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Rough sets in neutrosophic approximation space

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Abstract. In this paper, we introduce the concepts of neutrosophic rough sets and investigate some of its properties. Further, as the characterisation of neutrosophic rough approximation operators, we introduce various notions of cut sets of neutrosophic rough sets.

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1. Introduction

Rough set theory [10], is an extension of set theory for the study of intelligent systems characterized by inexact, uncertain or insufficient information. Rough set theory has found many interesting applications. The rough set approach seems to be of fundamental importance to artificial intelligence and cognitive sciences, especially in the areas of machine learning, knowledge acquisition, decision analysis, knowledge discovery from databases, expert systems, inductive reasoning and pattern recognition [4, 11, 15]. The basic idea of rough set is based upon the approximation of sets by a pair of sets known as the lower approximation and the upper approximation of a set [7, 8]. Any subset of a universe can be characterized by two definable or observable subsets called lower and upper approximations. Zadeh [19] introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. Now fuzzy sets are combined with rough sets in a fruitful way and defined by rough fuzzy sets and fuzzy rough sets [1, 9, 14]. Atanassov [2] introduced the degree of nonmembership/falsehood (f) and defined the intuitionistic fuzzy sets. One of the interesting generalizations of the theory of fuzzy sets and intuitionistic fuzzy sets is the theory of neutrosophic sets introduced by F. Smarandache Atanassov [12, 13] which deals with the degree of indeterminacy/neutrality (i) as independent component. Neutrosophy is a branch of philosophy that studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. The idea of neutrosophy is applied in many fields in order to solve problems related to indeterminacy. Neutrosophic sets are described by three functions: Truth function, indeterminacy function and false function that are independently related. The theories of neutrosophic set have achieved great success in various areas such as medical diagnosis, database, topology, image processing, and decision making problem [5, 6, 17, 18]. While the neutrosophic set is a powerful tool to deal with indeterminate and inconsistent data and the theory of rough sets is a powerful mathematical tool to deal with incompleteness. Neutrosophic sets and rough sets are two different topics, none conflicts the other. Recently many researchers had applied the notion of neutrosophic sets to relations, group theory, ring theory, soft set theory and so on.

In this paper we combine the mathematical tools rough sets and neutrosophic sets and introduce a new class of rough sets in neutrosophic approximation space. First we review some basic notions related to rough sets and neutrosophic sets and then we construct the neutrosophic rough approximation operators and introduce neutrosophic rough sets and discuss some of their interesting properties.

2. Preliminaries

Definition 2.1 ([12])**.** A neutrosophic set A on the universe of discourse X is defined as $A = \{\langle\, x, T_A(x), I_A(x), F_A(x)\rangle, x \in X\}$ where $T, F, I : X \longrightarrow [0,1]$ and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 2.2 ([12])**.** If $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \;/x \in X\}$ and $B = \{\langle x, T_B(x), I_B(x), F_B(x)\rangle \;/x \in X\}$ are any two neutrosophic sets of X, then

(i) $A \subseteq B \;\Leftrightarrow T_A(x) \leq T_B(x);\; I_A(x) \leq I_B(x)$ and $F_A(x) \geq F_B(x)$,

(ii) $A = B \;\Leftrightarrow T_A(x) = T_B(x);\; I_A(x) = I_B(x)$ and $F_A(x) = F_B(x)\ \forall x \in X$,

(iii) $\sim A = \{\langle x, F_A(x), 1 - I_A(x), T_A(x)\rangle \;/x \in X\}$,

(iv) $A \cap B = \{\langle x, T_{(A\cap B)}(x),\quad I_{(A\cap B)}(x), F_{(A\cap B)}(x)\rangle / x \in X\}$,

where $T_{A\cap B}(x) = \min\{T_A(x), T_B(x)\}, \quad I_{A\cap B}(x) = \min\{I_A(x), I_B(x)\}$,
$F_{A\cap B}(x) = \max\{F_A(x), F_B(x)\}$,

(v) $A \cup B = \{\langle x, T_{(A\cup B)}(x),\quad I_{(A\cup B)}(x), F_{(A\cup B)}(x)\rangle / x \in X\}$

where $T_{A\cup B}(x) = \max\{T_A(x), T_B(x)\}, \quad I_{A\cup B}(x) = \max\{I_A(x), I_B(x)\}$,
$F_{A\cup B}(x) = \min\{F_A(x), F_B(x)\}$.

Definition 2.3 ([8])**.** Let $R \subseteq U \times U$ be a crisp binary relation on U.

(i) R is referred to as reflexive, if $(x,x) \in R$, for all $x \in U$.

(ii) R is referred to as symmetric, if for all $(x,y) \in U$, $(x,y) \in R$ implies $(y,x) \in R$.

(iii) R is referred to as transitive, if for all $x,y,z \in U$, $(x,y) \in R$ and $(y,z) \in R$ implies $(x,z) \in R$.

Definition 2.4 ([8])**.** Let U be a non empty universe of discourse and $R \subseteq U \times U$, an arbitrary crisp relation on U. Denote

$$xR = \{y \in U/(x,y) \in R\}, x \in U,$$

where xR is called the R-after set of x [3] or successor neighbourhood of x with respect to R [16]. The pair (U,R) is called a crisp approximation space. For any $A \subseteq U$ the upper and lower approximation of A with respect to (U,R) denoted by

$\overline{R}(A)$ and $\underline{R}(A)$ are respectively defined as follows:

$$\overline{R}(A) = \{x \in U/xR \cap A \neq \varphi\}, \quad \underline{R}(A) = \{x \in U/xR \subseteq A\}.$$

The pair $(\underline{R}(A), \overline{R}(A))$ is referred to as crisp rough set of A with respect to (U, R) and $\overline{R}$, $\underline{R} : \rho(U) \to \rho(U)$ are referred to upper and lower crisp approximation operator respectively.

The crisp approximation operator satisfies the following properties for all $A, B \in \rho(U)$.

(L_1) $\underline{R}(A) = \sim \overline{R}(\sim A)$, (U_1) $\overline{R}(A) = \sim \underline{R}(\sim A)$,
(L_2) $\underline{R}(U) = U$, (U_2) $\overline{R}(\varphi) = \varphi$,
(L_3) $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$, (U_3) $\overline{R}(A \cap B) = \overline{R}(A) \cup \overline{R}(B)$,
(L_4) $A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$, (U_4) $A \subseteq B = \overline{R}(A) \subseteq \overline{R}(B)$,
(L_5) $\underline{R}(A \cup B) \supseteq \underline{R}(A) \cup \underline{R}(B)$, (U_5) $\overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B)$.

Properties $(L_1)$and(U_1) show that the approximation operators $\underline{R}$ and $\overline{R}$ are dual to each other. Properties with the same number may be considered as a dual properties. If R is equivalence relation in U then the pair (U, R) is called a Pawlak approximation space and $(\underline{R}(A), \overline{R}(A))$ is a Pawlak rough set, in such a case the approximation operators have additional properties.

3. Neutrosophic rough sets

In this section, we introduce neutrosophic approximation space and neutrosophic approximation operators induced from the same. Further we define a new type of set called neutrosophic rough set and investigate some of its properties.

