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Degrees in intuitionistic fuzzy graphs

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Abstract. In this paper, the concept of strong degree, total strong degree of a vertex, strong size and strong order in an intuitionistic fuzzy graph (IFG) has been introduced and its properties in IFGs and complete IFGs are analysed. The vertex intuitionistic fuzzy sequence has been introduced.

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Keywords: Strong degree of a vertex, Total strong degree of a vertex, Vertex membership and non-membership sequence, Strong size and strong order.

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1. Introduction

The fuzzy set theory was developed by Lotfi A Zadeh [13] in 1965. Bhutani and Rosenfeld [2] have analyzed strong arcs in fuzzy graphs. Nagoorgani and Radha [9] defined degree of a vertex, regular and totally regular fuzzy graphs. Manjusha and Sunitha [7] introduced connected domination in fuzzy graphs using strong arcs. The theory of intuitionistic fuzzy graphs (IFGs) was introduced by Krassimir T Atanassov ([1], [12]). Karunambigai and Parvathi [3] introduced intuitionistic fuzzy graph as a special case of Atanassov's IFG. Parvathi and Thamizhendhi [11] introduced and analyzed the theory of domination on join, cartesian product, lexicographic product, tenson product and strong product of two intuitionistic fuzzy graphs. Nagoorgani and Shajitha Begum [10] defined degree, order and size in intuitionistic fuzzy graphs. Degree, total degree of a vertex, constant IFG, totally constant IFG, size and order in intuitionistic fuzzy graphs are defined by Karunambigai et.al. [4]. The edges in intuitionistic fuzzy graphs are classified into α-strong, β-strong and δ-weak depending on the strength of connectedness between two vertices by Karunambigai et.al. [5]. Karunambigai and Buvaneswari [6] introduced the strong and superstrong vertices in intuitionistic fuzzy graphs. These concepts along with the concept of vertex-strength sequence which is given [8] for complete fuzzy graphs motivated the author to analyse these in complete IFGs. So here this paper has been organised as follows. Preliminaries required for this study are given in Section 2. In Section 3, strong degree, total strong degree and relationship between total degree of a vertex, minimum and maximum total degree of complete IFGs have been studied. In Section 4, vertex intuitionistic fuzzy sequence is defined and its properties in complete IFGs are discussed.

2. Preliminaries

Definition 2.1 ([3])**.** Minmax intuitionistic fuzzy graph (IFG) is of the form $G = (V, E)$, where

(i) $V = \{v_1, v_2, \dots v_n\}$ such that $\mu_i : V \to [0,1]$ and $\nu_i : V \to [0,1]$ denote the degrees of membership and non-membership of the element $v_i \in V$, respectively and $0 \le \mu_i + \nu_i \le 1$, for every $v_i \in V$ $(i = 1, 2, \dots, n)$,

(ii) $E \subset V \times V$, where $\mu_{ij} : V \times V \to [0,1]$ and $\nu_{ij} : V \times V \to [0,1]$ are such that

$$\mu_{ij} \le min[\mu_i, \mu_j],$$

$$\nu_{ij} \le max[\nu_i, \nu_j]$$

and $0 \le \mu_{ij} + \nu_{ij} \le 1$, for every $e_{ij} \in E$.

Here the triple (v_i, μ_i, ν_i) denotes the degrees of membership and non-membership of the vertex v_i. The triple $(e_{ij}, \mu_{ij}, \nu_{ij})$ denotes the degrees of membership and non-membership of the edge $e_{ij} = (v_i, v_j)$ on $V \times V$.

For each IFG G, the degree of hesitancy (hesitation degree) of the vertex $v_i \in V$ is $\Pi_i = 1 - \mu_i - \nu_i$ and the degree of hesitancy of an edge $e_{ij} \in E$ is $\Pi_{ij} = 1 - \mu_{ij} - \nu_{ij}$.

Notation: Here after an IFG, $G = (V, E)$ means a minmax IFG $G = (V, E)$.

Note 1. If $\mu_{ij} = \nu_{ij} = 0$, for some i and j, then there is no edge between v_i and v_j, and it is indexed by $\langle 0, 1\rangle$. Otherwise there exists an edge between v_i and v_j.

Example 2.2. Figure 1 shows an IFG $G = \langle V, E\rangle$, where $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{e_{12}, e_{14}, e_{15}, e_{16}, e_{23}, e_{24}, e_{25}, e_{34}, e_{45}, e_{56}\}$.

Definition 2.3 ([3])**.** An IFG G is said to be complete, if $\mu_{ij} = min(\mu_i, \mu_j)$ and $\nu_{ij} = max(\nu_i, \nu_j)$, for every $v_i, v_j \in V$.

Definition 2.4 ([3])**.** A path P in an IFG is a sequence of distinct vertices $v_1, v_2, ..., v_n$ for all $i, j = 1, 2, ..., n$ such that either one of the following conditions is satisfied:

(i) $\mu_{ij} > 0$ and $\nu_{ij} = 0$ for some i and j,

(ii) $\mu_{ij} > 0$ and $\nu_{ij} > 0$ for some i and j.

A path P between two vertices v_i and v_j is denoted by $[v_i, v_j]$-path.

Definition 2.5 ([4])**.** The μ-strength of a path $P = v_1v_2....v_n$ is defined as $min\,\{\mu_{ij}\}$, for all $i, j = 1, 2, ...n$ and it is denoted by S_μ.

Definition 2.6 ([4])**.** The ν-strength of a path $P = v_1v_2....v_n$ is defined as $max\,\{\nu_{ij}\}$, for all $i, j = 1, 2, ...n$ and it is denoted by S_ν.

Note 2. If same edge possess both the values S_μ and S_ν, then the value is the strength of the path P and is denoted by S_P.

