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Characterization of regular semigroup by $(\in, \in \vee(k^*, q_k))$-fuzzy ideals

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Abstract. In this article we define the notion of $(\in, \in \vee(k^*, q_k))$-fuzzy left (right) ideals, $(\in, \in \vee(k^*, q_k))$-fuzzy (generalized) bi-ideals, $(\in, \in \vee(k^*, q_k))$-fuzzy interior ideals and $(\in, \in \vee(k^*, q_k))$-fuzzy quasi-ideals in semigroups. Finally we characterized regular semigroup by the properties of these ideals.

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Keywords: $(\in, \in \vee(k^*, q_k))$-fuzzy (generalized) bi-ideals, $(\in, \in \vee(k^*, q_k))$-fuzzy interior ideals, $(\in, \in \vee(k^*, q_k))$-fuzzy quasi-ideals.

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1. Introduction

There has been a quick development worldwide in the importance of fuzzy set theory and its applications for the past several years. Evidence of this can be found in the increasing number of high quality research articles on fuzzy sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences held every year. It seems that the fuzzy set theory deals with the applications of fuzzy technology in information processing. The application of fuzzy technology in information processing is important and its significance will certainly increase in the future. The basic concept of fuzzy set was first proposed by Zadeh in his inspiring paper [20], in which he discussed the relationships between fuzzy set theory and probability theory. This paper has opened up keen insights and applications in an inclusive range of scientific fields. After the Zadeh's fuzzy sets many researches conveyed the researchers on the generalizations of the of fuzzy sets ideas, with huge applications in computer science, artificial intelligence, control engineering, expert, robotics, automat theory, finite state machine, graph theory logics and many branches of pure and applied mathematics. In, 1971, Rosenfeld introduced the theory of fuzzy groups [15]. In 1979, Kuroki originated the theory of fuzzy Semigroups in his definitive papers [9, 10], the author defined fuzzy bi-ideal of semigroups and discussed some of its properties. One can find theoretical results on fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines and fuzzy Languages in a systematic exposition of fuzzy semigroups by Mordeson et al. [11]. Fuzziness has a natural place in the field of formal languages. In [12], Mordeson and Malik, deal with the applications of fuzzy approach to the ideas of automata and formal languages. In 2004, the concept of a fuzzy point belonging to fuzzy subset under a natural equivalence on fuzzy subset was initiated by Murali [13]. The 1980, the idea of quasi-coincidence of a fuzzy point with a fuzzy set was defined by Pu [14]. These two ideas play a vital role in generating some different types of fuzzy subgroups. Bhakat and Das [1, 2], gave the concepts of (α, β)-fuzzy subgroups by using the "belongs to" relation $(\in)$ and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroup. In particular, $(\in, \in \vee q)$-fuzzy subgroup is an important and valuable generalization of the Rosenfeld's fuzzy subgroup. It is now natural to study similar type of generalizations of the existing fuzzy subsystems of other algebraic structures. In [3], $(\in, \in \vee q)$-fuzzy subrings and ideals are defined. In [5], Davvaz defined $(\in, \in \vee q)$-fuzzy subnearrings and ideals of a near ring. In [6], Jun and Song initiated the study of (α, β)-fuzzy interior ideals of a semigroup. In [8], Kazanci and Yamak studied $(\in, \in \vee q)$-fuzzy bi-ideals of a semigroup. In [4], Bhakat defined $(\in \vee q)$-level subset of a fuzzy set. Shabir et al. in [16], characterized semigroups by the properties of (α, β)-fuzzy ideals. Generalizing the concept of a quasi-coincidence of a fuzzy point with a fuzzy subset Jun [7], defined $(\in, \in \vee q_k)$-fuzzy subgroups and $(\in, \in \vee q_k)$-fuzzy subalgebras in BCK/BCI-algebras, respectively. In [17], $(\in, \in \vee q_k)$-fuzzy subsemigroup, $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideal, $(\in, \in \vee q_k)$-fuzzy interior ideal, $(\in, \in \vee q_k)$-fuzzy quasi ideal and $(\in, \in \vee q_k)$-fuzzy bi-ideal are defined and different classes of semigroups are characterized in terms of these notions. In [19], Thillaigovindan et al introduced the notion of characterized near-ring by interval valued (α, β)-fuzzy ideals. Recently in [18], Tang et al. initiated the notion of interval valued $(\in, \in \vee (k^*, q_k))$-fuzzy of ordered semigroups.

In this paper we introduce the concepts of $(\in, \in \vee (k^*, q_k))$-fuzzy left (right) ideal, $(\in, \in \vee (k^*, q_k))$-fuzzy generalized bi-ideal, $(\in, \in \vee (k^*, q_k))$-fuzzy bi-ideal, $(\in, \in \vee (k^*, q_k))$-fuzzy interior ideal, $(\in, \in \vee (k^*, q_k))$-fuzzy quasi-ideal and characterize regular and intra-regular semigroups by the properties of these ideals. The concepts of an $(\in, \in \vee (k^*, q_k))$-fuzzy left (right) ideal, $(\in, \in \vee (k^*, q_k))$-fuzzy generalized bi-ideal, $(\in, \in \vee (k^*, q_k))$-fuzzy bi-ideal, $(\in, \in \vee (k^*, q_k))$-fuzzy interior ideal, $(\in, \in \vee (k^*, q_k))$-fuzzy quasi-ideal are the generalization of the concepts studied in [16, 17]. If we take $k^* = 1$, then we get, an $(\in, \in \vee q_k)$-fuzzy left (right) ideal, $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal, $(\in, \in \vee q_k)$-fuzzy bi-ideal, $(\in, \in \vee q_k)$-fuzzy interior ideal, $(\in, \in \vee q_k)$-fuzzy quasi-ideal of semigroup [17]. If we take $k^* = 1$ and $k = 0$, then we get, an $(\in, \in \vee q)$-fuzzy left (right) ideal, $(\in, \in \vee q)$-fuzzy generalized bi-ideal, $(\in, \in \vee q)$-fuzzy bi-ideal, $(\in, \in \vee q)$-fuzzy interior ideal, $(\in, \in \vee q)$-fuzzy quasi-ideal [16], which means that these fuzzy ideals become a special case of an $(\in, \in \vee (k^*, q_k))$-fuzzy left (right) ideal, $(\in, \in \vee (k^*, q_k))$-fuzzy generalized bi-ideal, $(\in, \in \vee (k^*, q_k))$-fuzzy bi-ideal, $(\in, \in \vee (k^*, q_k))$-fuzzy interior ideal, $(\in, \in \vee (k^*, q_k))$-fuzzy quasi-ideal of a semigroup.

2. Preliminaries

Throughout this paper, S will denote a semigroup unless otherwise stated. A non-empty subset A of S is said to be a subsemigroup of S, if $A^2 \subseteq A$. A non-empty subset I of a S is said to be a left(right) ideal of S, if $SI \subseteq I(IS \subseteq I)$. A non-empty subset I of S is said to be an ideal, if it is both left and right ideal of S. A non-empty subset B of S is said to be a generalized bi-ideal of S, if $BSB \subseteq B$. A non-empty subset B of S is said to be a bi-ideal of S, if it is both a subsemigroup and a generalized bi-ideal of S. A subsemigroup I of S is said to be an interior ideal of S, if $SIS \subseteq I$. A nonempty subset Q of S is said to be a quasi-ideal of S, if $QS \cap SQ \subseteq Q$. An element a of S is said to be a regular element, if there exists an element x in S such that $a = axa$. S is called regular, if every element of S is regular. An element a of S is called intra-regular, if there exist elements $x, y \in S$, such that $a = xa^2y$. A semigroup S is called intra-regular, if every element of S is intra-regular.