Definition 3.1. A constant neutrosophic set on U is defined as follows:

$$\widehat{(\alpha, \beta, \gamma)} = \{ \langle x, \alpha, \beta, \gamma \rangle / \ x \in U\},$$

where $0 \leq \alpha, \beta, \gamma \leq 1$ and $\alpha + \beta + \gamma \leq 3$.
And we introduce a special neutrosophic set (neutrosophic singleton set) $)1_y$ for $y \in U$ as follows:

$$T_{1_y}(x) = \begin{cases} 1, & if \quad x = y \\ 0, & if \quad x \neq y \end{cases},$$

$$T_{1_{(U-(y)}}(x) = \begin{cases} 0, & if \quad x = y \\ 1, & if \quad x \neq y \end{cases},$$

$$I_{1_y} = \begin{cases} 1, & if \quad x = y \\ 0, & if \quad x \neq y \end{cases},$$

$$I_{1_{(U-(y)}}(x) = \begin{cases} 0, & if \quad x = y \\ 1, & if \quad x \neq y \end{cases},$$

$$F_{1_y}(x) = \begin{cases} 0, & if \quad x = y \\ 1, & if \quad x \neq y \end{cases},$$

$$F_{1_{(U-(y))}} = \begin{cases} 1, & if \quad x = y \\ 0, & if \quad x \neq y \end{cases}.$$

Definition 3.2. A neutrosophic relation on U is a neutrosophic set

$$R = \{\langle x,y\rangle, T_R(x,y), I_R(x,y), F_R(x,y)/x,y \in U\}$$

and

$$T_R : U \times U \longrightarrow [0,1]; \quad I_R : U \times U \longrightarrow [0,1]; \quad F_R : U \times U \longrightarrow [0,1]$$

satisfies $0 \leq T_R(x,y) + I_R(x,y) + F_R(x,y) \leq 3$, for all $(x,y) \in U \times U$.

We denote the family of all neutrosophic relation on U by $N(U \times U)$.

Definition 3.3. Let U be a nonempty universe of discourse. For an arbitrary neutrosophic relation R over $U \times U$ the pair (U,R) is called neutrosophic approximation space. For any $A \in N(U)$, we define the upper and lower approximations with respect to (U,R), denoted by $\underline{R}(A)$ and $\overline{R}(A)$ respectively:

$$\overline{R}(A) = \{\langle x, T_{\overline{R}(A)}(x), I_{\overline{R}(A)}(x), F_{\overline{R}(A)}(x)\rangle/x \in U\},$$
$$\underline{R}(A) = \{\langle x, T_{\underline{R}(A)}(x), I_{\underline{R}(A)}(x), F_{\underline{R}(A)}(x)\rangle/x \in U\},$$

where

$$T_{\overline{R}(A)}(x) = \bigvee_{y \in U} [\ T_R(x,y) \wedge T_A(y)\ ],$$
$$I_{\overline{R}(A)}(x) = \bigvee_{y \in U} [\ I_R(x,y) \wedge I_A(y)\ ],$$
$$F_{\overline{R}(A)}(x) = \bigwedge_{y \in U} [\ F_R(x,y) \vee F_A(y)\ ],$$
$$T_{\underline{R}(A)}(x) = \bigwedge_{y \in U} [\ F_R(x,y) \vee T_A(y)\ ],$$
$$I_{\underline{R}(A)}(x) = \bigwedge_{y \in U} [\ (1 - I_R(x,y)) \vee I_A(y)\ ],$$
$$F_{\underline{R}(A)}(x) = \bigvee_{y \in U} [\ T_R(x,y) \wedge F_A(y)\ ].$$

The pair $(\underline{R}(A), \overline{R}(A))$ is called neutrosophic rough set of A with respect to (U,R) and $\underline{R}, \overline{R} : N(U) \longrightarrow N(U)$ are referred to as upper and lower neutrosophic rough approximation operators respectively.

Remark 3.4. If R is an intuitionistic fuzzy relation on U and (U,R) is an intuitionistic fuzzy approximation space then neutrosophic rough operators are induced from an intuitionistic fuzzy approximation space as follows:

$$\overline{R}(A) = \{\langle x, T_{\overline{R}(A)}(x), I_{\overline{R}(A)}(x), F_{\overline{R}(A)}(x)\rangle/x \in U\}\ A \in N(U),$$
$$\underline{R}(A) = \{\langle x, T_{\underline{R}(A)}(x), I_{\underline{R}(A)}(x), F_{\underline{R}(A)}(x)\rangle/x \in U\}\ A \in N(U),$$

where

$$T_{\overline{R}(A)}(x) = \bigvee_{y \in U} [\ \mu_R(x,y) \wedge T_A(y)\ ],$$
$$I_{\overline{R}(A)}(x) = \bigvee_{y \in U} [\ 1 - (\mu_R(x,y) + \gamma_R(x,y))\ \wedge I_R(y)],$$
$$F_{\overline{R}(A)}(x) = \bigwedge_{y \in U} [\ \gamma_R(x,y) \vee F_A(y)\ ],$$
$$T_{\underline{R}(A)}(x) = \bigwedge_{y \in U} [\ \gamma_R(x,y) \vee T_A(y)\ ],$$
$$I_{\underline{R}(A)}(x) = \bigwedge_{y \in U} [\ (\mu_R(x,y) + \gamma_R(x,y)) \vee I_A(y)\ ],$$
$$F_{\underline{R}(A)}(x) = \bigvee_{y \in U} [\ \mu_R(x,y) \wedge F_A(y)\ ].$$

Remark 3.5. If R is a crisp binary relation on U and (U, R) is a crisp approximation space, then neutrosophic rough approximation operators are induced from crisp approximation space, such that $\forall A \in N(U)$,

$$\overline{R}(A) = \{\langle x, T_{\overline{R}(A)}(x), I_{\overline{R}(A)}(x), F_{\overline{R}(A)}(x)\rangle / x \in U\},$$
$$\underline{R}(A) = \{\langle x, T_{\underline{R}(A)}(x), I_{\underline{R}(A)}(x), F_{\underline{R}(A)}(x)\rangle / U \in U\},$$

where

$$T_{\overline{R}(A)}(x) = \bigvee_{y\in[x]_R} T_A(y), I_{\overline{R}(A)}(x) = \bigvee_{y\in[x]_R} I_A(y), F_{\overline{R}(A)}(x) = \bigwedge_{y\in[x]_R} F_A(y),$$
$$T_{\underline{R}(A)}(x) = \bigwedge_{y\in[x]_R} T_A(y), I_{\underline{R}(A)}(x) = \bigwedge_{y\in[x]_R} I_A(y), F_{\underline{R}(A)}(x) = \bigvee_{y\in[x]_R} F_A(y).$$

Theorem 3.6. *Let (U, R) be a neutrosophic approximation space. Then the lower and upper neutrosophic rough approximation operators induced from (U, R) satisfy the following properties.* $\forall\ A, B \in N(U)$, $\forall\ \alpha, \beta, \gamma \in [0, 1]$ *with* $\alpha + \beta + \gamma \ \leq\ 3$.

