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The cyclic and strongly connected of the products of intuitionistic fuzzy Mealy machines

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Abstract. This paper introduces an intuitionistic fuzzy Mealy machine(ifm) and various kinds of products of intuitionistic fuzzy Mealy machines such as intuitionistic cartesian product, intuitionistic full direct product, intuitionistic restricted direct product, intuitionistic cascade product and intuitionistic wreath product are analyzed. These products preserve the properties of cyclic, intuitionistic retrievable and strongly connected of intuitionistic fuzzy Mealy machines are examined.

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1. Introduction

The theory of fuzzy sets was proposed by Zadeh [22] as an effective generalization of classical sets which has been widely used in dealing with problems with imprecision and uncertainty. Much later, Santos [20] and W.G. Wee [21] first initiated the mathematical formation of a fuzzy automaton in 1969. Out of several higher order fuzzy sets, intuitionistic fuzzy sets(ifs) introduced by Atanassov [1, 2, 3, 4, 5] has been found to be highly useful to deal with vagueness, since the ifs are characterized by two functions expressing the degree of belongingness and the degree of non-belongingness. But, Burillo and Bustince [6, 7] showed that the notion of vague sets coincides with the intuitionistic fuzzy sets. Using the notions of intuitionistic fuzzy sets, Jun [12, 13, 14] introduced the concept of intuitionistic fuzzy finite state machines. Choubey Alka et al.[11] presented the concept of intuitionistic fuzzy finite state machines as a generalization of fuzzy finite state machines. Malik and Mordeson [15, 16, 17, 18, 19] illustrated the concepts developed in fuzzy finite state automata, fuzzy transformation, semigroups coverings, direct products, cascade products and wreath products based on Wee's concept of fuzzy automata by applying the algebraic techniques. Chaudhari et al.[8, 9, 10] discussed the equivalence of fuzzy Mealy machine and fuzzy Moore machine and they investigated the relation between the products of fuzzy Mealy machines with the help of topology. Intuitionistic fuzzy Mealy machine(ifm)is a kind of fuzzy automaton with outputs capabilities based on both the current state and the current input strings.

In this paper, various products of intuitionistic fuzzy Mealy machines such as intuitionistic cartesian product, intuitionistic full direct product, intuitionistic restricted direct product, intuitionistic cascade product and intuitionistic wreath product are introduced. All these products preserve the properties of cyclic, intuitionistic retrievable, intuitionistic connected and strongly connected of ifm.

2. Preliminaries

In this section, we recall Σ^* denote the set of all strings of finite length over Σ, λ denotes the empty string and $|x|$ denotes the length of x.

Definition 2.1 ([1]). Given a nonempty set Σ. Intuitionistic fuzzy sets(ifs) in Σ is an object having the form $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in \Sigma\}$, where $\mu_A : \Sigma \to [0,1]$ and $\nu_A : \Sigma \to [0,1]$ denote the degree of membership and the degree of non-membership of each element $x \in \Sigma$ to the set A, respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in \Sigma$. For the sake of simplicity, use the notation $A = (\mu_A, \nu_A)$ instead of $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in \Sigma\}$.

We introduce the following definitions.

Definition 2.2 ([11]). An intuitionistic fuzzy Mealy machine(ifm) $\mathscr{A} = (Q, \Sigma, X, A)$, where Q is a finite nonempty set of states, Σ is a finite nonempty set of input symbols, X is a finite nonempty set of output symbols and $A = (\mu_A, \nu_A)$, where each component is an intuitionistic fuzzy subset of $Q\times\Sigma\times Q\times X$, i.e., $\mu_A : Q\times\Sigma\times Q\times X \to [0,1]$ denotes the degree of membership values and $\nu_A : Q \times \Sigma \times Q \times X \to [0,1]$ denotes the degree of non-membership values.

The extensions of μ_A^*, ν_A^* to $Q \times \Sigma^* \times Q \times X^*$ are defined by

$$\mu_A^*(q, \lambda, p, \lambda) = \begin{cases} 1, \text{if } q = p \\ 0, \text{if } q \neq p \end{cases} \quad \text{and } \nu_A^*(q, \lambda, p, \lambda) = \begin{cases} 0, \text{if } q = p \\ 1, \text{if } q \neq p, \end{cases}$$

where $\mu_A^*(q, \lambda, q, b) = 0$ and $\nu_A^*(q, \lambda, q, b) = 1$,

$$\mu_A^*(q, xa, p, yb) = \vee\{\mu_A^*(q, x, r, y) \wedge \mu_A(r, a, p, b) \mid r \in Q\}$$

and

$$\nu_A^*(q, xa, p, yb) = \wedge\{\nu_A^*(q, x, r, y) \vee \nu_A(r, a, p, b) \mid r \in Q\},$$

$\forall q, p \in Q, \forall a \in \Sigma,\ \forall b \in X,\ \forall x \in \Sigma^*,\ \forall y \in X^*$ and for the empty string λ.
Then $\forall q, p \in Q,\ \forall x, u \in \Sigma^*,\ \forall y, v \in X^*,\ \exists\, |x| = |y|$ and $|u| = |v|$. Thus

$$\mu_A^*(q, xu, p, yv) = \vee\{\mu_A^*(q, x, r, y) \wedge \mu_A^*(r, u, p, v) \mid r \in Q\}$$

and

$$\nu_A^*(q, xu, p, yv) = \wedge\{\nu_A^*(q, x, r, y) \vee \nu_A^*(r, u, p, v) \mid r \in Q\}.$$

So $\forall p, q \in Q,\ \forall x \in \Sigma^*,\ y \in X^*$, if $|x| \neq |y|$, then $\mu_A^*(q, x, p, y) = 0$ and $\nu_A^*(q, x, p, y) = 1$.

Definition 2.3 ([16]). Let $\mathscr{A} = (Q, \Sigma, X, A)$ be an ifm. Let $q, p \in Q$.

(i) p is called an intuitionistic immediate successor of q, if there exists $a \in \Sigma$ and $b \in X$ such that $\mu_A(q, a, p, b) > 0$ and $\nu_A(q, a, p, b) < 1$.

(ii) p is called intuitionistic successor of q, if $x \in \Sigma^*$ and $y \in X^*$ such that $\mu_A^*(q, x, p, y) > 0$ and $\nu_A^*(q, x, p, y) < 1$.

Notation 2.4. Let $\mathscr{A} = (Q, \Sigma, X, A)$ be an ifm and $q \in Q$. We shall denote $S(q)$ the set of all intuitionistic successors of q.

Definition 2.5 ([17]). Let $\mathscr{A} = (Q, \Sigma, X, A)$ be an ifm and $T \subseteq Q$. The set of all intuitionistic successors of T, denoted by $S(T)$, is defined to be the set
$S(T) = \cup\{S(q) \mid q \in T\}$.

3. Products of intuitionistic fuzzy Mealy machines

This section is an introduction of various products of ifm and their interrelationship in terms of cyclicness, intuitionistic retrievable and intuitionistic connected are discussed.

Definition 3.1 ([11]). Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be intuitionistic fuzzy Mealy machines(ifm), $i = 1, 2$. Then the ifm $\mathscr{A}_1 \times \mathscr{A}_2 = (Q, \Sigma, X, A_1 \times A_2)$ is called

(i) the intuitionistic Cartesian product of ifm of $\mathscr{A}_1$ and $\mathscr{A}_2$ with
$\Sigma_1 \cap \Sigma_2 = \phi,\ X_1 \cap X_2 = \phi$, if $Q = Q_1 \times Q_2,\ \Sigma = \Sigma_1 \cup \Sigma_2$,
$X = X_1 \cup X_2, A_1 \times A_2 = (\mu_{A_1} \times \mu_{A_2}, \nu_{A_1} \times \nu_{A_2})$, where

$$(\mu_{A_1} \times \mu_{A_2})\Big((q_1, q_2), a, (p_1, p_2), b\Big) = \begin{cases} \mu_{A_1}(q_1, a, p_1, b), & \text{if } a \in \Sigma_1, b \in X_1, \\ & \text{and } \ q_2 = p_2 \\ \mu_{A_2}(q_2, a, p_2, b), & \text{if } a \in \Sigma_2, b \in X_2, \\ & \text{and } \ q_1 = p_1 \\ 0, \text{otherwise} \end{cases}$$

and

$$(\nu_{A_1} \times \nu_{A_2})\Big((q_1, q_2), a, (p_1, p_2), b\Big) = \begin{cases} \nu_{A_1}(q_1, a, p_1, b), & \text{if } a \in \Sigma_1, b \in X_1, \\ & \text{and } \ q_2 = p_2 \\ \nu_{A_2}(q_2, a, p_2, b), & \text{if } a \in \Sigma_2, b \in X_2, \\ & \text{and } \ q_1 = p_1 \\ 0, \text{otherwise} \end{cases}$$

$\forall (p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2,\ a \in \Sigma_1 \cup \Sigma_2,\ b \in X_1 \cup X_2$.

(ii) the intuitionistic full direct product of ifm of $\mathscr{A}_1$ and $\mathscr{A}_2$,
if $Q = Q_1 \times Q_2,\ \Sigma = \Sigma_1 \times \Sigma_2,\ X = X_1 \times X_2,\ A_1 \times A_2 = (\mu_{A_1} \times \mu_{A_2}, \nu_{A_1} \times \nu_{A_2})$, where

$$\begin{aligned}&(\mu_{A_1} \times \mu_{A_2})\Big((q_1, q_2), (a_1, a_2), (p_1, p_2), (b_1, b_2)\Big) \\ &= \mu_{A_1}(q_1, a_1, p_1, b_1) \wedge \mu_{A_2}(q_2, a_2, p_2, b_2)\end{aligned}$$

and

$$\begin{aligned}&(\nu_{A_1} \times \nu_{A_2})\Big((q_1, q_2), (a_1, a_2), (p_1, p_2), (b_1, b_2)\Big) \\ &= \nu_{A_1}(q_1, a_1, p_1, b_1) \vee \nu_{A_2}(q_2, a_2, p_2, b_2),\end{aligned}$$

$\forall (q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2,\ \forall (a_1, a_2) \in \Sigma_1 \times \Sigma_2,\ \forall (b_1, b_2) \in X_1 \times X_2$.

(iii) the intuitionistic restricted direct product of ifm of $\mathscr{A}_1$ and $\mathscr{A}_2$,
if $Q = Q_1 \times Q_2,\ \Sigma = \Sigma_1 = \Sigma_2,\ X = X_1 = X_2,\ A_1 \times A_2 = (\mu_{A_1} \times \mu_{A_2}, \nu_{A_1} \times \nu_{A_2})$, where

$$(\mu_{A_1} \times \mu_{A_2})\Big((q_1, q_2), a, (p_1, p_2), b\Big) = \mu_{A_1}(q_1, a, p_1, b) \wedge \mu_{A_2}(q_2, a, p_2, b)$$

and

$$(\nu_{A_1} \times \nu_{A_2})\Big((q_1, q_2), a, (p_1, p_2), b\Big) = \nu_{A_1}(q_1, a, p_1, b) \vee \nu_{A_2}(q_2, a, p_2, b)$$

$\forall (q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2,\ \forall a \in \Sigma,\ \forall b \in X.$

(iv) the intuitionistic cascade product of ifm of $\mathscr{A}_1$ and $\mathscr{A}_2$,
if $Q = Q_1 \times Q_2,\ \Sigma = \Sigma_2,\ X = X_2, A_1 \times A_2 = (\mu_{A_1} \times \mu_{A_2}, \nu_{A_1} \times \nu_{A_2})$, where

$$(\mu_{A_1} \times \mu_{A_2})\Big((q_1, q_2), a_2, (p_1, p_2), b_2\Big)$$
$$= \mu_{A_1}\Big(q_1, \omega_x(q_2, a_2), p_1, \omega_y(q_2, b_2)\Big) \wedge \mu_{A_2}(q_2, a_2, p_2, b_2)$$

and

$$(\nu_{A_1} \times \nu_{A_2})\Big((q_1, q_2), a_2, (p_1, p_2), b_2\Big)$$
$$= \nu_{A_1}\Big(q_1, \omega_x(q_2, a_2), p_1, \omega_y(q_2, b_2)\Big) \vee \nu_{A_2}(q_2, a_2, p_2, b_2),$$

$\forall (q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2,\ \forall a_2 \in \Sigma_2,\ \forall b_2 \in X_2$ and $\omega_x : Q_2 \times \Sigma_2 \to \Sigma_1,\ \omega_y : Q_2 \times X_2 \to X_1$.