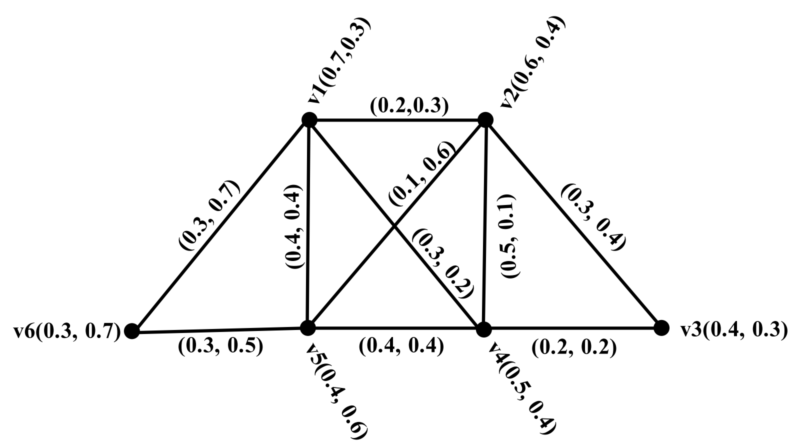

FIGURE 1. Intuitionistic fuzzy graph

Definition 2.7 ([4]). The μ-strength of connectedness between two vertices v_i and v_j is defined as $CONN_{\mu(G)}(v_i, v_j) = max\{S_\mu\}$ and ν-strength of connectedness between two vertices v_i and v_j is $CONN_{\nu(G)}(v_i, v_j) = min\{S_\nu\}$ of all possible paths between v_i and v_j.

Note 3. The notation $\langle CONN_{\mu(G)-e_{ij}}(v_i, v_j), CONN_{\nu(G)-e_{ij}}(v_i, v_j)\rangle$ is used to denote the strength of connectedness between v_i and v_j in the IFG obtained from G by deleting the edge e_{ij}.

Definition 2.8 ([4]). The order of an IFG G is defined as $O(G) = \langle O_\mu(G), O_\nu(G)\rangle$, where $O_\mu(G) = \sum_{v_i \in V} \mu_i$ and $O_\nu(G) = \sum_{v_i \in V} \nu_i$.

Definition 2.9 ([4]). The size of an IFG G is defined to be $S(G) = \langle S_\mu(G), S_\nu(G)\rangle$, where $S_\mu(G) = \sum_{v_i \neq v_j} \mu_{ij}$ and $S_\nu(G) = \sum_{v_i \neq v_j} \nu_{ij}$.

Definition 2.10. A scalar (say n) multiple of $S(G)$ is defined as

$$nS(G) = \langle nS_\mu(G), nS_\nu(G)\rangle .$$

Definition 2.11 ([4]). An IFG G is said to be connected, if there exists a path between every pair of vertices $v_i, v_j \in V$.

Definition 2.12 ([4]). Let G be an IFG.

(i) The μ-degree of a vertex v_i is

$$d_{\mu_i} = \sum_{e_{ij} \in E} \mu_{ij}.$$

(ii) The ν-degree of a vertex v_i is $d_{\nu_i} = \sum_{e_{ij} \in E} \nu_{ij}$.

(iii) The degree of a vertex is

$$d(v_i) = \left\langle \sum_{e_{ij} \in E} \mu_{ij}, \sum_{e_{ij} \in E} \nu_{ij} \right\rangle.$$

Definition 2.13 ([4]). (i)The minimum μ-degree is $\delta_\mu(G) = \wedge\{d_{\mu_i}/v_i \in V\}$.
(ii) The minimum ν-degree is $\delta_\nu(G) = \wedge\{d_{\nu_i}/v_i \in V\}$.
(iii) The maximum μ-degree is $\Delta_\mu(G) = \vee\{d_{\mu_i}/v_i \in V\}$.
(iv) The maximum ν-degree is $\Delta_\nu(G) = \vee\{d_{\nu_i}/v_i \in V\}$.

Definition 2.14 ([4]). Let G be an IFG. If $d_{\mu_i} = k_1$ and $d_{\nu_i} = k_2$ for all $v_i \in V$, then the graph is called $\langle k_1, k_2\rangle$-constant IFG (or) constant IFG of degree $\langle k_1, k_2\rangle$.

Definition 2.15 ([4]). Let G be an IFG. The total degree of a vertex $v_i \in V$ is defined as $td(v_i) = \langle d_{\mu_i} + \mu_i, d_{\nu_i} + \nu_i\rangle$.

Definition 2.16 ([4]). (i)The minimum μ-total degree is $\delta_{td_\mu}(G) = \wedge\{td_\mu(v_i)/v_i \in V\}$.
(ii) The minimum ν-total degree is $\delta_{td_\nu}(G) = \wedge\{td_\nu(v_i)/v_i \in V\}$.
(iii) The maximum μ-total degree is $\Delta_{td_\mu}(G) = \vee\{td_\mu(v_i)/v_i \in V\}$.
(iv) The maximum ν-total degree is $\Delta_{td_\nu}(G) = \vee\{td_\nu(v_i)/v_i \in V\}$.

Example 2.17. Consider an IFG, $G = \langle V, E\rangle$. Figure 2 shows that the $td(v_1) = \langle 1.7, 1.1\rangle$, $td(v_2) = \langle 2.0, 1.9\rangle$, $td(v_3) = \langle 1.3, 1.2\rangle$, $td(v_4) = \langle 1.3, 1.8\rangle$, $td(v_5) = \langle 1.5, 2.3\rangle$, and $td(v_6) = \langle 1.1, 2.1\rangle$. Minimum μ-total degree is $\delta_{td_\mu}(G) = 1.1$, minimum ν-total degree is $\delta_{td_{nu}}(G) = 1.1$, maximum μ-total degree is $\Delta_{td_\mu}(G) = 2.3$ and maximum ν-total degree is $\Delta_{td_\nu}(G) = 2.3$.

