Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 13, No. 3, (March 2017), pp. 379–391
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Solution of intuitionistic fuzzy fractional differential equations

M. Elomari, S. Melliani , L. S. Chadli

Received 8 March 2016; Revised 21 May 2016; Accepted 30 August 2016

Abstract. In this work, we studies the solution concept of fractional differential equations with intuitionistic fuzzy initial data under generalized fuzzy Caputo derivative.

2010 AMS Classification: 03E72, 08A72

Keywords: Generalized intuitionistic fuzzy derivative, Caputo fractional derivative, Hukuhara difference.

1. Introduction

In this paper, we studies the solution of fractional differential equation with initial value are intuitionistic fuzzy numbers.

$$\begin{cases} \left({}^{c}_{gH}D^{\delta}x(t)\right) = f\left(t,\ {}^{c}_{gH}D^{\gamma}x(t)\right), & t \in I = [0,T] \\ x^{(i)}(0) = x_i \in \mathbb{F}_1, & \forall i \in \{0,1\}. \end{cases} \tag{1.1}$$

where $1 < \delta < 2$, $0 < \gamma < 1$, the operator ${}^{c}_{gH}D^{\gamma}$ denote the Caputo fractional generalized derivative of order γ, $f : I \times \mathbb{F}_1(\mathbb{R}) \longrightarrow \mathbb{F}_1(\mathbb{R})$.

The concept of intuitionistic fuzzy sets is intoduced by K. Atanassov [2]. The autors in [3] built the concept of intuitionistic fuzzy metric space and intuionistic fuzzy numbers. In [4] S. Melliani introduce the extension of Hukuhara difference in the intuitionistic fuzzy case. T. Allahviranloo, A. Armand and Z. Gouyandeh in [1] solve the fuzzy fractional differential equations under generalized fuzzy Caputo derivative. From this end idea we introduce in this paper the concept of generalized intuitionistic fuzzy caputo derivative, and we give an integral solution of an intuitionistic fuzzy fractional equation.

This paper is organized as follows. In section 2 we recall some concept concerning the intuitionistic fuzzy numbers. The concept of generalized intuitionistic fuzzy derivative and generalized intuitionistic fuzzy Caputo derivative, takes place in section

3. The integral solution has descused in section 4. Finally in section 5 we illustrate by an example.

2. Preliminaries

Definition 2.1 ([3])**.** The set of all intuitionistic fuzzy numbers is given by

$$\mathbb{F}_1 = \mathbb{F}_1(\mathbb{R}) = \Big\{ < u, v >: \mathbb{R} \longrightarrow [0,1]^2,\ 0 \leq u + v \leq 1 \Big\}$$

with the following conditions:

(i) For Each $< u, v > \in \mathbb{F}_1$ is normal, i.e, $\exists x_0, x_1 \in \mathbb{R}$, such that $u(x_0) = 1$ and $v(x_1) = 1$.

(ii) For Each $< u, v > \in \mathbb{F}_1$ is a convex intuitionistic set, i.e, u is fuzzy convex and v is fuzzy concave.

(iii) For Each $< u, v > \in \mathbb{F}_1$, u is a lower continuous and v is appear continuous.

(iv) $cl \{x \in \mathbb{R},\ v(x) \leq \alpha\}$ is bounded.

Definition 2.2 ([3])**.** For $\alpha \in [0,1]$, we define the appear and lower α-cut by

$$[< u, v >]_\alpha = \{x \in \mathbb{R},\ u(x) \geq \alpha\},$$
$$[< u, v >]^\alpha = \{x \in \mathbb{R},\ v(x) \leq 1 - \alpha\}.$$

Definition 2.3. The intuitionistic fuzzy zero is intuitionistic fuzzy set defined by

$$\widetilde{0}(x) = \begin{cases} (1,0) & x = 0 \\ (0,1) & x \neq 0. \end{cases}$$

Proposition 2.4 ([3])**.** *We can write*

$$[< u, v >]_\alpha = \left[[< u, v >]_l^+(\alpha), [< u, v >]_r^+(\alpha)\right],$$
$$[< u, v >]^\alpha = \left[[< u, v >]_l^-(\alpha), [< u, v >]_r^-(\alpha)\right].$$

Remark 2.5. We can write $[< u, v >]_\alpha = [u]^\alpha$ and $[< u, v >]^\alpha = [1 - v]^\alpha$, in the fuzzy case

Proposition 2.6 ([3])**.** *For all* $< u, v >$, $< u', v' > \in \mathbb{F}_1$*, we have*

$$< u, v > = < u', v' > \Longleftrightarrow [< u, v >]_\alpha \ \textit{and}\ [< u, v >]^\alpha,\ \forall t \in [0,1].$$

We define two operations on $\mathbb{F}_1$ by

$$< u, v > \oplus < u', v' > = < u \vee v, u' \wedge v' >,\ \forall < u, v >, < u', v' > \in \mathbb{F}_1,$$

$$\lambda < u, v > = < \lambda u, \lambda v >,\ \forall \lambda \in \mathbb{R},\ \forall < u, v > \in \mathbb{F}_1.$$