$(FNL1)\underline{R}(A) =\sim \overline{R}(\sim A)$,
$(FNL2)\underline{R}(A \cup (\widehat{\alpha, \beta, \gamma})) = \underline{R}(A) \cup (\widehat{\alpha, \beta, \gamma})$,
$(FNL3)\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$,
$(FNL4)A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$,
$(FNL5)\underline{R}(A \cup B) \supseteq \underline{R}(A) \cup \underline{R}(B)$,
$(FNL6)R_1 \subseteq R_2 \Rightarrow \underline{R_1}(A) \supseteq \underline{R_2}(A)$,
$(FNU1)\overline{R}(A) = \overline{R}(A) =\sim \underline{R}(\sim A)$,
$(FNU2)\overline{R}(A \cap (\widehat{\alpha, \beta, \gamma}) = \overline{R}(A) \cap (\widehat{\alpha, \beta, \gamma})$,
$(FNU3)\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$,
$(FNU4)A \subseteq B \Rightarrow \overline{R}(A) \subseteq \overline{R}(B)$,
$(FNU5)\overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B)$,
$(FNU6)R_1 \subseteq R_2 \Rightarrow \overline{R_1}(A) \subseteq \overline{R_2}(A)$.

Proof. We only prove the properties of lower neutrosophic rough approximation operator $\underline{R}(A)$. The upper rough neutrosophic approximation operator $\overline{R}(A)$ can be proved similarly.

(FNL1) By Definition 3.3 , we have

$$\sim A = \{\langle x, T_{\sim A}(x), I_{\sim A}(x), F_{\sim A}(x) / x \in U\},$$
$$\begin{aligned}\overline{R}(\sim A) &= \bigvee_{y\in U} [\ T_R(x, y) \wedge T_{\sim A}(y)\ ],\ \bigvee_{y\in U} [\ I_R(x, y) \wedge I_{\sim A}(y)\ ],\\ &\qquad \bigwedge_{y\in U} [\ F_R(x, y) \vee F_{\sim A}(y)\ ]\\ &= \bigvee_{y\in U} [\ T_R(x, y) \wedge F_A(y)\ ],\ \bigvee_{y\in U} [\ I_R(x, y) \wedge (1 - I_A(y))\ ],\\ &\qquad \bigwedge_{y\in U} [F_R(x, y) \vee T_A(y)],\end{aligned}$$
$$\begin{aligned}\sim (\overline{R}(\sim A)) &= \bigwedge_{y\in U} [F_R(x, y) \vee T_A(y)],\ \bigwedge_{y\in U} [(1 - I_R(x, y)) \vee I_A(y)],\\ &\qquad \bigvee_{y\in U} [T_R(x, y) \wedge F_A(y)\ ]\\ &= \underline{R}(A).\end{aligned}$$

(FNL2) It can be easily verified by definition of $\underline{R}(A)$.

(FNL3) $\underline{R}(A \cap B)$
$= \{\langle x, T_{\underline{R}(A\cap B)}(x), I_{\underline{R}(A\cap B)}(x), F_{\underline{R}(A\cap B)}(x) | x \in U\rangle\}$

$= \{\langle x, \bigwedge_{y\in U} T_{(A\cap B)}(y), \bigwedge_{y\in U} I_{(A\cap B)}(y), \bigvee_{y\in U} F_{(A\cap B)}(y) | x \in U\rangle\}$
$= \{\langle x, \bigwedge_{y\in U} (T_A(y) \bigwedge T_B(y)), \bigwedge_{y\in U} (I_A(y) \bigwedge I_B(y)), \bigvee_{y\in U} (F_A(y) \bigvee F_B(y)) | x \in U\}$
$= \{\langle, T_{\underline{R}(A)}(x) \bigwedge T_{\underline{R}(B)}(x), I_{\underline{R}(A)}(x) \bigwedge I_{\underline{R}(B)}(x), F_{\underline{R}(A)}(x) \bigwedge F_{\underline{R}(B)}(x) | x \in U\rangle\}$
$= \underline{R}(A) \cap \underline{R}(B)$.

(FNL4) It is straightforward. Similarly, we can prove the properties of the upper rough neutrosophic approximation operators. □

Remark 3.7. The properties (FNL1) and (FNU1) shows that neutrosophic rough approximation operators $\underline{R}$ and $\overline{R}$ are dual to each other and the properties (FNL2) and (FNU2) imply, following properties $(FNL2)^{'}$ and $(FNU2)^{'}$.
$(FNL2)^{'}$ $\underline{R}(U) = U$ $\quad (FNU2)^{'} = \overline{R}(\varphi) = \varphi$.

Example 3.8. Let (U, R) be a neutrosophic approximation space where $U = \{x_1, x_2, x_3\}$ and $R \in N(U \times U)$ is defined as
$R = \{\langle (x_1, x_1) 0.8, 0.7, 0.1\rangle \ \langle (x_1, x_2), 0.2, 0.5, 0.4\rangle \ \langle (x_1, x_3) 0.6, 0.5, 0.7\rangle$

$\langle (x_2, x_1) 0.4, 0.6, 0.3\rangle \ \langle (x_2, x_2) 0.7, 0.8, 0.1\rangle \ \langle (x_2, x_3) 0.5, 0.3, 0.1\rangle$

$\langle (x_3, x_1) 0.6, 0.2, 0.1\rangle \ \langle (x_3, x_2) 0.7, 0.8, 0.1\rangle \ \langle (x_3, x_3) 1, 0.9, 0.1\rangle\}$.

If a neutrosophic set

$$A = \{\langle x_1, 0.8, 0.9, 0.1\rangle \ \langle x_2, 0.5, 0.4, 0.3\rangle \ \langle x_3, 0.5, 0.4, 0.7\rangle\},$$

we can calculate,

$$\overline{R}(A) = \{\langle x_1, 0.8, 0.7, 0.1\rangle \ \langle x_2, 0.7, 0.6, 0.3\rangle \ \langle x_3, 0.6, 0.4, 0.1\rangle\},$$

$$\underline{R}(A) = \{\langle x_1, 0.5, 0.5, 0.4\rangle \ \langle x_2, 0.5, 0.4, 0.3\rangle \ \langle x_3, 0.5, 0.5, 0.7\rangle\},$$

the upper and lower approximations of A respectively.

Definition 3.9. Let $A \in N(U)$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma \leq 3$ and (α, β, γ) level set of A denoted by $A^{(\alpha\beta\gamma)}$ is defined as:

$$A^{(\alpha\beta\gamma)} = \{x \in U / T_A(x) \geq \alpha, I_A(x) \geq \beta, F_A(x) \leq \gamma\}.$$

We define

$A_\alpha = \{x \in U / T_A(x) \geq \alpha\}$ and $A_{\alpha+} = \{x \in U / T_A(x) > \alpha\}$: the α level cut and strong α level cut of truth value function generated by A,

$A\beta = \{x \in U / I_A(x) \geq \beta\}$ and $A\beta+ = \{x \in U / I_A(x) > \beta\}$: the β level cut and strong β level cut of indeterminacy function generated by A and

$A^\gamma = \{x \in U / F_A(x) \leq \gamma\}$ and $A^{\gamma+} = \{x \in U / F_A(x) < \gamma\}$: the γ level cut and strong γ level cut of false value function generated by A.