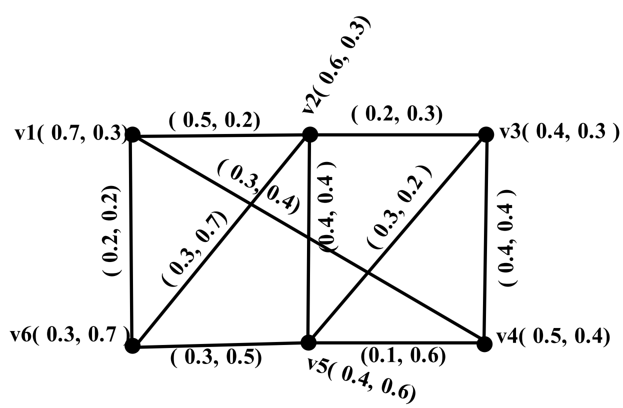

FIGURE 2. Total degrees in intuitionistic fuzzy graph

Definition 2.18 ([4]). Let G be an IFG. If $td_\mu(v_i) = r_1$ and $td_\nu(v_i) = r_2$ for all $v_i \in V$, then G is said to be an IFG of total degree $\langle r_1, r_2\rangle$ or a $\langle r_1, r_2\rangle$-totally constant IFG.

Definition 2.19 ([5]). An edge e_{ij} is said to be strong edge, if $\mu_{ij} \geq CONN_{\mu(G)-e_{ij}}(v_i, v_j)$ and $\nu_{ij} \leq CONN_{\nu(G)-e_{ij}}(v_i, v_j)$, for every $v_i, v_j \in V$.

Definition 2.20 ([5]). An edge e_{ij} is said to be weak edge, if $\mu_{ij} < CONN_{\mu(G)-e_{ij}}(v_i, v_j)$ and $\nu_{ij} > CONN_{\nu(G)-e_{ij}}(v_i, v_j)$, for every $v_i, v_j \in V$.

Definition 2.21 ([6]). Let G be an IFG. A vertex $v_i \in V$ is said to be strong, if e_{ij} is a strong edge, for all v_j incident with v_i.

3. Strong degrees in intuitionistic fuzzy graphs

Definition 3.1. Let $G = (V, E)$ be an IFG. The μ-strong degree of a vertex $v_i \in V$ is defined as $d_{s(\mu)}(v_i) = \sum_{e_{ij} \in E} \mu_{ij}$, where e_{ij} are strong edges incident at v_i.

Definition 3.2. Let $G = (V, E)$ be an IFG. The ν-strong degree of a vertex $v_i \in V$ is defined as $d_{s(\nu)}(v_i) = \sum_{e_{ij} \in E} \nu_{ij}$, where e_{ij} are strong edges incident at v_i.

Definition 3.3. Let $G = (V, E)$ be an IFG. The strong degree of a vertex $v_i \in V$ is defined as $d_s(v_i) = \left[\sum_{e_{ij} \in E} \mu_{ij}, \sum_{e_{ij} \in E} \nu_{ij} \right]$, where e_{ij} are strong edges incident at v_i.

Example 3.4. In Figure 3, the edges $e_{12}, e_{18}, e_{25}, e_{23}, e_{47}, e_{78}$ are strong and $d_s(v_1) = \langle 0.7, 1.0 \rangle$, $d_s(v_2) = \langle 1.1, 1.2 \rangle$, $d_s(v_3) = \langle 0.3, 0.3 \rangle$, $d_s(v_4) = \langle 0.3, 0.3 \rangle$, $d_s(v_5) = \langle 0.4, 0.4 \rangle$,$d_s(v_7) = \langle 0.7, 0.5 \rangle$, $d_s(v_8) = \langle 0.7, 0.7 \rangle$. The strong degree of vertex v_6 is $\langle 0.0, 0.0 \rangle$, since the incident edges at v_6 are not strong.

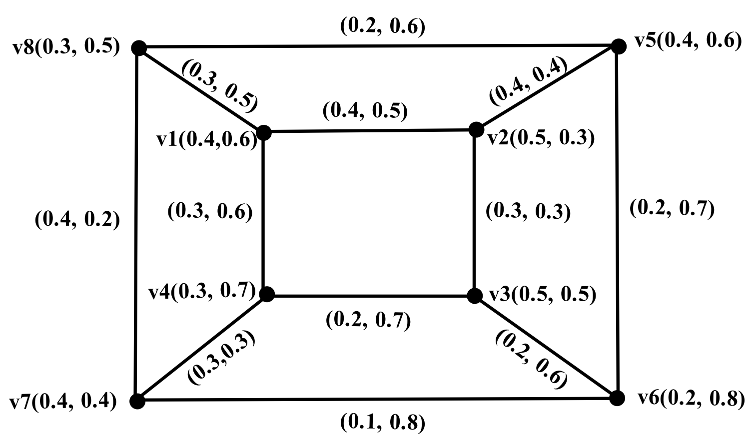

Figure 3. Strong degree of a vertex in IFG G

Definition 3.5. (i) The minimum μ-strong degree of G is

$$\delta_{s(\mu)}(G) = \wedge \left\{ d_{s(\mu)}(v_i)/v_i \in V \right\}$$

and minimum ν-strong degree of G is

$$\delta_{s(\nu)}(G) = \wedge \left\{ d_{s(\nu)}(v_i)/v_i \in V \right\}.$$

(ii) The maximum μ-strong degree of G is

$$\Delta_{s(\mu)}(G) = \vee \left\{ d_{s(\mu)}(v_i)/v_i \in V \right\}$$

and maximum ν-strong degree of G is

$$\Delta_{s(\nu)}(G) = \vee \left\{ d_{s(\nu)}(v_i)/v_i \in V \right\}.$$

Definition 3.6. Let G be an IFG. The μ-total strong degree of a vertex $v_i \in V$ in G is defined as $td_{s(\mu)}(v_i) = d_{s(\mu)}(v_i) + \mu_i$.

