The same conclusions hold for fuzzy ideals.

3. Regular and weakly regular ordered ternary semigroups

Recall that an ordered ternary semigroup S is called regular if every $a \in S$ is regular in the sense of $a \in (aSa]$ and is called left weakly regular if every $a \in S$ is left weakly regular in the sense of $a \in ((SSa)^3]$. Right weak regularity is defined similarly. By weak regularity we mean both left and right weak regularity. Regularity and weak regularity do not imply each other (see [1]).

Lemma 3.1 ([1])**.** *Let S be an ordered ternary semigroup. For $a \in S$ the following are equivalent:*

(1) *a is left weakly regular,*
(2) *$a \in (SaSSa]$,*
(3) *$a \in (ST(a)L(a)]$.*

Lemma 3.2. *Let S be an ordered ternary semigroup. If S is regular and satisfies $a \in SaS$ for all $a \in S$, then S is weakly regular.*

Proof. Since $a \in (aSa] \subseteq ((SaS)Sa]$, for all $a \in S$, S is left weakly regular by Lemma 3.1 and similarly, S is also right weakly regular. □

Lemma 3.2 implies that a regular ordered ternary semigroup with identity is weakly regular.

A weakly regular ordered ternary semigroup satisfying $a \in SaS$ for all $a \in S$ is not necessarily regular, for example, a simple semigroup with identity is always weakly regular and satisfies $a \in SaS$ for all $a \in S$ but not necessarily regular.

In this section we will characterize a regular and weakly regular ordered ternary semigroup, which includes the case of regular ordered semigroups, since regular ordered semigroup is weakly regular.

Lemma 3.3. *Let S be an ordered ternary semigroup and $a \in S$. Then the following are equivalent:*

(1) *a is regular and left weakly regular (resp. right weakly regular, weakly regular),*
(2) *$a \in (aSa] \cap (SaS]$,*
(3) *$a \in (aSSaSSa]$,*
(4) *$a \in (R(a)T(a)L(a)]$.*

Proof. (1) $\Rightarrow$ (2): The prof follows from Lemma 3.1.

(2) $\Rightarrow$ (3): The proof follows from the fact that $a \in (aSa] \cap (SaS]$ implies $a \in (aSaSa] \subseteq (aS(SaS)Sa]$.

(3) $\Rightarrow$ (4): The proof is trivial.

(4) $\Rightarrow$ (1): Since $a \in (R(a)T(a)L(a)]$ implies $a \in (aSS] \cap (SSa]$, we have

$$SSa \cup aSS \subseteq (SSaSS].$$

It follows that $R(a) = (aSS]$, $L(a) = (SSa]$ and $T(a) = (SSaSS]$. Thus

$$\begin{aligned} a \in (R(a)T(a)L(a)] &= (aSS(SSaSS]SSa] \subseteq (aSSaSSa] \\ &\subseteq (aSa] \cap (SaSSa]. \end{aligned}$$

So a is left weakly regular by Lemma 3.1.

The rest follows from the symmetry of (3). □

Lemma 3.4. *Let S be an ordered ternary semigroup. Let $\mathcal{A}$ be a collection of subsets of S containing all left ideals of S, $\mathcal{A}'$ a collection of subsets of S containing all right ideals of S and $\mathcal{A}''$ a collection of subsets of S containing all two-sided ideals of S. Then the following are equivalent:*

(1) *S is regular and weakly regular (resp. left weakly regular, right weakly regular),*
(2) *$a \in (aSSaSSa]$, for any $a \in S$,*

(3) *$R \cap R' \cap A \subseteq (RR'A]$, for any right ideals R, R' and any $A \in \mathcal{A}$,*
(4) *$A' \cap L' \cap L \subseteq (A'L'L]$, for any left ideal L', L and any $A' \in \mathcal{A}'$,*
(5) *$R \cap A'' \cap L \subseteq (RA''L]$, for any right ideal R, left ideal L and any $A'' \in \mathcal{A}''$,*
(6) *$A' \cap T \cap A \subseteq (A'TA]$, for any two-sided ideal T, any $A' \in \mathcal{A}'$ and $A \in \mathcal{A}$,*
(7) *$R \cap T \cap L \subseteq (RTL]$ for any right ideal R, two-sided ideal T and left ideal L,*
(8) *$R \cap L \subseteq (RLSRL]$, for any right ideal R and left ideal L.*

Proof. Equivalence between (1) and (2) follows from Lemma 3.3.

