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Homomorphism and anti homomorphism of cubic ideals of near-rings

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Abstract. In this paper, we discuss some characterizations of cubic ideals of near-rings with examples. Also investigate cubic ideals of near-rings using homomorphism and anti homomorphism.

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1. Introduction

Fuzzy set was first introduced by Zadeh[12]. After ten years he[13] defined new idea of interval valued fuzzy subsets, where the values of the membership functions are the intervals instead of numbers. Biswas[2] presented the concept of fuzzy subgroups and anti fuzzy subgroups. Abou-Zaid[1] introduced the new concept of fuzzy subnear-rings and ideals. Thillaigovindan et al.[11] studied the concept of interval valued fuzzy ideals of near-rings. Chandrasekhara Rao et al.[3] discussed the concept of anti homomorphism of near-rings. Jun et al.[6] initiated the new idea cubic set by using two sets interval valued fuzzy set and a fuzzy set. Further, Jun et al.[4] studied the cubic subalgebras and ideals over BCK/BCI algebras. Again Jun et al.[5] studied the concept of cubic q-ideals of BCI-algebras. Jun et al.[7] applied the structure of cubic ideals of BCI-algebras. Further Jun et al.[8] studied about cubic ideals of semigroups. Satyanarayana and Bindu Madhavi[10] introduced the notion Cubic H-ideals in BCK-Algebras. Kim et al.[9] initiated the new idea of anti fuzzy ideals in near-rings. In this paper, we discuss some characterizations of cubic ideals of near-rings with examples. Also investigate cubic ideals of near-rings using homomorphism and anti homomorphism.

2. Preliminaries

Throughout this paper R will denote a left near-ring. In this section, we present some basic definitions and results used in this paper.

Definition 2.1 ([9])**.** A near-ring is an algebraic system $(R,+,\cdot)$ consisting of a non empty set R together with two binary operations $+$ and $\cdot$ such that $(R,+)$ is a group, not necessarily abelian and $(R,\cdot)$ is a semigroup in which the distributive law: $x\cdot(y+z)=x\cdot y+x\cdot z$ holds for all $x,y,z\in R$. We will use the word 'near-ring 'to mean 'left near-ring '. We denote xy instead of $x\cdot y$.

An ideal I of a near-ring R is a subset of R such that
(i) $(I,+)$ is a normal subgroup of $(R,+)$,
(ii) $RI\subseteq I$,
(iii) $(x+a)y-xy\in I$, for any $a\in I$ and $x,y\in R$.

Note that I is a left ideal of R, if I satisfies (i) and (ii), and a right ideal of R, if it satisfies (i) and (iii).

Definition 2.2 ([11])**.** By an interval number $\overline{a}$, we mean an interval $[a^-,a^+]$ such that $0\leq a^-\leq a^+\leq 1$ where a^- and a^+ are the lower and upper limits of $\overline{a}$ respectively. The set of all closed subintervals of $[0,1]$ is denoted by $D[0,1]$. We also identify the interval $[a,a]$ by the number $\overline{a}\in D[0,1]$. For any interval numbers $\overline{a}_j=[a_j^-,a_j^+],\overline{b}_j=[b_j^-,b_j^+]\in D[0,1], j\in J$, we define

$$\max\{\overline{a}_j,\overline{b}_j\}=[\max\{a_j^-,b_j^-\},\max\{a_j^+,b_j^+\}],$$
$$\min\{\overline{a}_j,\overline{b}_j\}=[\min\{a_j^-,b_j^-\},\min\{a_j^+,b_j^+\}],$$
$$\inf\overline{a}_j=\left[\bigcap_{j\in I}a_j^-,\bigcap_{j\in I}a_j^+\right],\sup\overline{a}_j=\left[\bigcup_{j\in I}a_j^-,\bigcup_{j\in I}a_j^+\right]$$

and put
(i) $\overline{a}\leq\overline{b}\Longleftrightarrow a^-\leq b^-$ and $a^+\leq b^+$,
(ii) $\overline{a}=\overline{b}\Longleftrightarrow a^-=b^-$ and $a^+=b^+$,
(iii) $\overline{a}<\overline{b}\Longleftrightarrow\overline{a}\leq\overline{b}$ and $\overline{a}\neq\overline{b}$,
(iv) $k\overline{a}=[ka^-,ka^+]$, whenever $0\leq k\leq 1$.

Definition 2.3 ([11])**.** Let X be a non-empty set. A mapping $\overline{\mu}:X\to D[0,1]$ is called an i-v fuzzy subset of X. For all $x\in X$, $\overline{\mu}(x)=[\mu^-(x),\mu^+(x)]$, where μ^- and μ^+ are fuzzy subsets of X such that $\mu^-(x)\leq\mu^+(x)$. Thus $\overline{\mu}(x)$ is an interval(a closed subinterval of $[0,1]$) and not a number from the interval $[0,1]$ as in the case of a fuzzy set.