(v) the intuitionistic wreath product of ifm of $\mathscr{A}_1$ and $\mathscr{A}_2$,
if $Q = Q_1 \times Q_2,\ \Sigma = \Sigma_1^{Q_2} \times \Sigma_2,\ X = X_1^{Q_2} \times X_2,\ A_1 \times A_2 = (\mu_{A_1} \times \mu_{A_2}, \nu_{A_1} \times \nu_{A_2})$, where

$$(\mu_{A_1} \times \mu_{A_2})\Big((q_1, q_2), (f, a_2), (p_1, p_2), (g, b_2)\Big)$$
$$= \mu_{A_1}\Big(q_1, f(q_2), p_1, g(q_2)\Big) \wedge \mu_{A_2}(q_2, a_2, p_2, b_2)$$

and

$$(\nu_{A_1} \times \nu_{A_2})\Big((q_1, q_2), (f, a_2), (p_1, p_2), (g, b_2)\Big)$$
$$= \nu_{A_1}\Big(q_1, f(q_2), p_1, g(q_2)\Big) \vee \nu_{A_2}(q_2, a_2, p_2, b_2),$$

$\forall (q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2,\ \forall a_2 \in \Sigma_2,\ \forall b_2 \in X_2$ and $\Sigma_1^{Q_2} = \{f : Q_2 \to \Sigma_1\}, X_1^{Q_2} = \{g : Q_2 \to X_1\}$.

Remark 3.2. (1) The intuitionistic restricted direct product of $\mathscr{A}_1$ and $\mathscr{A}_2$ is a special case of their intuitionistic cascade product $\Sigma_1 = \Sigma_2, X_1 = X_2$ and $\omega_x : Q_2 \times \Sigma_2 \to \Sigma_1,\ \omega_y : Q_2 \times X_2 \to X_1$ both are projection functions.

(2) The intuitionistic cascade product of $\mathscr{A}_1$ and $\mathscr{A}_2$ is a special case of their intuitionistic wreath product, when $\Sigma_1^{Q_2} = \{\omega_x\}$ and $X_1^{Q_2} = \{\omega_y\}$.

Theorem 3.3. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be ifm, $i = 1, 2$ and let $X_1 \cap X_2 = \phi$ and $\Sigma_1 \cap \Sigma_2 = \phi$. Let $\mathscr{A}_1 \times \mathscr{A}_2$ be the intuitionistic Cartesian product of $\mathscr{A}_1$ and $\mathscr{A}_2$. Then $\forall x \in \Sigma_1^* \cup \Sigma_2^*,\ \ x \neq \lambda,\ \forall y \in X_1^* \cup X_2^*,\ y \neq \lambda$ and $A_1 \times A_2 = (\mu_{A_1} \times \mu_{A_2}, \nu_{A_1} \times$*

$\nu_{A_2})$, *where*

$$(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), x, (p_1, p_2), y\Big) = \begin{cases} \mu^*_{A_1}(q_1, x, p_1, y), & \textit{if } x \in \Sigma_1^*, y \in X_1^*, \\ & \textit{and } \ q_2 = p_2 \\ \mu^*_{A_2}(q_2, x, p_2, y), & \textit{if } x \in \Sigma_2^*, y \in X_2^*, \\ & \textit{and } \ q_1 = p_1 \\ 0, \textit{otherwise} \end{cases}$$

and

$$(\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), x, (p_1, p_2), y\Big) = \begin{cases} \nu^*_{A_1}(q_1, x, p_1, y), & \textit{if } x \in \Sigma_1^*, y \in X_1^*, \\ & \textit{and } \ q_2 = p_2 \\ \nu^*_{A_2}(q_2, x, p_2, y), & \textit{if } x \in \Sigma_2^*, y \in X_2^*, \\ & \textit{and } \ q_1 = p_1 \\ 0, \textit{otherwise} \end{cases}$$

$\forall (p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2$.

Proof. Let $x \in \Sigma_1^* \cup \Sigma_2^*$, $x \neq \lambda$, $y \in X_1^* \cup X_2^*$, $y \neq \lambda$ and $|x| = |y| = n$. Let $x \in \Sigma_1^*$ and $y \in X_1^*$. If $n = 1$, then the result is true. Suppose the result is true $\forall u \in \Sigma_1^*$, $|u| = n-1, n > 1$ and $\forall v \in X_1^*$, $|v| = n-1, n > 1$. Let $x = au$, where $a \in \Sigma_1$ and $u \in \Sigma_1^*$ and $y = bv$, where $b \in X_1$ and $v \in X_1^*$. Then,

$$\begin{aligned} & (\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), au, (p_1, p_2), bv\Big) \\ = & \vee\Big\{(\mu_{A_1} \times \mu_{A_2})\Big((q_1, q_2), a, (r_1, r_2), b\Big) \wedge \\ & (\mu_{A_1} \times \mu_{A_2})^*\Big((r_1, r_2), u, (p_1, p_2), v\Big) \mid (r_1, r_2) \in Q_1 \times Q_2\Big\} \\ & (\text{Since } \forall a \in \Sigma_1 \text{ and } u \in \Sigma_1^*, b \in X_1 \text{ and } v \in X_1^*) \\ = & \begin{cases} \vee\{\mu_{A_1}(q_1, a, r_1, b) \wedge \mu^*_{A_1}(r_1, u, p_1, v) \mid r_1 \in Q_1\}, \text{if } q_2 = p_2 \\ 0, \text{otherwise} \end{cases} \\ = & \begin{cases} \mu^*_{A_1}(q_1, au, p_1, bv), \text{if } q_2 = p_2 \\ 0, \text{otherwise} \end{cases} \\ = & \begin{cases} \mu^*_{A_1}(q_1, x, p_1, y), \text{if } q_2 = p_2 \\ 0, \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} & (\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), au, (p_1, p_2), bv\Big) \\ = & \wedge\Big\{(\nu_{A_1} \times \nu_{A_2})\Big((q_1, q_2), a, (r_1, r_2), b\Big) \vee \\ & (\nu_{A_1} \times \nu_{A_2})^*\Big((r_1, r_2), u, (p_1, p_2), v\Big) \mid (r_1, r_2) \in Q_1 \times Q_2\Big\} \\ & (\text{Since } \forall a \in \Sigma_1 \text{ and } u \in \Sigma_1^*, b \in X_1 \text{ and } v \in X_1^*) \end{aligned}$$

$$= \begin{cases} \wedge\{\nu_{A_1}(q_1, a, r_1, b) \vee \nu^*_{A_1}(r_1, u, p_1, v) \mid r_1 \in Q_1\}, \text{if } q_2 = p_2 \\ 0, \text{otherwise} \end{cases}$$

$$= \begin{cases} \nu^*_{A_1}(q_1, au, p_1, bv), \text{if } q_2 = p_2 \\ 0, \text{otherwise} \end{cases}$$

$$= \begin{cases} \nu^*_{A_1}(q_1, x, p_1, y), \text{if } q_2 = p_2 \\ 0, \text{otherwise.} \end{cases}$$

Thus the result follows by induction. Similarly, if $x \in \Sigma_2^*$ and $y \in X_2^*$, then one can prove the other case. □

Theorem 3.4. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be ifm, $i = 1, 2$. Let $\mathscr{A}_1 \times \mathscr{A}_2$ be the intuitionistic full direct product of ifm. Then $\forall x_1 \in \Sigma_1^*, x_2 \in \Sigma_2^*$, $x_1, x_2 \neq \lambda$, $\forall y_1 \in X_1^*, y_2 \in X_2^*, y_1, y_2 \neq \lambda$,*

$$(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), (x_1, x_2), (p_1, p_2), (y_1, y_2)\Big)$$
$$= \mu^*_{A_1}(q_1, x_1, p_1, y_1) \wedge \mu^*_{A_2}(q_2, x_2, p_2, y_2)$$

and

$$(\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), (x_1, x_2), (p_1, p_2), (y_1, y_2)\Big)$$
$$= \nu^*_{A_1}(q_1, x_1, p_1, y_1) \vee \nu^*_{A_2}(q_2, x_2, p_2, y_2).$$

Proof. Let $x_1 \in \Sigma_1^*, x_2 \in \Sigma_2^*, x_1, x_2 \neq \lambda$, $y_1 \in X_1^*, y_2 \in X_2^*, y_1, y_2 \neq \lambda$ and $|x_1| = |x_2| = n = |y_1| = |y_2|$. Then Clearly, the result is true for $n = 1$.

Suppose the result is true for all $u_1 \in \Sigma_1^*$, $u_2 \in \Sigma_2^*$; $|u_1| = n - 1 = |u_2|$, $n > 1$ and for all $v_1 \in X_1^*$, $v_2 \in X_2^*$; $|v_1| = n - 1 = |v_2|$, $n > 1$. Let $x_1 = a_1u_1$, $x_2 = a_2u_2$, where $a_1 \in \Sigma_1$, $a_2 \in \Sigma_2$ and $u_1 \in \Sigma_1^*$, $u_2 \in \Sigma_2^*$ and $y_1 = b_1v_1$, $y_2 = b_2v_2$, where $b_1 \in X_1$, $b_2 \in X_2$ and $v_1 \in X_1^*$, $v_2 \in X_2^*$. Then,

$$(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), (a_1u_1, a_2u_2), (p_1, p_2), (b_1v_1, b_2v_2)\Big)$$
$$= (\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), a_1u_1, (p_1, p_2), b_1v_1\Big) \wedge$$
$$(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), a_2u_2, (p_1, p_2), b_2v_2\Big)$$
$$= \Big\{ \vee \{(\mu_{A_1} \times \mu_{A_2})\Big((q_1, q_2), a_1, (r_1, r_2), b_1\Big) \wedge$$
$$(\mu_{A_1} \times \mu_{A_2})^*\Big((r_1, r_2), u_1, (p_1, p_2), v_1\Big) \mid (r_1, r_2) \in Q_1 \times Q_2\}\Big\} \wedge$$
$$\Big\{ \vee \{(\mu_{A_1} \times \mu_{A_2})\Big((q_1, q_2), a_2, (s_1, s_2), b_2\Big) \wedge$$
$$(\mu_{A_1} \times \mu_{A_2})^*\Big((s_1, s_2), u_2, (p_1, p_2), v_2\Big) \mid (s_1, s_2) \in Q_1 \times Q_2\}\Big\}$$
$$= \Big\{ \vee \{\mu_{A_1}(q_1, a_1, r_1, b_1) \wedge \mu^*_{A_1}(r_1, u_1, p_1, v_1) \mid r_1 \in Q_1\}\Big\} \wedge$$
$$\Big\{ \vee \{\mu_{A_2}(q_2, a_2, s_2, b_2) \wedge \mu^*_{A_2}(s_2, u_2, p_2, v_2) \mid s_2 \in Q_2\}\Big\}$$

$$= \mu^*_{A_1}(q_1, a_1u_1, p_1, b_1v_1) \wedge \mu^*_{A_2}(q_2, a_2u_2, p_2, b_2v_2)$$
$$= \mu^*_{A_1}(q_1, x_1, p_1, y_1) \wedge \mu^*_{A_2}(q_2, x_2, p_2, y_2)$$

and

$$\begin{aligned}
&(\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), (a_1u_1, a_2u_2), (p_1, p_2), (b_1v_1, b_2v_2)\Big)\\
&= (\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), a_1u_1, (p_1, p_2), b_1v_1\Big) \vee\\
&\quad (\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), a_2u_2, (p_1, p_2), b_2v_2\Big)\\
&= \Big\{ \wedge \{(\nu_{A_1} \times \nu_{A_2})\Big((q_1, q_2), a_1, (r_1, r_2), b_1\Big) \vee\\
&\quad (\nu_{A_1} \times \nu_{A_2})^*\Big((r_1, r_2), u_1, (p_1, p_2), v_1\Big) \mid (r_1, r_2) \in Q_1 \times Q_2\}\Big\} \vee\\
&\quad \Big\{ \wedge \{(\nu_{A_1} \times \nu_{A_2})\Big((q_1, q_2), a_2, (s_1, s_2), b_2\Big) \vee\\
&\quad (\nu_{A_1} \times \nu_{A_2})^*\Big((s_1, s_2), u_2, (p_1, p_2), v_2\Big) \mid (s_1, s_2) \in Q_1 \times Q_2\}\Big\}\\
&= \Big\{ \wedge \{\nu_{A_1}(q_1, a_1, r_1, b_1) \vee \nu^*_{A_1}(r_1, u_1, p_1, v_1) \mid r_1 \in Q_1\}\Big\} \vee\\
&\qquad \Big\{ \wedge \{\nu_{A_2}(q_2, a_2, s_2, b_2) \vee \nu^*_{A_2}(s_2, u_2, p_2, v_2) \mid s_2 \in Q_2\}\Big\}\\
&= \nu^*_{A_1}(q_1, a_1u_1, p_1, b_1v_1) \vee \nu^*_{A_2}(q_2, a_2u_2, p_2, b_2v_2)\\
&= \nu^*_{A_1}(q_1, x_1, p_1, y_1) \vee \nu^*_{A_2}(q_2, x_2, p_2, y_2).
\end{aligned}$$

□

Theorem 3.5. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be ifm, $i = 1, 2$. Let $\mathscr{A}_1 \times \mathscr{A}_2$ be the intuitionistic restricted direct product of ifm. Then $\forall (q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$, $\forall x \in \Sigma^*$, $\forall y \in X^*$,*

$$(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), x, (p_1, p_2), y\Big) = \mu^*_{A_1}(q_1, x, p_1, y) \wedge \mu^*_{A_2}(q_2, x, p_2, y)$$

and

$$(\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), x, (p_1, p_2), y\Big) = \nu^*_{A_1}(q_1, x, p_1, y) \vee \nu^*_{A_2}(q_2, x, p_2, y).$$

Proof. We prove the result by induction on $|x| = n = |y|$.