(2) $\Rightarrow$ (3): The proof follows from the fact that $a \in ((aSS)(aSS)a] \subseteq (RR'A]$, for any $a \in R \cap R' \cap A$.

(2) $\Rightarrow$ (4)-(7) can be proved in similar ways. Clearly any one of (3) through (6) implies (7).

(7)$\Rightarrow$ (1): The proof follows from Lemma 3.3.

(1) $\Rightarrow$ (8): Let $a \in R \cap L$. Then by Lemma 3.3, we have

$$a \in (aSSaSSa] \subseteq (a(SSa)(SSaSS)(aSS)a] \subseteq (RLSRL].$$

(8) $\Rightarrow$ (1): For any $a \in S$, we have

$$a \in R(a) \cap L(a) \subseteq (R(a)L(a)SR(a)L(a)] \subseteq (R(a)T(a)L(a)].$$

Thus (1) follows from Lemma 3.3. □

Theorem 3.5. *Let S be an ordered ternary semigroup. Let $\mathcal{U}$ be a collection of fuzzy subsets of S containing all fuzzy left ideals of S, $\mathcal{U}'$ a collection of fuzzy subsets of S containing all fuzzy right ideals of S and $\mathcal{U}''$ a collection of fuzzy subsets of S containing all fuzzy two-sided ideals of S. Then the following are equivalent:*

(1) *S is regular and weakly regular (resp. left weakly regular, right weakly regular),*
(2) *$g \wedge g' \wedge u \preceq g \circ g' \circ u$, for any fuzzy right ideals g, g' and any $u \in \mathcal{U}$,*
(3) *$u' \wedge f' \wedge f \preceq u' \circ f' \circ f$, for any fuzzy left ideal f', f and any $u' \in \mathcal{U}'$,*
(4) *$g \wedge u'' \wedge f \preceq g \circ u'' \circ f$, for any fuzzy right ideal g, any left ideal f and any $u'' \in \mathcal{U}''$,*
(5) *$u' \wedge h \wedge u \preceq u' \circ h \circ u$, for any fuzzy two-sided ideal h, any $u' \in \mathcal{U}'$ and any $u \in \mathcal{U}$,*
(6) *$g \wedge h \wedge f \preceq g \circ h \circ f$, for any fuzzy right ideal g, fuzzy two-sided ideal h and fuzzy left ideal f,*
(7) *$g \wedge f \preceq g \circ f \circ 1 \circ g \circ f$, for any fuzzy right ideal g and fuzzy left ideal f.*

Proof. (1) $\Rightarrow$ (2): Let $a \in S$. Then by Lemma 3.4(2), we have

$a \leq astaxya$ for some $s, t, x, y \in S$.

Thus

$$(g \circ g' \circ u)(a) \geq \min\{g(ast), g'(axy), u(a)\} \geq \min\{g(a), g'(a), u(a)\} = (g \wedge g' \wedge u)(a).$$

(1) $\Rightarrow$ (3)-(6): The proofs can be proved in the way similar to (1) $\Rightarrow$ (2).

(1) $\Rightarrow$ (7): Let $a \in S$. Then by Lemma 3.4(2), we have

$a \leq astaxya$ for some $s, t, x, y \in S$.

Thus $a \leq astaxyastaxya$. So

$$\begin{aligned}(g \circ f \circ 1 \circ g \circ f)(a) &\geq \min\{g(a), f(sta), 1(xyast), g(axy), f(a)\} \\ &\geq \min\{g(a), f(a), 1, g(a), f(a)\} \\ &= \min\{g(a), f(a)\} \\ &= (g \wedge f)(a).\end{aligned}$$

The rest of the proof follows from Lemma 3.4, by considering characteristic functions. □

Corollary 3.6. *Let S be an ordered ternary semigroup. Then the following are equivalent:*
(1) *S is regular and weakly regular,*
(2) *$g \wedge h \wedge f = g \circ h \circ f$, for any fuzzy right ideal g, fuzzy two-sided ideal h and fuzzy left ideal f,*
(3) *$g \wedge f = g \circ f \circ 1 \circ g \circ f$, for any fuzzy right ideal g and fuzzy left ideal f.*

It is interesting to compare our results with characterizations of regular ordered ternary semigroups presented in [4, 26] (see [5, 10, 11] for regular ternary semigroups), where for example a typical result states that an ordered ternary semigroup S is regular if and only if $g \wedge m \wedge f = g \circ m \circ f$ for any fuzzy right ideal g, fuzzy lateral ideal m and fuzzy left ideal f.