Similarly, we can define the level cuts sets, such as
$A^{(\alpha+,\beta+,\gamma+)} = \{x \in U / T_A(x) > \alpha, I_A(x) > \beta, F_A(x) < \gamma\}$ is $(\alpha+, \beta+, \gamma+)$,
$A^{(\alpha+,\beta,\gamma)} = \{x \in U / T_A(x) > \alpha, I_A(x) \geq \beta, F_A(x) \leq \gamma\}$ is $(\alpha+, \beta, \gamma)$,
$A^{(\alpha,\beta+,\gamma)} = \{x \in U / T_A(x) \geq \alpha, I_A(x) > \beta, F_A(x) \leq \gamma\}$ is $(\alpha, \beta+, \gamma)$,
$A^{(\alpha,\beta,\gamma+)} = \{x \in U / T_A(x) \geq \alpha, I_A(x) \geq \beta, F_A(x) < \gamma\}$ is $(\alpha, \beta, \gamma+)$:
level cut set of A, respectively.

Like wise other level cuts can also be defined.

Theorem 3.10. *The level cut sets of neutrosophic sets satisfy the following properties:* $\forall\ A, B \in N(U)$, $\alpha, \beta, \gamma \in [0,1]$ *with* $\alpha+\beta+\gamma \leq 3$, $\alpha_1, \beta_1, \gamma_1 \in [0,1]$ *with* $\alpha_1+\beta_1+\gamma_1 \leq 3$ *and* $\alpha_2, \beta_2, \gamma_2 \in [0,1]$ *with* $\alpha_2+\beta_2+\gamma_2 \leq 3$:

(1) $A^{(\alpha,\beta,\gamma)} = A_\alpha \bigcap A\beta \bigcap A^\gamma$,

(2) $(\sim A)_\alpha = \sim A^{\alpha+}; (\sim A)\beta = \sim A(1-\beta+); (\sim A)^\gamma = \sim A_{\gamma+}$,

(3) $\left(\bigcap_{i\in J} A_i\right)_\alpha = \bigcap_{i\in J}(A_i)_\alpha,\ \left(\bigcap_{i\in J} A_i\right)\beta = \bigcap_{i\in J}(A_i)\beta,\ \left(\bigcap_{i\in J} A_i\right)^\gamma = \bigcap_{i\in J}(A_i)^\gamma$,

(4) $\left(\bigcup_{i\in J} A_i\right)_\alpha = \bigcup_{i\in J}(A_i)_\alpha,\ \left(\bigcup_{i\in J} A_i\right)\beta = \bigcup_{i\in J}(A_i)\beta,\ \left(\bigcup_{i\in J} A_i\right)^\gamma = \bigcup_{i\in J}(A_i)^\gamma$,

(5) $\left(\bigcup_{i\in J} A_i\right)^{(\alpha,\beta,\gamma)} \supseteq \bigcup_{i\in J}(A_i)^{(\alpha,\beta,\gamma)}$,

(6) $\left(\bigcap_{i\in J} A_i\right)^{(\alpha,\beta,\gamma)} \supseteq \bigcap_{i\in J}(A_i)^{(\alpha,\beta,\gamma)}$,

(7) *For* $\alpha_1 \geq \alpha_2, \beta_1 \geq \beta_2, \gamma_1 \leq \gamma_2$,

$$A_{\alpha_1} \subseteq A_{\alpha_2}, A\beta_1 \subseteq A\beta_2, A^{\gamma_1} \subseteq A^{\gamma_2}, A^{(\alpha_1,\beta_1,\gamma_1)} \subseteq A^{(\alpha_2,\beta_2,\gamma_2)}.$$

Proof. (1) and (3) follow directly from Definition 3.9.

(2) Since $(\sim A) = \{\langle x, F_A(x), 1-I_A(x), T_A(x)\rangle / x \in U\}$. Then $(\sim A)_\alpha = \{x \in U/F_A(x) \geq \alpha\}$. Thus by definition, $A^{\alpha+} = \{x \in U/F_A(x) < \alpha\}$ and $\sim A^{\alpha+} = \{x \in U/F_A(x) \geq \alpha\}$. So $(\sim A)_\alpha = \sim A^{\alpha+}$.

Similarly we can prove $(\sim A)\beta = \sim A(1-\beta+)$ and $(\sim A)^\gamma = \sim A_{\gamma+}$.

(4) $\bigcap_{i\in J} A_i = \left\{\langle x, \bigwedge_{i\in J} T_{A_i}(x), \bigwedge_{i\in J} I_{A_i}(x), \bigvee_{i\in J} F_{A_i}(x)\rangle / x \in U\right\}$. Then we have

$$\left(\bigcap_{i\in J} A_i\right)_\alpha = \left\{x \in U/\bigwedge_{i\in J} T_{A_i}(x) \geq \alpha\right\} = \{x \in U/T_{A_i}(x) \geq \alpha\} = \bigcap_{i\in J}(A_i)_\alpha.$$

Similarly,

$$\left(\bigcap_{i\in J} A_i\right)\beta = \left\{x \in U/\bigwedge_{i\in J} I_{A_i}(x) \geq \beta\right\} = \{x \in U/I_{A_i}(x) \geq \beta \forall i \in J\} = \bigcap_{i\in J}(A_i)\beta$$

and

$$\left(\bigcap_{i\in J} A_i\right)^\gamma = \left\{x \in U/\bigvee_{i\in J} F_{A_i}(x) \leq \gamma\right\} = \{x \in U/F_{A_i}(x) \leq \gamma \forall i \in J\} = \bigcap_{i\in J}(A_i)_\gamma.$$

Thus

$$\begin{aligned}\left(\bigcap_{i\in J} A_i\right)^{\alpha,\beta,\gamma} &= \left(\bigcap_{i\in J} A_i\right)_\alpha \cap \left(\bigcap_{i\in J} A_i\right)\beta \cap \left(\bigcap_{i\in J} A_i\right)^\gamma \\ &= \bigcap_{i\in J}((A_i)_\alpha \cap (A_i)\beta \cap (A_i)^\gamma) \\ &= \bigcap_{i\in J}(A_i)^{(\alpha,\beta,\gamma)}.\end{aligned}$$

(5) We can easily prove followings:

$$\bigcup_{i\in J}(A_i) = \left\{\langle x, \bigvee_{i\in J} T_{A_i}(x), \bigvee_{i\in J} I_{A_i}(x), \bigwedge_{i\in J} F_{A_i}(x)\rangle / x \in U\right\},$$

$$\begin{aligned}\left(\bigcup_{i\in J} A_i\right)_\alpha &= \left\{x \in U/\bigvee_{i\in J} T_{A_i}(x) \geq \alpha\right\} \\ &= \left\{x \in U/\bigvee_{i\in J} T_{A_i}(x) \geq \alpha,\ \exists i \in J\right\}\end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{i\in J}(A_i)_\alpha,\\
\left(\bigcup_{i\in J} A_i\right)\beta &= \left\{x\in U/\bigvee_{i\in J} I_{A_i}(x)\geq\beta\right\}\\
&= \{x\in U/I_{A_i}(x)\geq\beta,\ \forall i\in J\}\\
&= \bigcup_{i\in J}(A_i)\beta,\\
\left(\bigcup_{i\in J} A_i\right)^\gamma &= \left\{x\in U/\bigwedge_{i\in J} F_{A_i}(x)\leq\gamma\right\}\\
&= \{x\in U/F_{A_i}(x)\leq\gamma,\ \forall i\in J\}\\
&= \bigcup_{i\in J}(A_i)^\gamma.
\end{aligned}
$$