If $n = 1$, then the result is obvious. Suppose the result is true for all $x \in \Sigma^*$ and $y \in X^*$. Let $x = au$, where $a \in \Sigma, u \in \Sigma^*$ and let $y = bv$, where $b \in X, v \in X^*$; $|u| = n - 1 = |v|$, $n > 1$. Then

$$\begin{aligned}
&(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), x, (p_1, p_2), y\Big)\\
&= (\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), au, (p_1, p_2), bv\Big)\\
&= \vee\Big\{(\mu_{A_1} \times \mu_{A_2})\Big((q_1, q_2), a, (r_1, r_2), b\Big) \wedge\\
&\qquad (\mu_{A_1} \times \mu_{A_2})^*\Big((r_1, r_2), u, (p_1, p_2), v\Big) \mid (r_1, r_2) \in Q_1 \times Q_2\Big\}
\end{aligned}$$

$$= \vee\{\mu_{A_1}(q_1, a, r_1, b) \wedge \mu_{A_2}(q_2, a, r_2, b) \wedge \mu^*_{A_1}(r_1, u, p_1, v) \wedge \mu^*_{A_2}(r_2, u, p_2, v) \mid (r_1, r_2) \in Q_1 \times Q_2\}$$
$$= \mu^*_{A_1}(q_1, au, p_1, bv) \wedge \mu^*_{A_2}(q_2, au, p_2, bv)$$
$$= \mu^*_{A_1}(q_1, x, p_1, y) \wedge \mu^*_{A_2}(q_2, x, p_2, y)$$

and

$$\begin{aligned}
&(\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), x, (p_1, p_2), y\Big)\\
&= (\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), au, (p_1, p_2), bv\Big)\\
&= \wedge\Big\{(\nu_{A_1} \times \nu_{A_2})\Big((q_1, q_2), a, (r_1, r_2), b\Big)\vee\\
&\quad (\nu_{A_1} \times \nu_{A_2})^*\Big((r_1, r_2), u, (p_1, p_2), v\Big) \mid (r_1, r_2) \in Q_1 \times Q_2\Big\}\\
&= \wedge\{\nu_{A_1}(q_1, a, r_1, b) \vee \nu_{A_2}(q_2, a, r_2, b) \vee \nu^*_{A_1}(r_1, u, p_1, v)\vee\\
&\quad \nu^*_{A_2}(r_2, u, p_2, v) \mid (r_1, r_2) \in Q_1 \times Q_2\}\\
&= \nu^*_{A_1}(q_1, au, p_1, bv) \vee \nu^*_{A_2}(q_2, au, p_2, bv)\\
&= \nu^*_{A_1}(q_1, x, p_1, y) \vee \nu^*_{A_2}(q_2, x, p_2, y).
\end{aligned}$$

This completes the proof. □

Theorem 3.6. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be ifm, $i = 1, 2$. Let $\omega_x : Q_2 \times \Sigma_2 \to \Sigma_1$ and $\omega_y : Q_2 \times X_2 \to X_1$. Let $\mathscr{A}_1 \times \mathscr{A}_2$ be the intuitionistic cascade product of ifm. Then $\forall (q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$ and $\forall x = x_1 \cdots x_n \in \Sigma_2^*, y = y_1 \cdots y_n \in X_2^*$,*

$$\begin{aligned}
&(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), x, (p_1, p_2), y\Big)\\
&= \vee\Big\{\mu^*_{A_1}\Big(q_1, \omega_x(q_2, x_1)\omega_x(q_2^{(1)}, x_2) \cdots \omega_x(q_2^{(n-1)}, x_n), p_1,\\
&\quad \omega_y(q_2, y_1)\omega_y(q_2^{(1)}, y_2) \cdots \omega_y(q_2^{(n-1)}, y_n)\Big)\\
&\quad \wedge \mu^*_{A_2}(q_2, x_1, q_2^{(1)}, y_1) \wedge \mu^*_{A_2}(q_2^{(1)}, x_2, q_2^{(2)}, y_2)\wedge\\
&\quad \cdots \wedge \mu^*_{A_2}(q_2^{(n-1)}, x_n, p_2, y_n) \mid q_2^{(i)} \in Q_2, i = 1, 2, \cdots, n-1\Big\}
\end{aligned}$$

and

$$\begin{aligned}
&(\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), x, (p_1, p_2), y\Big)\\
&= \wedge\Big\{\nu^*_{A_1}\Big(q_1, \omega_x(q_2, x_1)\omega_x(q_2^{(1)}, x_2) \cdots \omega_x(q_2^{(n-1)}, x_n), p_1,\\
&\quad \omega_y(q_2, y_1)\omega_y(q_2^{(1)}, y_2) \cdots \omega_y(q_2^{(n-1)}, y_n)\Big)\\
&\quad \vee \nu^*_{A_2}(q_2, x_1, q_2^{(1)}, y_1) \vee \nu^*_{A_2}(q_2^{(1)}, x_2, q_2^{(2)}, y_2)\wedge\\
&\quad \cdots \vee \nu^*_{A_2}(q_2^{(n-1)}, x_n, p_2, y_n) \mid q_2^{(i)} \in Q_2, i = 1, 2, \cdots, n-1\Big\}.
\end{aligned}$$

Proof. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$ and let $x = x_1 \cdots x_n \in \Sigma_2^*,\ y = y_1 \cdots y_n \in X_2^*$. Then

$$\begin{aligned}
&(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), x, (p_1, p_2), y\Big) \\
&= \vee\Big\{(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), x_1, (q_1^{(1)}, q_2^{(1)}), y_1\Big) \\
&\quad \wedge (\mu_{A_1} \times \mu_{A_2})^*\Big((q_1^{(1)}, q_2^{(1)}), x_2, (q_1^{(2)}, q_2^{(2)}), y_2\Big) \\
&\quad \wedge \cdots \wedge (\mu_{A_1} \times \mu_{A_2})^*\Big((q_1^{(n-1)}, q_2^{(n-1)}), x_n, (p_1, p_2), y_n\Big) \mid (q_1^{(1)}, q_2^{(1)}), \\
&\qquad\qquad \cdots, (q_1^{(n-1)}, q_2^{(n-1)}) \in Q_1 \times Q_2\Big\} \\
&= \vee\Big\{\mu_{A_1}^*\Big(q_1, \omega_x(q_2, x_1), q_1^{(1)}, \omega_y(q_2, y_1)\Big) \wedge \mu_{A_2}^*(q_2, x_1, q_2^{(1)}, y_1) \wedge \\
&\quad \mu_{A_1}^*\Big(q_1^{(1)}, \omega_x(q_2^{(1)}, x_2), q_1^{(2)}, \omega_y(q_2^{(1)}, y_2)\Big) \wedge \mu_{A_2}^*(q_2^{(1)}, x_2, q_2^{(2)}, y_2) \\
&\quad \wedge \cdots \wedge \mu_{A_1}^*\Big(q_1^{(n-1)}, \omega_x(q_2^{(n-1)}, x_n), p_1, \omega_y(q_2^{(n-1)}, y_n)\Big) \wedge \\
&\quad \mu_{A_2}^*(q_2^{(n-1)}, x_n, p_2, y_n) \mid (q_1^{(1)}, q_2^{(1)}), \cdots, (q_1^{(n-1)}, q_2^{(n-1)}) \in Q_1 \times Q_2\Big\} \\
&= \vee\Big\{\mu_{A_1}^*\Big(q_1, \omega_x(q_2, x_1), q_1^{(1)}, \omega_y(q_2, y_1)\Big) \wedge \mu_{A_1}^*\Big(q_1^{(1)}, \omega_x(q_2^{(1)}, x_2), q_1^{(2)}, \omega_y(q_2^{(1)}, y_2)\Big) \wedge \\
&\quad \cdots \wedge \mu_{A_1}^*\Big(q_1^{(n-1)}, \omega_x(q_2^{(n-1)}, x_n), p_1, \omega_y(q_2^{(n-1)}, y_n)\Big) \wedge \\
&\quad \mu_{A_2}^*(q_2, x_1, q_2^{(1)}, y_1) \wedge \mu_{A_2}^*(q_2^{(1)}, x_2, q_2^{(2)}, y_2) \wedge \cdots \wedge \\
&\quad \mu_{A_2}^*(q_2^{(n-1)}, x_n, p_2, y_n) \mid (q_1^{(1)}, q_2^{(1)}), \cdots, (q_1^{(n-1)}, q_2^{(n-1)}) \in Q_1 \times Q_2\Big\} \\
&= \vee\Big\{\mu_{A_1}^*\Big(q_1, \omega_x(q_2, x_1)\omega_x(q_2^{(1)}, x_2), \cdots, \\
&\quad \omega_x(q_2^{(n-1)}, x_n), p_1, \omega_y(q_2, y_1)\omega_y(q_2^{(1)}, y_2), \cdots, \omega_y(q_2^{(n-1)}, y_n)\Big) \wedge \\
&\quad \mu_{A_2}^*(q_2, x_1, q_2^{(1)}, y_1) \wedge \mu_{A_2}^*(q_2^{(1)}, x_2, q_2^{(2)}, y_2) \wedge \cdots \\
&\quad \wedge \mu_{A_2}^*(q_2^{(n-1)}, x_n, p_2, y_n) \mid q_2^{(i)} \in Q_2, i = 1, 2, \cdots, n-1\Big\}
\end{aligned}$$

and

$$\begin{aligned}
&(\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), x, (p_1, p_2), y\Big) \\
&= \wedge\Big\{(\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), x_1, (q_1^{(1)}, q_2^{(1)}), y_1\Big) \vee \\
&\quad (\nu_{A_1} \times \nu_{A_2})^*\Big((q_1^{(1)}, q_2^{(1)}), x_2, (q_1^{(2)}, q_2^{(2)}), y_2\Big) \vee \\
&\quad \cdots \vee (\nu_{A_1} \times \nu_{A_2})^*\Big((q_1^{(n-1)}, q_2^{(n-1)}), x_n, (p_1, p_2), y_n\Big) \mid (q_1^{(1)}, q_2^{(1)}), \\
&\qquad\qquad \cdots, (q_1^{(n-1)}, q_2^{(n-1)}) \in Q_1 \times Q_2\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \wedge\Big\{\nu^*_{A_1}\Big(q_1,\omega_x(q_2,x_1),q_1^{(1)},\omega_y(q_2,y_1)\Big)\vee\nu^*_{A_2}(q_2,x_1,q_2^{(1)},y_1)\vee\\
&\quad \nu^*_{A_1}\Big(q_1^{(1)},\omega_x(q_2^{(1)},x_2),q_1^{(2)},\omega_y(q_2^{(1)},y_2)\Big)\vee\nu^*_{A_2}(q_2^{(1)},x_2,q_2^{(2)},y_2)\wedge\cdots\\
&\quad \vee\nu^*_{A_1}\Big(q_1^{(n-1)},\omega_x(q_2^{(n-1)},x_n),p_1,\omega_y(q_2^{(n-1)},y_n)\Big)\vee\\
&\quad \nu^*_{A_2}(q_2^{(n-1)},x_n,p_2,y_n)\mid(q_1^{(1)},q_2^{(1)}),\cdots,(q_1^{(n-1)},q_2^{(n-1)})\in Q_1\times Q_2\Big\}\\
&= \wedge\Big\{\nu^*_{A_1}\Big(q_1,\omega_x(q_2,x_1),q_1^{(1)},\omega_y(q_2,y_1)\Big)\vee\\
&\quad \nu^*_{A_1}\Big(q_1^{(1)},\omega_x(q_2^{(1)},x_2),q_1^{(2)},\omega_y(q_2^{(1)},y_2)\Big)\vee\\
&\quad \cdots\vee\nu^*_{A_1}\Big(q_1^{(n-1)},\omega_x(q_2^{(n-1)},x_n),p_1,\omega_y(q_2^{(n-1)},y_n)\Big)\vee\\
&\quad \nu^*_{A_2}(q_2,x_1,q_2^{(1)},y_1)\vee\nu^*_{A_2}(q_2^{(1)},x_2,q_2^{(2)},y_2)\vee\cdots\vee\\
&\quad \nu^*_{A_2}(q_2^{(n-1)},x_n,p_2,y_n)\mid(q_1^{(1)},q_2^{(1)}),\cdots,(q_1^{(n-1)},q_2^{(n-1)})\in Q_1\times Q_2\Big\}\\
&= \wedge\Big\{\nu^*_{A_1}\Big(q_1,\omega_x(q_2,x_1)\omega_x(q_2^{(1)},x_2)\cdots\omega_x(q_2^{(n-1)},x_n),p_1,\\
&\quad \omega_y(q_2,y_1)\omega_y(q_2^{(1)},y_2)\cdots\omega_y(q_2^{(n-1)},y_n)\Big)\vee\\
&\quad \nu^*_{A_2}(q_2,x_1,q_2^{(1)},y_1)\vee\nu^*_{A_2}(q_2^{(1)},x_2,q_2^{(2)},y_2)\vee\\
&\quad \cdots\vee\nu^*_{A_2}(q_2^{(n-1)},x_n,p_2,y_n)\mid q_2^{(i)}\in Q_2, i=1,2,\cdots,n-1\Big\}.
\end{aligned}
$$