Definition 3.7. Let G be an IFG. The ν-total strong degree of a vertex $v_i \in V$ in G is defined as $td_{s(\nu)}(v_i) = d_{s(\nu)}(v_i) + \nu_i$.

Definition 3.8. Let G be an IFG. The total strong degree of a vertex $v_i \in V$ in G is defined as $td_s(v_i) = \left[td_{s(\mu)}(v_i), td_{s(\nu)}(v_i)\right]$.

Definition 3.9. (i) The minimum μ-total strong degree of G is

$$\delta_{ts(\mu)}(G) = \wedge \left\{td_{s(\mu)}(v_i)/v_i \in V\right\}$$

and minimum ν-total strong degree of G is

$$\delta_{ts(\nu)}(G) = \wedge \left\{td_{s(\nu)}(v_i)/v_i \in V\right\}.$$

(ii) The maximum μ-total strong degree of G is

$$\Delta_{ts(\mu)}(G) = \vee \left\{td_{s(\mu)}(v_i)/v_i \in V\right\}$$

and maximum ν-total strong degree of G is

$$\Delta_{ts(\nu)}(G) = \vee \left\{td_{s(\nu)}(v_i)/v_i \in V\right\}.$$

Definition 3.10. The μ-strong size of an IFG is defined as

$$S_{s(\mu)}(G) = \sum_{v_i \neq v_j} \mu_{ij},$$

where μ_{ij} is the membership of strong edge $e_{ij} \in E$.

Definition 3.11. The ν-strong size of an IFG is defined as

$$S_{s(\nu)}(G) = \sum_{v_i \neq v_j} \nu_{ij},$$

where ν_{ij} is the non-membership of strong edge $e_{ij} \in E$.

Definition 3.12. The strong size of an IFG G is defined as

$$S_s(G) = \left[S_{s(\mu)}(G), S_{s(\nu)}(G)\right].$$

Definition 3.13. The μ-strong order of an IFG is defined as

$$O_{s(\mu)}(G) = \sum_{v_i \in V} \mu_i,$$

where v_i is the strong vertex in G.

Definition 3.14. The ν-strong order of an IFG is defined as

$$O_{s(\nu)}(G) = \sum_{v_i \in V} \nu_i,$$

where v_i is the strong vertex in G.

Definition 3.15. The strong order of an IFG G is defined as

$$O_s(G) = \left[O_{s(\mu)}(G), O_{s(\nu)}(G)\right].$$

Definition 3.16. Let G be an IFG. If $d_{s(\mu)}(v_i) = k_1$ and $d_{s(\nu)}(v_i) = k_2$, for all $v_i \in V$, then the IFG is called as $\langle k_1, k_2\rangle$-strong constant IFG (or) strong constant IFG of degree $\langle k_1, k_2\rangle$.

Definition 3.17. Let G be an IFG. If $td_{s(\mu)}(v_i) = r_1$ and $td_{s(\nu)}(v_i) = r_2$, for all $v_i \in V$, then the IFG is called as $\langle r_1, r_2\rangle$-totally strong constant IFG (or) totally strong constant IFG of degree $\langle r_1, r_2\rangle$.

Proposition 3.18. *In an IFG G,*

$$2S_{s(\mu)}(G) = \sum_{i=1}^{n} d_{s(\mu)}(v_i)$$

and

$$2S_{s(\nu)}(G) = \sum_{i=1}^{n} d_{s(\nu)}(v_i).$$

Proposition 3.19. *In a connected IFG G,*

(1) $d_{s(\mu)}(v_i) \leq d_{\mu_i}$ and $d_{s(\nu)}(v_i) \leq d_{\nu_i}$.
(2) $td_{s(\mu)}(v_i) \leq td_{\mu_i}$ and $td_{s(\nu)}(v_i) \leq td_{\nu_i}$.

Proposition 3.20. *Let G be an IFG where crisp graph G^* is an odd cycle. Then G is strong constant if and only if $\langle \mu_{ij}, \nu_{ij}\rangle$ is a constant function for every $e_{ij} \in E$.*

Proposition 3.21. *Let G be an IFG where crisp graph G^* is an even cycle. Then G is strong constant if and only if either $\langle \mu_{ij}, \nu_{ij}\rangle$ is a constant function or alternate edges have same membership and non-membership values for every $e_{ij} \in E$.*

Remark 3.22. The above Proposition 3.20 and Proposition 3.21 hold for totally strong constant IFG, if $\langle \mu_i, \nu_i\rangle$ is a constant function.

Remark 3.23. A complete IFG need not be a strong constant IFG and also totally strong constant IFG.

Remark 3.24. A strong IFG need not be a strong constant IFG and also totally strong constant IFG.

Remark 3.25. For a strong vertex $v_i \in V$,

(1) $d_\mu(v_i) = d_{s(\mu)}(v_i)$ and $d_\nu(v_i) = d_{s(\nu)}(v_i)$,
(2) $td_\mu(v_i) = td_{s(\mu)}(v_i)$ and $td_\nu(v_i) = td_{s(\nu)}(v_i)$.