(6) For any $x\in A_\alpha$, according to Definition 3.9 we have for $T_A(x)\geq\alpha_1\geq\alpha_2$, we obtain $A_{\alpha_1}\subseteq A_{\alpha_2}$. Similarly, for $\beta_1\geq\beta_2$ and $\gamma_1\leq\gamma_2$, we obtain $A\beta_1\subseteq A\beta_2$ and $A^{\gamma_1}\subseteq A^{\gamma_2}$. Then we have $A^{(\alpha_1,\beta_1,\gamma_1)}\subseteq A^{(\alpha_2,\beta_2,\gamma_2)}$. □

Corollary 3.11. *Assume that R is a neutrosophic relation in U,*
$R_\alpha=\{(x,y)\in U\times U/T_R(x,y)\geq\alpha\}$, $R_\alpha(x)=\{y\in U/T_R(x,y)\geq\alpha\}$,
$R_{\alpha+}=\{(x,y)\in U\times U/T_R(x,y)>\alpha\}$, $R_{\alpha+}(x)=\{y\in U/T_R(x,y)>\alpha\}$,
$R\beta=\{(x,y)\in U\times U/I_R(x,y)\geq\beta\}$, $R\beta(x)=\{y\in U/I_R(x,y)\geq\beta\}$,
$R\beta+=\{(x,y)\in U\times U/I_R(x,y)>\beta\}$, $R\beta+(x)=\{y\in U/I_R(x,y)>\beta\}$,
$R^\gamma=\{(x,y)\in U\times U/F_R(x,y)\leq\gamma\}$, $R^\gamma(x)=\{y\in U/F_R(x,y)\leq\gamma\}$,
$R^{\gamma+}=\{(x,y)\in U\times U/F_R(x,y)<\gamma\}$, $R^{\gamma+}(x)=\{y\in U/F_R(x,y)<\alpha\}$,
$R^{(\alpha,\beta,\gamma)}=\{(x,y)\in U\times U/T_R(x,y)\geq\alpha, I_R(x,y)\geq\beta, F_R(x,y)\leq\gamma\}$,
$R^{(\alpha,\beta,\gamma)}(x)=\{y\in U/T_R(x,y)\geq\alpha, I_R(x,y)\geq\beta, F_R(x,y)\leq\gamma\}$.
Then for all $R_\alpha, R_{\alpha+}, R\beta, R\beta+, R^\gamma, R^{\gamma+}, R^{(\alpha\beta\gamma)}$ are crisp relation in U and
(1) *if R is reflexive, then the above level cuts are reflexive,*
(2) *If R is symmetric, then the above level cuts are symmetric,*
(3) *if R is transitive then the above level cuts are transitive.*

Proof. (1) Suppose R is reflexive. Then $\forall\ x\in U$,

$$T_R(x,x)=1,\quad I_R(x,x)=1,\quad F_R(x,x)=0,\quad \forall\ x\in U.$$

Now, we have R_α is a crisp binary relation in U and $x\in U$, $(x,x)\in R_\alpha$. Then R_α is reflexive.

Suppose R is symmetric. Then $\forall\ x,y\in U$, we have $(x,y)\in R_\alpha\Rightarrow(y,x)\in R_\alpha$. Thus R_α is symmetry. Similarly, we can prove $R\beta$ and R^γ are symmetric.

Suppose R is transitive. Then $\forall\ x,y,z\in U$ and $\alpha,\beta,\gamma\in[0,1]$,

$$T_R(x,z)\geq T_R(x,y)\wedge T_R(y,z),$$
$$I_R(x,z)\geq I_R(x,y)\wedge I_R(y,z),$$
$$F_R(x,z)\leq F_R(x,y)\vee F_R(y,z).$$

Let $(x,y)\in R_\alpha$, $(y,z)\in R_\alpha$, $(x',y')\in R\beta$, $(y',z')\in R\beta$, $(x'',y'')\in R^\gamma$ and $(y'',z'')\in R^\gamma$. Then

$$T_R(x,y)\geq\alpha,\ \ T_R(y,z)\geq\alpha\ \ \Rightarrow T_R(x,z)\geq\alpha,$$
$$I_R(x',y')\geq\beta,\ \ I_R(y',z')\geq\beta\ \ \Rightarrow I_R(x',z')\geq\beta,$$
$$F_R(x'',y'')\leq\gamma,\ \ F_R(y'',z'')\leq\gamma\ \ \Rightarrow F_R(x'',z'')\leq\gamma.$$

Thus $R_\alpha, R\beta, R^\gamma$ are transitive. So $R^{(\alpha,\beta,\gamma)}$ is transitive.

Similarly, we can prove other level cuts sets are transitive. □

Theorem 3.12. *Let (U, R) be a neutrosophic approximation space and $A \in N(U)$, then the upper neutrosophic approximation operator can be represented as follows $\forall\ x \in U$,*

(1) $T_{\overline{R}(A)}(x) = \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \overline{R}_\alpha(A_\alpha)(x)\right] = \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \overline{R}_\alpha(A_{\alpha+})(x)\right]$
$= \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \overline{R}_{\alpha+}(A_\alpha)(x)\right] = \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \overline{R}_{\alpha+}(A_{\alpha+})(x)\right]$,

(2) $I_{\overline{R}(A)}(x) = \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \overline{R}\alpha(A\alpha)(x)\right] = \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \overline{R}\alpha(A\alpha+)(x)\right]$
$= \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \overline{R}\alpha+(A\alpha)(x)\right] = \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \overline{R}\alpha+(A\alpha+)(x)\right]$,

(3) $F_{\overline{R}(A)}(x) = \bigwedge_{\alpha\in[0,1]} \left[\alpha \vee \overline{R}^\alpha(A^\alpha)(x)\right] = \bigwedge_{\alpha\in[0,1]} \left[\alpha \vee \overline{R}^\alpha(A^{\alpha+})(x)\right]$
$= \bigwedge_{\alpha\in[0,1]} \left[\alpha \vee \overline{R}^{\alpha+}(A^\alpha)(x)\right] = \bigwedge_{\alpha\in[0,1]} \left[\alpha \vee \overline{R}^{\alpha+}(A^{\alpha+})(x)\right]$

and more over for any $\alpha \in [0, 1]$,

(4) $\left[\overline{R}(A)\right]_{\alpha+} \subseteq \overline{R}_{\alpha+}(A_{\alpha+}) \subseteq \overline{R}_\alpha(A_\alpha) \subseteq \left[\bar{R}(A)\right]_\alpha$,
(5) $\left[\overline{R}(A)\right]\alpha+ \subseteq \overline{R}\alpha+(A\alpha+) \subseteq \overline{R}\alpha(A\alpha) \subseteq \left[\overline{R}(A)\right]\alpha$,
(6) $\left[\overline{R}(A)\right]^{\alpha+} \subseteq \overline{R^{\alpha+}}(A^{\alpha+}) \subseteq \overline{R^\alpha}(A^\alpha) \subseteq \left[\overline{R}(A)\right]^\alpha$,
(7) $\left[\overline{R}(A)\right]_{\alpha+} \subseteq \overline{R}_{\alpha+}(A_{\alpha+}) \subseteq \overline{R}_\alpha(A_\alpha) \subseteq \left[\overline{R}(A)\right]_\alpha$,
(8) $\left[\overline{R}(A)\right]\alpha+ \subseteq \overline{R}\alpha+(A\alpha+) \subseteq \overline{R}\alpha(A\alpha) \subseteq \left[\overline{R}(A)\right]\alpha$,
(9) $\left[\overline{R}(A)\right]^{\alpha+} \subseteq \overline{R^{\alpha+}}(A^{\alpha+}) \subseteq \overline{R^\alpha}(A^\alpha) \subseteq \left[\overline{R}(A)\right]^\alpha$.