□

Definition 3.7 ([11])**.** Let $\mathscr{A}=(Q,\Sigma,X,A)$ be an ifm. $\mathscr{A}$ is singly generated or cyclic if $\exists\ q\in Q$ such that $\mathscr{A}=\langle\{q\}\rangle$. In this case q is called a generator of $\mathscr{A}$ and we say that $\mathscr{A}$ is generated by q. Hence $\mathscr{A}$ is singly generated by $q\in Q$ iff $\mathscr{A}=\langle\{q\}\rangle$.

Definition 3.8 ([11])**.** Let $\mathscr{A}=(Q,\Sigma,X,A)$ be an ifm. Then $\mathscr{A}$ is called strongly connected if $\forall p,q\in Q,\ p\in S(q)$.

Theorem 3.9. *Let $\mathscr{A}_i=(Q_i,\Sigma_i,X_i,A_i)$ be ifm, $i=1,2$. Then intuitionistic Cartesian product of ifm $\mathscr{A}_1\times\mathscr{A}_2$ is cyclic iff $\mathscr{A}_1$ and $\mathscr{A}_2$ are cyclic.*

Proof. Let $\times$ be intuitionistic Cartesian product. Suppose $\mathscr{A}_1$ and $\mathscr{A}_2$ are cyclic, say $Q_1=S(q_1)$ and $Q_2=S(q_2)$ for some $q_1\in Q_1$ and $q_2\in Q_2$. Let $(p_1,p_2)\in Q_1\times Q_2$. Then there exists $x_1\in\Sigma_1^*, x_2\in\Sigma_2^*, y_1\in X_1^*, y_2\in X_2^*$ such that

$$\mu^*_{A_1}(q_1,x_1,p_1,y_1)>0,\ \mu^*_{A_2}(q_2,x_2,p_2,y_2)>0$$

and

$$\nu^*_{A_1}(q_1,x_1,p_1,y_1)<1,\ \nu^*_{A_2}(q_2,x_2,p_2,y_2)<1.$$

Thus

$$
\begin{aligned}
&(\mu_{A_1}\times\mu_{A_2})^*\Big((q_1,q_2),x_1x_2,(p_1,p_2),y_1y_2\Big)\\
&=\mu^*_{A_1}(q_1,x_1,p_1,y_1)\wedge\mu^*_{A_2}(q_2,x_2,p_2,y_2)>0
\end{aligned}
$$

and

$$(\nu_{A_1} \times \nu_{A_2})^* \Big((q_1, q_2), x_1x_2, (p_1, p_2), y_1y_2\Big)$$
$$= \nu^*_{A_1}(q_1, x_1, p_1, y_1) \vee \nu^*_{A_2}(q_2, x_2, p_2, y_2) < 1.$$

That is, $(p_1, p_2) \in S(q_1, q_2)$. So, $Q_1 \times Q_2 = S((q_1, q_2))$. Hence $\mathscr{A}_1 \times \mathscr{A}_2$ is cyclic.

Conversely, suppose $\mathscr{A}_1 \times \mathscr{A}_2$ is cyclic. Let $Q_1 \times Q_2 = S((q_1, q_2))$, for some $(q_1, q_2) \in Q_1 \times Q_2$. Let $p_1 \in Q_1$ and $p_2 \in Q_2$. Then $\exists x \in (\Sigma_1 \cup \Sigma_2)^*$ and $\exists y \in (X_1 \cup X_2)^*$ such that

$$(\mu_{A_1} \times \mu_{A_2})^* \Big((q_1, q_2), x, (p_1, p_2), y\Big) > 0$$

and

$$(\nu_{A_1} \times \nu_{A_2})^* \Big((q_1, q_2), x, (p_1, p_2), y\Big) < 1.$$

Thus by theorem 3.3, $\exists x_1 \in \Sigma_1^*,\ x_2 \in \Sigma_2^*$, and $\exists y_1 \in X_1^*,\ y_2 \in X_2^*$ such that

$$\mu^*_{A_1}(q_1, x_1, p_1, y_1) \wedge \mu^*_{A_2}(q_2, x_2, p_2, y_2)$$
$$= (\mu_{A_1} \times \mu_{A_2})^* \Big((q_1, q_2), x, (p_1, p_2), y\Big) > 0$$

and

$$\nu^*_{A_1}(q_1, x_1, p_1, y_1) \vee \nu^*_{A_2}(q_2, x_2, p_2, y_2)$$
$$= (\nu_{A_1} \times \nu_{A_2})^* \Big((q_1, q_2), x, (p_1, p_2), y\Big) < 1.$$

That is, $\exists x_1 \in \Sigma_1^*,\ x_2 \in \Sigma_2^*$ and $y_1 \in X_1^*,\ y_2 \in X_2^*$ such that
$\mu^*_{A_1}(q_1, x_1, p_1, y_1) > 0,\ \mu^*_{A_2}(q_2, x_2, p_2, y_2) > 0$
and
$\nu^*_{A_1}(q_1, x_1, p_1, y_1) < 1,\ \nu^*_{A_2}(q_2, x_2, p_2, y_2) < 1$.
Thus, $p_1 \in S(q_1)$ and $p_2 \in S(q_2)$. So, $Q_1 = S(q_1)$ and $Q_2 = S(q_2)$. Hence $\mathscr{A}_1$ and $\mathscr{A}_2$ are cyclic. □

Theorem 3.10. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be ifm, $i = 1, 2$. Then intuitionistic full direct product of ifm $\mathscr{A}_1 \times \mathscr{A}_2$ is cyclic iff $\mathscr{A}_1$ and $\mathscr{A}_2$ are cyclic.*

Proof. Let $\times$ be intuitionistic full direct product. Suppose $\mathscr{A}_1$ and $\mathscr{A}_2$ are cyclic, say $Q_1 = S(q_1)$ and $Q_2 = S(q_2)$ for some $q_1 \in Q_1$ and $q_2 \in Q_2$. Let $(p_1, p_2) \in Q_1 \times Q_2$. Then
$\mu^*_{A_1}(q_1, x_1, p_1, y_1) > 0, \mu^*_{A_2}(q_2, x_2, p_2, y_2) > 0$
and
$\nu^*_{A_1}(q_1, x_1, p_1, y_1) < 1,\ \nu^*_{A_2}(q_2, x_2, p_2, y_2) < 1.$
Thus

$$(\mu_{A_1} \times \mu_{A_2})^* \Big((q_1, q_2), (x_1, x_2), (p_1, p_2), (y_1, y_2)\Big)$$
$$= \mu^*_{A_1}(q_1, x_1, p_1, y_1) \wedge \mu^*_{A_2}(q_2, x_2, p_2, y_2) > 0$$

and

$$(\nu_{A_1} \times \nu_{A_2})^* \Big((q_1, q_2), (x_1, x_2), (p_1, p_2), (y_1, y_2)\Big)$$
$$= \nu^*_{A_1}(q_1, x_1, p_1, y_1) \vee \nu^*_{A_2}(q_2, x_2, p_2, y_2) < 1.$$

That is, $(p_1, p_2) \in S((q_1, q_2))$. So, $Q_1 \times Q_2 = S(q_1, q_2)$. Hence $\mathscr{A}_1 \times \mathscr{A}_2$ is cyclic.

Conversely, suppose $\mathscr{A}_1 \times \mathscr{A}_2$ is cyclic. Let $Q_1 \times Q_2 = S((q_1, q_2))$, for some $(q_1, q_2) \in Q_1 \times Q_2$. Let $p_1 \in Q_1$ and $p_2 \in Q_2$. Then

$$(\mu_{A_1} \times \mu_{A_2})^* \Big((q_1, q_2), (x_1, x_2), (p_1, p_2), (y_1, y_2)\Big)$$
$$= \mu^*_{A_1}(q_1, x_1, p_1, y_1) \wedge \mu^*_{A_2}(q_2, x_2, p_2, y_2) > 0$$

and

$$(\nu_{A_1} \times \nu_{A_2})^* \Big((q_1, q_2), (x_1, x_2), (p_1, p_2), (y_1, y_2)\Big)$$
$$= \nu^*_{A_1}(q_1, x_1, p_1, y_1) \vee \nu^*_{A_2}(q_2, x_2, p_2, y_2) < 1.$$

This implies that
$$\mu^*_{A_1}(q_1, x_1, p_1, y_1) > 0,\ \mu^*_{A_2}(q_2, x_2, p_2, y_2) > 0$$
and
$$\nu^*_{A_1}(q_1, x_1, p_1, y_1) < 1,\ \nu^*_{A_2}(q_2, x_2, p_2, y_2) < 1.$$
Thus, $p_1 \in S(q_1)$ and $p_2 \in S(q_2)$. So, $Q_1 = S(q_1)$ for some $q_1 \in Q_1$ and $Q_2 = S(q_2)$ for some $q_2 \in Q_2$. Hence $\mathscr{A}_1$ and $\mathscr{A}_2$ are cyclic. □

Theorem 3.11. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be ifm, $i = 1, 2$. If intuitionistic restricted direct product of ifm $\mathscr{A}_1 \times \mathscr{A}_2$ is cyclic, then $\mathscr{A}_1$ and $\mathscr{A}_2$ are cyclic.*

Proof. Let $\times$ be intuitionistic restricted direct product. Suppose $\mathscr{A}_1 \times \mathscr{A}_2$ is cyclic. Let $Q_1 \times Q_2 = S((q_1, q_2))$, for some $(q_1, q_2) \in Q_1 \times Q_2$. Let $p_1 \in Q_1$ and $p_2 \in Q_2$. Then

$$(\mu_{A_1} \times \mu_{A_2})^* \Big((q_1, q_2), x, (p_1, p_2), y\Big)$$
$$= \mu^*_{A_1}(q_1, x, p_1, y) \wedge \mu^*_{A_2}(q_2, x, p_2, y) > 0$$

and

$$(\nu_{A_1} \times \nu_{A_2})^* \Big((q_1, q_2), x, (p_1, p_2), y\Big)$$
$$= \nu^*_{A_1}(q_1, x, p_1, y) \vee \nu^*_{A_2}(q_2, x, p_2, y) < 1.$$

This implies that $\mu^*_{A_1}(q_1, x, p_1, y) > 0,\ \mu^*_{A_2}(q_2, x, p_2, y) > 0$
and
$$\nu^*_{A_1}(q_1, x, p_1, y) < 1,\ \nu^*_{A_2}(q_2, x, p_2, y) < 1.$$
Thus, $p_1 \in S(q_1)$ and $p_2 \in S(q_2)$. So, $Q_1 = S(q_1)$ for some $q_1 \in Q_1$ and $Q_2 = S(q_2)$ for some $q_2 \in Q_2$. Hence $\mathscr{A}_1$ and $\mathscr{A}_2$ are cyclic. □

Theorem 3.12. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be ifm, $i = 1, 2$. If intuitionistic cascade product of ifm $\mathscr{A}_1 \times \mathscr{A}_2$ is cyclic, then $\mathscr{A}_1$ and $\mathscr{A}_2$ are cyclic.*