Theorem 3.26. *Let G be a complete IFG with $V = \{v_1, v_2, v_3, ..., v_n\}$ such that $\mu_1 \leq \mu_2 \leq \mu_3 \leq ..... \leq \mu_n$ and $\nu_1 \geq \nu_2 \geq \nu_3 \geq ..... \geq \nu_n$. Then,*

(1) *μ_{1j} is the minimum edge membership* and *ν_{1j} is the maximum edge non-membership of e_{ij} emits from v_1 for all $j = 2, 3, ..., n$,*

(2) *μ_{in} is the maximum edge membership* and *ν_{in} is the minimum edge non-membership of among all edges emits from v_i to v_n for all $i = 1, 2, 3, ..., n-1$,*

(3) $td_\mu(v_1) = \delta_{td_\mu}(G) = n.\mu_1$ and $td_\nu(v_1) = \Delta_{td_\nu}(G) = n.\nu_1$,

(4) $td_\mu(v_n) = \Delta_{td_\mu}(G) = \sum_{i=1}^{n} \mu_i$ and $td_\nu(v_n) = \delta_{td_\nu}(G) = \sum_{i=1}^{n} \nu_i$.

Proof. Throughout the proof, suppose that

$\mu_1 < \mu_2 \leq \mu_3 \leq .... \leq \mu_n$ and $\nu_1 > \nu_2 \geq \nu_3 \geq .... \geq \nu_n$.

(1) To prove that μ_{1j} is minimum edge membership and ν_{1j} is maximum edge non-membership for $j = 2, 3, ...n$, assume the contrary. That is, let e_{1l} is not an edge of minimum membership and maximum non-membership emits from v_l. Also, let e_{kl}, $2 \leq k \leq n$, $k \neq l$ be an edge with minimum membership and maximum non-membership emits from v_k. Being a complete IFG, $\mu_{1l} = min \langle \mu_1, \mu_l \rangle$ and $\nu_{1l} = max \langle \nu_1, \nu_l \rangle$. Then, $\mu_{kl} = min \langle \mu_k, \mu_l \rangle$ and $\nu_{kl} = max \langle \nu_k, \nu_l \rangle$. Since $\mu_{kl} < \mu_{1l}$, $min \langle \mu_k, \mu_l \rangle < min \langle \mu_1, \mu_l \rangle$. Thus, either $\mu_k < \mu_1$ or $\mu_l < \mu_1$.

Also since $\nu_{kl} > \nu_{1l}$, $max \langle \nu_k, \nu_l \rangle > max \langle \nu_1, \nu_l \rangle$. So, either $\nu_k > \nu_1$ or $\nu_l > \nu_1$. Since $l, k \neq 1$, this is a contradiction to our assumption that μ_1 is the unique minimum vertex membership and ν_1 is the maximum vertex non-membership. Hence, μ_{1j} is minimum edge membership and ν_{1j} is maximum edge non-membership emits from v_1 to v_j for all $j = 2, 3, ..., n$.

(2) On the contrary, assume that e_{kn} is not an edge with maximum membership and minimum non-membership emits from v_k, for $1 \leq k \leq (n-1)$. On the other hand, let e_{kr} be an edge with maximum membership and minimum non-membership emits from v_r, for $1 \leq r \leq (n-1)$, $k \neq r$. Then, $\mu_{kr} > \mu_{kn}$. Thus $min \langle \mu_k, \mu_r \rangle > min \langle \mu_k, \mu_n \rangle = \mu_k$. So $\mu_r > \mu_k$.

Similarly, $\nu_{kr} < \nu_{kn}$. Then $max \langle \nu_k, \nu_r \rangle < max \langle \nu_k, \nu_n \rangle = \nu_k$. Thus $\nu_r < \nu_k$. So, $\mu_{kr} = \mu_k = \mu_{kn}$ and $\nu_{kr} = \nu_k = \nu_{kn}$ which is a contradiction. Hence, e_{kn} is an edge with maximum membership and minimum non-membership among all edges emit from v_k to v_n.

(3) Now,

$$
\begin{aligned}
td_\mu(v_1) &= d_\mu(v_1) + \mu_1 \\
&= \sum_{e_{1j} \in E} \mu_{1j} + \mu_1 = \sum_{j=2}^{n} \mu_{1j} + \mu_1 = (n-1).\mu_1 + \mu_1 = n.\mu_1 - \mu_1 + \mu_1 = n.\mu_1.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
td_\nu(v_1) &= d_\nu(v_1) + \nu_1 \\
&= \sum_{e_{1j} \in E} \nu_{1j} + \nu_1 = \sum_{j=2}^{n} \nu_{1j} + \nu_1 = (n-1).\nu_1 + \nu_1 = n.\nu_1 - \nu_1 + \nu_1 = n.\nu_1.
\end{aligned}
$$

Suppose that $td_\mu(v_1) \neq \delta_{td_\mu}(G)$ and let v_k, $k \neq 1$ be a vertex in G with minimum μ-total degree. Then, $td_\mu(v_1) > td_\mu(v_k)$. Thus $\sum_{i=2}^{n} \mu_{1i} + \mu_1 > \sum_{k \neq 1, j \neq k} \mu_{kj} + \mu_k$. So,

$$
\sum_{i=2}^{n} \mu_1 \wedge \mu_i + \mu_1 > \sum_{k \neq 1, j \neq k} \mu_k \wedge \mu_j + \mu_k.
$$

Since $\mu_1 \wedge \mu_i = \mu_1$ for $i = 1, 2, 3, ..., n$, and for all other indices j, $\mu_k \wedge \mu_j > \mu_1$, it follows that,

$$
(n-1).\mu_1 + \mu_1 > \sum_{k \neq 1, j \neq k} \mu_k \wedge \mu_j + \mu_k > (n-1)\mu_1 + \mu_1.
$$

Hence, $td_\mu(v_1) > td_\mu(v_1)$, a contradiction. Therefore, $td_\mu(v_1) = \delta_{td_\mu}(G)$.