Proof. (1) For $x \in U$, we have

$$\begin{aligned}
\bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \overline{R}_\alpha(A_\alpha)(x)\right] &= Sup\left\{\alpha \in [0,1]/x \in \overline{R}_\alpha(A_\alpha)\right\} \\
&= Sup\left\{\alpha \in [0,1]/R_\alpha(x) \cap A_\alpha \neq \varphi\right\} \\
&= Sup\left\{\alpha \in [0,1]/\exists y \in U(y \in R_\alpha(x), y \in A_\alpha)\right\} \\
&= Sup\left\{\alpha \in [0,1]/\exists y \in U[T_R(x,y) \geq \alpha, T_A(y) \geq \alpha]\right\} \\
&= \bigvee_{y\in U} [T_R(x,y) \wedge T_A(y)] = T_{\overline{R}(A)}(x).
\end{aligned}$$

(2)

$$\begin{aligned}
\bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \overline{R}\alpha(A\alpha)(x)\right] &= Sup\left\{\alpha \in [0,1]/x \in \overline{R}\alpha(A\alpha)\right\} \\
&= Sup\left\{\alpha \in [0,1]/R\alpha(x) \cap A\alpha \neq \varphi\right\} \\
&= Sup\left\{\alpha \in [0,1]/\exists y \in U(y \in R\alpha(x), y \in A\alpha)\right\} \\
&= Sup\left\{\alpha \in [0,1]/\exists y \in U[I_R(x,y) \geq \alpha, I_A(y) \geq \alpha]\right\} \\
&= \bigvee_{y\in U} [I_R(x,y) \wedge I_A(y)] = I_{\overline{R}(A)}(x).
\end{aligned}$$

(3)

$$\begin{aligned}\bigvee_{\alpha\in[0,1]}\left[\alpha\wedge\overline{R}^{\alpha}(A^{\alpha})(x)\right] &= inf\,\{\alpha\in[0,1]/R^{\alpha}(x)\cap A^{\alpha}\neq\varphi\}\\ &= inf\,\{\alpha\in[0,1]/R^{\alpha}(x)\cap A^{\alpha}\neq\varphi\}\\ &= inf\,\{\alpha\in[0,1]/\exists y\in U(y\in R^{\alpha}(x),y\in A^{\alpha})\}\\ &= inf\,\{\alpha\in[0,1]/\exists y\in U[F_R(x,y)\leq\alpha,F_A(y)\leq\alpha]\}\\ &= \bigwedge_{y\in U}[F_R(x,y)\vee F_A(y)]=F_{\overline{R}(A)}(x).\end{aligned}$$

Like wise, we can conclude

$$\begin{aligned}T_{\overline{R}(A)}(x) &= \bigvee_{\alpha\in[0,1]}\left[\alpha\wedge\overline{R}_{\alpha}(A_{\alpha+})(x)\right]\\ &= \bigvee_{\alpha\in[0,1]}\left[\alpha\wedge\overline{R}_{\alpha+}(A_{\alpha})(x)\right]\\ &= \bigvee_{\alpha\in[0,1]}\left[\alpha\wedge\overline{R}_{\alpha+}(A_{\alpha+})(x)\right],\\ I_{\overline{R}(A)}(x) &= \bigvee_{\alpha\in[0,1]}\left[\alpha\wedge\overline{R}\alpha(A\alpha+)(x)\right]\\ &= \bigvee_{\alpha\in[0,1]}\left[\alpha\wedge\overline{R}\alpha+(A\alpha)(x)\right]\\ &= \bigvee_{\alpha\in[0,1]}\left[\alpha\wedge\overline{R}\alpha+(A\alpha+)(x)\right],\\ F_{\overline{R}(A)}(x) &= \bigwedge_{\alpha\in[0,1]}\left[\alpha\vee\overline{R}^{\alpha}(A^{\alpha+})(x)\right]\\ &= \bigwedge_{\alpha\in[0,1]}\left[\alpha\vee\overline{R}^{\alpha+}(A^{\alpha})(x)\right]\\ &= \bigwedge_{\alpha\in[0,1]}\left[\alpha\vee\overline{R}^{\alpha+}(A^{\alpha+})(x)\right].\end{aligned}$$

(4) Since $\overline{R}_{\alpha+}(A_{\alpha+})\subseteq\overline{R}_{\alpha+}(A_{\alpha})\subseteq\overline{R_{\alpha}}(A_{\alpha})$, We prove only

$$[\overline{R}(A)]_{\alpha+}\subseteq\overline{R}_{\alpha+}(A_{\alpha+})$$

and

$$\overline{R}_{\alpha}(A_{\alpha})\subseteq[R(A)]_{\alpha}.$$

Let $x\in[\overline{R}(A)]_{\alpha+}$, $T_{\overline{R}(A)}>\alpha$. Then $\bigvee_{y\in U}[T_R(x,y)\wedge T_A(y)]>\alpha$. Thus $\exists\ y^{'}\in U$ such that $T_R(x,y^{'})\wedge T_R(y^{"})>\alpha$. So $y^{'}\in R_{\alpha+}(x)$ and $y^{'}\in A_{\alpha+}$. Hence $R_{\alpha+}(x)\cap A_{\alpha+}\neq\varphi$. From the definition of upper crisp approximation operator, we have $x\in\overline{R}_{\alpha+}(A_{\alpha+})$. Therefore $[\overline{R}(A)]_{\alpha+}\subseteq\overline{R}_{\alpha+}(A_{\alpha+})$.

Next, to prove $\overline{R}_{\alpha}(A_{\alpha})\subseteq[R(A)]_{\alpha}$, let $x\in\overline{R}_{\alpha}(A_{\alpha})$. Then $R_{\alpha}(A_{\alpha})(x)=1$. If $\exists\ \beta$, then $T_{\overline{R}(A)}(x)=\bigvee_{\beta\in[0,1]}\left[\beta\wedge\overline{R}_{\beta}(A_{\beta})(x)\right]\geq\alpha\wedge\overline{R}_{\alpha}(A_{\alpha})(x)=\alpha$. Thus we obtain $x\in[\overline{R}(A)]_{\alpha}$ $\overline{R}_{\alpha}(A_{\alpha})\subseteq[\overline{R}(A)]_{\alpha}$.