Proof. Let $\times$ be intuitionistic cascade product. Suppose $\mathscr{A}_1 \times \mathscr{A}_2$ is cyclic. Let $Q_1 \times Q_2 = S((q_1, q_2))$, for some $(q_1, q_2) \in Q_1 \times Q_2$. Let $p_1 \in Q_1$ and $p_2 \in Q_2$. Then

$$(\mu_{A_1} \times \mu_{A_2})^* \Big((q_1, q_2), x_2, (p_1, p_2), y_2\Big)$$
$$= \mu^*_{A_1}\Big(q_1, \omega_x(q_2, x_2), p_1, \omega_y(q_2, y_2)\Big) \wedge \mu^*_{A_2}(q_2, x_2, p_2, y_2) > 0$$

and

$$(\nu_{A_1} \times \nu_{A_2})^* \Big((q_1, q_2), x_2, (p_1, p_2), y_2\Big)$$
$$= \nu^*_{A_1}\Big(q_1, \omega_x(q_2, x_2), p_1, \omega_y(q_2, y_2)\Big) \vee \nu^*_{A_2}(q_2, x_2, p_2, y_2) < 1.$$

This implies that

$$\mu^*_{A_1}\Big(q_1, \omega_x(q_2, x_2), p_1, \omega_y(q_2, y_2)\Big) > 0,\ \mu^*_{A_2}(q_2, x_2, p_2, y_2) > 0$$

and

$\nu^*_{A_1}\Big(q_1, \omega_x(q_2, x_2), p_1, \omega_y(q_2, y_2)\Big) < 1,\ \nu^*_{A_2}(q_2, x_2, p_2, y_2) < 1$. Thus, $p_1 \in S(q_1)$ and $p_2 \in S(q_2)$. So, $Q_1 = S(q_1)$, for some $q_1 \in Q_1$ and $Q_2 = S(q_2)$, for some $q_2 \in Q_2$. Hence $\mathscr{A}_1$ and $\mathscr{A}_2$ are cyclic. □

Theorem 3.13. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be ifm, $i = 1, 2$. If intuitionistic wreath product of ifm $\mathscr{A}_1 \times \mathscr{A}_2$ is cyclic, then $\mathscr{A}_1$ and $\mathscr{A}_2$ are cyclic.*

Proof. Let $\times$ be intuitionistic wreath product. Suppose $\mathscr{A}_1 \times \mathscr{A}_2$ is cyclic. Let $Q_1 \times Q_2 = S((q_1, q_2))$,for some $(q_1, q_2) \in Q_1 \times Q_2$. Let $p_1 \in S(q_1)$ and $p_2 \in S(q_2)$. Then

$$(\mu_{A_1} \times \mu_{A_2})^* \Big((q_1, q_2), (f, x_2), (p_1, p_2), (g, y_2)\Big)$$
$$= \mu^*_{A_1}\Big(q_1, f(q_2), p_1, g(q_2)\Big) \wedge \mu^*_{A_2}(q_2, x_2, p_2, y_2) > 0$$

and

$$(\nu_{A_1} \times \nu_{A_2})^* \Big((q_1, q_2), (f, x_2), (p_1, p_2), (g, y_2)\Big)$$
$$= \nu^*_{A_1}\Big(q_1, f(q_2), p_1, g(q_2)\Big) \vee \nu^*_{A_2}(q_2, x_2, p_2, y_2) < 1.$$

This implies that

$$\mu^*_{A_1}\Big(q_1, f(q_2), p_1, g(q_2)\Big) > 0, \mu^*_{A_2}(q_2, x_2, p_2, y_2) > 0$$

and $\nu^*_{A_1}\Big(q_1, f(q_2), p_1, g(q_2)\Big) < 1,\ \nu^*_{A_2}(q_2, x_2, p_2, y_2) < 1.$

Thus, $p_1 \in S(q_1)$ and $p_2 \in S(q_2)$. So, $Q_1 = S(q_1)$, for some $q_1 \in Q_1$ and $Q_2 = S(q_2)$, for some $q_2 \in Q_2$. Hence $\mathscr{A}_1$ and $\mathscr{A}_2$ are cyclic. □

Note 3.14. The converse of Theorems 3.11 to 3.13 is true when individual ifm are strongly connected.

Theorem 3.15. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be a strongly connected ifm, $i = 1, 2$. Then intuitionistic restricted direct product of ifm $\mathscr{A}_1 \times \mathscr{A}_2$ is cyclic.*

Proof. Let $\times$ be intuitionistic restricted direct product. By theorem 3.11, $\mathscr{A}_1$ and $\mathscr{A}_2$ are cyclic, one has $Q_1 = S(q_1)$ and $Q_2 = S(q_2)$ for some $q_1 \in Q_1$ and $q_2 \in Q_2$. Let $(p_1, p_2) \in Q_1 \times Q_2$. Then
$\mu^*_{A_1}(q_1, x, p_1, y) > 0,\ \mu^*_{A_2}(q_2, x, p_2, y) > 0$
and $\nu^*_{A_1}(q_1, x, p_1, y) < 1,\ \nu^*_{A_2}(q_2, x, p_2, y) < 1.$
Thus,

$$(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), x, (p_1, p_2), y\Big)$$
$$= \mu^*_{A_1}(q_1, x, p_1, y) \wedge \mu^*_{A_2}(q_2, x, p_2, y) > 0$$

and

$$(\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), x, (p_1, p_2), y\Big)$$
$$= \nu^*_{A_1}(q_1, x, p_1, y) \vee \nu^*_{A_2}(q_2, x, p_2, y) < 1.$$

That is, $(p_1, p_2) \in S((q_1, q_2))$. So $Q_1 \times Q_2 = S(q_1, q_2)$. Hence $\mathscr{A}_1 \times \mathscr{A}_2$ is cyclic. □

Theorem 3.16. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be a strongly connected ifm, $i = 1, 2$. Then intuitionistic cascade product of ifm $\mathscr{A}_1 \times \mathscr{A}_2$ is cyclic.*

Proof. Let $\times$ be intuitionistic cascade product. Since $\mathscr{A}_1$ and $\mathscr{A}_2$ are cyclic, one has $Q_1 = S(q_1)$ and $Q_2 = S(q_2)$ for some $q_1 \in Q_1,\ q_2 \in Q_2$. Let $(p_1, p_2) \in Q_1 \times Q_2$. Then
$\mu^*_{A_1}\Big(q_1, \omega_x(q_2, x_2), p_1, \omega_y(q_2, y_2)\Big) > 0,\ \mu^*_{A_2}(q_2, x_2, p_2, y_2) > 0$
and
$\nu^*_{A_1}\Big(q_1, \omega_x(q_2, x_2), p_1, \omega_y(q_2, y_2)\Big) < 1,\ \nu^*_{A_2}(q_2, x_2, p_2, y_2) < 1.$
Thus

$$(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), x_2, (p_1, p_2), y_2\Big)$$
$$= \mu^*_{A_1}\Big(q_1, \omega_x(q_2, x_2), p_1, \omega_y(q_2, y_2)\Big) \wedge \mu^*_{A_2}(q_2, x_2, p_2, y_2) > 0$$

and

$$(\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), x_2, (p_1, p_2), y_2\Big)$$
$$= \nu^*_{A_1}\Big(q_1, \omega_x(q_2, x_2), p_1, \omega_y(q_2, y_2)\Big) \vee \nu^*_{A_2}(q_2, x_2, p_2, y_2) < 1.$$

That is, $(p_1, p_2) \in S((q_1, q_2))$. So $Q_1 \times Q_2 = S(q_1, q_2)$. Hence $\mathscr{A}_1 \times \mathscr{A}_2$ is cyclic. □

Theorem 3.17. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be a strongly connected ifm, $i = 1, 2$. Then intuitionistic wreath product of ifm $\mathscr{A}_1 \times \mathscr{A}_2$ is cyclic.*

Proof. Let $\times$ be intuitionistic wreath product. Since $\mathscr{A}_1$ and $\mathscr{A}_2$ are cyclic, one has $Q_1 = S(q_1)$ and $Q_2 = S(q_2)$ for some $q_1 \in Q_1,\ q_2 \in Q_2$. Let $(p_1, p_2) \in Q_1 \times Q_2$. Then
$\mu^*_{A_1}\Big(q_1, f(q_2), p_1, g(q_2)\Big) > 0,\ \mu^*_{A_2}(q_2, x_2, p_2, y_2) > 0$
and

$\nu^*_{A_1}\Big(q_1, f(q_2), p_1, g(q_2)\Big) < 1,\ \nu^*_{A_2}(q_2, x_2, p_2, y_2) < 1.$

Thus,

$$(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), (f, x_2), (p_1, p_2), (g, y_2)\Big)$$
$$= \mu^*_{A_1}\Big(q_1, f(q_2), p_1, g(q_2)\Big) \wedge \mu^*_{A_2}(q_2, x_2, p_2, y_2) > 0$$

and

$$(\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), (f, x_2), (p_1, p_2), (g, y_2)\Big)$$
$$= \nu^*_{A_1}\Big(q_1, f(q_2), p_1, g(q_2)\Big) \vee \nu^*_{A_2}(q_2, x_2, p_2, y_2) < 1.$$

That is, $(p_1, p_2) \in S((q_1, q_2))$. So $Q_1 \times Q_2 = S(q_1, q_2)$. Hence $\mathscr{A}_1 \times \mathscr{A}_2$ is cyclic. □

Definition 3.18. An ifm $\mathscr{A} = (Q, \Sigma, X, A)$ is said to be intuitionistic retrievable, if $\forall q \in Q,\ \forall x \in \Sigma^*,\ \forall y \in X^*,\ \exists p \in Q$ such that

$\mu^*_A(q, x, p, y) > 0$ and $\nu^*_A(q, x, p, y) < 1,$

then $\exists u \in \Sigma^*,\ v \in X^*$ such that

$\mu^*_A(p, u, q, v) > 0$ and $\nu^*_A(p, u, q, v) < 1.$

Theorem 3.19. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be ifm, $i = 1, 2$. Then intuitionistic Cartesian product of ifm $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic retrievable iff $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic retrievable.*

Proof. Let $\times$ be intuitionistic Cartesian product. Suppose $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic retrievable. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$ and $x \in (\Sigma_1 \cup \Sigma_2)^*,\ y \in (X_1 \cup X_2)^*$ be such that

$(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), x, (p_1, p_2), y\Big) > 0$

and

$(\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), x, (p_1, p_2), y\Big) < 1.$

Let $x_1^* = x_1x_2$ be the standard form of $x,\ x_1 \in \Sigma_1$ and $x_2 \in \Sigma_2$ and $y_1^* = y_1y_2$ be the standard form of $y,\ y_1 \in X_1$ and $y_2 \in X_2$. Then

$$(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), x, (p_1, p_2), y\Big)$$
$$= (\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), x_1x_2, (p_1, p_2), y_1y_2\Big)$$
$$= \mu^*_{A_1}(q_1, x_1, p_1, y_1) \wedge \mu^*_{A_2}(q_2, x_2, p_2, y_2)$$

and

$$(\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), x, (p_1, p_2), y\Big)$$
$$= (\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), x_1x_2, (p_1, p_2), y_1y_2\Big)$$
$$= \nu^*_{A_1}(q_1, x_1, p_1, y_1) \vee \nu^*_{A_2}(q_2, x_2, p_2, y_2).$$

Thus, $\mu^*_{A_1}(q_1, x_1, p_1, y_1) > 0,\ \mu^*_{A_2}(q_2, x_2, p_2, y_2) > 0$

and

$\nu^*_{A_1}(q_1, x_1, p_1, y_1) < 1,\ \nu^*_{A_2}(q_2, x_2, p_2, y_2) < 1.$

Since $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic retrievable, $\exists u_1 \in \Sigma_1^*,\ u_2 \in \Sigma_2^*,\ v_1 \in X_1^*, v_2 \in X_2^*$ such that

$\mu^*_{A_1}(p_1, u_1, q_1, v_1) > 0,\ \mu^*_{A_1}(p_2, u_2, q_2, v_2) > 0$
and
$\nu^*_{A_1}(p_1, u_1, q_1, v_1) < 1,\ \nu^*_{A_1}(p_2, u_2, q_2, v_2) < 1.$
So $(\mu_{A_1} \times \mu_{A_2})^*\Big((p_1, p_2), u_1u_2, (q_1, q_2), v_1v_2\Big) > 0$
and
$(\nu_{A_1} \times \nu_{A_2})^*\Big((p_1, p_2), u_1u_2, (q_1, q_2), v_1v_2\Big) < 1.$
That is, $(\mu_{A_1} \times \mu_{A_2})^*\Big((p_1, p_2), u, (q_1, q_2), v\Big) > 0$
and
$(\nu_{A_1} \times \nu_{A_2})^*\Big((p_1, p_2), u, (q_1, q_2), v\Big) < 1.$
Hence $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic retrievable.