A fuzzy subset f of a universe X is a function from X into the unit closed interval $[0,1]$, that is $f : X \to [0,1]$.

Let f be a fuzzy set of a semigroup S and $r \in [0,1]$, the set $U(f;r) = \{x \in S | f(x) \geq r\}$ is called a level subset of the fuzzy set f.

Let f and g be two fuzzy subset of S. Then the product $f \circ g$ is defined by

$$(f \circ g)(x) = \begin{cases} \bigvee\limits_{x=ab} \{f(a) \wedge g(b)\}, & \text{if there exists } a,\ b \in S, \text{ such that } x = ab \\ 0 & \text{otherwise.} \end{cases}$$

Let A be a non-empty subset of S. We denote by f_A, the characteristic function of A, that is the mapping of S into $[0,1]$ defined by

$$f_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A. \end{cases}$$

Obviously f_A is a fuzzy subset of S.

A fuzzy subset in a universe X of the form

$$f(x) = \begin{cases} r \in (0,1] & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is called a fuzzy point with support x and value r and is denoted by x_r.

The following theorems are well known in semigroups.

Lemma 2.1. *For a semigroup S, the following are equivalent:*
(1) *S is regular.*
(2) *$R \cap L = RL$ for every right ideal R and every left ideal L of S.*
(3) *$QSQ = Q$ for every quasi-ideal Q of S.*

Lemma 2.2. *For a semigroup S, the following are equivalent:*
(1) *S is intra-regular.*
(2) *$R \cap L \subseteq RL$ for every right ideal R and every left ideal L of S.*

Lemma 2.3. *For a semigroup S, the following are equivalent:*
(1) *S is both regular and intra-regular.*
(2) *Every quasi-ideal of S is idempotent.*
(3) *Every bi-ideal of S is idempotent.*

Theorem 2.4. *For a semigroup S the following conditions are equivalent:*
(1) *S is regular.*
(2) *$B \cap A \cap L \subseteq BAL$ for every bi-ideal B, interior ideal A and left ideal L of a semigroup S.*
(3) *$B[x] \cap A[x] \cap L[x] \subseteq B[x]A[x]L[x]$, for some x in S.*

3. Characterization of Regular Semigroups in Term of $(\in, \in \vee(k^*, q_k))$-fuzzy Ideals

Generalizing the concept of $x_r q_k f$, we define $x_r(k^*, q_k)f$, where $k \in [0, 1)$, and $k^* \in (0, 1]$ and $0 \leq k < k^* \leq 1$ as $x_r(k^*, q_k)f$, if $f(x) + r + k > k^*$ and $x_r \in \vee(k^*, q_k)f$, if $x_r \in f$ or $x_r(k^*, q_k)f$. To say that $x_r\overline{(k^*, q_k)}f$ means that $x_r(k^*, q_k)f$ does not hold. In this section we characterize regular semigroups by the properties of $(\in, \in \vee(k^*, q_k))$-fuzzy left(right) ideal, $(\in, \in \vee(k^*, q_k))$-fuzzy generalized bi-ideal, $(\in, \in \vee(k^*, q_k))$-fuzzy bi-ideal, $(\in, \in \vee(k^*, q_k))$-fuzzy interior-ideal, $(\in, \in \vee(k^*, q_k))$-fuzzy quasi-ideal

Let S be a semigroup and $0 \leq k < k^* \leq 1$. For a fuzzy point x_r, we say that
(i) $x_r(k^*, q_k)f$, if $f(x) + r + k > k^*$,
(ii) $x_r \in \vee(k^*, q_k)f$ if $x_r \in f$ or $x_r(k^*, q_k)f$,
(iii) $x_r\overline{\alpha}f$, if $x_r\alpha f$ does not hold for $\alpha \in \{(k^*, q_k), \in \vee(k^*, q_k)\}$.

Definition 3.1. A fuzzy subset f of S is called an $(\in, \in \vee(k^*, q_k))$-fuzzy subsemigroup of S, if for all $r, s \in (0, 1]$ and $x, y \in S$,

$$x_r \in f, y_s \in f \Rightarrow (xy)_{\min\{r,s\}} \in \vee(k^*, q_k)f.$$

Example 3.2. Let $S = \{1, 2, 3, 4, 5\}$ and " $\cdot$ " be a binary operation defined on S in the following table:

$\cdot$	1	2	3	4	5
1	1	1	1	2	2
2	1	1	1	2	1
3	1	1	1	1	1
4	1	1	1	1	1
5	1	1	1	1	1

Then $(S, \cdot)$ is a semigroup. Define a fuzzy subset $f : S \to [0, 1]$ by $f(1) = 0.7 = f(3)$, $f(2) = 0.2 = f(5)$, $f(4) = 0.6$. If we take $k = 0.2$ and $k^* = 0.8$, then f is an $(\in, \in \vee(0.8, q_{0.2}))$-fuzzy subsemigroup of S. But
(1) f is not a fuzzy subsemigroup of S, $f(1 \cdot 4) = f(2) = 0.2 \ngeq 0.6 = f(1) \wedge f(4)$.
(2) f is not an $(\in, \in \vee q)$-fuzzy subsemigroup of S, $f(1 \cdot 4) = f(2) = 0.2 \ngeq 0.5 = f(1) \wedge f(4) \wedge 0.5$.
(3) f is not an $(\in, \in \vee q_{0.2})$-fuzzy subsemigroup of S, $f(1 \cdot 4) = f(2) = 0.2 \ngeq 0.4 = f(1) \wedge f(4) \wedge \frac{1-k}{2}$.

Definition 3.3. A fuzzy subset f of S is said to be an $(\in, \in \vee(k^*, q_k))$-fuzzy left (right) ideal of S, if the following condition holds:

$$y_r \in f \Rightarrow (xy)_r \in \vee(k^*, q_k) f \quad (y_r \in f \Rightarrow (yx)_r \in \vee(k^*, q_k) f).$$

Definition 3.4. A fuzzy subset of S is said to be an $(\in, \in \vee(k^*, q_k))$-fuzzy ideal of S, if it is both an $(\in, \in \vee(k^*, q_k))$-fuzzy left and an $(\in, \in \vee(k^*, q_k))$-fuzzy right ideals of S.

Example 3.5. From example 3.2, if we define a fuzzy subset $f : S \to [0,1]$ by $f(1) = 0.8$, $f(2) = 0.3$, $f(3) = 0.4$, $f(4) = 0.5 = f(5)$. Let $k = 0.2$ and $k^* = 0.6$, then f is an $(\in, \in \vee(0.6, q_{0.2}))$-fuzzy ideal of S. But

(1) f is not a fuzzy ideal of S,

$$f(1 \cdot 4) = f(2) = 0.3 \ngeq 0.5 = f(4)$$

and

$$(f(1 \cdot 4) = f(2) = 0.3 \ngeq 0.8 = f(1))$$

(2) f is not an $(\in, \in \vee q)$-fuzzy ideal of S,

$$f(1 \cdot 4) = f(2) = 0.3 \ngeq 0.5 = f(4) \wedge 0.5.$$

(3) f is not an $(\in, \in \vee q_{0.2})$-fuzzy ideal of S,

$$f(1 \cdot 4) = f(2) = 0.3 \ngeq 0.4 = f(4) \wedge \frac{1-k}{2}.$$

Lemma 3.6. *Let f be a fuzzy subset of S. Then f is an $(\in, \in \vee(k^*, q_k))$-fuzzy subsemigroup (left, right, two sided ideal) of S if and only if $U(f, r)(\neq \phi)$ is a subsemigroup (left, right, two sided ideal) of S for all $r \in (0, \frac{k^*-k}{2}]$.*