Since regular ordered semigroups are weakly regular, Theorem 3.5 and its corollary generalize several results on regular ordered semigroups in the literature, for example [16, 32].

Lemma 3.7 ([1])**.** *Let S be a weakly regular ordered ternary semigroup. Then a subset Q of S is a quasi-ideal of S if and only if $Q = (SSQ] \cap (QSS]$.*

Lemma 3.8 ([27])**.** *An ordered ternary semigroup S is left (resp. right) weakly regular if and only if every fuzzy left (resp. right) ideal f is idempotent.*

Lemma 3.9 ([1])**.** *Let S be a weakly regular ordered ternary semigroup. Then a fuzzy subset f of S is a fuzzy quasi-ideal of S if and only if $f = (1 \circ 1 \circ f) \wedge (f \circ 1 \circ 1)$.*

Lemma 3.10 ([1])**.** *Let S be an ordered ternary semigroup. If S is left weakly regular then $f \preceq 1 \circ 1 \circ f \preceq 1 \circ f \circ 1 \preceq 1 \circ 1 \circ f \circ 1 \circ 1$ for any fuzzy subset f of S. In addition, a similar conclusions holds if S is right weakly regular.*

Lemma 3.11 ([1])**.** *Let S be an ordered ternary semigroup and f be a fuzzy subset of S. Then $1 \circ 1 \circ f$ (resp. $f \circ 1 \circ 1$, $1 \circ f \circ 1$ and $1 \circ 1 \circ f \circ 1 \circ 1$) is a fuzzy left (resp. right, two-sided) ideal of S.*

Theorem 3.12. *Let S be an ordered ternary semigroup. Then S is regular and weakly regular if and only if fuzzy left ideals and fuzzy right ideals are all idempotent and $g \circ 1 \circ f$ is a fuzzy quasi-ideal for any fuzzy right ideal g and fuzzy left ideal f.*

Proof. ($\Rightarrow$): Idempotence of fuzzy left ideals and fuzzy right ideals follows from Lemma 3.8. Set $h = g \circ 1 \circ f$, $h_1 = h \circ 1 \circ 1$ and $h_2 = 1 \circ 1 \circ h$. Then by Lemma 3.10, $h \preceq h_1 \wedge h_2$. Since h_1 is a fuzzy right ideal and h_2 is a fuzzy left ideal by Lemma 3.11, Corollary 3.6 implies

$$h_1 \wedge h_2 = h_1 \circ h_2 \circ 1 \circ h_1 \circ h_2 = g \circ 1 \circ f \circ 1 \circ 1 \circ h_2 \circ 1 \circ h_1 \circ 1 \circ 1 \circ g \circ 1 \circ f \preceq g \circ 1 \circ f = h.$$

Thus $h = h_1 \wedge h_2$. So h is a fuzzy quasi-ideal by Lemma 3.9.

($\Leftarrow$): Weak regularity follows form Lemma 3.8. For any $a \in S$, let $R = (aSS]$ and $L = (SSa]$. By the assumption, $f_R \circ 1 \circ f_L = f_{(RSL]}$ is a fuzzy quasi-ideal. Then $(aSSSSSa] = (aSa]$ is a quasi-ideal. By Lemma 4.5, $(SaS] = (SSaSS]$. Thus by Lemma 3.7, $(aSa] = (aSaSS] \cap (SSaSa] = (aSSaSSS] \cap (SSSaSSa] = (aSSaS] \cap (SaSSa] \ni a$ by Lemma 3.1. So S is regular. □

4. Intra-regular and weakly regular ordered ternary semigroups

Recall that a semigroup S is said to be intra-regular if $a \in Sa^2S$ for all $a \in S$ ([3]). Due to its importance intra-regularity is also studied in the setting of ordered semigroups, ternary semigroups and ordered ternary semigroups.

An ordered semigroup S is called intra-regular if $a \in (Sa^2S]$ for all $a \in S$ ([13]). A ternary semigroup S is called intra-regular if $a = Sa^3S$ for all $a \in S$ ([5, 10]).

An intra-regular ordered ternary semigroup S may be defined twofold:

(a) $a \in (Sa^3S]$, for all $a \in S$ or
(b) $a \in SSaSaSS$ for all $a \in S$ ([25]).