Let $\overline{\mu},\overline{\nu}$ be i-v fuzzy subsets of X. Then
(i) $\overline{\mu}\leq\overline{\nu}\Leftrightarrow\overline{\mu}(x)\leq\overline{\nu}(x)$,
(ii) $\overline{\mu}=\overline{\nu}\Leftrightarrow\overline{\mu}(x)=\overline{\nu}(x)$,
(iii) $(\overline{\mu}\cup\overline{\nu})(x)=\max\{\overline{\mu}(x),\overline{\nu}(x)\}$,
(iv) $(\overline{\mu}\cap\overline{\nu})(x)=\min\{\overline{\mu}(x),\overline{\nu}(x)\}$,
(v) $\overline{\mu}^c(x)=\overline{1}-\overline{\mu}(x)=[1-\mu^+(x),1-\mu^-(x)]$.

Definition 2.4 ([4])**.** Let X be a nonempty set. A cubic set $\mathscr{A}$ of X is a structure $\mathscr{A}(x)=\{(x,\overline{\mu}_A(x),\lambda_A(x)):x\in X\}$ which is briefly denoted by $\mathscr{A}=(\overline{\mu}_A,\lambda_A)$, where $\overline{\mu}_A=[\mu_A^-,\mu_A^+]$ is an i-v fuzzy subset of X and λ is a fuzzy set of X.

In this case, we will use $\mathscr{A}(x) = (\overline{\mu}_A(x), \lambda_A(x)) = ([\mu_A^-(x), \mu_A^+(x)], \lambda_A(x))$ for all $x \in X$.

Definition 2.5 ([7])**.** Let $\mathscr{A} = (\overline{\mu}_A, \lambda_A)$ be a cubic set of X. For any $r \in [0,1]$ and $[s,t] \in D[0,1]$, we define $U(\mathscr{A};[s,t],r)$ as follows:

$$U(\mathscr{A};[s,t],r) = \{x \in X | \overline{\mu}_A(x) \geq [s,t], \lambda_A(x) \leq r\}$$

and we say it is a cubic level set of $\mathscr{A} = (\overline{\mu}_A, \lambda_A)$.

For any non-empty subset G of a set X, the characteristic cubic set of G of X is defined to be a structure $\chi_G = \{(x, \overline{\mu}_{\chi_G}(x), \lambda_{\chi_G}(x)) : x \in X\}$ which is briefly denoted by $\chi_G(x) = (\overline{\mu}_{\chi_G}(x), \lambda_{\chi_G}(x))$, where

$$\overline{\mu}_{\chi_G}(x) = \begin{cases} [1,1] \text{ if } x \in G, \\ [0,0] \text{ otherwise}, \end{cases} \qquad \lambda_{\chi_G}(x) = \begin{cases} 0 \text{ if } x \in G, \\ 1 \text{ otherwise}. \end{cases}$$

Definition 2.6 ([4])**.** For two cubic sets $\mathscr{A} = (\overline{\mu}_A, \lambda_A)$ and $\mathscr{B} = (\overline{\mu}_B, \lambda_B)$ of a near-ring R we define $\mathscr{A} \sqsubseteq \mathscr{B} \Leftrightarrow \overline{\mu}_A \leq \overline{\mu}_B, \lambda_A \geq \lambda_B$.

Definition 2.7 ([4])**.** Let $\mathscr{A} = (\overline{\mu}_A, \lambda_A)$ and $\mathscr{B} = (\overline{\mu}_B, \lambda_B)$ be two cubic sets of X.

(i) The intersection of $\mathscr{A}$ and $\mathscr{B}$, denoted by $\mathscr{A} \sqcap \mathscr{B}$, is the cubic set

$$\mathscr{A} \sqcap \mathscr{B} = (\overline{\mu}_A \cap \overline{\mu}_B, \lambda_A \vee \lambda_B),$$

where $(\overline{\mu}_A \cap \overline{\mu}_B)(x) = \min\{\overline{\mu}_A(x), \overline{\mu}_B(x)\}$ and $(\lambda_A \vee \lambda_B)(x) = \max\{\lambda_A(x), \lambda_B(x)\}$, for all $x \in X$.

(ii) The union of $\mathscr{A}$ and $\mathscr{B}$, denoted by $\mathscr{A} \sqcup \mathscr{B}$ is the cubic set

$$\mathscr{A} \sqcup \mathscr{B} = (\overline{\mu}_A \cup \overline{\mu}_B, \lambda_A \wedge \lambda_B),$$

where $(\overline{\mu}_A \cup \overline{\mu}_B)(x) = \max\{\overline{\mu}_A(x), \overline{\mu}_B(x)\}$ and $(\lambda_A \wedge \lambda_B)(x) = \min\{\lambda_A(x), \lambda_B(x)\}$, for all $x \in X$.