Lemma 3.7. *Let I be a non-empty subset of S. Then I is a left(right) ideal of S if and only if C_I the characteristic function of I is an $(\in, \in \vee(k^*, q_k))$-fuzzy left(right) ideal of S.*

Definition 3.8. A fuzzy subset f of S is called an $(\in, \in \vee(k^*, q_k))$-fuzzy generalized bi-ideal of S, if for all $r, s \in (0,1]$ and $x, y, z \in S$,

$$x_r \in f, z_s \in f \Rightarrow (xyz)_{\min\{r,s\}} \in \vee(k^*, q_k) f.$$

Definition 3.9. A fuzzy subset of S is said to be an $(\in, \in \vee(k^*, q_k))$-fuzzy bi-ideal of S, if it is both $(\in, \in \vee(k^*, q_k))$-fuzzy subsemigroup and $(\in, \in \vee(k^*, q_k))$-fuzzy generalized bi-ideal of S.

Definition 3.10. Let f be a fuzzy subset of S. Then f is said to be an $(\in, \in \vee(k^*, q_k))$-fuzzy interior ideal of S if it satisfies the following conditions:

(i) $x_r \in f$ and $y_s \in f \Rightarrow (xy)_{\min\{r,s\}} \in \vee(k^*, q_k) f$,

(ii) $x_r \in f$ and $z_s \in f \Rightarrow (xyz)_{\min\{r,s\}} \in \vee(k^*, q_k) f$,

for all $x, y, z \in S$ and $r, s \in (0,1]$, $k \in [0,1)$ and $k^* \in (0,1]$, where $0 \leq k < k^* \leq 1$.

Definition 3.11. Let f be a fuzzy subset of a semigroup S. Then f is said to be an $(\in, \in \vee(k^*, q_k))$-fuzzy quasi-ideal of S, if the following condition is satisfies

$$f(x) = \min\left\{(f \circ w)(x), (w \circ f)(x), \frac{k^* - k}{2}\right\},$$

where w is a fuzzy subset of S mapping every element of S on 1.

Definition 3.12. Let f be a fuzzy subset of S. Then the (k^*, k)-upper part of f is denoted by $\left(f_k^{k^*}\right)^+$ and is defined as

$$\left(f_k^{k^*}\right)^+(x) = \max\left\{f(x), \frac{k^*-k}{2}\right\} = f(x) \vee \frac{k^*-k}{2},$$

for all $x \in S$, where $k^* \in (0,1], k \in [0,1)$ and $0 \leq k < k^* \leq 1$.

Clearly, $\left(f_k^{k^*}\right)^+$ is a fuzzy subset of S. Let A be a non-empty subset of S and f a fuzzy subset of S. Then the (k^*, k)-upper part of the characteristic function f_A, is denoted by $\left(f_k^{k^*}\right)_A^+$.

Definition 3.13. Let f be a fuzzy subset of S. Then the (k^*, k)-lower part of f is denoted by $\left(f_k^{k^*}\right)^-$ and is defined as

$$\left(f_k^{k^*}\right)^-(x) = \min\left\{f(x), \frac{k^*-k}{2}\right\} = f(x) \wedge \frac{k^*-k}{2},$$

for all $x \in S$, where $k^* \in (0,1], k \in [0,1)$ and $0 \leq k < k^* \leq 1$.

Clearly, $\left(f_k^{k^*}\right)^-$ is a fuzzy subset of S. Let A be a non-empty subset of S and f a fuzzy subset of S. Then the (k^*, k)-lower part of the characteristic function f_A, is denoted by $\left(f_k^{k^*}\right)_A^-$.

Lemma 3.14. *A non-empty subset I of S is quasi-ideal if and only if $\left(C_k^{k^*}\right)_I^-$ is an $(\in, \in \vee (k^*, q_k))$-fuzzy quasi-ideal of S.*

Lemma 3.15. *A non-empty subset Q of S is quasi-ideal if and only if $\left(C_k^{k^*}\right)_Q^-$ is an $(\in, \in \vee (k^*, q_k))$-fuzzy quasi-ideal of S.*

Lemma 3.16. *For a semigroup S the following conditions are equivalent:*
(1) *S is regular.*
(2) *$\left(f \wedge_k^{k^*} g\right)^- = \left(f \circ_k^{k^*} g\right)^-$, for every $(\in, \in \vee(k^*, q_k))$-fuzzy right ideal f and every $(\in, \in \vee(k^*, q_k))$-fuzzy left ideal g of S.*

Theorem 3.17. *For a semigroup S, the following conditions are equivalent:*
(1) *S is regular.*
(2) *$\left(f \circ_k^{k^*} g \circ_k^{k^*} h\right)^- \geq \left(f \wedge_k^{k^*} g \wedge_k^{k^*} h\right)^-$, for every $(\in, \in \vee(k^*, q_k))$-fuzzy bi-ideal f, $(\in, \in \vee(k^*, q_k))$-fuzzy interior ideal g and $(\in, \in \vee(k^*, q_k))$-fuzzy left ideal h of a semigroup S.*
(3) *$\left(f \circ_k^{k^*} g \circ_k^{k^*} h\right)^- \geq \left(f \wedge_k^{k^*} g \wedge_k^{k^*} h\right)^-$, for every $(\in, \in \vee(k^*, q_k))$-fuzzy generalized bi-ideal f, $(\in, \in \vee(k^*, q_k))$-fuzzy interior g and $(\in, \in \vee(k^*, q_k))$-fuzzy left ideal h of a semigroup S.*

Proof. (1)$\Rightarrow$(3): Let f, g and h be any $(\in, \in \vee(k^*, q_k))$-fuzzy bi-ideal, $(\in, \in \vee(k^*, q_k))$-fuzzy interior ideal and $(\in, \in \vee(k^*, q_k))$-fuzzy left ideal of S. Since S is regular, for each $x \in S$, there exists $a \in S$ such that

$$x = xax = xaxax = xaxaxaxax = (xax)(axa)(xax).$$

Then

$$\begin{aligned}
(f\circ_k^{k^*}\circ_k^{k^*}g\circ_k^{k^*}h)^-(x) &= (f\circ g\circ h)(x)\wedge\frac{k^*-k}{2}\\
&= \left(\bigvee_{x=bc}\{f(b)\wedge(g\circ h)(c))\}\right)\wedge\frac{k^*-k}{2}\\
&\geq f(xax)\wedge(g\circ h)(axa)(xax)\wedge\frac{k^*-k}{2}\\
&= f(x)\wedge\left(\bigvee_{(axa)(xax)=pq}\{g(p)\wedge h(q))\}\right)\wedge\frac{k^*-k}{2}\\
&\geq f(x)\wedge g(axa)\wedge h(xax)\wedge\frac{k^*-k}{2}\\
&\geq f(x)\wedge g(x)\wedge h(ax)\wedge\frac{k^*-k}{2}\\
&\geq f(x)\wedge g(x)\wedge h(x)\wedge\frac{k^*-k}{2}\\
&= \left(f\wedge_k^{k^*}g\wedge_k^{k^*}h\right)^-(x).
\end{aligned}$$

(3)⇒ (2): It is straightforward.