Also, suppose that $td_\nu(v_1) \neq \Delta_{td_\nu}(G)$ and let v_k, $k \neq 1$ be a vertex in G with maximum ν-total degree. Then, $td_\nu(v_1) < td_\mu(v_k) \sum\limits_{i=2}^{n} \nu_{1i} + \nu_1 < \sum\limits_{k\neq 1, j\neq k} \nu_{kj} + \nu_k$.

Thus, $\sum\limits_{i=2}^{n} \nu_1 \vee \nu_i + \nu_1 < \sum\limits_{k\neq 1, j\neq k} \nu_k \vee \nu_j + \nu_k$. Since $\nu_1 \vee \nu_i = \nu_1$ for $i = 1, 2, 3, ..., n$ and for all other indices j, $\nu_k \vee \nu_j < \mu_1$, it follows that,

$$(n-1).\nu_1 + \nu_1 < \sum_{k\neq 1, j\neq k} \nu_k \vee \nu_j + \nu_k < (n-1).\nu_1 + \nu_1.$$

So, $td_\nu(v_1) < td_\mu(v_1)$, a contradiction. Hence, $td_\nu(v_1) = \Delta_{td_\nu}(G)$. Therefore, $td_\mu(v_1) = \delta_{td_\mu}(G) = n.\mu_1$ and $td_\nu(v_1) = \Delta_{td_\nu}(G) = n.\nu_1$.

(4) Since $\mu_n > \mu_i$, $\nu_n < \nu_i$, for $i = 1, 2, ..., n-1$ and G is complete,

$$\mu_{ni} = \mu_n \wedge \mu_i = \mu_i \text{ and } \nu_{ni} = \nu_n \vee \mu_i = \mu_i.$$

Then, $td_\mu(v_n) = \sum\limits_{i=1}^{n-1} \mu_{ni} + \mu_n = \sum\limits_{i=1}^{n-1} (\mu_n \wedge \mu_i) + \mu_n = \sum\limits_{i=1}^{n-1} (\mu_i) + \mu_n = \sum\limits_{i=1}^{n} \mu_i$.

And, $td_\nu(v_n) = \sum\limits_{i=1}^{n-1} \nu_{ni} + \nu_n = \sum\limits_{i=1}^{n-1} (\nu_n \vee \nu_i) + \nu_n = \sum\limits_{i=1}^{n-1} \nu_i + \nu_n = \sum\limits_{i=1}^{n} \nu_i$.

Suppose that $td_\mu(v_n) \neq \Delta_{td_\mu}(G)$. Let v_l, $1 \leq l \leq n-1$ be a vertex in G such that

$$td_\mu(v_l) = \Delta_{td_\mu}(G) \text{ and } td_\mu(v_n) < td_\mu(v_l).$$

On one hand,

$$\begin{aligned} td_\mu(v_l) &= \left[\sum_{i=1}^{l-1} \mu_{il} + \sum_{i=l+1}^{n-l} \mu_{il} + \mu_{nl}\right] + \mu_l \\ &\leq \left[\sum_{i=1}^{l-1} \mu_i + (n-l)\mu_l + \mu_l\right] + \mu_l \\ &\leq \sum_{i=1}^{n-1} \mu_i + \mu_l \\ &\leq \sum_{i=1}^{n} \mu_i = td_\mu(v_n). \end{aligned}$$

Thus, $td_\mu(v_l) \leq td_\mu(v_n)$, a contradiction. So, $td_\mu(v_n) = \Delta_{td_\mu}(G) = \sum\limits_{i=1}^{n} \mu_i$.

Also, suppose that $td_\nu(v_n) \neq \delta_{td_\nu}(G)$. Let v_l, $1 \leq l \leq n-1$ be a vertex in G such that $td_\nu(v_l) = \delta_{td_\nu}(G)$ and $td_\nu(v_n) > td_\nu(v_l)$.

On the other hand,

$$\begin{aligned} td_\nu(v_l) &= \left[\sum_{i=1}^{l-1} \nu_{il} + \sum_{i=l+1}^{n-l} \nu_{il} + \nu_{nl}\right] + \nu_l \\ &\geq \left[\sum_{i=1}^{l-1} \nu_i + (n-l)\nu_l + \nu_l\right] + \nu_l \\ &\geq \sum_{i=1}^{n-1} \nu_i + \nu_l \\ &\geq \sum_{i=1}^{n} \nu_i = td_\nu(v_n). \end{aligned}$$

Thus, $td_\nu(v_l) \geq td_\nu(v_n)$, a contradiction to our assumption. So $td_\nu(v_n) = \delta_{td_\nu}(G) = \sum\limits_{i=1}^{n} \nu_i$. Hence the lemma is proved. □

Remark 3.27. In a complete IFG G,

(1) there exists at least one pair of vertices v_i and v_j such that $d_{\mu_i} = d_{\mu_j} = \Delta_\mu(G)$ and $d_{\nu_i} = d_{\nu_j} = \delta_\nu(G)$,

(2) $td_\mu(v_i) = O_\mu(G) = \Delta_{td_\mu}(G)$ and $td_\nu(v_i) = O_\nu(G) = \delta_{td_\nu}(G)$ for a vertex $v_i \in V$,

(3) $\sum\limits_{i=1}^{n} td_\mu(v_i) = 2S_\mu(G) + O_\mu(G)$ and $\sum\limits_{i=1}^{n} td_\nu(v_i) = 2S_\nu(G) + O_\nu(G)$.

4. Vertex membership and non-membership sequences in intuitionistic fuzzy graphs

In this section, vertex membership and non-membership sequences are defined in IFGs.

Definition 4.1. Let G be an IFG with $|V| = n$. The vertex membership sequence of G is defined to be $\{p_i\}_{i=1}^n$ with $p_1 \leq p_2 \leq p_3 \leq, ..., \leq p_n$ where p_i, $0 < p_i \leq 1$, is the membership value of the vertex v_i when vertices are arranged so that their membership values are non-decreasing.