Definition 2.8 ([9])**.** Let R and S be near-rings. A map $\theta : R \to S$ is called a (near-ring)homomorphism, if $\theta(x+y) = \theta(x) + \theta(y)$ and $\theta(xy) = \theta(x)\theta(y)$ for all $x, y \in R$.

Definition 2.9 ([3])**.** Let R and S be near-rings. A map $\theta : R \to S$ is called a (near-ring) anti-homomorphism, if $\theta(x+y) = \theta(y) + \theta(x)$ and $\theta(xy) = \theta(y)\theta(x)$, for all $x, y \in R$.

3. Some characterizations of cubic ideals of near-rings

In this section, we introduce the notion of cubic ideals of near-rings and establish some of their properties.

Definition 3.1. A cubic set $\mathscr{A} = (\overline{\mu}_A, \lambda_A)$ of a near-ring R is called a cubic subnear-ring of R, if

(i) $\overline{\mu}_A(x-y) \geq \min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}$,
(ii) $\overline{\mu}_A(xy) \geq \min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}$,
(iii) $\lambda_A(x-y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$,
(iv) $\lambda_A(xy) \leq \max\{\lambda_A(x), \lambda_A(y)\}$, for all $x, y \in R$.

Definition 3.2. Let $\mathscr{A} = (\overline{\mu}_A, \lambda_A)$ be a cubic set of R. We say that $\mathscr{A}$ is a cubic ideal of R, if it satisfies the following:
(i) $\overline{\mu}_A(x - y) \geq \min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}$,
(ii) $\overline{\mu}_A(y + x - y) \geq \overline{\mu}_A(x)$,
(iii) $\overline{\mu}_A(xy) \geq \overline{\mu}_A(y)$,
(iv) $\overline{\mu}_A((x + z)y - xy) \geq \overline{\mu}_A(z)$,
(v) $\lambda_A(x - y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$,
(vi) $\lambda_A(y + x - y) \leq \lambda_A(x)$,
(vii) $\lambda_A(xy) \leq \lambda_A(y)$,
(viii) $\lambda_A((x + z)y - xy) \leq \lambda_A(z)$, for all $x, y \in R$.

Example 3.3. Let $R = \{a, b, c, d\}$ be a set with two binary operations defined as follows:

$+$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	b	c	d

Then $(R, +, \cdot)$ is a near-ring. Define a cubic set $\mathscr{A} = (\overline{\mu}_A, \lambda_A)$ by
$$\overline{\mu}_A(a) = [0.8, 0.9], \overline{\mu}_A(b) = [0.6, 0.7], \overline{\mu}_A(c) = [0.5, 0.5] = \overline{\mu}_A(d)$$
and
$$\lambda_A(a) = 0.2, \lambda_A(b) = 0.6, \lambda_A(c) = 0.8 = \lambda_A(d).$$
Then, $\mathscr{A} = (\overline{\mu}_A, \lambda_A)$ is a cubic ideal of R.

Lemma 3.4. *Let $\mathscr{A}$ be a cubic ideal of R. If $\mathscr{A}(x) \sqsubset \mathscr{A}(y)$, i.e., $\overline{\mu}_A(x) < \overline{\mu}_A(y)$ and $\lambda_A(x) > \lambda_A(y)$. Then*
$$\overline{\mu}_A(x - y) = \overline{\mu}_A(x) = \overline{\mu}_A(y - x)$$
and
$$\lambda_A(x - y) = \lambda_A(x) = \lambda_A(y - x).$$

Proof. Let $\mathscr{A}$ be a cubic ideal of R. Let $x, y \in R$. Then
$$\overline{\mu}_A(x - y) \geq \min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\} = \overline{\mu}_A(x).$$
Now,
$$\begin{aligned}\overline{\mu}_A(x) &= \overline{\mu}_A(y + x - y)\\ &= \overline{\mu}_A(y - (x - y))\\ &\geq \min\{\overline{\mu}_A(y), \overline{\mu}_A(x - y)\}\\ &= \overline{\mu}_A(x - y).\end{aligned}$$
Thus $\overline{\mu}_A(x - y) = \overline{\mu}_A(x)$.