(5) The proof is similar to (4). It is enough to prove

$$\overline{R\alpha+}(A\alpha+)\subseteq\overline{R\alpha+}(A\alpha)\subseteq\overline{R\alpha}(A\alpha).$$

(i) In order to show that $[\overline{R}(A)]\alpha+ \subseteq \overline{R\alpha+}(A\alpha+)$, let $x \in [\overline{R}(A)]\alpha+$. Then $I_{\overline{R}(A)}(x) > \alpha$, i.e., $\bigvee_{y\in U}[I_R(x,y) \wedge I_A(y)] > \alpha$. Thus $\exists\ y^{'} \in U$ such that $I_R(x,y^{'}) \wedge I_A(y^{'}) > \alpha$, i.e., $I_R(x,y^{'})\alpha$ and $I_A(y^{'})\alpha$. So $y^{'} \in R\alpha + (x)$ and $y^{'} \in A\alpha+$, i.e., $y^{'} \in R(x) \cap A\alpha+$. Hence $R\alpha + (x) \cap A\alpha+ \neq \varphi$. By the definition of crisp approximation operator, we have $x \in \overline{R\alpha+}(A\alpha+)$. Therefore $[\overline{R}(A)]\alpha+ \subseteq \overline{R\alpha+}(A\alpha+)$.

(ii) To prove that $\overline{R\alpha}(A\alpha) \subseteq [\overline{R}(A)]\alpha$, let $x \in \overline{R\alpha}(A\alpha)$. Then $\overline{R}\alpha(A\alpha)(x) = 1$. If there exists β, then $T_{\overline{R}(A)}(x) = \bigvee_{\beta\in[0,1]}\left[\beta \wedge \overline{R}_\beta(A_\beta)(x)\right] \geq \alpha \wedge \overline{R}\alpha(A\alpha)(x) = \alpha$. Thus we obtain $x \in [\overline{R}(A)]\alpha$. So $\overline{R\alpha}(A\alpha) \subseteq [\overline{R}(A)]\alpha$.

(6) Since the proof of (6) is similar to (4) and (5), we need to prove only

$$[\overline{R}(A)]^{\alpha+} \subseteq \overline{R^{\alpha+}}(A^{\alpha+}) \text{ and } \overline{R^{\alpha}}(A^{\alpha}) \subseteq [\overline{R}(A)]^{\alpha}.$$

Let $x \in [\overline{R}(A)]^{\alpha+}$. Then $F_{\overline{R}(A)}(x) < \alpha$, i.e., $\bigwedge_{y\in U}[F_R(x,y) \vee F_A(y)] < \alpha$ and $\exists\ y^{'} \in U$ such that $F_R(x,y^{'}) \vee F_A(y^{'}) < \alpha$. Thus $F_R(x,y^{'}) < \alpha, T_A(y^{'}) < \alpha$, i.e., $y^{'} \in R^{\alpha+}(x)$ and $y^{'} \in A^{\alpha+}$. So $R^{\alpha+}(x) \cap A^{\alpha+} \neq \phi$. Hence $x \in \overline{R^{\alpha+}}(A^{\alpha+})$ and $[\overline{R}(A)]^{\alpha+} \subseteq \overline{R^{\alpha+}}(A^{\alpha+})$.

Next for any $x \in \overline{R^{\alpha}}(A^{\alpha})$ note $\overline{R^{\alpha}}(A^{\alpha})(x) = 1$. Then we have

$$F_{\overline{R}(A)}(x) = \bigwedge_{\beta\in[0,1]}\left[\beta \vee \overline{R^{\beta}}(A^{\beta})(x)\right] \leq \alpha \vee \overline{R^{\alpha}}(A^{\alpha})(x) = \alpha.$$

Thus $x \in [\overline{R}(A)]^{\alpha}$. So $\overline{R^{\alpha}}(A^{\alpha}) \subseteq [\overline{R}(A)]^{\alpha}$.

The proofs of (7), (8), (9) can be obtained similar to (4), (5), (6). □

Theorem 3.13. *Let (U,R) be neutrosophic approximation space and $A \in N(U)$ then $\forall x \in U$ and for $\alpha \in [0,1]$,*

(1) $T_{\underline{R}(A)}(x) = \bigwedge_{\alpha\in[0,1]}[\alpha \vee (1 - \underline{R^{\alpha}}(A_{\alpha+})(x))]$
$= \bigwedge_{\alpha\in[0,1]}[\alpha \vee (1 - R^{\alpha}(A_{\alpha})(x))], \bigwedge_{\alpha\in[0,1]}[\alpha \vee (1 - \underline{R^{\alpha+}}(A_{\alpha+})(x))]$
$= \bigwedge_{\alpha\in[0,1]}[\alpha \vee (1 - R^{\alpha+}(A_{\alpha})(x))]$,

(2) $I_{\underline{R}(A)}(x)$
$= \bigwedge_{\alpha\in[0,1]}\left[\alpha \vee (1 - \underline{R(1-\alpha)}(A\alpha+)(x))\right]$
$= \bigwedge_{\alpha\in[0,1]}[\alpha \vee (1 - R(1-\alpha)(A\alpha)(x))], \bigwedge_{\alpha\in[0,1]}\left[\alpha \vee (1 - \underline{R(1-\alpha+)}(A\alpha+)(x))\right]$
$= \bigwedge_{\alpha\in[0,1]}[\alpha \vee (1 - R(1-\alpha+)(A\alpha)(x))]$,

(3) $F_{\underline{R}(A)}(x) = \bigvee_{\alpha\in[0,1]}[\alpha \wedge (1 - \underline{R_{\alpha}}(A^{\alpha+})(x))]$
$= \bigvee_{\alpha\in[0,1]}[\alpha \wedge (1 - R_{\alpha}(A^{\alpha})(x))], \bigvee_{\alpha\in[0,1]}\left[\alpha \wedge (1 - \underline{R_{\alpha+}}(A^{\alpha+})(x))\right]$
$= \bigvee_{\alpha\in[0,1]}[\alpha \wedge (1 - R_{\alpha+}(A^{\alpha})(x))]$

(4) $[\underline{R}(A)]_{\alpha+} \subseteq \underline{R}^{\alpha}(A_{\alpha+}) \subseteq \underline{R^{\alpha+}}(A_{\alpha+}) \subseteq \underline{R^{\alpha+}}(A_{\alpha}) \subseteq [\underline{R}(A)]_{\alpha}$,

(5) $[\underline{R}(A)]\,\alpha+ \subseteq R1-\alpha(A\alpha+) \subseteq R1-\alpha+(A\alpha+) \subseteq R1-\alpha+(A\alpha) \subseteq [\underline{R}(A)]\alpha$,

(6) $[\underline{R}(A)]^{\alpha+} \subseteq \underline{R}_\alpha(A^{\alpha+}) \subseteq \underline{R_{\alpha+}}(A^{\alpha+}) \subseteq \underline{R_{\alpha+}}(A^\alpha) \subseteq [\underline{R}(A)]^\alpha$,
(7) $[\underline{R}(A)]_\alpha \subseteq \underline{R}^\alpha(A_{\alpha+}) \subseteq \underline{R^\alpha(A_\alpha)} \subseteq \underline{R^{\alpha+}(A_\alpha)} \subseteq [\underline{R}(A)]_\alpha$,
(8) $[\underline{R}(A)]\,\alpha+ \subseteq \underline{R1-\alpha}(A\alpha+) \subseteq \underline{R\alpha}(A\alpha) \subseteq \underline{R(1-\alpha+)}(A\alpha) \subseteq [\underline{R}(A)]^\alpha$,
(9) $[\underline{R}(A)]^{\alpha+} \subseteq \underline{R}_\alpha(A^{\alpha+}) \subseteq \underline{R_\alpha}(A^\alpha) \subseteq \underline{R_{\alpha+}}(A^\alpha) \subseteq [\underline{R}(A)]^\alpha$.