In particular, p_1 is the smallest vertex membership value and p_n is largest vertex membership value in G.

Note 4. If vertex membership sequence p_i is repeated more than once in G, say $r \neq 1$ times, then it is denoted by p_i^r in the sequence.

Example 4.2. In Figure 4, the vertex membership sequence of G is
$\{0.2, 0.2, 0.3, 0.3, 0.5, 0.8\}$ or $\{0.2^2, 0.3^2, 0.5, 0.8\}$.

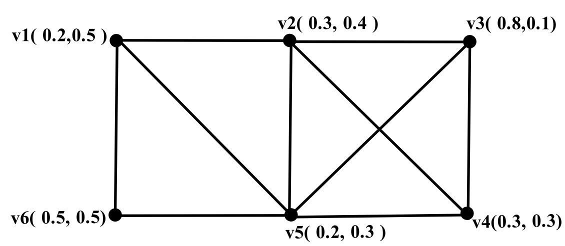

Figure 4. Vertex membership sequence

Definition 4.3. Let G be an IFG with $|V| = n$. The vertex non-membership sequence of G is defined to be $\{q_i\}_{i=1}^n$ with $q_1 \leq q_2 \leq q_3 \leq, ..., \leq q_n$, where q_i, $0 < q_i \leq 1$, is the non-membership value of the vertex v_i, when vertices are arranged so that their non-membership values are non-increasing.

In particular, q_1 is the largest vertex non-membership value and q_n is smallest vertex non-membership value in G.

Note 5. If vertex non-membership sequence q_i is repeated more than once in G, say $r \neq 1$ times, then it is denoted by q_i^r in the sequence.

Example 4.4. In Figure 5, the vertex non-membership sequence of G is
$\{0.8, 0.4, 0.4, 0.4, 0.3, 0.3\}$ or $\{0.8, 0.4^3, 0.3^2\}$.

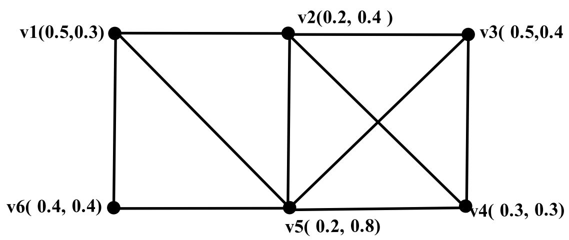

Figure 5. Vertex non-membership sequence

Definition 4.5. If an IFG G with $|V| = n$ has both vertex membership sequence $\{p_i\}_{i=1}^n$ and vertex non-membership sequence $\{q_i\}_{i=1}^n$ in the same order, then it is said to have vertex intuitionistic fuzzy sequence and is denoted by $\{\langle p_i, q_i\rangle\}_{i=1}^n$.

Example 4.6. In Figure 6, the vertex membership and non-membership sequence of G is $\{\langle 0.4, 0.4\rangle, \langle 0.4, 0.4\rangle, \langle 0.5, 0.3\rangle, \langle 0.5, 0.3\rangle, \langle 0.6, 0.2\rangle, \langle 0.7, 0.1\rangle\}$
or $\left\{\left\langle 0.4^2, 0.4^2\right\rangle, \left\langle 0.5^2, 0.3^2\right\rangle, \langle 0.6, 0.2\rangle, \langle 0.7, 0.1\rangle\right\}$.

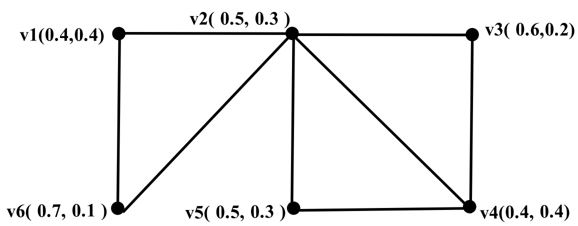

Figure 6. Vertex intuitionistic fuzzy sequence

The properties of the vertex membership and non-membership sequences of complete IFGs are discussed below.

Theorem 4.7. *Let $G = (V, E)$ be a complete IFG with $|V| = n$. Then,*

(1) *If the vertex membership sequence of G is of the form $\left\{p_1^{n-1}, p_2\right\}$ and vertex non-membership sequence of G is of the form $\left\{q_1^{n-1}, q_2\right\}$, then*

(a) $\delta_{td_\mu}(G) = n.\mu_1$ and $\Delta_{td_\mu}(G) = \sum\limits_{i=1}^{n} \mu_i$.

(b) $\Delta_{td_\nu}(G) = n.\nu_1$ and $\delta_{td_\nu}(G) = \sum\limits_{i=1}^{n} \nu_i$.

(2) *If the vertex membership sequence of G is of the form $\left\{p_1^{r_1}, p_2^{n-r_1}\right\}$* and *vertex non-membership sequence of G is of the form $\left\{q_1^{r_1}, q_2^{n-r_1}\right\}$ with $0 < r_1 \leq n-2$, then there exist exactly r_1 vertices with minimum μ-total degree $\delta_{td_\mu}(G)$* and *maximum ν-total degree $\Delta_{td_\nu}(G)$* and *exactly $(n - r_1)$ vertices with maximum μ-total degree $\Delta_{td_\mu}(G)$* and *minimum ν-total degree $\delta_{td_\nu}(G)$.*

(3) *If the vertex membership sequence of* G *is of the form* $\{p_1^{r_1}, p_2^{r_2}, ..., p_k^{r_k}\}$ and *the vertex non-membership sequence is of the form* $\{q_1^{r_1}, q_2^{r_2}, ..., q_k^{r_k}\}$ *with* $r_k > 1$ and $k > 2$, *then there exists exactly* r_1 *vertices with minimum* μ*-total degree* $\delta_{td_\mu}(G)$ and *maximum* ν*-total degree* $\Delta_{td_\nu}(G)$. *Also, there exist exactly* r_k *vertices with maximum* μ*-total degree* $\Delta_{td_\mu}(G)$ and *minimum* ν*-total degree* $\delta_{td_\nu}(G)$.