On the other hand $\overline{\mu}_A(y - x) \geq \min\{\overline{\mu}_A(y), \overline{\mu}_A(x)\} = \overline{\mu}_A(x)$ and
$$\begin{aligned}\overline{\mu}_A(x) &= \overline{\mu}_A(y + x - y)\\ &= \overline{\mu}_A(y - (y - x)))\\ &\geq \min\{\overline{\mu}_A(y), \overline{\mu}_A(y - x)\}\\ &= \overline{\mu}_A(y - x).\end{aligned}$$

So $\overline{\mu}_A(y-x) = \overline{\mu}_A(x)$. Hence $\overline{\mu}_A(x-y) = \overline{\mu}_A(x) = \overline{\mu}_A(y-x)$.
Similarly, we have to prove the other. □

Theorem 3.5. *If $\mathscr{A}$ is a cubic ideal of R, then the set $R_{\mathscr{A}} = \{x \in R| \ \ \mathscr{A}(x) = \mathscr{A}(0)\}$ is an ideal of R.*

Proof. Let $\mathscr{A}$ be a cubic ideal of R. Let $x, y \in R_{\mathscr{A}}$. Then $\overline{\mu}_A(x) = \overline{\mu}_A(0), \overline{\mu}_A(y) = \overline{\mu}_A(0)$ and $\lambda_A(x) = \lambda_A(0), \lambda_A(y) = \lambda_A(0)$. Thus

$$\begin{aligned}\overline{\mu}_A(x-y) &\geq \min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}\\ &= \min\{\overline{\mu}_A(0), \overline{\mu}_A(0)\}\\ &= \overline{\mu}_A(0).\end{aligned}$$

and

$$\begin{aligned}\lambda_A(x-y) &\leq \max\{\lambda_A(x), \lambda_A(y)\}\\ &= \max\{\lambda_A(0), \lambda_A(0)\}\\ &= \lambda_A(0).\end{aligned}$$

So $\overline{\mu}_A(x-y) = \overline{\mu}_A(0)$ and $\lambda_A(x-y) = \lambda_A(0)$. Hence $x - y \in R_{\mathscr{A}}$.
Let $y \in R$ and $x \in R_{\mathscr{A}}$. Then we have $\overline{\mu}_A(y+x-y) \geq \overline{\mu}_A(x) = \overline{\mu}_A(0)$ and $\lambda_A(y+x-y) \leq \lambda_A(x) = \lambda_A(0)$. Thus $y + x - y \in R_{\mathscr{A}}$.
Let $x \in R$ and $y \in R_{\mathscr{A}}$. Then $\overline{\mu}_A(xy) \geq \overline{\mu}_A(y) = \overline{\mu}_A(0)$ and $\lambda_A(xy) \leq \lambda_A(y) = \lambda_A(0)$. Thus $xy \in R_{\mathscr{A}}$.
Similarly, we have to prove $(x+z)y - xy \in R_{\mathscr{A}}$.
Therefore $R_{\mathscr{A}}$ is an ideal of R. □

Definition 3.6. Let I be an ideal of a near-ring R. If for each $a+I, b+I$ of the factor group R/I, we define $(a+I)+(b+I) = (a+b)+I$ and $(a+I)(b+I) = (ab)+I$, for all $a, b \in R$. Then R/I is a near-ring which we shall call the residue class near-ring of R with respect to I.

Theorem 3.7. *Let I be an ideal of a near-ring R. If $\mathscr{A}$ is a cubic ideal of R, then the cubic set $\mathscr{A}$ of R/I defined by*

$$\overline{\mu}^*_A(a+I) = \sup_{x \in I} \overline{\mu}_A(a+x)$$

and

$$\lambda^*_A(a+I) = \inf_{x \in I} \lambda_A(a+x), \text{ for all } x \in I$$

is a cubic ideal of the residue class near-ring R/I of R with respect to I.

Proof. Let $a, b \in R$ be such that $a + I = b + I$. Then $b = a + y$ for some $y \in I$. Thus

$$\begin{aligned}\overline{\mu}^*_A(b+I) &= \sup_{x\in I} \overline{\mu}_A(b+x) = \sup_{x\in I} \overline{\mu}_A(a+y+x)\\ &= \sup_{x+y=z\in I} \overline{\mu}_A(a+z) = \overline{\mu}^*_A(a+I).\end{aligned}$$

and

$$\begin{aligned}\lambda^*_A(b+I) &= \inf_{x\in I} \lambda_A(b+x) = \inf_{x\in I} \lambda_A(a+y+x)\\ &= \inf_{x+y=z\in I} \lambda_A(a+z) = \lambda^*_A(a+I).\end{aligned}$$

So A is well defined. One the other hand

$$\begin{aligned}\overline{\mu}^*_A((x+I)-(y+I)) &= \overline{\mu}^*_A((x-y)+I) = \sup_{z\in I}\overline{\mu}_A((x-y)+z)\\ &= \sup_{z=u-v\in I}\overline{\mu}_A((x-y)+(u-v))\\ &= \sup_{u,v\in I}\overline{\mu}_A((x+u)-(y+v))\\ &\geq \sup_{u,v\in I}\{\min\{\overline{\mu}_A(x+u),\overline{\mu}_A(y+v)\}\}\\ &= \min\{\sup_{u\in I}\{\overline{\mu}_A(x+u)\},\sup_{v\in I}\{\overline{\mu}_A(y+v)\}\}\\ &= \min\{\overline{\mu}^*_A(x+I),\overline{\mu}^*_A(y+I)\},\end{aligned}$$