According to Zadeh extension, we have

$$[< u, v > \oplus < u', v' >]_\alpha = [< u, v >]_\alpha + [< u', v' >]_\alpha,$$
$$[< u, v > \oplus < u', v' >]^\alpha = [< u, v >]^\alpha + [< u', v' >]^\alpha,$$
$$[\lambda < u, v >]_\alpha = \lambda [< u, v >]_\alpha,$$
$$[\lambda < u, v >]^\alpha = \lambda [< u, v >]^\alpha.$$

Theorem 2.7 ([3]). *Let $\mathcal{M} = \{M_\alpha, M^\alpha, \alpha \in [0,1]\}$ be a family of subsets in $\mathbb{R}$ stisfying the following conditions:*

(1) *$\alpha \leq s \Longrightarrow M_s \subset M_\alpha$ and $M^s \subset M^\alpha$, for each $\alpha, s \in [0,1]$.*

(2) *M_α and M_s are nonempty compact convex sets in $\mathbb{R}$ for each $\alpha \in [0,1]$.*

(3) *For any nondecreasing sequence $\alpha_i \longrightarrow \alpha$ on $[0,1]$, we have*

$$M_\alpha \in [0,1] = \bigcap_i M_{\alpha_i} \text{ and } M^\alpha = \bigcap_i M^{\alpha_i}.$$

We define u and v by

$$u(x) = \begin{cases} 0, & x \notin M_0 \\ \sup\limits_{\alpha\in[0,1]} M_\alpha & x \in M_0, \end{cases}$$

$$v(x) = \begin{cases} 1, & x \notin M^0 \\ 1 - \sup\limits_{\alpha\in[0,1]} M_\alpha & x \in M^0. \end{cases}$$

Then $< u, v > \in I\!\!F_1$ with $M_\alpha = [< u, v >]_\alpha$ and $M^\alpha = [< u, v >]^\alpha$.

Remark 2.8 ([3]). (1) The family $\{[< u, v >]_\alpha, [< u, v >]^\alpha,\ \alpha \in [0,1]\}$ satisfying (1)-(3) of the previous theorem.

(2) For all $\alpha \in [0,1]$, $[< u, v >]_\alpha \subset [< u, v >]^\alpha$.

Theorem 2.9 ([3]). *On $I\!\!F_1$, we can define the metric*

$$\begin{aligned} d_\infty\Big((u,v),(z,w)\Big) = & \tfrac{1}{4}\sup_{0<\alpha\leq 1}\left|\Big[(u,v)\Big]_r^+(\alpha) - \Big[(z,w)\Big]_r^+(\alpha)\right| \\ & +\tfrac{1}{4}\sup_{0<\alpha\leq 1}\left|\Big[(u,v)\Big]_l^+(\alpha) - \Big[(z,w)\Big]_l^+(\alpha)\right| \\ & +\tfrac{1}{4}\sup_{0<\alpha\leq 1}\left|\Big[(u,v)\Big]_r^-(\alpha) - \Big[(z,w)\Big]_r^-(\alpha)\right| \\ & +\tfrac{1}{4}\sup_{0<\alpha\leq 1}\left|\Big[(u,v)\Big]_l^-(\alpha) - \Big[(z,w)\Big]_l^-(\alpha)\right| \end{aligned}$$

and

$$\begin{aligned} d_p\left(< u, v >, < u', v' >\right) = \Bigg(& \tfrac{1}{4}\int_0^1 \left|[< u, v >]_l^+(\alpha) - [< u', v' >]_l^+(\alpha)\right|^p dt \\ & +\tfrac{1}{4}\int_0^1 \left|[< u, v >]_r^+(\alpha) - [< u', v' >]_r^+(\alpha)\right|^p dt \\ & +\tfrac{1}{4}\int_0^1 \left|[< u, v >]_l^-(\alpha) - [< u', v' >]_l^-(\alpha)\right|^p dt \\ & +\tfrac{1}{4}\int_0^1 \left|[< u, v >]_r^-(\alpha) - [< u', v' >]_r^-(\alpha)\right|^p dt\Bigg)^{\frac{1}{p}}. \end{aligned}$$

For $p \in [1,\infty)$, we have $\Big(I\!\!F_1, d_p\Big)$ is a complete metric space.

3. The generalized Hukuhara derivative of an intuitionistic fuzzy-valued function

The concept of intuitionistic fuzzy Hukuhra difference is introduced by the autors in [4], in this paper we will give the definition of generalized Hukuhara difference betwen two intuitionistic fuzzy number.

Definition 3.1. The generalized Hukuhara difference of two fuzzy number $< u, v >$, $< u', v' > \in I\!\!F_1$ is defined as follows:

$$< u, v > -_{gH} < u', v' >=< z, w >$$
$$\Longleftrightarrow < u, v >=< u', v' > + < z, w > \text{ or } < u', v' >=< u, v > +(-1) < z, w > .$$

Note that the (α, β)-level representation of fuzzy-valued function $f : [0, T] \longrightarrow \mathbb{F}_1$ expressed by $[f]_\alpha = [f_{\alpha,l}, f_{\alpha,r}]$ and $[f]^\alpha = \left[f^{\alpha,l}, f^{\alpha,r}\right]$.