Now we introduce another intra-regularity for ordered ternary semigroups.

Definition 4.1. An ordered ternary semigroup S is called intra-regular if its every element a is intra-regular in the sense of $a \in (SSaaS]$.

In the case of an ordered semigroup, the intra-regularities defined by Definition 4.1 and (a) are the same as the original one by ([13, Remark 1]) and are strictly stronger than that in the sense of (b) since a regular semigroup as an ordered semigroup with the trivial ordering satisfies (b) but not necessarily (a). In general, Definition 4.1 is clearly weaker than (a), but we do not know whether they are equivalent.

Lemma 4.2. *For an element a of an ordered ternary semigroup S, the following are equivalent:*

(1) *a is intra-regular,*
(2) $a \in (SaaSS]$,
(3) $a \in (SSaaSaaSS]$,
(4) $a \in (SaaSaaS]$.

Proof. (1) ⇒ (2): Since $a \in (SSaaS]$, we have

$$(SSaaS] \subseteq (SSa(SSaaS)S] = ((SSaSS)aaSS] \subseteq (SaaSS].$$

Then (2) holds.

(2) ⇒ (3): Since $a \in (SaaSS]$, we have

$$(SaaSS] \subseteq (S(SaaSS)(SaaSS)SS] \subseteq (SSaaSaaSS].$$

Then (3) holds.

(3) ⇒ (4): Since $a \in (SSaaSaaSS]$, we have

$$\begin{aligned}(SSaaSaaSS] &\subseteq (SS(SSaaSaaSS]aSa(SSaaSaaSS]SS] \\ &\subseteq ((SSaaS)aa(SSaSaSS)aa(SaaSS)] \\ &\subseteq (SaaSaaS].\end{aligned}$$

(4) ⇒ (1): It follows from the fact that $(SaaSaaS] \subseteq (SSaaS]$. □

Lemma 4.3 ([1])**.** *Let S be an ordered ternary semigroup and $a \in S$. If $a \in (SaS]$, then $M(a) = T(a) = I(a) = (SaS] = (SSaSS]$.*

Lemma 4.4. *Let a be an element of an ordered ternary semigroup S. Then the following are equivalent:*

(1) *a is intra-regular and left weakly regular,*
(2) $a \in (SaaSa]$,
(3) $a \in (SSaaSSa]$,

(4) $a \in (L(a)R(a)L(a)]$,
(5) $a \in (SL(a)B(a)]$.

Proof. (1) $\Rightarrow$ (2): Since a is left weakly regular and intra-regular, by Lemma 4.2, we have $a \in ((SSa)^3] \subseteq (SS(SaaSS)(SSa)(SSa)] \subseteq (SaaSSaSSa)] \subseteq (SaaSa]$.

(2) $\Rightarrow$ (3): It follows from

$$a \in (SaaSa] \subseteq (SaaS(SaaSa)] \subseteq (S(SaaSa)aS(SaaSa)] \subseteq (SSaaSSa].$$

(3) $\Rightarrow$ (4): It follows from the fact that $(SSaaSSa] \subseteq (L(a)R(a)L(a)]$.

(4) $\Rightarrow$ (5): By $a \in (L(a)R(a)L(a)]$, we get that $a \in (SS(a \cup SSa)] \subseteq (SSa]$ and $a \in (SR(a)S] = (S(a \cup aSS)S] \subseteq (SaS]$. Then

$$a \in (L(a)R(a)L(a)] = (SSa(a \cup aSS)SSa] \subseteq (SSaaSSa].$$

Thus $a \in (SSa(SSaaSSa)SSa] = (S(SaSSa)(aSSaSSa)] \subseteq (SL(a)B(a)]$, since $(SSaSS] = (SaS]$ by Lemma 4.3.