Conversely, suppose $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic retrievable. Let $q_1, p_1 \in Q_1,\ x \in \Sigma_1^*$ and $y \in X_1^*$ be such that $\mu^*_{A_1}(q_1, x_1, p_1, y_1) > 0$ and $\nu^*_{A_1}(q_1, x_1, p_1, y_1) < 1$. Then $\forall q_2 \in Q_2$,
$(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), x_1, (p_1, q_2), y_1\Big) > 0$
and $(\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), x_1, (p_1, q_2), y_1\Big) < 1.$
Thus, $\exists u \in (\Sigma_1 \cup \Sigma_2)^*$ and $\exists v \in (X_1 \cup X_2)^*$ such that
$(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), u, (p_1, q_2), v\Big) > 0$
and
$(\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), u, (p_1, q_2), v\Big) < 1.$
Let $u = u_1u_2,\ v = v_1v_2$ be the standard form of u and v respectively, where $u_1 \in \Sigma_1^*,\ u_2 \in \Sigma_2^*$ and $v_1 \in X_1^*,\ v_2 \in X_2^*$. Then

$$\begin{aligned}&(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), u_1u_2, (p_1, q_2), v_1v_2\Big)\\ &= \mu_{A_1}(q_1, u_1, p_1, v_1) \wedge \mu_{A_2}(q_2, u_2, q_2, v_2)\end{aligned}$$

and

$$\begin{aligned}&(\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), u_1u_2, (p_1, q_2), v_1v_2\Big)\\ &= \nu_{A_1}(q_1, u_1, p_1, v_1) \vee \nu_{A_2}(q_2, u_2, q_2, v_2).\end{aligned}$$

Thus, $\mu_{A_1}(q_1, u_1, p_1, v_1) > 0$ and $\nu_{A_1}(q_1, u_1, p_1, v_1) < 1$. So $\mathscr{A}_1$ is intuitionistic retrievable. Similarly, $\mathscr{A}_2$ is intuitonistic retrievable □

Theorem 3.20. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be ifm, $i = 1, 2$. Then intuitionistic full direct product of ifm $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic retrievable iff $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic retrievable.*

Proof. Let $\times$ be intuitionistic full direct product. Suppose $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic retrievable. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$ and $(x_1, x_2) \in (\Sigma_1 \times \Sigma_2)^*,\ (y_1, y_2) \in (X_1 \times X_2)^*$ be such that

$$\begin{aligned}&(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), (x_1, x_2), (p_1, p_2), (y_1, y_2)\Big)\\ &= \mu^*_{A_1}(q_1, x_1, p_1, y_1) \wedge \mu^*_{A_2}(q_2, x_2, p_2, y_2) > 0\end{aligned}$$

and

$$(\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), (x_1, x_2), (p_1, p_2), (y_1, y_2)\Big)$$
$$= \nu^*_{A_1}(q_1, x_1, p_1, y_1) \vee \nu^*_{A_2}(q_2, x_2, p_2, y_2) < 1.$$

Since $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic retrievable. $\exists u_1 \in \Sigma_1^*,\ v_1 \in X_1^*$ and $\exists u_2 \in \Sigma_2^*,\ v_1 \in X_2^*$ such that

$\mu^*_{A_1}(p_1, u_1, q_1, v_1) > 0,\ \mu^*_{A_2}(p_2, u_2, q_2, v_2) > 0$

and

$\nu^*_{A_1}(p_1, u_1, q_1, v_1) < 1,\ \nu^*_{A_2}(p_2, u_2, q_2, v_2) < 1.$

Then,

$$\mu^*_{A_1}(p_1, u_1, q_1, v_1) \wedge \mu^*_{A_2}(p_2, u_2, q_2, v_2)$$
$$= (\mu_{A_1} \times \mu_{A_2})^*\Big((p_1, p_2), (u_1, u_2), (q_1, q_2), (v_1, v_2)\Big) > 0$$

and

$$\nu^*_{A_1}(p_1, u_1, q_1, v_1) \vee \nu^*_{A_2}(p_2, u_2, q_2, v_2)$$
$$= (\nu_{A_1} \times \nu_{A_2})^*\Big((p_1, p_2), (u_1, u_2), (q_1, q_2), (v_1, v_2)\Big) < 1.$$

Thus $\mathscr{A}_1 \times \mathscr{A}_2$ are intuitionistic retrievable.

Conversely, suppose $\mathscr{A}_1 \times \mathscr{A}_2$ are intuitionistic retrievable. Let $(q_1, q_2) \in Q_1 \times Q_2$ and $(x_1, x_2) \in (\Sigma_1 \times \Sigma_2)^*$ and $(y_1, y_2) \in (X_1 \times X_2)^*$, $\exists (p_1, p_2) \in Q_1 \times Q_2$ such that

$(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), (x_1, x_2), (p_1, p_2), (y_1, y_2)\Big) > 0$

and

$(\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), (x_1, x_2), (p_1, p_2), (y_1, y_2)\Big) < 1.$

Then $\exists (u_1, u_2) \in (\Sigma_1 \times \Sigma_2)^*,\ (v_1, v_2) \in (X_1 \times X_2)^*$ such that

$$(\mu_{A_1} \times \mu_{A_2})^*\Big((p_1, p_2), (u_1, u_2), (q_1, q_2), (v_1, v_2)\Big)$$
$$= \mu^*_{A_1}(p_1, u_1, q_1, v_1) \wedge \mu^*_{A_2}(p_2, u_2, q_2, v_2) > 0$$

and

$$(\nu_{A_1} \times \nu_{A_2})^*\Big((p_1, p_2), (u_1, u_2), (q_1, q_2), (v_1, v_2)\Big)$$
$$= \nu^*_{A_1}(p_1, u_1, q_1, v_1) \vee \nu^*_{A_2}(p_2, u_2, q_2, v_2) < 1.$$

Hence $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic retrievable □

Theorem 3.21. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be ifm, $i = 1, 2$. Then intuitionistic restricted direct product of ifm $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic retrievable then $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic retrievable.*

Proof. Let $\times$ be intuitionistic restricted direct product. Suppose $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic retrievable. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2,\ x \in \Sigma$ and $y \in X$ such that

$(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), x, (p_1, p_2), y\Big) > 0$

and

$(\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), x, (p_1, p_2), y\Big) < 1.$
Then $\exists u \in \Sigma^*,\ v \in X^*$ such that

$$(\mu_{A_1} \times \mu_{A_2})^*\Big((p_1, p_2), u, (q_1, q_2), v\Big)$$
$$= \mu^*_{A_1}(p_1, u, q_1, v) \wedge \mu^*_{A_2}(p_2, u, q_2, v) > 0$$

and

$$(\nu_{A_1} \times \nu_{A_2})^*\Big((p_1, p_2), u, (q_1, q_2), v\Big)$$
$$= \nu^*_{A_1}(p_1, u, q_1, v) \vee \nu^*_{A_2}(p_2, u, q_2, v) < 1.$$

Thus $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic retrievable. □

Theorem 3.22. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be ifm, $i = 1, 2$. If intuitionistic cascade product of ifm $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic retrievable then $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic retrievable.*

Proof. Let $\times$ be intuitionistic cascade product. Suppose $\mathscr{A}_1 \times \mathscr{A}_2$ are intuitionistic retrievable. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2,\ x_2 \in \Sigma_2$ and $y_2 \in X_2$ such that
$(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), x_2, (p_1, p_2), y_2\Big) > 0$
and
$(\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), x_2, (p_1, p_2), y_2\Big) < 1.$
Then $\exists u_2 \in \Sigma_2^*,\ v_2 \in X_2^*$ such that

$$(\mu_{A_1} \times \mu_{A_2})^*\Big((p_1, p_2), u_2, (q_1, q_2), v_2\Big)$$
$$= \mu^*_{A_1}\Big(p_1, \omega_u(p_2, u_2), q_1, \omega_v(p_2, v_2)\Big) \wedge \mu^*_{A_2}(p_2, u_2, q_2, v_2) > 0$$

and

$$(\nu_{A_1} \times \nu_{A_2})^*\Big((p_1, p_2), u_2, (q_1, q_2), v_2\Big)$$
$$= \nu^*_{A_1}\Big(p_1, \omega_u(p_2, u_2), q_1, \omega_v(p_2, v_2)\Big) \vee \nu^*_{A_2}(p_2, u_2, q_2, v_2) < 1.$$

thus $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic retrievable. □

Theorem 3.23. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be ifm, $i = 1, 2$. If intuitionistic wreath product of ifm $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic retrievable then $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic retrievable.*

Proof. Let $\times$ be intuitionistic wreath product. Suppose $\mathscr{A}_1 \times \mathscr{A}_2$ are intuitionistic retrievable. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2,\ x_2 \in \Sigma_2^*$ and $y_2 \in X_2^*$ such that
$(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), (f, x_2), (p_1, p_2), (g, y_2)\Big) > 0$
and

$$(\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), (f, x_2), (p_1, p_2), (g, y_2)\Big) < 1.$$

then $\exists u_2 \in \Sigma_2^*,\ v_2 \in X_2^*$ such that

$$(\mu_{A_1} \times \mu_{A_2})^*\Big((p_1, p_2), (f, u_2), (q_1, q_2), (g, v_2)\Big)$$
$$= \mu_{A_1}^*\Big(p_1, f(p_2), q_1, g(p_2)\Big) \wedge \mu_{A_2}^*(p_2, u_2, q_2, v_2) > 0$$

and

$$(\nu_{A_1} \times \nu_{A_2})^*\Big((p_1, p_2), (f, u_2), (q_1, q_2), (g, v_2)\Big)$$
$$= \nu_{A_1}^*\Big(p_1, f(p_2), q_1, g(p_2)\Big) \vee \nu_{A_2}^*(p_2, u_2, q_2, v_2) < 1.$$

Thus $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic retrievable. □

Note 3.24. The converse of Theorems 3.21 to 3.23 is true when ifm are strongly connected.

Theorem 3.25. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be a strongly connected ifm, $i = 1, 2$. Then intuitionistic restricted direct product of ifm $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic retrievable.*

Proof. Let $\times$ be intuitionistic restricted direct product. By that theorem 3.21, $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic retrievable. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2,\ x \in \Sigma^*,\ y \in X^*$ be such that

$$(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), x, (p_1, p_2), y\Big)$$
$$= \mu_{A_1}^*(q_1, x, p_1, y) \wedge \mu_{A_2}^*(q_2, x, p_2, y) > 0$$

and

$$(\nu_{A_1} \times \nu_{A_2}^*)^*\Big((q_1, q_2), x, (p_1, p_2), y\Big)$$
$$= \nu_{A_1}^*(q_1, x, p_1, y) \vee \nu_{A_2}^*(q_2, x, p_2, y) < 1.$$

Since $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic retrievable, $\exists u \in \Sigma^*,\ v \in X^*$ such that

$$\mu_{A_1}^*(p_1, u, q_1, v) > 0,\ \mu_{A_2}^*(p_2, u, q_2, v) > 0$$

and

$$\nu_{A_1}^*(p_1, u, q_1, v) < 1,\ \nu_{A_2}^*(p_2, u, q_2, v) < 1.$$

Thus

$$\mu_{A_1}^*(p_1, u, q_1, v) \wedge \mu_{A_2}^*(p_2, u, q_2, v)$$
$$= (\mu_{A_1} \times \mu_{A_2})^*\Big((p_1, p_2), u, (q_1, q_2), v\Big) > 0$$

and

$$\nu_{A_1}^*(p_1, u, q_1, v) \vee \nu_{A_2}^*(p_2, u, q_2, v)$$
$$= (\nu_{A_1} \times \nu_{A_2})^*\Big((p_1, p_2), u, (q_1, q_2), v\Big) < 1.$$

So $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic retrievable. □

Theorem 3.26. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be a strongly connected ifm, $i = 1, 2$. Then intuitionistic cascade product of ifm $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic retrievable.*

Proof. Let $\times$ be intuitionistic cascade product. Suppose $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic retrievable. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$ and $x_2 \in \Sigma_2^*,\ y_2 \in X_2^*$ be such that

$$(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), x_2, (p_1, p_2), y_2\Big)$$
$$= \mu^*_{A_1}\Big(q_1, \omega_x(q_2, x_2), p_1, \omega_y(q_2, y_2)\Big) \wedge \mu^*_{A_2}(q_2, x_2, p_2, y_2) > 0$$

and

$$(\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), x_2, (p_1, p_2), y_2\Big)$$
$$= \nu^*_{A_1}\Big(q_1, \omega_x(q_2, x_2), p_1, \omega_y(q_2, y_2)\Big) \vee \nu^*_{A_2}(q_2, x_2, p_2, y_2) < 1.$$