Definition 3.2. The generalized Hukuhara derivative of a intuitionistic fuzzy-valued function $f : [0, T] \longrightarrow \mathbb{F}_1$ at t_0 is defined as

$$f'_{gH}(t_0) = \lim_{t \to t_0} \frac{f(t) -_{gH} f(t_0)}{t - t_0}$$

if $f'_{gH}(t_0) \in \mathbb{F}_1$ and we say that f is generalized Hukuhara differentiable at t_0

Also we say that f is $[(i) - gH]$-differentiable at t_0, if

$$\begin{cases} \left(f'_{gH}\right)_\alpha = \left[(f_{\alpha,l})', (f_{\alpha,r})'\right] \\ \left(f'_{gH}\right)^\beta = \left[(f^{\beta,l})', (f^{\beta,r})'\right]. \end{cases}$$

And that f is $[(ii) - gH]$-differentiable at t_0, if

$$\begin{cases} \left(f'_{gH}\right)_\alpha = \left[(f_{\alpha,r})', (f_{\alpha,l})'\right] \\ \left(f'_{gH}\right)^\beta = \left[(f^{\beta,r})', (f^{\beta,l})'\right]. \end{cases}$$

Remark 3.3. We can defined the generalized derivative of higher order by

$$\begin{cases} f^{(0)} = f \\ f^{(n)}_{gH} = \left(f^{(n-1)}\right)'_{gH}, \quad \forall n \in \mathbb{N}. \end{cases} \tag{3.1}$$

Definition 3.4. Let $f : (0, T) \longrightarrow \mathbb{F}_1$. We say that f of classe $\mathcal{C}^m$, $m \in \mathbb{N}$, if $f^{(m)}_{gh}$ exists and continues, by respect to metric d_∞.

Now if the α-levels of $f : (0, T] \longrightarrow \mathbb{F}_1$, are given by $[f]_\alpha = [f_{\alpha,l}, f_{\alpha,r}]$ and $[f]^\beta = \left[f^{\beta,l}, f^{\beta,r}\right]$ and $f_{\alpha,l}$, $f_{\alpha,r}$, $f^{\beta,l}$, $f^{\beta,r}$ are Riemann integrable on $[0, T]$. Since the family

$$\left\{ [f_{\alpha,l}, f_{\alpha,r}], \ \left[f^{\beta,l}, f^{\beta,r}\right] \ \right\}$$

built an intuitionistic element and the integrale preserve the monotony, by the Theorem 2.7, the family

$$\left\{ \left[\int_{[0,T]} f_{\alpha,l}, \int_{[0,T]} f_{\alpha,r} \right], \left[\int_{[0,T]} f^{\beta,l}, \int_{[0,T]} f^{\beta,r} \right] \right\}$$

define an intuitionistic fuzzy element, which is the integral of f on $[0, T]$ and we denote $\int_{[0,T]} f$

Definition 3.5. Let $f : [0, T] \longrightarrow \mathbb{F}_1$ be a intuitionistic fuzzy-valued function, we say that f is integrable on $[0, T]$, if $f_{\alpha,l}$, $f_{\alpha,r}$, $f^{\beta,l}$, $f^{\beta,r}$ defined in the previous are integrable on $[0, T]$

4. Intuitionistic fuzzy generalized caputo-derivative

Let $f : [0,T] \longrightarrow \mathbb{IF}_1$ be a intuitionistic fuzzy-valued integrable function on $[0,T]$, and $\delta \in (m-1,m]$ and $m \in \mathbb{N}^*$. Then it's (α,β)-levels are defined by

$$[f]_\alpha = [f_{\alpha,l}, f_{\alpha,r}] \text{ and } [f]^\beta = \left[f^{\beta,l}, f^{\beta,r}\right],$$

where $f_{\alpha,l},\ f_{\alpha,r},\ f^{\beta,l},\ f^{\beta,r} \in \mathcal{C}^m([0,T])$.
We set

$$M_\alpha = \left[\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-m-1}\Big(f_{\alpha,l}\Big)^{(m)}(s), \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-m-1}\Big(f_{\alpha,r}\Big)^{(m)}(s)\right]$$

and

$$M^\beta = \left[\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-m-1}\Big(f^{\beta,l}\Big)^{(m)}(s), \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-m-1}\Big(f^{\beta,r}\Big)^{(m)}(s)\right].$$

Proposition 4.1. *The the family $\left\{M_\alpha, M^\beta,\ \alpha,\ \beta \in [0,1]\right\}$ defined an intuitionistic fuzzy element*