$$\begin{aligned}\lambda^*_A((x+I)-(y+I)) &= \lambda^*_A((x-y)+I) = \inf_{z\in I}\lambda_A((x-y)+z)\\ &= \inf_{z=u-v\in I}\lambda_A((x-y)+(u-v))\\ &= \inf_{u,v\in I}\lambda_A((x+u)-(y+v))\\ &\leq \inf_{u,v\in I}\{\max\{\lambda_A(x+u),\lambda_A(y+v)\}\}\\ &= \max\{\inf_{u\in I}\{\lambda_A(x+u)\},\inf_{v\in I}\{\lambda_A(y+v)\}\}\\ &= \max\{\lambda^*_A(x+I),\lambda^*_A(y+I)\},\end{aligned}$$

$$\begin{aligned}\overline{\mu}^*_A((y+I)+(x+I)-(y+I)) &= \overline{\mu}^*_A((y+x-y)+I)\\ &= \sup_{z\in I}\overline{\mu}_A((y+x-y)+z)\\ &\geq \sup_{z\in I}\overline{\mu}_A(x+z)\\ &= \overline{\mu}^*_A(x+I),\end{aligned}$$

$$\begin{aligned}\lambda^*_A((y+I)+(x+I)-(y+I)) &= \lambda^*_A((y+x-y)+I)\\ &= \inf_{z\in I}\lambda_A((y+x-y)+z)\\ &\leq \inf_{z\in I}\lambda_A(x+z)\\ &= \lambda^*_A(x+I),\end{aligned}$$

$$\begin{aligned}\overline{\mu}^*_A((x+I)(y+I)) &= \overline{\mu}^*_A((xy)+I) = \sup_{z\in I}\overline{\mu}_A((xy)+z)\\ &\geq \sup_{z\in I}\overline{\mu}_A(y+z) = \overline{\mu}^*_A(y+I),\end{aligned}$$

$$\begin{aligned}\lambda^*_A((x+I)(y+I)) &= \lambda^*_A((xy)+I)\\ &= \inf_{z\in I}\lambda_A((xy)+z)\\ &\leq \inf_{z\in I}\lambda_A(y+z) = \lambda^*_A(y+I),\end{aligned}$$

$$\begin{aligned}\overline{\mu}^*_A((x+I)(i+I))(y+I)-(x+I)(y+I) &= \overline{\mu}^*_A((x+i)y-xy)+I)\\ &= \sup_{z\in I}\overline{\mu}_A((x+i)y-xy)+z)\\ &\geq \sup_{z\in I}\overline{\mu}_A(i+z) = \overline{\mu}^*_A(i+I),\end{aligned}$$

$$\begin{aligned}\lambda^*_A((x+I)+(i+I))(y+I)-(x+I)(y+I)) &= \lambda^*_A((x+i)y-xy)+I)\\ &= \inf_{z\in I}\lambda_A((x+i)y-xy)+z)\\ &\leq \inf_{z\in I}\lambda_A(i+z)=\lambda^*_A(i+I).\end{aligned}$$

Hence $\mathscr{A}$ is a cubic ideal of R/I. □

4. Homomorphism and anti homomorphism of cubic ideals of near-rings

Definition 4.1. Let f be a mapping from a set R to a set S. Let $\mathscr{A}=(\overline{\mu}_A,\lambda_A)$ be a cubic set of R and $\mathscr{B}=(\overline{\mu}_B,\lambda_B)$ be a cubic set of S. Then

(i) The pre-image $f^{-1}(\mathscr{B})=(f^{-1}(\overline{\mu}_B),f^{-1}(\lambda_B))$ is a cubic set of R defined by

$$f^{-1}(\mathscr{B})(x)=(f^{-1}(\overline{\mu}_B)(x),f^{-1}(\lambda_B)(x))=(\overline{\mu}_B(f(x)),\lambda_B(f(x))).$$

(ii) The image $f(\mathscr{A})=(f(\overline{\mu}_A),f(\lambda_A))$ is a cubic set of S defined by

$$f(\overline{\mu}_A)(x)=\begin{cases}\sup\limits_{y\in f^{-1}(x)}\overline{\mu}_A(y) & \text{if } f^{-1}(x)\neq\varnothing\\ 0 & \text{otherwise}\end{cases}$$

$$f(\lambda_A)(x)=\begin{cases}\inf\limits_{y\in f^{-1}(x)}\lambda_A(y) & \text{if } f^{-1}(x)\neq\varnothing\\ 1 & \text{otherwise}\end{cases}$$

Theorem 4.2. *Let $f:R\to S$ be an onto near-ring homomorphism. If $\mathscr{A}=(\overline{\mu}_A,\lambda_A)$ is a cubic ideal of R, then $f(\mathscr{A})=(f(\overline{\mu}_A),f(\lambda_A))$ is a cubic ideal of S.*

Proof. Let $\mathscr{A}=(\overline{\mu}_A,\lambda_A)$ be a cubic ideal of R. Since $f(\overline{\mu}_A)(x')=\sup\limits_{f(x)=x'}\overline{\mu}_A(x)$ for $x'\in S$ and $f(\lambda_A)(x')=\inf\limits_{f(x)=x'}\lambda_A(x)$, for $x'\in S$, $f(\mathscr{A})=(f(\overline{\mu}_A),f(\lambda_A)$ is nonempty.