Proof. The proofs of (1) and (2) are obvious.

(3) Let $v_i^{(j)}$ be the set of vertices in G, for $j = 1, 2, ...r_i$, $1 \leq i \leq k$. Then by Theorem 3.26,

$$td_\mu(v_1^{(j)}) = \delta_{td_\mu}(G) = n.\mu_1 = n.p_1 \text{ and } td_\nu(v_1^{(j)}) = \Delta_{td_\nu}(G) = n.\nu_1 = n.q_1,$$

for $j = 1, 2, ...r_1$. Since $\mu(v_i^{(j)}, v_{i+1}^{(l)}) = \mu(v_i^{(j)}) > p_1$ for $2 \leq i \leq k$, $j = 1, 2, ..., r_i$, $l = 1, 2, ..., r_{i+1}$, no vertex with membership more than p_1 can have degree $\delta_{td_\mu}(G)$. Also since $\nu(v_i^{(j)}, v_{i+1}^{(l)}) = \nu(v_i^{(j)}) < q_1$ for $2 \leq i \leq k$, $j = 1, 2, ..., r_i$, $l = 1, 2, ..., r_{i+1}$, no vertex with non-membership less than q_1 can have degree $\Delta_{td_\nu}(G)$. Thus, there exist exactly r_1 vertices with degree $\delta_{td_\mu}(G)$ and $\Delta_{td_\nu}(G)$.

To prove $td_\mu(v_k^{(t)}) = \Delta_{td_\mu}(G)$ and $td_\nu(v_k^{(t)}) = \delta_{td_\mu}(G)$, $t = 1, 2, ..., r_k$.
Since, $\mu(v_k^t)$ is maximum vertex membership, $\mu(v_k^t, v_k^j) = p_k$, $t \neq j$, $t, j = 1, 2, ..., r_k$ and $\mu(v_k^t, v_i^j) = min\left\{\mu(v_k^t), \mu(v_i^j)\right\} = \mu(v_i^j)$ for $t = 1, 2, ..., r_k$, $j = 1, 2, ..., r_i$, $i = 1, 2, ..., k-1$. Thus, for $t = 1, 2, ..., r_k$,

$$\begin{aligned} td_\mu(v_k^t) &= \sum_{i=1}^{k}\sum_{j=1}^{r_i} \mu(v_i^j) + (r_k - 1)p_k \\ &= \sum_{i=1}^{n} \mu_i \\ &= \Delta_{td_\mu}(G) \quad \text{by Theorem 3.26} \end{aligned}$$

Now, if v_m is a vertex such that $\mu_m = p_{k-1}$, then

$$\begin{aligned} td_\mu(v_m) &= \sum_{i=1}^{k-2}\sum_{j=1}^{r_i} \mu(v_m, v_i^j) + (r_{k-1} - 1 + r_k)p_{k-1} + \mu_m \\ &= \sum_{i=1}^{k-2}\sum_{j=1}^{r_i} \mu(v_i^j) + \sum_{j=1}^{r_{k-1}} \mu(v_{k-1}^j) + (r_k - 1)p_{k-1} + \mu_m \\ &< \sum_{i=1}^{k-2}\sum_{j=1}^{r_i} \mu(v_i^j) + \sum_{j=1}^{r_{k-1}} \mu(v_{k-1}^j) + (r_k - 1)p_k + \mu_m = \Delta_{td_\mu}(G). \end{aligned}$$

Thus, there exist exactly r_k vertices with degree $\Delta_{td_\mu}(G)$.

Similarly, it can be proved that $td_\nu(v_k^{(t)}) = \delta_\nu(G)$, $t = 1, 2, ..., r_k$. Since $\nu(v_k^t)$ is minimum vertex non-membership, $\nu(v_k^t, v_i^j) = q_k$, $t \neq j$, $t, j = 1, 2, ..., r_k$ and $\nu(v_k^t, v_i^j) = max\left\{\nu(v_k^t), \nu(v_i^j)\right\} = \nu(v_i^j)$, for $t = 1, 2, ..., r_k$, $j = 1, 2, ..., r_i$, $i = 1, 2, ..., k-1$. Thus, for $t = 1, 2, ..., r_k$,

$$\begin{aligned} td_\nu(v_k^t) &= \sum_{i=1}^{k}\sum_{j=1}^{r_i} \nu(v_i^j) + (r_k - 1)q_k \\ &= \sum_{i=1}^{n} \nu_i \\ &= \delta_{td_\nu}(G) \ \text{ by Theorem 3.26} \end{aligned}$$

Now, if v_m is a vertex such that $\nu_m = q_{k-1}$, then

$$\begin{aligned} td_\nu(v_m) &= \sum_{i=1}^{k-2}\sum_{j=1}^{r_i} \nu(v_m, v_i^j) + (r_{k-1} - 1 + r_k)q_{k-1} + \nu_m \\ &= \sum_{i=1}^{k-2}\sum_{j=1}^{r_i} \nu(v_i^j) + \sum_{j=1}^{r_{k-1}} \nu(v_{k-1}^j) + (r_k - 1)q_{k-1} + \nu_m \\ &< \sum_{i=1}^{k-2}\sum_{j=1}^{r_i} \nu(v_i^j) + \sum_{j=1}^{r_{k-1}} \mu(v_{k-1}^j) + (r_k - 1)q_k + \nu_m = \delta_{td_\nu}(G). \end{aligned}$$

So, there exist exactly r_k vertices with degree $\delta_{td_\nu}(G)$. □

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