Proof. Just use the Theorem 2.7 □

Definition 4.2. The intuitionistic fuzzy preceding item is called the generalized caputo derivative of f, we denote $D^\alpha f$.
we say that f is ${}^{cf}[(i)-gH]$-differentiable at t_0, if

$$\left[\ _{gH}D^\delta f\right]_\alpha = \left[D^\delta f_{\alpha,l}, D^\delta f_{\alpha,r}\right]$$
$$\left[\ _{gH}D^\delta f\right]^\beta = \left[D^\delta f^{\beta,l}, D^\delta f^{\beta,r}\right]$$

and that f is ${}^{cf}[(ii)-gH]$-differentiable at t_0, if

$$\left[\ _{gH}D^\delta f\right]_\alpha = \left[D^\delta f_{\alpha,r}, D^\delta f_{\alpha,l}\right]$$
$$\left[\ _{gH}D^\delta f\right]^\beta = \left[D^\delta f^{\beta,r}, D^\delta f^{\beta,l}\right].$$

As in the previuos definition, we will give the difinition of intuitionistic fuzyy fractional Riemann-Liouville integral. If the (α,β)-levels of $f : (0,T] \longrightarrow \mathbb{IF}_1$, are given by $[f]_\alpha = [f_{\alpha,l}, f_{\alpha,r}]$ and $[f]^\beta = \left[f^{\beta,l}, f^{\beta,r}\right]$ and $f_{\alpha,l}, f_{\alpha,r}, f^{\beta,l}, f^{\beta,r}$ are Riemann integrable on $(0,T]$. Since the family

$$\left\{[f_{\alpha,l}, f_{\alpha,r}],\ \left[f^{\beta,l}, f^{\beta,r}\right]\right\}$$

built an intuitionistic element and the integrale preserve the monotony, by Theorem 2.7, the family

$$\left\{\mathcal{A}_\alpha, \mathcal{A}^\beta\ :\ \alpha,\ \beta \in [0,1]\right\},$$

where

$$\mathcal{A}_\alpha = \left[\frac{1}{\Gamma(\delta)}\int_{(0,t)} (t-s)^{\delta-1} f_{\alpha,l}(s), \frac{1}{\Gamma(\delta)}\int_{(0,t)} (t-s)^{\delta-1} f_{\alpha,r}(s)\right]$$

and

$$\mathcal{A}^\beta = \left[\frac{1}{\Gamma(\delta)}\int_{(0,t)} (t-s)^{\delta-1} f^{\beta,l}(s), \frac{1}{\Gamma(\delta)}\int_{(0,t)} (t-s)^{\delta-1} f^{\beta,r}(s)\right]$$

define an intuitionistic fuzzy element, which is the Riemann-liouville fractional integral of f on $(0,T)$ and we denote $\frac{1}{\Gamma(\delta)}\int_{(0,t)} (t-s)^{\delta-1} f(s)ds$.

Definition 4.3. The Riemann-liouville fractional integral of f on $(0,T)$, defined as

$$I^{\delta}f(t)=\frac{1}{\Gamma(\delta)}\int_{(0,t)}(t-s)^{\delta-1}f(s)ds,$$

where $\delta\in(m-1,m)$

5. Intuitionistic fuzzy generalized integral solution

Lemma 5.1. *If $a\in \mathbb{F}_1$, then*

$$\int_{(0,t)} ads = ta$$

Proof. We set $[a]_{\alpha}=[a_-,a_+]$ and $[a]^{\alpha}=[a^-,a^+]$. Then we have

$$[\int_{(0,t)} ads]_{\alpha}=[ta_-,ta_+]=t[a_-,a_+]=t[a]_{\alpha}$$

and

$$[\int_{(0,t)} ads]^{\alpha}=[ta^-,ta^+]=t[a^-,a^+]=t[a]^{\alpha}.$$

□

Lemma 5.2. *Let $u\in\mathcal{C}^2\Big([0,T],\mathbb{F}_1\Big)$, we have*

(1) $D^{\delta}u(t)=D^{\delta-1}u'(t)_{gH}$
(2) $I^{\delta+\beta}=I^{\delta}I^{\beta}$

Proof. (1) In the classical case, we have

$$D^{\delta}u_{\alpha,l}(t)=D^{\delta-1}{u_{\alpha,l}}'(t),\ D^{\delta}u_{\alpha,r}(t)=D^{\delta-1}{u_{\alpha,r}}'(t),$$
$$D^{\delta}u^{\beta,l}(t)=D^{\delta-1}{u^{\beta,l}}'(t),\ D^{\delta}u^{\beta,r}(t)=D^{\delta-1}{u^{\beta,r}}'(t).$$

Then

$$\big[D^{\delta}u(t)\big]_{\alpha}=\big[D^{\delta-1}u'(t)_{gH}\big]_{\alpha}$$

and

$$\big[D^{\delta}u(t)\big]^{\beta}=\big[D^{\delta-1}u'(t)_{gH}\big]^{\beta}.$$

Thus

$$D^{\delta}u(t)=D^{\delta-1}u'(t)_{gH}.$$

(2) See [3]. □

Define *sgn(x)* by

$$sgn(x)=\begin{cases}+ & \text{if x is } {}^{cf}\ \text{[(i)-gH]-differentiable}\\ \ominus(-1) & \text{if x is } {}^{cf}\ \text{[(ii)-gH]-differentiable.}\end{cases}\tag{5.1}$$