Since $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic retrievable, $\exists u_2 \in \Sigma_2^*,\ v_2 \in X_2^*$ such that

$$\mu^*_{A_1}\Big(p_1, \omega_u(p_2, u_2), q_1, \omega_v(p_2, v_2)\Big) > 0,\ \mu^*_{A_2}(p_2, u_2, q_2, v_2) > 0$$

and

$$\nu^*_{A_1}\Big(p_1, \omega_u(p_2, u_2), q_1, \omega_v(p_2, v_2)\Big) < 1,\ \nu_{A_2}(p_2, u_2, q_2, v_2) < 1.$$

Thus

$$\mu^*_{A_1}\Big(p_1, \times_u(q_2, u_2), q_1, \times_v(q_2, v_2)\Big) \wedge \mu^*_{A_2}(p_2, u_2, q_2, v_2)$$
$$= (\mu_{A_1} \times \mu_{A_2})^*\Big((p_1, p_2), u_2, (q_1, q_2), v_2\Big) > 0$$

and

$$\nu^*_{A_1}\Big(p_1, \times_u(q_2, u_2), q_1, \times_v(q_2, v_2)\Big) \vee \nu^*_{A_2}(p_2, u_2, q_2, v_2)$$
$$= (\nu_{A_1} \times \nu_{A_2})^*\Big((p_1, p_2), u_2, (q_1, q_2), v_2\Big) < 1.$$

So, $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic retrievable. □

Theorem 3.27. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be a strongly connected ifm, $i = 1, 2$. Then intuitionistic wreath product of ifm $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic retrievable.*

Proof. Let $\times$ be intuitionistic wreath product. Suppose $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic retrievable. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$ and $x_2 \in \Sigma_2^*,\ y_2 \in X_2^*$ be such that

$$(\mu_{A_1} \times \mu_{A_2})^*\Big((q_1, q_2), (f, x_2), (p_1, p_2), (g, y_2)\Big)$$
$$= \mu^*_{A_1}\Big(p_1, f(p_2), q_1, g(p_2)\Big) \wedge \mu^*_{A_2}(q_2, x_2, p_2, y_2) > 0$$

and

$$(\nu_{A_1} \times \nu_{A_2})^*\Big((q_1, q_2), (f, x_2), (p_1, p_2), (g, y_2)\Big)$$
$$= \nu^*_{A_1}\Big(p_1, f(p_2), q_1, g(p_2)\Big) \vee \nu^*_{A_2}(q_2, x_2, p_2, y_2) < 1.$$

Since $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic retrievable, $\exists u_2 \in \Sigma_2^*,\ v_2 \in X_2^*$ such that

$$\mu^*_{A_1}\Big(p_1, f(p_2), q_1, g(p_2)\Big) > 0,\ \mu^*_{A_2}(p_2, u_2, q_2, v_2) > 0$$

and

$$\nu^*_{A_1}\Big(p_1, f(p_2), q_1, g(p_2)\Big) < 1,\ \nu^*_{A_2}(p_2, u_2, q_2, v_2) < 1.$$

Thus,

$$\mu^*_{A_1}\Big(p_1, f(p_2), q_1, g(p_2)\Big) \wedge \mu^*_{A_2}(p_2, u_2, q_2, v_2)$$
$$= (\mu_{A_1} \times \mu_{A_2})^*\Big((p_1, p_2), (g, u_2), (q_1, q_2), (g, v_2)\Big) > 0$$

and

$$\nu^*_{A_1}\Big(p_1, f(p_2), q_1, g(p_2)\Big) \vee \nu^*_{A_2}(p_2, u_2, q_2, v_2)$$
$$= (\nu_{A_1} \times \nu_{A_2})^*\Big((p_1, p_2), (g, u_2), (q_1, q_2), (g, v_2)\Big) < 1.$$

So, $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic retrievable. □

Definition 3.28 ([17]). Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be ifm, $i = 1, 2$. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$. Then $(q_1, q_2), (p_1, p_2)$ are called intuitionistic connected, if either $q_1 = p_1$ or $\exists q_{10}, q_{11}, \cdots, q_{1k} \in Q_1,\ q_1 = q_{10}, p_1 = q_{1k}$ and $\exists a_{10}, a_{11}, \cdots, a_{1k} \in \Sigma_1$ and $\exists b_{11}, b_{12}, b_{13}, \cdots, b_{1k} \in X_1$ such that $\forall i = 1, 2, \cdots, k$

either $\mu_{A_1}(q_{1i-1}, a_{1i}, q_{1i}, b_{1i}) > 0$ or $\mu_{A_1}(q_{1i}, a_{1i}, q_{1i-1}, b_{1i}) > 0$

and

either $\nu_{A_1}(q_{1i-1}, a_{1i}, q_{1i}, b_{1i}) < 1$ or $\nu_{A_1}(q_{1i}, a_{1i}, q_{1i-1}, b_{1i}) < 1$.

Theorem 3.29. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be ifm, $i = 1, 2$. Then intuitionistic Cartesian product of ifm $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic connected iff $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic connected.*

Proof. Let $\times$ be intuitionistic Cartesian product. Suppose $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic connected. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$. Then $\exists\ q_{10}, q_{11}, q_{12}, \cdots, q_{1n} \in Q_1,\ q_1 = q_{10},\ p_1 = q_{1n}$ and $\exists a_{11}, a_{12}, a_{13}, \cdots, a_{1n} \in \Sigma_1$ and $\exists b_{11}, b_{12}, b_{13}, \cdots, b_{1n} \in X_1$ such that $\forall i = 1, 2, \cdots, n$

either $\mu_{A_1}(q_{1i-1}, a_{1i}, q_{1i}, b_{1i}) > 0$ or $\mu_{A_1}(q_{1i}, a_{1i}, q_{1i-1}, b_{1i}) > 0$

and

either $\nu_{A_1}(q_{1i-1}, a_{1i}, q_{1i}, b_{1i}) < 1$ or $\nu_{A_1}(q_{1i}, a_{1i}, q_{1i-1}, b_{1i}) < 1$

and

$\exists\ q_{20}, q_{21}, q_{22}, \cdots, q_{2m} \in Q_2,\ q_2 = q_{20},\ p_2 = q_{2m}$

and

$\exists a_{21}, a_{22}, a_{23}, \cdots, a_{2m} \in \Sigma_2$

and

$\exists\ b_{21}, b_{22}, b_{23}, \cdots, b_{2m} \in X_2\ \forall i = 1, 2, \cdots, m,$

either $\mu_{A_2}(q_{2i-1}, a_{2i}, q_{2i}, b_{2i}) > 0$ or $\mu_{A_2}(q_{2i}, a_{2i}, q_{2i-1}, b_{2i}) > 0$

and

either $\nu_{A_2}(q_{2i-1}, a_{2i}, q_{2i}, b_{2i}) < 1$ or $\nu_{A_2}(q_{2i}, a_{2i}, q_{2i-1}, b_{2i}) < 1$.

Consider the sequence of states

$$(q_1, q_2) = (q_{10}, q_{20}), (q_{11}, q_{20}), \cdots, (q_{1n}, q_{20}), (q_{1n}, q_{21}), \cdots, (q_{1n}, q_{2m})$$
$$= (p_1, p_2) \in Q_1 \times Q_2$$

and

the sequence $a_{11}, a_{12}, a_{13}, \cdots, a_{1n},\ a_{21}, a_{22}, a_{23}, \cdots, a_{2m} \in \Sigma_1 \cup \Sigma_2$

and

$b_{11}, b_{12}, b_{13}, \cdots, b_{1n},\ b_{21}, b_{22}, b_{23}, \cdots, b_{2m} \in X_1 \cup X_2,\ \forall i = 1, 2, \cdots, n$

either $(\mu_{A_1} \times \mu_{A_2})\Big((q_{1i-1}, q_{20}), a_{1i}, (q_{1i}, q_{20}), b_{1i}\Big) > 0$ or

$(\mu_{A_1} \times \mu_{A_2})\Big((q_{1i}, q_{20}), a_{1i}, (q_{1i-1}, q_{20}), b_{1i}\Big) > 0$
and either $(\nu_{A_1} \times \nu_{A_2})\Big((q_{1i-1}, q_{20}), a_{1i}, (q_{1i}, q_{20}), b_{1i}\Big) < 1$ or
$(\nu_{A_1} \times \nu_{A_2})\Big((q_{1i}, q_{20}), a_{1i}, (q_{1i-1}, q_{20}), b_{1i}\Big) < 1$
and $\forall j = 1, 2, \cdots, m$ either $(\mu_{A_1} \times \mu_{A_2})\Big((q_{1n}, q_{2j-1}), a_{2j}, (q_{1n}, q_{2j}), b_{2j}\Big) > 0$ or
$(\mu_{A_1} \times \mu_{A_2})\Big((q_{1n}, q_{2j}), a_{2j}, (q_{1n}, q_{2j-1}), b_{2j}\Big) > 0$
and either $(\nu_{A_1} \times \nu_{A_2})\Big((q_{1n}, q_{2j-1}), a_{2j}, (q_{1n}, q_{2j}), b_{2j}\Big) < 1$ or
$(\nu_{A_1} \times \nu_{A_2})\Big((q_{1n}, q_{2j}), a_{2j}, (q_{1n}, q_{2j-1}), b_{2j}\Big) < 1.$
Then (q_1, q_2) and (p_1, p_2) are intuitionistic connected. Thus, $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic connected.

Conversely, suppose $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic connected. Let $q_1, p_1 \in Q_1$ and let $r_2 \in Q_2$.

If $p_1 = q_1$, then p_1 and q_1 are connected.

Suppose $p_1 \neq q_1$. Then $\exists$ $(q_1, r_2) = (q_{10}, r_{20}), (q_{11}, r_{21}), \cdots, (q_{1n}, r_{2n}) = (p_1, r_2) \in Q_1 \times Q_2$ and $a_1, a_2, \cdots, a_n \in \Sigma_1 \cup \Sigma_2$ such that $\forall i = 1, 2, \cdots, n$,
either $(\mu_{A_1} \times \mu_{A_2})\Big((q_{1i-1}, r_{2i-1}), a_i, (q_{1i}, q_{2i}), b_i\Big) > 0$ or
$(\mu_{A_1} \times \mu_{A_2})\Big((q_{1i}, r_{2i}), a_i, (q_{1i-1}, q_{2i-1}), b_i\Big) > 0$
and
either $(\nu_{A_1} \times \nu_{A_2})\Big((q_{1i-1}, r_{2i-1}), a_i, (q_{1i}, q_{2i}), b_i\Big) < 1$ or
$(\nu_{A_1} \times \nu_{A_2})\Big((q_{1i}, r_{2i}), a_i, (q_{1i-1}, q_{2i-1}), b_i\Big) < 1.$

Clearly, if $q_{1i-1} \neq q_{1i}$, then $r_{2i-1} = r_{2i}$ and if $r_{2i-1} \neq r_{2i}$, then $q_{1i-1} = q_{1i}$, $\forall i = 1, 2, \cdots, n$.

Let $\{q_1 = q'_{11}, q'_{12}, \cdots, q'_{1k} = p_1\}$ be the set of all distinct $q'_{1i} \in \{q_{10}, q_{11}, \cdots, q_{1n}\}$ and let $a'_1, a'_2, \cdots, a'_k \in \Sigma_1$ and $b'_1, b'_2, \cdots, b'_k \in X_1$ be the corresponding $a'_i s$ and $b'_i s$ respectively and $\forall j = 1, 2, \cdots, k$,

either $\mu_{A_1}(q'_{1j-1}, a'_j, q'_{1j}, b'_j) > 0$ or $\mu_{A_1}(q'_{1j}, a'_j, q'_{1j-1}, b'_j) > 0$
and
either $\nu_{A_1}(q'_{1j-1}, a'_j, q'_{1j}, b'_j) < 1$ or $\nu_{A_1}(q'_{1j}, a'_j, q'_{1j-1}, b'_j) < 1.$
Thus, p_1 and q_1 are intuitionistic connected. So $\mathscr{A}_1$ is intuitionistic connected. Similarly, we can prove that $\mathscr{A}_2$ is intuitionistic connected. □

Theorem 3.30. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be ifm, $i = 1, 2$. Then intuitionistic full direct product of ifm $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic connected iff $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic connected.*

Proof. Let $\times$ be intuitionistic full direct product.