(5) $\Rightarrow$ (1): Note that $a \in (SL(a)B(a)]$ implies $a \in (SSa]$. Thus

$$\begin{aligned}(SL(a)B(a)] &= (SSSa(a \cup a^3 \cup aSaSa)] \\ &\subseteq (Saa \cup SaaSa \cup SaaSaSa] \\ &\subseteq (Saa \cup SaaSa].\end{aligned}$$

If $a \in (Saa]$, then $a \in (SSaaa] \subseteq (SSSaaaa] \subseteq (SaaSa]$. Thus $a \in (SaaSa]$. So a is intra-regular and left weakly regular by Lemma 4.4. □

Lemma 4.5 ([1]). *Let S be an ordered ternary semigroup. If S is left weakly regular then $a \in (SSa] \subseteq (SaS] = (SSaSS]$ for any $a \in S$. Moreover, a similar conclusions holds if S is right weakly regular.*

Lemma 4.6. *Let S be an ordered ternary semigroup and $\mathcal{A}$ be a collection of subsets of S containing all left ideals of S. Then S is intra-regular and left weakly regular if and only if one of the following assertions holds:*

(1) *$L \cap B \cap B' \subseteq (LBB']$, for any left ideal L and any bi-ideals B, B',*

(2) *$L \cap A \cap L' \subseteq (LAL']$, for any left ideals L', L and any subset (resp. bi-ideal, quasi-ideal) A,*

(3) *$L \cap R \cap A \subseteq (LRA]$, for any left ideal L, any right ideal R and any $A \in \mathcal{A}$,*

(4) *$T \cap A \cap B \subseteq (TAB]$, for any two-sided ideal T, any bi-ideal B and any $A \in \mathcal{A}$*

(5) *$A \cap B \subseteq (SAB]$, for any bi-ideal B and any $A \in \mathcal{A}$.*

Proof. It is enough to prove the necessity for any subset A.

(1) Let $a \in L \cap B \cap B'$. Then by Lemma 4.4, we have

$$a \in (SaaSa] \subseteq (S(SaaSa)(SaaSa)Sa] = (SSa(aSaSa)(aSaSa)] \subseteq (LBB'].$$

(2) Let $a \in L \cap A \cap L'$. Then by Lemma 4.4, we have

$$a \in (SSaaSSa] = ((SSa)a(SSa)] \subseteq (LAL'].$$

(3) Let $a \in L \cap R \cap A$. Then by Lemma 4.4, we have

$$a \in (SSaaSSa] = ((SSa)(aSS)a] \subseteq (LRA].$$

(4) Let $a \in T \cap A \cap B$. Then by Lemma 4.5, we have $(SaS] = (SSaSS]$. It follows from Lemma 4.4 that

$$a \in (SaaSa] \subseteq (Sa(SaaSa)Sa] = ((SaS)a(aSaSa)] \subseteq (TAB].$$

(5) It follows by taking $T = S$ in (4).

To prove the sufficiency, it is enough to consider the cases of (1) and (2), for any quasi-ideal A and (3)-(5), for any left ideal A. In these cases, either of (1) and (4) implies (5) and (2) implies (3).

Assume that (5) holds. Then for any $a \in S$, we have $a \in L(a) \cap B(a) \subseteq (SL(a)B(a)]$. Thus Lemma 4.4 implies that a is intra-regular and left weakly regular.

Assume that (3) holds. Then for any $a \in S$, we have $a \in L(a) \cap R(a) \cap L(a) \subseteq (L(a)R(a)L(a)]$. Thus Lemma 4.4 implies that a is intra-regular and left weakly regular. □

Theorem 4.7. *Let S be an ordered ternary semigroup and $\mathcal{U}$ be a collection of fuzzy subsets of S containing all fuzzy left ideals of S. Then S is intra-regular and left weakly regular if and only if one of the following assertions holds:*

(1) $f \wedge g \wedge g' \preceq f \circ g \circ g'$, *for any fuzzy left ideal f and any bi-ideals g, g',*

(2) $f \wedge u \wedge f' \preceq f \circ u \circ f'$, *for any fuzzy left ideals f, f' and any fuzzy subset (resp. bi-ideal, quasi-ideal) u,*

(3) $f \wedge h \wedge u \preceq f \circ h \circ u$, *for any fuzzy left ideal f, any fuzzy right ideal h and any $u \in \mathcal{U}$,*

(4) $h \wedge u \wedge g \preceq h \circ u \circ g$, *for any fuzzy two-sided ideal h, any fuzzy bi-ideal g and any $u \in \mathcal{U}$,*

(5) $u \wedge g \preceq 1 \circ u \circ g$, *for any fuzzy bi-ideal g and any $u \in \mathcal{U}$.*

Proof. The sufficiency follows from Lemma 4.6 by considering characteristic functions. Now we prove the necessity. It is enough to do this for (1) and (2)-(5) for any fuzzy subset u.