Theorem 5.3. *The initial value problem (1.1) is equivalent to the integral equation:*

$$x(t)=x_0+sign(x)\Big(tx_1+sign(x')\frac{1}{\Gamma(\delta-1)}\int_0^t y(s)ds\Big),$$

where

$$y(s)=\int_0^s(s-\tau)^{\delta-2}f\left(\tau,\frac{1}{\Gamma(1-\beta)}\int_0^{\tau}(\tau-\eta)^{-\beta}y(\eta)d\eta\right)d\tau.$$

Proof. We have $D^{\delta}x(t) = f\left(t, D^{\gamma}x(t)\right)$. Then $D^{\delta-1}x'(t) = f\left(t, D^{\gamma}x(t)\right)$. Thus

$$I^{\delta-1}D^{\delta-1}x'(t) = I^{\delta-1}f\left(t, D^{\gamma}x(t)\right).$$

So

$$x'(t) -_{gH} x_1 = I^{\delta-1}f\left(t, D^{\gamma}x(t)\right).$$

If x' is ${}^{cf}[(i) - gH]$-differentiable, then
$x'(t) = x_1 + I^{\delta-1}f\left(t, D^{\gamma}x(t)\right)$.
If x' is ${}^{cf}[(ii) - gH]$-differentiable, then $x'(t) = x_1 \ominus (-1)\Big(I^{\delta-1}f\left(t, D^{\gamma}x(t)\right)\Big)$.
Now use the formula in Theorem 3.1 [5]

$$x(t) = x_0 + sign(x)\int_0^t y(s)ds,$$

where

$$y(s) = x_1 + sign(x')\frac{1}{\Gamma(\delta-1)}\int_0^s (s-\tau)^{\delta-2} f\left(\tau, \frac{1}{\Gamma(1-\gamma)}\int_0^\tau (\tau-\eta)^{-\gamma}y(\eta)d\eta\right)d\tau.$$

□

Now we make the following assumptions:

- H_1: $f : [0, T] \longrightarrow \mathbb{F}_1$ is a continuously gH-differentiable function.
- H_2: there exist nonnegative functions a_1 and a_2 such that

$$d_\infty\left(f(t, < u, v >), \widetilde{0}\right) \le a_1(t) + a_2(t)d_\infty\left(< u, v >, \widetilde{0}\right).$$

- H_3: $\forall t \in [0, T]$ and $x,\ y \in \mathcal{C}^1\Big([0, T], \mathbb{F}_1\Big)$,

$$d_\infty\Big(f(t, x), f(t, y)\Big) \le M d_\infty\left(x, y\right)),$$

  where $M > 0$.

Theorem 5.4. *Suppose that H_1 - H_3 hold, then the integral equation*

$$x(t) = x_0 + tx_1 + \int_0^t y(s)ds,$$

where

$$y(s) = sign(x')\frac{1}{\Gamma(\delta-1)}\int_0^s (s-\tau)^{\delta-2} f\left(\tau, \frac{1}{\Gamma(1-\gamma)}\int_0^\tau (\tau-\eta)^{-\gamma}y(\eta)d\eta\right)d\tau$$

has a solution in $\mathcal{C}\Big([0, T], \mathbb{F}_1\Big)$ provided

$$A = \frac{1}{\Gamma(\delta-1)\Gamma(2-\gamma)}\sup_{t\in[0,T]}\int_0^t (t-s)^{\delta-2}s^{1-\beta}a_2(s)ds < 1,$$

$$B = D(x_0 + tx_1, 0) + \sup_{t\in[0,T]}\frac{1}{\Gamma(\delta-1)}\int_0^t (t-s)^{\delta-2}a_1(s)ds < \infty,$$

$$\frac{MT^{\delta-\gamma}B(\delta-1, -\gamma)}{\Gamma(1-\gamma)} < 1,$$

where

$$B(\delta-1, 2-\gamma) = \int_0^1 t^{1-\gamma}(1-t)^{\delta-2}dt.$$

Proof. The space $\mathcal{C}\Big([0,T], \mathbb{F}_1\Big)$ endowed with the following metric

$$D(x,y) = \sup_{t\in[0,T]} d_\infty\left(x(t), y(t)\right)$$

and the mapping F is define by

$$Fx(t) = x_0 + tx_1 + \frac{1}{\Gamma(\delta-1)}\int_0^t (t-s)^{\delta-2} f\Bigg(\tau, \frac{1}{\Gamma(1-\gamma)}\int_0^s (s-\tau)^{-\gamma}x(\tau)d\tau\Bigg)ds.$$

We prove the existence of and uniqueness of fixed point of F on $\mathcal{C}\Big([0,T], \mathbb{F}_1\Big)$.