Let $x',y'\in S$. Then we have

$$\{x|x\in f^{-1}(x'-y')\}\supseteq\{x-y|x\in f^{-1}(x'),y\in f^{-1}(y')\}$$

and

$$\{x|x\in f^{-1}(x'y')\}\supseteq\{xy|x\in f^{-1}(x'),y\in f^{-1}(y')\}.$$

One the other hand

$$\begin{aligned}f(\overline{\mu}_A)(x'-y') &= \sup_{f(z)=x'-y'}\{\overline{\mu}_A(z)\}\geq\sup_{f(x)=x',f(y)=y'}\{\overline{\mu}_A(x-y)\}\\ &\geq \sup_{f(x)=x',f(y)=y'}\{\min\{\overline{\mu}_A(x),\overline{\mu}_A(y)\}\}\\ &= \min\{\sup_{f(x)=x'}\{\overline{\mu}_A(x)\},\sup_{f(y)=y'}\{\overline{\mu}_A(y)\}\}\\ &= \min\{f(\overline{\mu}_A)(x'),f(\overline{\mu}_A)(y')\},\end{aligned}$$

$$\begin{aligned}f(\lambda_A)(x'-y') &= \inf_{f(z)=x'-y'}\{\lambda_A(z)\}\leq\inf_{f(x)=x',f(y)=y'}\{\lambda_A(x-y)\}\\ &\leq \inf_{f(x)=x',f(y)=y'}\{\max\{\lambda_A(x),\lambda_A(y)\}\}\\ &= \max\{\inf_{f(x)=x'}\{\lambda_A(x)\},\inf_{f(y)=y'}\{\lambda_A(y)\}\}\\ &= \max\{f(\lambda_A)(x'),f(\lambda_A)(y')\},\end{aligned}$$

$$
\begin{aligned}
f(\overline{\mu}_A)(y'+x'-y') &= \sup_{f(z)=y'+x'-y'}\{\overline{\mu}_A(z)\} \geq \sup_{f(x)=x',f(y)=y'}\{\overline{\mu}_A(y+x-y)\}\\
&\geq \sup_{f(x)=x'}\{\overline{\mu}_A(x)\} = f(\overline{\mu}_A)(x'),\\
f(\lambda_A)(y'+x'-y') &= \inf_{f(z)=y'+x'-y'}\{\lambda_A(z)\} \leq \inf_{f(x)=x',f(y)=y'}\{\lambda_A(y+x-y)\}\\
&\leq \inf_{f(x)=x'}\{\lambda_A(x)\} = f(\lambda_A)(x'),\\
f(\overline{\mu}_A)(x'y') &= \sup_{f(z)=x'y'}\{\overline{\mu}_A(z)\} \geq \sup_{f(x)=x',f(y)=y'}\{\overline{\mu}_A(xy)\}\\
&\geq \sup_{f(y)=y'}\{\overline{\mu}_A(y)\} = f(\overline{\mu}_A)(y'),\\
f(\lambda_A)(x'y') &= \inf_{f(z)=x'y'}\{\lambda_A(z)\} \leq \inf_{f(x)=x',f(y)=y'}\{\lambda_A(xy)\}\\
&\leq \inf_{f(y)=y'}\{\lambda_A(y)\} = f(\lambda_A)(y')\\
f(\overline{\mu}_A)((x'+i')y'-x'y') &= \sup_{f(z)=(x'+i')y'-x'y'}\{\overline{\mu}_A(z)\},\\
&\geq \sup_{f(x)=x',f(i)=i',f(y)=y'}\{\overline{\mu}_A((x+i)y-xy)\}\\
&\geq \sup_{f(i)=i'}\{\overline{\mu}_A(i)\} = f(\overline{\mu}_A)(i'),\\
f(\lambda_A)((x'+i')y'-x'y') &= \inf_{f(z)=(x'+i')y'-x'y'}\{\lambda_A(z)\}\\
&\leq \inf_{f(x)=x',f(i)=i',f(y)=y'}\{\lambda_A((x+i)y-xy)\}\\
&\leq \inf_{f(i)=i'}\{\lambda_A(i)\} = f(\lambda_A)(i').
\end{aligned}
$$