(2)⇒ (1): Let $B[x]$, $A[x]$ and $L[x]$ be bi-ideal, interior ideal and left ideal of S generated by x, respectively. Then $\left(C_k^{k^*}\right)^-_{B[x]}$, $\left(C_k^{k^*}\right)^-_{A[x]}$ and $\left(C_k^{k^*}\right)^-_{L[x]}$ are $(\in,\in\vee(k^*,q_k))$-fuzzy bi-ideal, $(\in,\in\vee(k^*,q_k))$-fuzzy interior ideal and $(\in,\in\vee(k^*,q_k))$-fuzzy left ideal of semigroup S, respectively. Let $x\in S$ and $y\in B[x]\cap A[x]\cap L[x]$. Then $y\in B[x]$, $y\in A[x]$ and $y\in L[x]$. Now

$$\begin{aligned}
\frac{k^*-k}{2} &\leq \left(C_k^{k^*}\right)^-_{B[x]\cap A[x]\cap L[x]}(y)\\
&= \left(C_k^{k^*}\right)^-_{B[x]}\wedge_k^{k^*}\left(C_k^{k^*}\right)^-_{A[x]}\wedge_k^{k^*}\left(C_k^{k^*}\right)^-_{L[x]}(y)\\
&\leq \left(C_k^{k^*}\right)^-_{B[x]}\circ_k^{k^*}\left(C_k^{k^*}\right)^-_{A[x]}\circ_k^{k^*}\left(C_k^{k^*}\right)^-_{L[x]}(y)\\
&= \left(C_k^{k^*}\right)^-_{B[x]A[x]L[x]}(y).
\end{aligned}$$

Thus, $y\in B[x]A[x]L[x]$. So $B[x]\cap A[x]\cap L[x]\subseteq B[x]A[x]L[x]$. Hence by Theorem 2.4, S is regular. □

Theorem 3.18. *For a semigroup S, the following are equivalent:*

(1) *S is regular.*

(2) $\left(f\wedge_k^{k^*}g\wedge_k^{k^*}h\right)^-\leq\left(f\circ_k^{k^*}g\circ_k^{k^*}h\right)^-$, *for every $(\in,\in\vee(k^*,q_k))$-fuzzy right ideal f, every $(\in,\in\vee(k^*,q_k))$-fuzzy generalized bi-ideal g and every $(\in,\in\vee(k^*,q_k))$-fuzzy left ideal h of S.*

(3) $\left(f\wedge_k^{k^*}g\wedge_k^{k^*}h\right)^-\leq\left(f\circ_k^{k^*}g\circ_k^{k^*}h\right)^-$, *for every $(\in,\in\vee(k^*,q_k))$-fuzzy right ideal f, every $(\in,\in\vee(k^*,q_k))$-fuzzy bi-ideal g and every $(\in,\in\vee(k^*,q_k))$-fuzzy left ideal h of S.*

(4) *$\left(f \wedge_k^{k^*} g \wedge_k^{k^*} h\right)^- \leq \left(f \circ_k^{k^*} g \circ_k^{k^*} h\right)^-$, for every $(\in, \in \vee (k^*, q_k))$-fuzzy right ideal f, every $(\in, \in \vee (k^*, q_k))$-fuzzy quasi-ideal g and every $(\in, \in \vee (k^*, q_k))$-fuzzy left ideal h of S.*

Proof. (1)$\Rightarrow$ (2): Let f, g and h be any $(\in, \in \vee (k^*, q_k))$-fuzzy right ideal, $(\in, \in \vee (k^*, q_k))$-fuzzy generalized bi-ideal and $(\in, \in \vee (k^*, q_k))$-fuzzy left ideal of S, respectively. Let $x \in S$. Since S is regular, there exists $a \in S$ such that $x = xax$. Then we have

$$\begin{aligned}
\left(f \circ_k^{k^*} g \circ_k^{k^*} h\right)^- (x) &= \left(\bigvee_{x=bc} \{f(b) \wedge (g \circ h)(c)\}\right) \wedge \frac{k^*-k}{2} \\
&\geq f(xa) \wedge (g \circ h)(x) \wedge \frac{k^*-k}{2} \\
&\geq \left(f(x) \wedge \frac{k^*-k}{2}\right) \wedge \left(\bigvee_{x=de} \{g(d) \wedge h(e)\}\right) \wedge \frac{k^*-k}{2} \\
&\geq \left(f(x) \wedge \frac{k^*-k}{2}\right) \wedge (\{g(x) \wedge h(ax)\}) \wedge \frac{k^*-k}{2} \\
&\geq f(x) \wedge g(x) \wedge h(x) \wedge \frac{k^*-k}{2} \\
&= \left(f \wedge_k^{k^*} g \wedge_k^{k^*} h\right)^- (x).
\end{aligned}$$

(2)$\Rightarrow$ (3)$\Rightarrow$ (4) are straightforward.

(4)$\Rightarrow$ (1): Let f and g be any $(\in, \in \vee (k^*, q_k))$-fuzzy right ideal and $(\in, \in \vee (k^*, q_k))$-fuzzy left ideal of S, respectively. Since w is an $(\in, \in \vee (k^*, q_k))$-fuzzy quasi-ideal of S, by hypothesis, we have

$$\begin{aligned}
\left(f \wedge_k^{k^*} g\right)^- (x) &= (f \wedge g)(x) \wedge \tfrac{k^*-k}{2} \\
&= (f \wedge w \wedge g)(x) \wedge \tfrac{k^*-k}{2} \\
&= \left(f \wedge_k^{k^*} w \wedge_k^{k^*} g\right)(x) \\
&\leq \left(f \circ_k^{k^*} w \circ_k^{k^*} g\right)(x) \\
&= (f \circ w \circ g)(x) \wedge \tfrac{k^*-k}{2} \\
&= \left(\bigvee_{x=bc} \{((f \circ w)(b) \wedge g(c))\}\right) \wedge \tfrac{k^*-k}{2} \\
&= \left(\bigvee_{x=bc} \left\{\left(\bigvee_{b=de} \{f(d) \wedge w(e)\}\right) \wedge g(c)\right\}\right) \wedge \tfrac{k^*-k}{2} \\
&= \left(\bigvee_{x=bc} \left\{\left(\bigvee_{b=de} \{f(d) \wedge 1\}\right) \wedge g(c)\right\}\right) \wedge \tfrac{k^*-k}{2} \\
&= \left(\bigvee_{x=bc} \left\{\left(\bigvee_{b=de} f(d)\right) \wedge g(c)\right\}\right) \wedge \tfrac{k^*-k}{2} \\
&= \left(\bigvee_{x=bc} \left\{\left(\bigvee_{b=de} \{f(d) \wedge g(c)\}\right) \wedge \tfrac{k^*-k}{2}\right\}\right) \wedge \tfrac{k^*-k}{2} \\
&= \left(\bigvee_{x=bc} \left(\bigvee_{b=de} \left\{f(d) \wedge \tfrac{k^*-k}{2}\right\}\right)\right\} \wedge g(c)\right) \wedge \tfrac{k^*-k}{2} \\
&\leq \left(\bigvee_{x=bc} \left\{\bigvee_{b=de} \{f(de) \wedge g(b)\}\right\}\right) \wedge \tfrac{k^*-k}{2}
\end{aligned}$$

$$= \left(\bigvee_{x=bc} \{f(b) \wedge g(c)\} \right) \wedge \frac{k^*-k}{2}$$
$$= f(x) \circ g(x) \wedge \frac{k^*-k}{2}$$
$$= \left(f \circ_k^{k^*} g\right)^- (x).$$