Step 1: We set

$$\mathbb{D}_{\frac{B}{1-A}} = \Big\{x \in \mathcal{C}\Big([0,T], \mathbb{F}_1\Big)\ ,\ D(x, \widetilde{0}) \leq \frac{B}{1-A}\Big\}.$$

Then for all $x \in \mathcal{C}\Big([0,T], \mathbb{F}_1\Big)$, we have

$$D(Fx, \widetilde{0}) = D\left(\rho, \widetilde{0}\right),$$

where

$$\rho = x_0 + tx_1 + \frac{1}{\Gamma(\delta-1)}\int_0^t (t-s)^{\delta-2} f\Bigg(\tau, \frac{1}{\Gamma(1-\gamma)}\int_0^s (s-\tau)^{-\gamma}x(\tau)d\tau\Bigg)ds.$$

Thus

$$D(Fx, \widetilde{0}) \leq D\left(x_0 + tx_1, \widetilde{0}\right) + D\left(\sigma, \widetilde{0}\right),$$

where

$$\sigma = \frac{1}{\Gamma(\delta-1)}\int_0^t (t-s)^{\delta-2} f\Bigg(\tau, \frac{1}{\Gamma(1-\gamma)}\int_0^s (s-\tau)^{-\gamma}x(\tau)d\tau\Bigg)ds.$$

So

$$D\left(\sigma, \widetilde{0}\right) \leq \frac{1}{\Gamma(\delta-1)}\int_0^t (t-s)^{\delta-2}\omega ds,$$

where

$$\omega = D\Bigg(f\Bigg(\tau, \frac{1}{\Gamma(1-\gamma)}\int_0^s (s-\tau)^{-\gamma}x(\tau)d\tau\Bigg), \widetilde{0}\Bigg).$$

Hence we have

$$\omega \leq f\Bigg(\tau, \frac{1}{\Gamma(1-\gamma)}\int_0^s (s-\tau)^{-\gamma}d\tau\Bigg)D(x, \widetilde{0}).$$

By using H_3, we have

$$\begin{aligned} D(Fx, \widetilde{0}) &\leq D\left(x_0 + tx_1\right) + \tfrac{1}{\Gamma(\delta-1)} \sup_{t\in[0,T]} \textstyle\int_0^t (t-s)^{\delta-2} a_1(s)ds \\ &\quad + \tfrac{D(x,\widetilde{0})}{\Gamma(\delta-1)\Gamma(1-\gamma)} \sup_{t\in[0,T]} \textstyle\int_0^t (t-s)^{\delta-2} \int_0^s (s-\tau)^{-\gamma} d\tau a_2(s)ds \\ &\leq B + AD(x, \widetilde{0}) \\ &\leq r. \end{aligned}$$

This shows that $Fx \in \mathbb{D}_r$.

step 2: It remain to prove that F is a contraction, let x, $y \in \mathcal{C}\Big([0,T], \mathbb{F}_1\Big)$. Then by H_3, we get

$$D(Fx, Fy) \leq \frac{M}{\Gamma(\delta-1)} \int_0^t (t-s)^{\delta-2} D\left(\mathcal{B}_x, \mathcal{B}_x\right) ds,$$

where

$$\mathcal{B}_z = f\left(\tau, \frac{1}{\Gamma(1-\beta)} \int_0^s (s-\tau)^{-\gamma} x(\tau) d\tau\right).$$

Thus

$$D\left(Fx, Fy\right) \leq \frac{M}{\Gamma(1-\beta)} \int_0^t (t-s)^{\delta-2} \int_0^s (s-\tau)^{-\gamma} d\tau ds D(x,y).$$

So

$$D(Fx, Fy) \leq \frac{MT^{\delta-\gamma} B(\delta-1, -\beta)}{(1-\gamma)\Gamma(1-\gamma)} D(x,y).$$

Since the completeness of d_∞ implies the completeness of D, F is a contraction. □

Now define the following metric on $\mathcal{C}\Big([0,T], \mathbb{F}_1\Big)$:

$$\mathcal{D}(x,y) = \sup_{t\in[0,T]} d_\infty\left(x(t), y(t)\right) e^{-\rho t},$$

where $\rho > 0$ fixed.

We define

$$\mathcal{D}_1(x,y) = \mathcal{D}(x,y) + \mathcal{D}(x', y') + \mathcal{D}(x'', y'')$$

with x' and x'' are the first and second derivative of x.

Proposition 5.5 ([5])**.** $\left(\mathcal{C}^2\left([0,T], \mathbb{F}_1\right), \mathcal{D}_1\right)$ *is a metric space*

We add the following condition.