(1) Let f be any fuzzy left ideal, and let g, g' be bi-ideals of S. Let $a \in S$. Then by Lemma 4.4, we have $a \leq saata$ for some $s, t \in S$. Thus

$$a \leq s(saata)(saata)ta = (ssa)(atasa)(atata).$$

It follows that

$$\begin{aligned}(f \circ g \circ g')(a) &\geq \min\{f(ssa), g(atasa), g'(atata)\}.\\ &\geq \min\{f(a), g(a) \wedge g(a) \wedge g(a), g'(a) \wedge g'(a) \wedge g'(a)\}\\ &\geq \min\{f(a), g(a), g'(a)\}\\ &= (f \wedge g \wedge g')(a).\end{aligned}$$

(2) Let f and f' be any fuzzy left ideals and u any fuzzy subset. Let $a \in S$. Then by Lemma 4.4, we have $a \leq (sta)a(xya)$, for some $s, t, x, y \in S$. Thus

$$\begin{aligned}(f \circ u \circ f')(a) &\geq \min\{f(sta), u(a), f'(xya)\}\\ &\geq \min\{f(a), u(a), f'(a)\}\\ &= (f \wedge u \wedge f')(a).\end{aligned}$$

(3) Let f be any fuzzy left ideal, h be any fuzzy right ideal and u be any fuzzy subset. Let $a \in S$. Then by Lemma 4.4, we have $a \leq (sta)(axy)a$, for some $s, t, x, y \in S$. Thus

$$\begin{aligned}(f \circ h \circ u)(a) &\geq \min\{f(sta), h(axy), u(a)\}\\ &= \min\{f(a), h(a), u(a)\}\\ &= (f \wedge h \wedge u)(a).\end{aligned}$$

(4) Let h be a fuzzy two-sided ideal, g a fuzzy bi-ideal and u a fuzzy subset of S. Let $a \in S$. Then by Lemma 4.4, we have

$$a \in (SaaSa] \subseteq (Sa(SaaSa)Sa] = (SaS)a(aSaSa)].$$

Thus $a \in (SaS]$. So $a \leq (stauv)a(axaya)$, for some $s, t, u, v, x, y \in S$. Hence

$$\begin{aligned}(h \circ u \circ g)(a) &\geq \min\{h(stauv), u(a), g(axaya)\}\\ &\geq \min\{h(a), u(a), g(a)\}\\ &= (h \wedge u \wedge g)(a).\end{aligned}$$

(5) It follows by (1) implying (4) and by taking $h = 1$ in (4). □

Theorem 4.7 generalizes corresponding results in [17, 18, 31, 32].

5. Regular and intra-regular ordered ternary semigroups

Lemma 5.1. *Let S be an ordered ternary semigroup. Then for $a \in S$ the following are equivalent:*

1. *a is regular and intra-regular,*
2. *$a \in ((aSa)^3]$,*
3. *$a \in (aSSaaSa]$,*
4. *$a \in (aSaaSSa]$,*
5. *$a \in (B(a)^3]$,*
6. *$a \in (B(a)R(a)L(a)]$.*

Proof. (1) ⇒ (2): By (1) and Lemma 4.2(3), we have

$$a \in aSa \subseteq aSaSa \subseteq aS(SSaaSaaSS)Sa \subseteq ((aSa)^3].$$

(2) ⇒ (3): The proof follows from $a \in ((aSa)^3] = ((aSa)(aSa)(aSa)] \subseteq (aSSaaSa]$.

(3) ⇒ (4): The proof follows from

$$a \in (aSSaaSa] \subseteq (aSS(aSSaaSa)aSa] = (a(SSaSS)aa(SaaS)a \subseteq (aSaaSSa].$$

(4) ⇒ (5): By (4), we have $a \in aSa$, implying $B(a) = aSa$. Thus

$$\begin{aligned}a &\in (aSaaSSa] \subseteq (aSa(aSaaSSa)SSa]\\ &= ((aSa)(aSa)(aSSaSSa]\\ &\subseteq (B(a)^3].\end{aligned}$$

(5) ⇒ (6): The proof follows from $B(a) \subseteq L(a)$.