It follows that $\left(f \wedge_k^{k^*} g\right)^- \leq \left(f \circ_k^{k^*} g\right)^-$, for every $(\in, \in \vee (k^*, q_k))$-fuzzy right ideal and every $(\in, \in \vee (k^*, q_k))$-fuzzy left ideal of S. But from Lemma 3.16, we see that $\left(f \circ_k^{k^*} g\right)^- \leq \left(f \wedge_k^{k^*} g\right)^-$. Thus $\left(f \circ_k^{k^*} g\right)^- = \left(f \wedge_k^{k^*} g\right)^-$. So S is regular. □

Theorem 3.19. *For a semigroup S, the following are equivalent:*

(1) *S is regular.*

(2) $\left(f_k^{k^*}\right)^- = \left(f \circ_k^{k^*} w \circ_k^{k^*} f\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy generalized bi-ideal f of S*

(3) $\left(f_k^{k^*}\right)^- = \left(f \circ_k^{k^*} w \circ_k^{k^*} f\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy bi-ideal f of S.*

(4) $\left(f_k^{k^*}\right)^- = \left(f \circ_k^{k^*} w \circ_k^{k^*} f\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy quasi-ideal f of S.*

Proof. (1)⇒ (2): Let f be an $(\in, \in \vee (k^*, q_k))$-fuzzy generalized bi-ideal of S and $x \in S$. As S is regular, there exists $a \in S$ such that $x = xax$. Then we have

$$\begin{aligned}
\left(f \circ_k^{k^*} w \circ_k^{k^*} f\right)^- (x) &= (f \circ w \circ f)(x) \wedge \frac{k^*-k}{2} \\
&= \left(\bigvee_{x=bc} \{(f \circ w)(b) \wedge f(c)\} \right) \wedge \frac{k^*-k}{2} \\
&\geq (f \circ w)(xa) \wedge f(x) \wedge \frac{k^*-k}{2} \\
&= \left(\left(\bigvee_{xa=de} \{f(d) \wedge w(e)\} \right) \wedge f(x) \right) \wedge \frac{k^*-k}{2} \\
&\geq (f(x) \wedge w(a)) \wedge f(x) \wedge \frac{k^*-k}{2} \\
&= (f(x) \wedge 1) \wedge f(x) \wedge \frac{k^*-k}{2} \\
&= f(x) \wedge f(x) \wedge \frac{k^*-k}{2} \\
&= f(x) \wedge \frac{k^*-k}{2} \\
&= \left(f_k^{k^*}\right)^- (x).
\end{aligned}$$

Thus, $\left(f_k^{k^*}\right)^- \leq \left(f \circ_k^{k^*} w \circ_k^{k^*} f\right)^-$. Since f is an $(\in, \in \vee (k^*, q_k))$-fuzzy generalized bi-ideal of S, we have

$$\begin{aligned}
\left(f \circ_k^{k^*} w \circ_k^{k^*} f\right)^- (x) &= (f \circ w \circ f)(x) \wedge \frac{k^*-k}{2} \\
&= \left(\bigvee_{x=bc} \{(f \circ w)(b) \wedge f(c)\} \right) \wedge \frac{k^*-k}{2} \\
&= \left(\bigvee_{x=bc} \left\{ \left(\bigvee_{b=de} \{f(d) \wedge w(e)\} \right) \wedge f(c) \right\} \right) \wedge \frac{k^*-k}{2} \\
&= \left(\bigvee_{x=bc} \left\{ \left(\bigvee_{b=de} \{f(d) \wedge 1\} \right) \wedge f(c) \right\} \right) \wedge \frac{k^*-k}{2} \\
&= \left(\bigvee_{x=bc} \left\{ \bigvee_{b=de} \{f(d) \wedge f(c)\} \right\} \right) \wedge \frac{k^*-k}{2} \\
&= \left(\bigvee_{x=bc} \left\{ \bigvee_{b=de} \left\{ f(d) \wedge f(c) \wedge \frac{k^*-k}{2} \right\} \right\} \right) \wedge \frac{k^*-k}{2}
\end{aligned}$$

$$\leq \bigvee_{x=(de)c} \left\{ f(dec) \wedge \tfrac{k^*-k}{2} \right\}$$

(Since, f is an $(\in, \in \vee (k^*, q_k))$-fuzzy generalized bi-ideal of S.)

$$\leq f(x) \wedge \tfrac{k^*-k}{2}$$

$$= \left(f_k^{k^*}\right)^-(x).$$

This shows that $\left(f_k^{k^*}\right)^- \geq \left(f \circ_k^{k^*} w \circ_k^{k^*} f\right)^-$. So $\left(f_k^{k^*}\right)^- = \left(f \circ_k^{k^*} w \circ_k^{k^*} f\right)^-$.

(2)⇒(3)⇒(4) are straightforward.

(4)⇒(1): Let Q be a quasi-ideal of S. Then C_Q is an $(\in, \in \vee (k^*, q_k))$-fuzzy quasi-ideal of S. Thus $\left(C_k^{k^*}\right)_Q^- = \left(C_Q \circ_k^{k^*} C_S \circ_k^{k^*} C_Q\right)^- = \left(C_k^{k^*}\right)_{QSQ}^-$. This shows that $Q = QSQ$. So from Lemma 2.1, it follows that S is regular. □

Theorem 3.20. *For a semigroup S, the following are equivalent:*

(1) *S is regular.*

(2) $\left(f \wedge_k^{k^*} g\right)^- = \left(f \circ_k^{k^*} g \circ_k^{k^*} f\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy quasi-ideal f and $(\in, \in \vee (k^*, q_k))$-fuzzy ideal g of S.*

(3) $\left(f \wedge_k^{k^*} g\right)^- = \left(f \circ_k^{k^*} g \circ_k^{k^*} f\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy quasi-ideal f and $(\in, \in \vee (k^*, q_k))$-fuzzy interior ideal g of S.*

(4) $\left(f \wedge_k^{k^*} g\right)^- = \left(f \circ_k^{k^*} g \circ_k^{k^*} f\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy bi-ideal f and $(\in, \in \vee (k^*, q_k))$-fuzzy ideal g of S.*

(5) $\left(f \wedge_k^{k^*} g\right)^- = \left(f \circ_k^{k^*} g \circ_k^{k^*} f\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy bi-ideal f and $(\in, \in \vee (k^*, q_k))$-fuzzy interior ideal g of S.*

(6) $\left(f \wedge_k^{k^*} g\right)^- = \left(f \circ_k^{k^*} g \circ_k^{k^*} f\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy generalized bi-ideal f and $(\in, \in \vee (k^*, q_k))$-fuzzy ideal g of S.*

(7) $\left(f \wedge_k^{k^*} g\right)^- = \left(f \circ_k^{k^*} g \circ_k^{k^*} f\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy generalized bi-ideal f and $(\in, \in \vee (k^*, q_k))$-fuzzy interior ideal g of S.*