Theorem 5.6. *If H_1 - H_3 are verified, then the problem* (1.1) *has unique solutionon [0,T] in each sense of differentiability.*

Proof. If H_3 is verified, then

$$\begin{aligned}
&d_\infty\bigg(x \ominus (-1)\Big(I^{\delta-1} f\left(t, D^\gamma x(t)\right)\Big), y \ominus (-1)\Big(I^{\delta-1} f\left(t, D^\gamma y(t)\right)\Big)\bigg) \\
&\leq \frac{1}{\Gamma(\delta-1)} \int_0^t (t-s)^{\delta-1} d_\infty(f(s,x(s)), f(s,y(s))) ds \\
&\leq M \frac{1}{\Gamma(\delta-1)} \int_0^t (t-s)^{\delta-1} d_\infty(f(s,x(s)), f(s,y(s))) ds \\
&\leq C_T \sup_{s\in[0,T]} d_\infty(x(s), y(s)),
\end{aligned}$$

where

$$C_T = M \frac{T^\delta}{\Gamma(\delta-1)}.$$

Thus by Theorem 3.2 [5], the problem

$$z' = x_1 \ominus (-1)\Big(I^{\delta-1} f\left(t, D^\gamma z(t)\right)\Big)$$

has a unique solution in each sense of differentiability.
That is x' is defined, which implies with Theorem 5.4 the existence and uniqueness of x. □

6. Example

We give an example to illustrate the efficiency of our main results, we take $f(t,x) = x$

$$\begin{cases} {}_{gH}D^{\frac{3}{2}}x(t) = (-2)\ {}_{gH}D^{\frac{1}{2}}x(t),\ t \in [0,1] \\ x_0 =< 1,2,3;0,2,4 >, \quad x_1 =< 0,2,4;-1,2,6 > \end{cases} \tag{6.1}$$

Here $\delta = \frac{3}{2}$, $\gamma = \frac{1}{2}$, $f(t,x) = (-2)x$, and $I = [0,1]$
For all α, $\beta \in [0,1]$, the upper and lower cuts of initial conditions are:

$$[x_0]_\alpha = \Big[1+\alpha, 3-\alpha\Big], \quad [x_0]^\beta = \Big[-3+2\beta, 1-2\beta\Big],$$
$$[x_1]_\alpha = \Big[2\alpha, 4-2\alpha\Big], \quad [x_1]^\beta = \Big[-5+4\beta, 2-3\beta\Big].$$

Applying $I^{\frac{1}{2}}$, we obtain

$$x'(t) \ominus_{gH} x_1 = (-2)(x(t) \ominus_{gH} x_0), \quad \forall t \in [0,1].$$

We get the following cases.

Case (i): The solution of the equation

$$x'(t) = -2x(t) + x_1 + 2x_0, \quad \forall t \in [0,1]$$

is given by

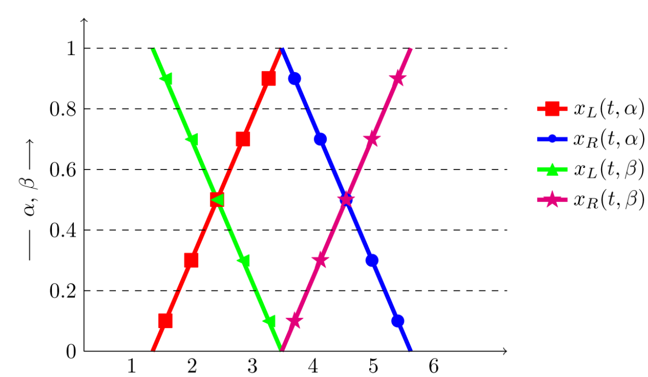
$$x(t) = e^{-2t}x_0 + \frac{1}{2}x_1 + x_0.$$

Then for $\alpha \in [0,1]$, we have

$$\begin{aligned} [x(t)]_\alpha &= \Big[x_L^+(t,\alpha), x_R^+(t,\alpha)\Big] \\ &= \Big[\left(e^{-2t}+1\right)(1+\alpha)+\alpha, \left(e^{-2t}+1\right)(3-\alpha)+2-\alpha\Big], \\ [x(t)]^\beta &= \Big[x_L^-(t,\beta), x_R^-(t,\beta)\Big] \\ &= \Big[\left(e^{-2t}+1\right)-\frac{5}{2}+2\beta, \left(e^{-2t}+1\right)(1-2\beta)+1-\frac{3}{2}\beta\Big]. \end{aligned}$$

Table 1 Value of $x_L(t,\alpha)$, $x_R(t,\alpha)$, $x_L(t,\beta)$ and $x_R(t,\beta)$ at $t=1$ for different α and β.

α	$x_L(t,\alpha)$	$x_R(t,\alpha)$	$x_L(t,\beta)$	$x_R(t,\beta)$
0	1.135	5.406	-1.635	-1.635
0.1	1.349	5.192	-2.062	-1.258
0.2	1.562	4.979	-2.489	-0.881
0.3	1.776	4.765	-2.917	-0.504
0.4	1.989	4.552	-3.344	-0.127
0.5	2.203	4.338	-3.771	0.250
0.6	2.417	4.125	-4.198	0.627
0.7	2.630	3.911	-4.625	1.004
0.8	2.844	3.698	-5.052	1.381
0.9	3.057	3.484	-5.479	1.758
1	3.271	3.271	-5.906	2.135



Graph 1. Graph of $x_L(t,\alpha)$, $x_R(t,\alpha)$, $x_L(t,\beta)$ and $x_R(t,\beta)$ at $t = 1$ for different α and β.