Thus $f(\mathscr{B}) = (f(\overline{\mu}_B), f(\lambda_B))$ is a cubic ideal of S. □

Theorem 4.3. *Let $f : R \to S$ be a homomorphism of near-rings R and S. If $\mathscr{B} = (\overline{\mu}_B, \lambda_B)$ is a cubic ideal of S, then $f^{-1}(\mathscr{B}) = (f^{-1}(\overline{\mu}_B), f^{-1}(\lambda_B))$ is a cubic ideal of R.*

Proof. Let $\mathscr{B} = (\overline{\mu}_B, \lambda_B)$ be a cubic ideal of S. Let $x, y \in R$. Then

$$
\begin{aligned}
f^{-1}(\overline{\mu}_B)(x-y) &= \overline{\mu}_B(f(x-y))\\
&= \overline{\mu}_B(f(x)-f(y))\\
&\geq \min\{\overline{\mu}_B(f(x)), \overline{\mu}_B(f(y))\}\\
&= \min\{f^{-1}(\overline{\mu}_B(x)), f^{-1}(\overline{\mu}_B(y))\},\\
f^{-1}(\lambda_B)(x-y) &= \lambda_B(f(x-y)) = \lambda_B(f(x)-f(y))\\
&\leq \max\{\lambda_B(f(x)), \lambda_B(f(y))\}\\
&= \max\{f^{-1}(\lambda_B(x)), f^{-1}(\lambda_B(y))\},\\
f^{-1}(\overline{\mu}_B)(y+x-y) &= \overline{\mu}_B(f(y+x-y))\\
&= \overline{\mu}_B(f(y)+f(x)-f(y))\\
&\geq \overline{\mu}_B(f(x))\\
&= f^{-1}(\overline{\mu}_B(x)),
\end{aligned}
$$

$$\begin{aligned}f^{-1}(\lambda_B)(y+x-y) &= \lambda_B(f(y+x-y)) \\ &= \lambda_B(f(y)+f(x)-f(y)) \\ &\leq \lambda_B(f(x)) = f^{-1}(\lambda_B(x)),\end{aligned}$$

$$\begin{aligned}f^{-1}(\overline{\mu}_B)(xy) &= \overline{\mu}_B(f(xy)) \\ &= \overline{\mu}_B(f(x)f(y)) \\ &\geq \overline{\mu}_B(f(y)) \\ &= f^{-1}(\overline{\mu}_B(y)),\end{aligned}$$

$$\begin{aligned}f^{-1}(\lambda_B)(xy) &= \lambda_B(f(xy)) \\ &= \lambda_B(f(x)f(y)) \\ &\leq \lambda_B(f(y)) = f^{-1}(\lambda_B(y)),\end{aligned}$$

$$\begin{aligned}f^{-1}(\overline{\mu}_B)((x+i)y-xy) &= \overline{\mu}_B(f((x+i)y-xy)) \\ &= \overline{\mu}_B((f(x)+f(i))f(y)-f(x)f(y)) \\ &\leq \overline{\mu}_B(f(i)) \\ &= f^{-1}(\overline{\mu}_B(i)),\end{aligned}$$

$$\begin{aligned}f^{-1}(\lambda_B)((x+i)y-xy) &= \lambda_B(f((x+i)y-xy)) \\ &= \lambda_B((f(x)+f(i))f(y)-f(x)f(y)) \\ &\leq \lambda_B(f(i)) = f^{-1}(\lambda_B(i)).\end{aligned}$$

Thus $f^{-1}(\mathscr{B}) = (f^{-1}(\overline{\mu}_B), f^{-1}(\lambda_B))$ is a cubic ideal of R. □

We can also state the converse of the Theorem 4.3 by we strengthening the condition on f as follows.

Theorem 4.4. *Let $f : R \to S$ be an epimorphism of near-rings R and S. If $\mathscr{B} = (\overline{\mu}_B, \lambda_B)$ is a cubic set of S, such that $f^{-1}(\mathscr{B}) = (f^{-1}(\overline{\mu}_B), f^{-1}(\lambda_B))$ is a cubic ideal of R, then $\mathscr{B} = (\overline{\mu}_B, \lambda_B)$ is a cubic ideal of S.*