Proof. (1) and (2). For any $x \in U$, by the duality of upper and lower crisp approximation operators and in terms of Theorem, we have

$$T_{\overline{R}(A)}(x) = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_\alpha}(A_\alpha)(x)],$$

$$I_{\overline{R}(A)}(x) = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R\alpha}(A_\alpha)(x)],$$

and

$$F_{\overline{R}(A)}(x) = \bigwedge_{\alpha\in[0,1]} [\alpha \vee \overline{R^\alpha}(A_\alpha)(x)].$$

Then

$$\begin{aligned} T_{\overline{R}(\sim A)}(x) &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_\alpha}(\sim A_\alpha)(x)] \\ &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_\alpha}(\sim A^{\alpha+})(x)] \\ &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R_\alpha}(\sim A^{\alpha+})(x))], \end{aligned}$$

$$\begin{aligned} I_{\overline{R}(\sim A)}(x) &= \bigvee \alpha\epsilon[0,1][\alpha \wedge \overline{R\alpha}(\sim A_\alpha)(x)] \\ &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R\alpha}(\sim A1-\alpha+)(x)] \\ &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge \underline{\sim R\alpha}(A1-\alpha+)(x)] \\ &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R\alpha}(\sim A1-\alpha+)(x))], \end{aligned}$$

and

$$\begin{aligned} F_{\overline{R}(\sim A)}(x) &= \bigwedge_{\alpha\in[0,1]} [\alpha \vee \overline{R^\alpha}(\sim A^\alpha)(x)] \\ &= \bigwedge_{\alpha\in[0,1]} [\alpha \vee \overline{R^\alpha}(\sim A_{\alpha+})(x)] \\ &= \bigwedge_{\alpha\in[0,1]} [\alpha \vee \underline{\sim R^\alpha}(A_{\alpha+})(x)] \\ &= \bigwedge_{\alpha\in[0,1]} [\alpha \vee (1 - \underline{R^\alpha}(\sim A_{\alpha+})(x))]. \end{aligned}$$

Thus by fixing $\underline{R}(A) =\sim \overline{R}(\sim A)$, we conclude

$$T_{\underline{R}(A)}(x) = T_{\overline{R}(\sim A)}(x) = \bigwedge_{\alpha\in[0,1]} [\alpha \vee (1 - \underline{R_\alpha}(\sim A^{\alpha+})(x))],$$

$$I_{\underline{R}(A)}(x) = I_{\overline{R}(\sim A)}(x) = \bigwedge_{\alpha\in[0,1]} [\alpha \vee (1 - \underline{R\alpha}(\sim A1 - \alpha+)(x)),$$

and

$$F_{\underline{R}(A)}(x) = F_{\overline{R}(\sim A)}(x) = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R^\alpha}(\sim A_{\alpha+})(x))].$$

Likewise, we can prove

$$\begin{aligned} T_{\underline{R}(A)}(x) &= T_{\overline{R}(\sim A)}(x) \\ &= \bigwedge_{\alpha\in[0,1]} [\alpha \vee (1 - \underline{R_\alpha}(\sim A^{\alpha})(x))] \\ &= T_{\overline{R}(\sim A)}(x) \\ &= \bigwedge_{\alpha\in[0,1]} [\alpha \vee (1 - \underline{R_{\alpha+}}(\sim A^{\alpha+})(x))] \\ &= T_{\overline{R}(\sim A)}(x) \\ &= \bigwedge_{\alpha\in[0,1]} [\alpha \vee (1 - \underline{R_{\alpha+}}(\sim A^{\alpha})(x))], \end{aligned}$$

$$\begin{aligned} I_{\underline{R}(A)}(x) &= I_{\overline{R}(\sim A)}(x) \\ &= \bigwedge_{\alpha\in[0,1]} [\alpha \vee (1 - \underline{R\alpha}(\sim A1 - \alpha)(x)) \\ &= I_{\overline{R}(\sim A)}(x) \\ &= \bigwedge_{\alpha\in[0,1]} [\alpha \vee (1 - \underline{R\alpha+}(\sim A1 - \alpha+)(x)) \\ &= I_{\overline{R}(\sim A)}(x) \\ &= \bigwedge_{\alpha\in[0,1]} [\alpha \vee (1 - \underline{R\alpha+}(\sim A1 - \alpha)(x)), \end{aligned}$$

and

$$\begin{aligned} F_{\underline{R}(A)}(x) &= F_{\overline{R}(\sim A)}(x) = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R^\alpha}(\sim A_\alpha)(x))] \\ &= F_{\overline{R}(\sim A)}(x) \\ &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R^{\alpha+}}(\sim A_{\alpha+})(x))] \\ &= F_{\overline{R}(\sim A)}(x) \\ &= \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R^{\alpha+}}(\sim A_\alpha)(x))]. \end{aligned}$$

It is easy to prove that $\underline{R^\alpha} A_{\alpha+} \subseteq \underline{R^{\alpha+}} A_{\alpha+} \subseteq \underline{R^{\alpha+}} A_\alpha$.

Now we tend to prove that $[\underline{R}(A)]_{\alpha+} \subseteq \underline{R^\alpha} A_{\alpha+}$ and $\underline{R^{\alpha+}} A_\alpha \subseteq [\underline{R}(A)]_\alpha$. For any $x \in \underline{R^\alpha} A_{\alpha+}$, we have $T_R(A)(x) > \alpha$. Then we have $\bigwedge_{y \in U} [\ F_R(x,y) \vee T_A(y)\ ] > \alpha$. Thus $[\ F_R(x,y) \vee T_A(y)\ ] > \alpha$, for any $y \in U$, i.e., if $F_R(x,y) \leq \alpha$, then $T_A(y) > \alpha$.

Alternatively, for any $y \epsilon U$, if $y \epsilon R^{\alpha^\delta}(x)$, then $y \epsilon A_{a+}$. Thus, $R^\alpha(x) \subseteq A_{\alpha+}$. By the definition of lower approximation operator, we have $x \epsilon \underline{R^\alpha}(A_{\alpha+})$. So we conclude $[\underline{R}(A)]_{\alpha+} \subseteq \underline{R^\alpha} A_{\alpha+}$. Also, for any $x \epsilon \underline{R^{\alpha+}}(A_\alpha)$, we have $\underline{R^{\alpha+}}(A_\alpha = 1$. Hence

$$\begin{aligned} T_{\underline{R}}(x) &= \bigwedge_{\alpha' \epsilon [0,1]} [\alpha' \vee \underline{R^{\alpha'+}}(A_{\alpha'})(x)] \\ &= \bigvee_{\alpha' \epsilon [0,1]} [\alpha' \wedge \underline{R^{\alpha'+}}(A_{\alpha'})(x)] \\ &= \geq \alpha \wedge \underline{R^{\alpha+}}(A_\alpha)(x) = \alpha. \end{aligned}$$

Therefore $x \epsilon \, [\underline{R}(A)]_\alpha$ and $\underline{R^{\alpha+}} A_\alpha \subseteq [\underline{R}(A)]_\alpha$.

Similarly, we can prove (5) and (6) and hence (7), (8) and (9) can be concluded. □

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