Proof. (1)⇒(7): Suppose S is regular semigroup. Let f and g be any two an $(\in, \in \vee (k^*, q_k))$-fuzzy generalized bi-ideal and $(\in, \in \vee (k^*, q_k))$-fuzzy interior ideal of S, respectively. Then for $x \in S$, we have

$$\begin{aligned}
\left(f \circ_k^{k^*} g \circ_k^{k^*} f\right)^-(x) &= (f \circ g \circ f)(x) \wedge \tfrac{k^*-k}{2} \\
&\leq (f \circ w \circ f)(x) \wedge \tfrac{k^*-k}{2} \\
&= \left(\bigvee_{x=ab} \{(f \circ w)(a) \wedge f(b)\} \right) \wedge \tfrac{k^*-k}{2} \\
&= \left(\bigvee_{x=ab} \left\{ \bigvee_{a=cd} \{(f(c) \wedge w(d))\} \wedge f(b) \right\} \right) \wedge \tfrac{k^*-k}{2} \\
&= \left(\bigvee_{x=ab} \left\{ \bigvee_{a=cd} \{(f(c) \wedge 1)\} \wedge f(b) \right\} \right) \wedge \tfrac{k^*-k}{2} \\
&= \left(\bigvee_{x=ab} \left\{ \bigvee_{a=cd} f(c) \wedge f(b) \right\} \right) \wedge \tfrac{k^*-k}{2} \\
&= \bigvee_{x=ab} \left\{ \bigvee_{a=cd} \left\{ f(c) \wedge f(b) \wedge \tfrac{k^*-k}{2} \right\} \right\} \\
&= \bigvee_{x=(cd)b} \left\{ f(c) \wedge f(b) \wedge \tfrac{k^*-k}{2} \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \bigvee_{x=(cd)b} \left\{ f(cdb) \wedge \tfrac{k^*-k}{2} \right\} \\
&= f(x) \wedge \tfrac{k^*-k}{2} \\
&= \left(f_k^{k^*}\right)^-(x).
\end{aligned}$$

Thus, $\left(f \circ_k^{k^*} g \circ_k^{k^*} f\right)^- \leq \left(f_k^{k^*}\right)^-$. Also,

$$\begin{aligned}
\left(f \circ_k^{k^*} g \circ_k^{k^*} f\right)^-(x) &\leq \left(w \circ_k^{k^*} g \circ_k^{k^*} w\right)^-(x) \\
&= (w \circ g \circ w)(x) \wedge \tfrac{k^*-k}{2} \\
&= \left(\bigvee_{x=ab} \{ (w \circ g)(a) \wedge w(b) \} \right) \wedge \tfrac{k^*-k}{2} \\
&= \left(\bigvee_{x=ab} \left\{ \left(\bigvee_{a=cd} \{ w(c) \wedge g(d) \} \right) \wedge w(b) \right\} \right) \wedge \tfrac{k^*-k}{2} 4 \\
&= \left(\bigvee_{x=ab} \left\{ \left(\bigvee_{a=cd} \{ 1 \wedge g(d) \} \right) \wedge 1 \right\} \right) \wedge \tfrac{k^*-k}{2} \\
&= \left(\bigvee_{x=ab} \left\{ \bigvee_{a=cd} g(d) \right\} \right) \wedge \tfrac{k^*-k}{2} \\
&= \bigvee_{x=(cd)b} \left\{ \bigvee_{a=cd} g(d) \wedge \tfrac{k^*-k}{2} \right\} \\
&\leq \bigvee_{x=(cd)b} \left\{ f(cdb) \wedge \left(f \circ_k^{k^*} g \circ_k^{k^*} f\right)^- \right\} \\
&= g(x) \wedge \tfrac{k^*-k}{2} \\
&= \left(g_k^{k^*}\right)^-(x).
\end{aligned}$$

So, $\left(f \circ_k^{k^*} g \circ_k^{k^*} f\right)^- \leq \left(f \circ_k^{k^*} g\right)^- = \left(g \circ_k^{k^*} f\right)^-$.

Now let $x \in S$. Since S is regular, there exists $a \in S$ such that $x = xax = xaxax$. Since f is an $(\in, \in \vee (k^*, q_k))$-fuzzy interior ideal of S, we have

$$\begin{aligned}
\left(f \circ_k^{k^*} g \circ_k^{k^*} f\right)^-(x) &= (f \circ g \circ f)(x) \wedge \frac{k^*-k}{2} \\
&= \bigvee_{x=bc} \{ f(b) \wedge (g \circ f)(c) \} \wedge \frac{k^*-k}{2} \\
&\geq f(x) \wedge (g \circ f)(axax) \wedge \frac{k^*-k}{2} \\
&= f(x) \wedge \left(\bigvee_{axax=de} (g(d) \wedge f(e)) \right) \wedge \frac{k^*-k}{2} \\
&\geq f(x) \wedge (g(axa) \wedge f(x)) \wedge \frac{k^*-k}{2} \\
&\geq f(x) \wedge \left(g(x) \wedge \frac{k^*-k}{2} \wedge f(x) \right) \wedge \frac{k^*-k}{2} \\
&\qquad \text{(Since } g \text{ is an } (\in, \in \vee (k^*, q_k))\text{-fuzzy interior ideal of } S.) \\
&= f(x) \wedge g(x) \wedge \frac{k^*-k}{2} \\
&= \left(f \wedge_k^{k^*} g\right)^-(x).
\end{aligned}$$

So, $\left(f \circ_k^{k^*} g \circ_k^{k^*} f\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$. Hence $\left(f \circ_k^{k^*} g \circ_k^{k^*} f\right)^- = \left(f \wedge_k^{k^*} g\right)^-$.

(7)$\Rightarrow$ (5)$\Rightarrow$ (3)$\Rightarrow$ (2) and (7)$\Rightarrow$ (6)$\Rightarrow$ (4)$\Rightarrow$ (2) are straightforward.

(4)$\Rightarrow$ (2): Let f be an $(\in, \in \vee (k^*, q_k))$-fuzzy quasi-ideal of S. Then, since S itself is an $(\in, \in \vee (k^*, q_k))$-fuzzy ideal of S, for $x \in S$, we have

$$\begin{aligned}
\left(f_k^{k^*}\right)^- (x) &= f(x) \wedge \frac{k^* - k}{2} \\
&= (f \wedge w)(x) \wedge \frac{k^* - k}{2} \\
&= \left(f \wedge_k^{k^*} w\right)^- (x) \\
&= \left(f \circ_k^{k^*} w \circ_k^{k^*} f\right)^- (x).
\end{aligned}$$

So by Lemma 3.19, it follows that S is regular. □

Theorem 3.21. *For a semigroup S, the following are equivalent:*

(1) *S is regular.*

(2) $\left(f \wedge_k^{k^*} g\right)^- \leq \left(f \circ_k^{k^*} g\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy generalized bi-ideal f and $(\in, \in \vee (k^*, q_k))$-fuzzy left ideal g of S.*

(3) $\left(f \wedge_k^{k^*} g\right)^- \leq \left(f \circ_k^{k^*} g\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy bi-ideal f and $(\in, \in \vee (k^*, q_k))$-fuzzy left ideal g of S.*

(4) $\left(f \wedge_k^{k^*} g\right)^- \leq \left(f \circ_k^{k^*} g\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy quasi-ideal f and $(\in, \in \vee (k^*, q_k))$-fuzzy left ideal g of S.*

Proof. (1)$\Rightarrow$(2): Let f be an $(\in, \in \vee (k^*, q_k))$-fuzzy generalized bi-ideal and g an $(\in, \in \vee (k^*, q_k))$-fuzzy left ideal of S. Let $x \in S$. Since, S is regular so there exists $a \in S$ such that $x = xax$. Thus we have

$$\begin{aligned}
\left(f \circ_k^{k^*} g\right)^- (x) &= (f \circ g)(x) \wedge \frac{k^* - k}{2} \\
&= \left(\bigvee_{x = bc} \{f(b) \wedge g(c)\}\right) \wedge \frac{k^* - k}{2} \\
&\geq f(x) \wedge g(ax) \wedge \frac{k^* - k}{2} \\
&\geq f(x) \wedge g(x) \wedge \frac{k^* - k}{2} \wedge \frac{k^* - k}{2} \\
&= f(x) \wedge g(x) \wedge \frac{k^* - k}{2} \\
&= \left(f \wedge_k^{k^*} g\right)^- (x).
\end{aligned}$$

So, $\left(f \circ_k^{k^*} g\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$.