Case (ii): The solution

$$x'(t) = -2x(t) + (-1)x_1 + 2x_0, \ \forall t \in [0,1]$$

is given by

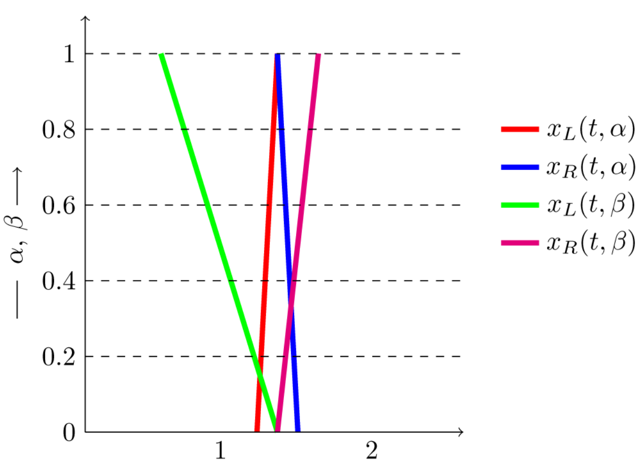
$$x(t) = e^{-2t}x_0 + (-1)\frac{1}{2}x_1 + x_0.$$

Then for $\forall t \in [0,1]$, the upper and lower solutions are

$$\begin{aligned}[x(t)]_\alpha &= \left[x_L^+(t,\alpha), x_R^+(t,\alpha)\right]\\ &= \left[(e^{-2t}+1)(1+\alpha)-\alpha, (e^{-2t}+1)(3-\alpha)-2+\alpha\right],\\ [x(t)]^\beta &= \left[x_L^-(t,\beta), x_R^-(t,\beta)\right]\\ &= \left[(e^{-2t}+1)(-3+2\beta)+\frac{5}{2}-2\beta, (e^{-2t}+1)(1-2\beta)-1+\frac{3}{2}\beta\right].\end{aligned}$$

Table 2 Value of $x_L^+(t,\alpha)$, $x_R^+(t,\alpha)$, $x_L^-(t,\beta)$ and $x_R^-(t,\beta)$ at $t=1$ for different α and β.

(α,β)	$x_L(t,\alpha)$	$x_R(t,\alpha)$	$x_L(t,\beta)$	$x_R(t,\beta)$
0	1.135	1.406	1.271	1.271
0.1	1.149	1.392	1.194	1.298
0.2	1.162	1.379	1.117	1.325
0.3	1.176	1.365	1.039	1.352
0.4	1.189	1.352	0.962	1.379
0.5	1.203	1.338	0.885	1.406
0.6	1.217	1.325	0.808	1.433
0.7	1.230	1.311	0.731	1.460
0.8	1.244	1.298	0.654	1.487
0.9	1.257	1.284	0.577	1.514
1	1.271	1.271	0.500	1.541



Graph 2. Graph of $x_L(t,\alpha)$, $x_R(t,\alpha)$, $x_L(t,\beta)$ and $x_R(t,\beta)$ at $t = 1$ for different α and β.

7. Conclusion

In this paper we solved fractional differential equation initial value as triangular Intuitionistic fuzzy number. We use upper and lower α-cut method for solving this equation. Then the fractional differential equation converted to a system of crisp differential equations, and then solve the differential equation.

Acknowledgements. The author would like to appreciate the anonymous reviewers for their helpful comments.

References

[1] T. Allahviranloo, A. Armand and Z.Gouyandeh , Fuzzy fractional differential equations under generalized fuzzy Caputo derivative, J. Intell. Fuzzy Syst. 26 (2014) 1481–1490.
[2] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy sets and Systems 20 (1986) 87–96.
[3] S. Melliani, M. Elomari, L. S. Chadli, R. Ettoussi, Intuitionistic fuzzy metric space, Notes IFS 21 (1) (2015) 43–53.
[4] S. Melliani, M. Elomari, L. S. Chadli and R. Ettoussi, Extension of Hukuhara difference in intuitionistic fuzzy set theory, Notes IFS 21 (4) (2015) 34–47.
[5] S. Salahshour, Nth-order fuzzy differential equations under Generalized differentiability, Journal of fuzzy set Valued Analysis, 2011 (1) (2011) 26–39.

M'hamed Elomari (m.elomari@usms.ma)
Laboratory of Applied Mathematics & Scientific Calculus, Sultan Moulay Slimane University, BP 523, 23000, Beni Mellal, Morocco

Said Melliani (said.melliani@gmail.com)
Laboratory of Applied Mathematics & Scientific Calculus, Sultan Moulay Slimane University, BP 523, 23000, Beni Mellal, Morocco

Lalla Saadia Chadli (sa.chadli@yahoo.fr)
Laboratory of Applied Mathematics & Scientific Calculus, Sultan Moulay Slimane University, BP 523, 23000, Beni Mellal, Morocco