Proof. Let $x, y, i \in S$. Then $f(a) = x$, $f(b) = y$ and $f(c) = i$ for some $a, b, c \in R$. It follows that

$$\begin{aligned}\overline{\mu}_B(x-y) &= \overline{\mu}_B(f(a)-f(b)) = \overline{\mu}_B(f(a-b)) \\ &= f^{-1}(\overline{\mu}_B)(a-b) \\ &\geq \min\{f^{-1}(\overline{\mu}_B)(a), f^{-1}(\overline{\mu}_B)(b)\} \\ &= \min\{\overline{\mu}_B(f(a)), \overline{\mu}_B(f(b))\} \\ &= \min\{\overline{\mu}_B(x), \overline{\mu}_B(y)\},\end{aligned}$$

$$\begin{aligned}\lambda_B(x-y) &= \lambda_B(f(a)-f(b)) = \lambda_B(f(a-b)) \\ &= f^{-1}(\lambda_B)(a-b) \\ &\leq \max\{f^{-1}(\lambda_B)(a), f^{-1}(\lambda_B)(b)\} \\ &= \max\{\lambda_B(f(a)), \lambda_B(f(b))\} \\ &= \max\{\lambda_B(x), \lambda_B(y)\},\end{aligned}$$

$$\begin{aligned}\overline{\mu}_B(y+x-y) &= \overline{\mu}_B(f(b)+f(a)-f(b))\\ &= \overline{\mu}_B(f(b+a-b))\\ &= f^{-1}(\overline{\mu}_B)(b+a-b)\\ &\geq f^{-1}(\overline{\mu}_B)(a)\\ &= \overline{\mu}_B(f(a)) = \overline{\mu}_B(x),\end{aligned}$$

$$\begin{aligned}\lambda_B(y+x-y) &= \lambda_B(f(b)+f(a)-f(b)) = \lambda_B(f(b+a-b))\\ &= f^{-1}(\lambda_B)(b+a-b)\\ &\leq f^{-1}(\lambda_B)(a) = \lambda_B(f(a))\\ &= \lambda_B(x),\end{aligned}$$

$$\begin{aligned}\overline{\mu}_B(xy) &= \overline{\mu}_B(f(a)f(b)) = \overline{\mu}_B(f(ab))\\ &= f^{-1}(\overline{\mu}_B)(ab)\\ &\leq f^{-1}(\overline{\mu}_B)(b)\\ &= \overline{\mu}_B(f(b)) = \overline{\mu}_B(y),\end{aligned}$$

$$\begin{aligned}\lambda_B(xy) &= \lambda_B(f(a)f(b)) = \lambda_B(f(ab))\\ &= f^{-1}(\lambda_B)(ab)\\ &\leq f^{-1}(\lambda_B)(b)\\ &= \lambda_B(f(b)) = \lambda_B(y),\end{aligned}$$

$$\begin{aligned}\overline{\mu}_B((x+i)y-xy) &= \overline{\mu}_B((f(a)+f(c))f(b)-f(x)f(b))\\ &= \overline{\mu}_B(f((a+c)b-ab))\\ &= f^{-1}(\overline{\mu}_B)((a+c)b-ab) \geq f^{-1}(\overline{\mu}_B)(c) = \overline{\mu}_B(f(c))\\ &= \overline{\mu}_B(i),\end{aligned}$$

$$\begin{aligned}\lambda_B((x+i)y-xy) &= \lambda_B((f(a)+f(c))f(b)-f(x)f(b))\\ &= \lambda_B(f((a+c)b-ab))\\ &= f^{-1}(\lambda_B)((a+c)b-ab))\\ &\leq f^{-1}(\lambda_B)(c) = \lambda_B(f(c)) = \lambda_B(i).\end{aligned}$$

Thus $\mathscr{B} = (\overline{\mu}_B, \lambda_B)$ is a cubic ideal of R. □

Theorem 4.5. *Let $f : R \to S$ be an onto anti-homomorphism of near-rings. If $\mathscr{A} = (\overline{\mu}_A, \lambda_A)$ is a cubic ideal of R, then $f(\mathscr{A}) = (f(\overline{\mu}_A), f(\lambda_A))$ is a cubic ideal of S.*

Proof. It follows from Theorem 4.2 and thus its proof is omitted. □

Theorem 4.6. *Let $f : R \to S$ be an anti-homomorphism of near-rings R and S. If $\mathscr{B} = (\overline{\mu}_B, \lambda_B)$ is a cubic ideal of S, then $f^{-1}(\mathscr{B}) = (f^{-1}(\overline{\mu}_B), f^{-1}(\lambda_B))$ is a cubic ideal of R.*

Proof. Follows from Theorem 4.3 and hence omitted. □

We can also state the converse of the Theorem 4.6 by strengthening the condition on f as follows.

Theorem 4.7. *Let $f : R \to S$ be an onto anti-homomorphism of near-rings R and S. If $\mathscr{B} = (\overline{\mu}_B, \lambda_B)$ is a cubic set of S, such that $f^{-1}(\mathscr{B}) = (f^{-1}(\overline{\mu}_B), f^{-1}(\lambda_B))$ is a cubic ideal of R, then $\mathscr{B} = (\overline{\mu}_B, \lambda_B)$ is a cubic ideal of S.*

Proof. It follows from Theorem 4.4 and thus its proof is omitted. □

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