(2)$\Rightarrow$(3)$\Rightarrow$(4) are straightforward.

(4)$\Rightarrow$(1): Let f be an $(\in, \in \vee (k^*, q_k))$-fuzzy right ideal and g an $(\in, \in \vee (k^*, q_k))$-fuzzy left ideal of S. Since every $(\in, \in \vee (k^*, q_k))$-fuzzy right ideal of S is an

$(\in, \in \vee (k^*, q_k))$-fuzzy quasi-ideal of S, So we have $\left(f \circ_k^{k^*} g\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$. Now

$$\begin{aligned}
\left(f \circ_k^{k^*} g\right)^- (x) &= (f \circ g)(x) \wedge \frac{k^* - k}{2} \\
&= \left(\bigvee_{x=bc} \{f(b) \wedge g(c)\}\right) \wedge \frac{k^* - k}{2} \\
&= \bigvee_{x=bc} \left(\{f(b) \wedge g(c)\} \wedge \frac{k^* - k}{2}\right) \\
&= \bigvee_{x=bc} \left(\left\{f(b) \wedge \frac{k^* - k}{2}\right\} \wedge \left\{g(c) \wedge \frac{k^* - k}{2}\right\}\right) \wedge \frac{k^* - k}{2} \\
&\leq \bigvee_{x=bc} \left(\{f(bc) \wedge g(bc)\} \wedge \frac{k^* - k}{2}\right) \\
&= f(x) \wedge g(x) \wedge \frac{k^* - k}{2} \\
&= \left(f \wedge_k^{k^*} g\right)^- (x).
\end{aligned}$$

Then, $\left(f \circ_k^{k^*} g\right)^- \leq \left(f \wedge_k^{k^*} g\right)^-$. Thus $\left(f \circ_k^{k^*} g\right)^- = \left(f \wedge_k^{k^*} g\right)^-$, for every $(\in, \in \vee (k^*, q_k))$-fuzzy right ideal f on S and every $(\in, \in \vee (k^*, q_k))$-fuzzy left ideal g of S. So by Lemma 3.16, S is regular. □

Theorem 3.22. *For a semigroup S, the following are equivalent:*

(1) *S is intra-regular.*

(2) $\left(f \wedge_k^{k^*} g\right)^- \leq \left(f \circ_k^{k^*} g\right)^-$*, for every $(\in, \in \vee (k^*, q_k))$-fuzzy left ideal f on S and every $(\in, \in \vee (k^*, q_k))$-fuzzy right ideal g of S.*

Proof. (1)⇒(2): Let f be an $(\in, \in \vee (k^*, q_k))$-fuzzy left ideal and g an $(\in, \in \vee (k^*, q_k))$-fuzzy right ideal of S. Since S is intra-regular, for $x \in S$, there exist $a, b \in S$ such that $x = ax^2b = axxb$. Then

$$\begin{aligned}
\left(f \circ_k^{k^*} g\right)^- (x) &= (f \circ g)(x) \wedge \frac{k^* - k}{2} \\
&= \left(\bigvee_{x=pq} \{f(p) \wedge g(q)\}\right) \wedge \frac{k^* - k}{2} \\
&\geq \{f(ax) \wedge g(xb)\} \wedge \frac{k^* - k}{2} \\
&\geq \left\{\left(f(x) \wedge \frac{k^* - k}{2}\right) \wedge \left(g(x) \wedge \frac{k^* - k}{2}\right)\right\} \wedge \frac{k^* - k}{2} \\
&= f(x) \wedge g(x) \wedge \frac{k^* - k}{2} \\
&= \left(f \wedge_k^{k^*} g\right)^- (x).
\end{aligned}$$

Thus, $\left(f \wedge_k^{k^*} g\right)^- \leq \left(f \circ_k^{k^*} g\right)^-$.

(2)⇒(1): Let R and L be right and left ideals of S. Then by Lemma 3.14, $\left(C_k^{k^*}\right)_L^-$ is an $(\in, \in \vee (k^*, q_k))$-fuzzy left ideal of S and $\left(C_k^{k^*}\right)_R^-$ is an $(\in, \in \vee (k^*, q_k))$-fuzzy right ideal of S. Thus by assumption, we have

$$\begin{aligned} \left(C_k^{k^*}\right)_{LR}^- &= \left(C_L \circ_k^{k^*} C_R\right) \\ &\geq \left(C_L \wedge_k^{k^*} C_R\right) \\ &= \left(C_k^{k^*}\right)_{R\cap L}^- . \end{aligned}$$

So $R \cap L \subseteq LR$. Hence from Lemma 2.2, it follows that S is intra-regular. □

Theorem 3.23. *For a semigroup S, the following are equivalent:*
(1) *S is both regular and intra-regular.*
(2) $\left(f \circ_k^{k^*} f\right)^- = \left(f_k^{k^*}\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy quasi-ideal f of S.*
(3) $\left(f \circ_k^{k^*} f\right)^- = \left(f_k^{k^*}\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy bi-ideal f of S.*
(4) $\left(f \circ_k^{k^*} g\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$, *for all $(\in, \in \vee (k^*, q_k))$-fuzzy quasi-ideal f and g of S.*
(5) $\left(f \circ_k^{k^*} g\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy quasi-ideal f and $(\in, \in \vee (k^*, q_k))$-fuzzy bi-ideal g of S.*
(6) $\left(f \circ_k^{k^*} g\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$, *for all $(\in, \in \vee (k^*, q_k))$-fuzzy bi-ideal f and g of S.*

Proof. (1)⇒(6): Let f and g be any $(\in, \in \vee (k^*, q_k))$-fuzzy bi-ideal f of S. Since S is both regular and intra-regular, for $x \in S$, there exist $a, b, c \in S$ such that $x = xax$ and $x = bx^2c$. Then $x = xax = xaxax = xa(bx^2)ax = (xabx)(xcax)$. Thus,

$$\begin{aligned} \left(f \circ_k^{k^*} g\right)^- (x) &= (f \circ g) \wedge \frac{k^* - k}{2} \\ &= \left(\bigvee_{x=pq} \{f(p) \wedge g(q)\}\right) \wedge \frac{k^* - k}{2} \\ &\geq (f(xabx) \wedge g(xcax)) \wedge \frac{k^* - k}{2} \\ &\geq \left(f(x) \wedge \frac{k^* - k}{2}\right) \wedge \left(g(x) \wedge \frac{k^* - k}{2}\right) \\ &= f(x) \wedge g(x) \wedge \frac{k^* - k}{2} \\ &= \left(f \wedge_k^{k^*} g\right)^- (x). \end{aligned}$$

So, $\left(f \circ_k^{k^*} g\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$, for every $(\in, \in \vee (k^*, q_k))$-fuzzy bi-ideal f and g of S.

(6)⇒(5)⇒(4) are straightforward.

(4)⇒(2): In (4), take $f = g$. Then we have $\left(f \circ_k^{k^*} f\right)^- \geq \left(f_k^{k^*}\right)^-$. Since every $(\in, \in \vee (k^*, q_k))$-fuzzy quai-ideal is an $(\in, \in \vee (k^*, q_k))$-fuzzy subsemigroup, $\left(f_k^{k^*}\right)^- \geq \left(f \circ_k^{k^*} f\right)^-$. Thus $\left(f_k^{k^*}\right)^- = \left(f \circ_k^{k^*} f\right)^-$.