(6) ⇒ (1): By (1), we have $a \in aSa$, implying a is regular, and $B(a) = aSa$, $R(a) = aSS$ and $L(a) = SSa$. Then (6) yields $a \in (aSaaSSSSa] \subseteq (SSaaS]$, implying a is intra-regular. □

Lemma 5.2. *Let S be an ordered ternary semigroup. Then the following conditions are equivalent:*

1. *S is regular and intra-regular,*
2. *every bi-ideal of S is idempotent,*
3. *every quasi-ideal of S is idempotent.*

Proof. (1) ⇒ (2): Suppose S is both regular and intra-regular. Let B be a bi-ideal of S. Let $a \in B$.Then Lemma 5.1, gives $a \in (B(a)^3] \subseteq (B^3]$. Thus $B \subseteq B^3$. The reverse inclusion follows from the fact that B is a ternary subsemigroup. So (2) is proved.

(2) $\Rightarrow$ (3): The proof follows from the fact that quasi-ideals are bi-ideals.

(3) $\Rightarrow$ (1): For any $a \in S$, note that $Q(a)^3 \subseteq aSa$. By Lemma 5.1, we have

$$a \in Q(a) = (Q(a)^3] = (Q(a)^5] \subseteq (B(a)R(a)L(a)].$$

By Lemma 5.1, S is regular and intra-regular. □

Theorem 5.3. *Let S be an ordered ternary semigroup. Then the following conditions are equivalent:*

(1) *S is regular and intra-regular.*
(2) *every fuzzy bi-ideal f of S is idempotent,*
(3) *every fuzzy quasi-ideal f of S is idempotent.*

Proof. (1) $\Rightarrow$ (2): Let f be a fuzzy bi-ideals of S. Let $a \in S$. Then by Lemma 5.1, there exists $u, x, y, z \in S$ such that $a \leq aua$ and $a \leq axaayaaza$. Thus

$$a \leq ax(aua)(aua)ya(aua)za = (axaua)(auaya)(auaza).$$

So

$$\begin{aligned}(f \circ f \circ f)(a) &\geq \min\{f(axaua), f(auaya), f(auaza)\} \\ &\geq \min\{f(a) \wedge f(a) \wedge f(a), f(a) \wedge f(a) \wedge f(a), f(a) \wedge f(a) \wedge f(a)\} \\ &= f(a).\end{aligned}$$

Hence, $f \preceq f \circ f \circ f$. Since every fuzzy bi-ideal is a fuzzy subsemigroup, we have $f \circ f \circ f \preceq f$. Therefore $f \circ f \circ f = f$.

(2) $\Rightarrow$ (3): The proof follows from the fact that fuzzy quasi-ideals are fuzzy bi-ideals.

(3) $\Rightarrow$ (1): The proof follows from Lemma 5.2. □

6. Conclusions

The study of a fuzzy set in ordered ternary semigroup was initiated by Chinram ([2]). We apply successfully this technique of fuzzifications to different classes of ordered ternary semigroups to generalize and unify the results of ordered semigroups and ternary semigroups in literature. Regular and weakly regular ordered ternary semigroups, intra-regular and weakly regular ordered ternary semigroups, and regular and intra-regular ordered ternary semigroups are characterized in terms of fuzzy ideals. It is interesting that a regular and intra-regular ordered ternary semigroup is characterized by idempotence of bi-ideals and quasi-ideals.

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References

[1] S. Bashir and X. Du, On weakly regular fuzzy ordered ternary semigroups, Appl. Math. Inf. Sci 10 (6) (2016) 1–8.
[2] R. Chinram and S. Saelee, Fuzzy ideals and fuzzy filters of ordered ternary semigroups, J. Math. Res. 2 (1) (2010) 9-97.
[3] A. H. Clifford and G. B. Preston, The algebraic theory of semigroups. Vol. I. Math. Surveys, No. 7. Amer. Math. Soc. Providence, R.I. 1961.

[4] V. R. Daddi and Y. S. Pawar, On ordered ternary semigroups, Kyungpook Math. J. 52 (4) (2012) 375–381.
[5] T. K. Dutta, S. Kar, B. K. Maity, On ideals in regular ternary semigroups, Discuss. Math. Gen. Algebra Appl. 28 (2008) 147-159.
[6] R. B. Feizullaer, Ternary semigroup locally homeomorphic mapping, Dokl. Akad. Nauk. USSR. 252 (5) (1980) 1063–1065.
[7] A. Iampan, Characterizing the minimality and maximality of ordered lateral ideals in ordered ternary semigroups, J. Korean Math. Soc. 46 (4) (2009) 775–784.
[8] A. Iampan, On ordered ideal extensions of ordered ternary semigroups, Lobachevskii J. Math. 31 (1) (2010) 13–17.
[9] Y. B. Jun, J. Meng, X. L. Xin, On ordered filters of implicative semigroups, Semigroup Forum. 54 (1997) 75-82.
[10] S. Kar and P. Sarkar, Fuzzy quasi-ideals and fuzzy bi-ideals of ternary semigroup, Ann. Fuzzy Math. Inform. 2 (4) (2012) 407–423.
[11] S. Kar and P. Sarkar, Fuzzy ideals of ternary semigroup, Fuzzy Inform. Eng. 2 (2012) 181–193.
[12] N. Kehayopulu, On weakly prime ideals of ordered semigroups, Semigroup Forum. 44 (1992) 341–346.
[13] N. Kehayopulu, On intra-regular ordered semigroups, Semigroup Forum. 46 (1993) 271–278.
[14] N. Kehayopulu and M. Tsingelis, Fuzzy sets in ordered semigroup s, Semigroup Forum 65 (2002) 128–132.
[15] N. Kehayopulu and M. Tsingelis, The embedding of an ordered semigroup into a poe-semigroup in terms of fuzzy sets, Inform Sci. 152 (2003) 231–236.
[16] N. Kehayopulu and M. Tsingelis, Regular ordered semigroups in terms of fuzzy subsets, Inform. Sci. 176 (2006) 3675–3693.
[17] N. Kehayopulu and M. Tsingelis, Intra-regular ordered semigroups in terms of fuzzy sets. Lobachevskii J. Math. 30 (1) (2009) 23–29.
[18] N. Kehayopulu, Left quasi-regular and intra-regular ordered semigroups using fuzzy ideals. Quasigroups Related Systems 20 (2) (2012) 239–249.
[19] H. K. Kim, W. R. Fred, Ternary semigroup associative on each pair of factor. Simon Stevin, 54 (1980) 65-74.
[20] N. Kuroki, Fuzzy bi-ideals in semigroups, Comment. Math. Univ. St. Pauli. 28 (1979) 17–21.
[21] N. Kuroki, On fuzzy ideals and fuzzy bi-ideals in semigroups, Fuzzy Sets and Systems. 5 (1981) 203-215.
[22] N. Kuroki, On fuzzy semigroups, Inform. Sci. 53 (1991) 203–236.
[23] N. Kuroki, Fuzzy generalized bi-ideals in semigroups, Inform. Sci. 66 (1992) 235–243.
[24] N. Kuroki, Fuzzy semiprime quasi-ideals in semigroups. Inform. Sci. 75 (1993) 201–211.
[25] S. Lekkoksung and N. Lekkoksung, On intra-regular ordered ternary semigroup, Int. J Math. Analysis 6 (2) (2012) 69–73.
[26] N. Lekkoksung and P. Jampachon, Characterizations of regular ordered ternary semigroups in terms of fuzzy subsets, Southeast-Asian J. Sci. 2 (2) (2013) 141–153.
[27] N. Lekkoksung and P. Jampachon, On right weakly regular ordered ternary semigroup, Quasigroups related systems, 22 (2014) 257–266.
[28] E. S. Lyapin, Realization of ternary semigroups in binary semigroups. Modern algebra, pp. 43-48, Leningrad. Univ., Leningrad 1981.
[29] M. Shabir and M. Bano, Prime bi-ideals in ternary semigroup, Quasigroups related systems 16 (2008) 239–256.
[30] M. Shabir, A. Khan, Characterizations of ordered semigroups by the properties of their fuzzy generalized bi-ideals, New Math. Nat. Comput. 4 (2) (2008) 237–250.
[31] M. Shabir and A. Khan, Characterizations of ordered semigroups by the properties of their fuzzy ideals, Comput. Math. Appl. 59 (2010) 539–549.
[32] M. Shabir, A. Khan, On fuzzy ordered semigroups, Inform. Sci. 274 (2014) 236-248.
[33] F. M. Sioson, On regular algebraic systems, Proc. Japan. Acad. Ser. A Math. Sci. 39 (1963) 283-286.
[34] J. Tang and X. Xie, Some characterizations of left weakly regular ordered semigroups, Commun. Math. Res. 27 (3) (2011) 253–267.

[35] J. Tang and X. Xie, Characterizations of regular ordered semigroups by generalized fuzzy ideals, J. Intell. Fuzzy Systems 26 (1) (2014) 239–252.
[36] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

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