(6)⇒(3): In (6). take $f = g$. Then we have $\left(f \circ_k^{k^*} f\right)^- \geq \left(f_k^{k^*}\right)^-$. Since every $(\in, \in \vee (k^*, q_k))$-fuzzy bi-ideal is an $(\in, \in \vee (k^*, q_k))$-fuzzy subsemigroup, $\left(f_k^{k^*}\right)^- \geq \left(f \circ_k^{k^*} f\right)^-$. Thus $\left(f_k^{k^*}\right)^- = \left(f \circ_k^{k^*} f\right)^-$.

(3)⇒(2) is straightforward.

(2)⇒(1): Let Q be a quasi-ideal of S. Then by Lemma 3.15, C_Q is an $(\in, \in \vee (k^*, q_k))$-fuzzy quasi-ideal of S. Thus, by assumption, we have $\left(C_Q \circ_k^{k^*} C_Q\right)^- = \left(C_k^{k^*}\right)_Q^-$. So $\left(C_k^{k^*}\right)_{QQ}^- = \left(C_Q \circ_k^{k^*} C_Q\right)^- = \left(C_k^{k^*}\right)_Q^-$. Hence $QQ = Q$. Therefore by Lemma 2.3, S is both regular and intra-regular. □

Theorem 3.24. *For a semigroup S, the following are equivalent:*

(1) *S both regular and intra-regular.*

(2) $\left(f \circ_k^{k^*} g\right)^- \wedge \left(g \circ_k^{k^*} f\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy right ideal f and every $(\in, \in \vee (k^*, q_k))$-fuzzy left ideal g of S.*

(3) $\left(f \circ_k^{k^*} g\right)^- \wedge \left(g \circ_k^{k^*} f\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy right ideal f and every $(\in, \in \vee (k^*, q_k))$-fuzzy quasi-ideal g of S.*

(4) $\left(f \circ_k^{k^*} g\right)^- \wedge \left(g \circ_k^{k^*} f\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy right ideal f and every $(\in, \in \vee (k^*, q_k))$-fuzzy bi-ideal g of S.*

(5) $\left(f \circ_k^{k^*} g\right)^- \wedge \left(g \circ_k^{k^*} f\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy right ideal f and every $(\in, \in \vee (k^*, q_k))$-fuzzy generalized bi-ideal g of S.*

(6) $\left(f \circ_k^{k^*} g\right)^- \wedge \left(g \circ_k^{k^*} f\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy left ideal f and every $(\in, \in \vee (k^*, q_k))$-fuzzy quasi-ideal g of S.*

(7) $\left(f \circ_k^{k^*} g\right)^- \wedge \left(g \circ_k^{k^*} f\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy left ideal f and every $(\in, \in \vee (k^*, q_k))$-fuzzy bi-ideal g of S.*

(8) $\left(f \circ_k^{k^*} g\right)^- \wedge \left(g \circ_k^{k^*} f\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy left ideal f and every $(\in, \in \vee (k^*, q_k))$-fuzzy generalized bi-ideal g of S.*

(9) $\left(f \circ_k^{k^*} g\right)^- \wedge \left(g \circ_k^{k^*} f\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy quasi-ideal f and g of S.*

(10) $\left(f \circ_k^{k^*} g\right)^- \wedge \left(g \circ_k^{k^*} f\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy quasi-ideal f and $(\in, \in \vee (k^*, q_k))$-fuzzy bi-ideal g of S.*

(11) $\left(f \circ_k^{k^*} g\right)^- \wedge \left(g \circ_k^{k^*} f\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy quasi-ideal f and $(\in, \in \vee (k^*, q_k))$-fuzzy generalized bi-ideal g of S.*

(12) $\left(f \circ_k^{k^*} g\right)^- \wedge \left(g \circ_k^{k^*} f\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy bi-ideal f and g of S.*

(13) $\left(f \circ_k^{k^*} g\right)^- \wedge \left(g \circ_k^{k^*} f\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy bi-ideal f and $(\in, \in \vee (k^*, q_k))$-fuzzy generalized bi-ideal g of S.*

(14) $\left(f \circ_k^{k^*} g\right)^- \wedge \left(g \circ_k^{k^*} f\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$, *for every $(\in, \in \vee (k^*, q_k))$-fuzzy generalized bi-ideal f and g of S.*

Proof. (1)⇒(14): Let f and g be any $(\in, \in \vee (k^*, q_k))$-fuzzy generalized bi-ideal of S. Since S is both regular and intra-regular, for $x \in S$, there exist $a, b, c \in S$ such that $x = xax$ and $x = bx^2c$. Then $x = xax = xaxax = xa(ba^2c)ax = (xabx)(xcax)$.

Thus,

$$\begin{aligned}
\left(f \circ_k^{k^*} g\right)^- (x) &= \left(\bigvee_{x=pq} \{f(q) \wedge g(q)\}\right) \wedge \frac{k^*-k}{2} \\
&\geq \{f(xabx) \wedge g(xcax)\} \wedge \frac{k^*-k}{2} \\
&\geq \left\{\left(f(x) \wedge \frac{k^*-k}{2}\right) \wedge \left(g(x) \wedge \frac{k^*-k}{2}\right)\right\} \wedge \frac{k^*-k}{2} \\
&= f(x) \wedge g(x) \wedge \frac{k^*-k}{2} \\
&= \left(f \wedge_k^{k^*} g\right)^- (x).
\end{aligned}$$

In similar, way we can prove that $\left(g \circ_k^{k^*} f\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$. So $\left(f \circ_k^{k^*} g\right)^- \wedge \left(g \circ_k^{k^*} f\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$.

(14)⇒(13)⇒(12)⇒(10)⇒(9)⇒(3)⇒(2), (14)⇒(9)⇒(10), (14)⇒(8)⇒(7)⇒(6)⇒(2) and (14)⇒(5)⇒(4)⇒(3)⇒(2) are straightforward.

(2)⇒(1): Let f be an $(\in, \in \vee (k^*, q_k))$-fuzzy right ideal and g an $(\in, \in \vee (k^*, q_k))$-fuzzy left ideal of S. Let $x \in S$. Then we have

$$\begin{aligned}
\left(f \circ_k^{k^*} g\right)^- (x) &= (f \circ g)(x) \wedge \frac{k^*-k}{2} \\
&= \left(\bigvee_{x=bc} \{f(b) \wedge g(c)\}\right) \wedge \frac{k^*-k}{2} \\
&= \bigvee_{x=bc} \left\{f(b) \wedge g(c) \wedge \frac{k^*-k}{2}\right\} \\
&= \bigvee_{x=bc} \left\{\left(f(b) \wedge \frac{k^*-k}{2}\right) \wedge \left(g(c) \wedge \frac{k^*-k}{2}\right) \wedge \frac{k^*-k}{2}\right\} \\
&\leq \bigvee_{x=bc} \left\{f(bc) \wedge g(bc) \wedge \frac{k^*-k}{2}\right\} \\
&= f(x) \wedge g(x) \wedge \frac{k^*-k}{2} \\
&= \left(f \wedge_k^{k^*} g\right)^- (x).
\end{aligned}$$

Thus, $\left(f \circ_k^{k^*} g\right)^- \leq \left(f \wedge_k^{k^*} g\right)^-$. So by assumption, $\left(f \circ_k^{k^*} g\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$ and thus $\left(f \circ_k^{k^*} g\right)^- = \left(f \wedge_k^{k^*} g\right)^-$. Hence by Lemma 3.16, S is regular. Also by assumption, $\left(f \circ_k^{k^*} g\right)^- \geq \left(f \wedge_k^{k^*} g\right)^-$. Therefore by Theorem 3.22, S is intra-regular. □

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