Case(i): Suppose $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic connected. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$. Then $\exists$ $q_{10}, q_{11}, q_{12}, \cdots, q_{1n} \in Q_1,\ q_1 = q_{10},\ p_1 = q_{1n}$
and $\exists$ $a_{11}, a_{12}, a_{13}, \cdots, a_{1n} \in \Sigma_1$
and $\exists$ $b_{11}, b_{12}, b_{13}, \cdots, b_{1n} \in X_1$ $\forall i = 1, 2, \cdots, n$,

$\mu_{A_1}(q_{1i-1}, a_{1i}, q_{1i}, b_{1i}) > 0,\ \nu_{A_1}(q_{1i-1}, a_{1i}, q_{1i}, b_{1i}) < 1$
and $\exists\ q_{20}, q_{21}, q_{22}, \cdots, q_{2m} \in Q_2,\ q_2 = q_{20},\ p_2 = q_{2m}$
and $\exists a_{21}, a_{22}, a_{23}, \cdots, a_{2m} \in \Sigma_2$
and $\exists\ b_{21}, b_{22}, b_{23}, \cdots, b_{2m} \in X_2\ \forall i = 1, 2, \cdots, m,$
$\mu_{A_2}(q_{2i-1}, a_{2i}, q_{2i}, b_{2i}) > 0,\ \ \nu_{A_2}(q_{2i-1}, a_{2i}, q_{2i}, b_{2i}) < 1.$

Without the loss of generality $m \le n$. Consider the sequence of states
$$(q_1, q_2) = (q_{10}, q_{20}), (q_{11}, q_{21}), \cdots, (q_{1m}, q_{2m}), (q_{1m+1}, q_{2m}), \cdots, (q_{1n}, q_{2m}) = (p_1, p_2)$$
and a sequence $(a_{11}, a_{21}), \cdots, (a_{1m}, a_{2m}), (a_{1m+1}, a_{2m+1}), \cdots, (a_{1n}, a_{2n})$,
where $a_{2k} = \lambda,\ \forall k = m + 1, \cdots, n$
and a sequence $(b_{11}, b_{21}), \cdots, (b_{1m}, b_{2m}), (b_{1m+1}, b_{2m+1}), \cdots, (b_{1n}, b_{2n})$,
where $b_{2k} = \lambda, \forall k = m + 1, \cdots, n$.
Thus $\forall i = 1, 2, \cdots, n,\ (\mu_{A_1} \times \mu_{A_2})\Big((q_{1i-1}, q_{2i-1}), (a_{1i}, a_{2i}), (q_{1i}, q_{2i}), (b_{1i}, b_{2i})\Big) > 0$
and $(\nu_{A_1} \times \nu_{A_2})\Big((q_{1i-1}, q_{2i-1}), (a_{1i}, a_{2i}), (q_{1i}, q_{2i}), (b_{1i}, b_{2i})\Big) < 1,$
where $q_{2i} = q_{2m},\ \forall i = m + 1, \cdots, n.$
So (q_1, q_2) and (p_1, p_2) are intuitionistic connected. Hence, $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic connected.

Case(ii): Suppose $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic connected. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$. Then $\exists\ q_{10}, q_{11}, q_{12}, \cdots, q_{1n} \in Q_1,\ q_1 = q_{10},\ p_1 = q_{1n}$
and $\exists a_{11}, a_{12}, a_{13}, \cdots, a_{1n} \in \Sigma_1$
and $\exists b_{11}, b_{12}, b_{13}, \cdots, b_{1n} \in X_1\ \forall i = 1, 2, \cdots, n,$
$\mu_{A_1}(q_{1i}, a_{1i}, q_{1i-1}, b_{1i}) > 0,\ \nu_{A_1}(q_{1i}, a_{1i}, q_{1i-1}, b_{1i}) < 1$
and $\exists\ q_{20}, q_{21}, q_{22}, \cdots, q_{2m} \in Q_2,\ q_2 = q_{20},\ p_2 = q_{2m}$
and $\exists\ a_{21}, a_{22}, a_{23}, \cdots, a_{2m} \in \Sigma_2$
and $\exists\ b_{21}, b_{22}, b_{23}, \cdots, b_{2m} \in X_2\ \forall i = 1, 2, \cdots, m,$
$\mu_{A_2}(q_{2i}, a_{2i}, q_{2i-1}, b_{2i}) > 0,\ \ \nu_{A_2}(q_{2i}, a_{2i}, q_{2i-1}, b_{2i}) < 1.$

Without the loss of generality $m \le n$. Consider the sequence of states
$$(p_1, p_2) = (q_{1n}, q_{2m}), (q_{1n-1}, q_{2m}), \cdots, (q_{1m}, q_{2m}), (q_{1m-1}, q_{2m-1}), \cdots, (q_{11}, q_{21}), (q_{10}, q_{20}) = (q_1, q_2)$$
and a sequence $(a_{1n}, a_{2n}), \cdots, (a_{1m}, a_{2m}), (a_{1m-1}, a_{2m-1}), \cdots, (a_{10}, a_{20})$,
where $a_{2k} = \lambda,\ \forall k = m + 1, \cdots, n$
and a sequence $(b_{1n}, b_{2n}), \cdots, (b_{1m}, b_2m), (b_{1m-1}, b_{2m-1}), \cdots, (b_{10}, b_{20})$,
where $b_{2k} = \lambda,\ \forall k = m + 1, \cdots, n.$
Thus $\forall i = n, n-1, \cdots, 1,\ (\mu_{A_1} \times \mu_{A_2})\Big((q_{1i}, q_{2i}), a_{1i}, (q_{1i}, q_{2i}), b_{1i}\Big) > 0$
and $(\nu_{A_1} \times \nu_{A_2})\Big((q_{1i}, q_{2i}), a_{1i}, (q_{1i}, q_{2i}), b_{1i}\Big) < 1,$
where $q_{2i} = q_{2m},\ \forall i = m + 1, \cdots, n.$
So (q_1, q_2) and (p_1, p_2) are intuitionistic connected. Hence, $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic connected.

Case(iii): Suppose $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic connected. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$. Then $\exists\ q_{10}, q_{11}, q_{12}, \cdots, q_{1n} \in Q_1,\ q_1 = q_{10},\ p_1 = q_{1n}$
and $\exists\ a_{11}, a_{12}, a_{13}, \cdots, a_{1n} \in \Sigma_1$
and $\exists b_{11}, b_{12}, b_{13}, \cdots, b_{1n} \in X_1\ \forall i = 1, 2, \cdots, n,$

$\mu_{A_1}(q_{1i-1}, a_{1i}, q_{1i}, b_{1i}) > 0,\ \nu_{A_1}(q_{1i-1}, a_{1i}, q_{1i}, b_{1i}) < 1$
and $\exists\ q_{20}, q_{21}, q_{22}, \cdots, q_{2m} \in Q_2,\ q_2 = q_{2m},\ p_2 = q_{20}$
and $\exists a_{21}, a_{22}, a_{23}, \cdots, a_{2m} \in \Sigma_2$
and $\exists\ b_{21}, b_{22}, b_{23}, \cdots, b_{2m} \in X_2\ \forall i = 1, 2, \cdots, m,$
$\mu_{A_2}(q_{2i}, a_{2i}, q_{2i-1}, b_{2i}) > 0, \nu_{A_2}(q_{2i}, a_{2i}, q_{2i-1}, b_{2i}) < 1.$

Without the loss of generality $m \le n$. Consider the sequence of states
(q_1, q_2)
$= (q_{10}, q_{20}), (q_{11}, q_{21}), \cdots, (q_{1m}, q_{2m}), (q_{1m+1}, q_{2m}), \cdots, (q_{1n}, q_{20})$
$= (p_1, p_2)$
and a sequence $(a_{11}, a_{2m}), \cdots, (a_{1m}, a_{21}), (a_{1m+1}, a_{2m+1}), \cdots, (a_{1n}, a_{2n}),$
where $a_{2k} = \lambda,\ \forall k = m + 1, \cdots, n$
and a sequence $(b_{11}, b_{2m}), \cdots, (b_{1m}, b_{21}), (b_{1m+1}, b_{2m+1}), \cdots, b_{1n}, b_{2n}),$
where $b_{2k} = \lambda,\ \forall k = m + 1, \cdots, n.$
Thus (q_1, q_2) and (p_1, p_2) are intuitionistic connected. So, $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic connected.

Case(iv): Suppose $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic connected. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$. Then $\exists\ q_{10}, q_{11}, q_{12}, \cdots, q_{1n} \in Q_1,\ p_1 = q_{10},\ q_1 = q_{1n}$
and $\exists\ a_{11}, a_{12}, a_{13}, \cdots, a_{1n} \in \Sigma_1$
and $\exists b_{11}, b_{12}, b_{13}, \cdots, b_{1n} \in X_1\ \forall i = 1, 2, \cdots, n,$
$\mu_{A_1}(q_{1i}, a_{1i}, q_{1i-1}, b_{1i}) > 0,\ \nu_{A_1}(q_{1i}, a_{1i}, q_{1i-1}, b_{1i}) < 1$
and $\exists\ q_{20}, q_{21}, q_{22}, \cdots, q_{2m} \in Q_2,\ q_2 = q_{20},\ p_2 = q_{2m}$
and $\exists\ a_{21}, a_{22}, a_{23}, \cdots, a_{2m} \in \Sigma_2$
and $\exists\ b_{21}, b_{22}, b_{23}, \cdots, b_{2m} \in X_2,\ \forall i = 1, 2, \cdots, m,$
$\mu_{A_2}(q_{2i-1}, a_{2i}, q_{2i}, b_{2i}) > 0$ and $\nu_{A_2}(q_{2i-1}, a_{2i}, q_{2i}, b_{2i}) < 1.$

Without the loss of generality $m \le n$. Consider the sequence of states
(q_1, q_2)
$= (q_{1n}, q_{20}), (q_{1n-1}, q_{20}), \cdots, (q_{1m+1}, q_{20}), (q_{1m}, q_{20}), (q_{1m-1}, q_{21}), \cdots, (q_{10}, q_{2m})$
$= (p_1, p_2)$
and a sequence $(a_{1n}, a_{2n}), \cdots, (a_{1m}, a_{21}), (a_{1m-1}, a_{22}), \cdots, (a_{1n}, a_{2m}),$
where $a_{2k} = \lambda,\ \forall k = m + 1, \cdots, n$
and a sequence $(b_{1n}, b_{2n}), \cdots, (b_{1m}, b_{21}), (b_{1m-1}, b_{22}), \cdots, (b_{1n}, b_{2m}),$
where $b_{2k} = \lambda,\ \forall k = m + 1, \cdots, n.$
Thus (q_1, q_2) and (p_1, p_2) are intuitionistic connected. So, $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic connected.

Conversely, suppose $\mathscr{A}_1 \times \mathscr{A}_2$ are intuitionistic connected. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$. Then $\exists$ a sequence of states
$\{(q_1, q_2) = (q_{10}, q_{20}), (q_{11}, q_{21}), \cdots, (q_{1n}, q_{2n}) = (p_1, p_2)\} \in Q_1 \times Q_2$
and the sequence $\{(a_{11}, a_{21}), (a_{12}, a_{22}), \cdots, (a_{1n}, a_{2n})\} \in \Sigma_1 \times \Sigma_2$
and $\{(b_{11}, b_{21}), (b_{12}, b_{22}), \cdots, (b_{1n}, b_{2n})\} \in X_1 \times X_2,\ \forall i = 1, 2, \cdots, n,$
either $(\mu_{A_1} \times \mu_{A_2})\Big((q_{1i-1}, q_{2i-1}), (a_{1i}, a_{2i}), (q_{1i}, q_{2i}), (b_{1i}, b_{2i})\Big) > 0$
or $(\mu_{A_1} \times \mu_{A_2})\Big((q_{1i}, q_{2i}), (a_{1i}, a_{2i}), (q_{1i-1}, q_{2i-1}), (b_{1i}, b_{2i})\Big) > 0$
and either $(\nu_{A_1} \times \nu_{A_2})\Big((q_{1i-1}, q_{2i-1}), (a_{1i}, a_{2i}), (q_{1i}, q_{2i}), (b_{1i}, b_{2i})\Big) < 1$
or $(\nu_{A_1} \times \nu_{A_2})\Big((q_{1i}, q_{2i}), (a_{1i}, a_{2i}), (q_{1i-1}, q_{2i-1}), (b_{1i}, b_{2i})\Big) < 1.$

Without loss of generality, suppose

$(\mu_{A_1} \times \mu_{A_2})\Big((q_{1i-1}, q_{2i-1}), (a_{1i}, a_{2i}), (q_{1i}, q_{2i}), (b_{1i}, b_{2i})\Big) > 0$
and
$(\nu_{A_1} \times \nu_{A_2})\Big((q_{1i-1}, q_{2i-1}), (a_{1i}, a_{2i}), (q_{1i}, q_{2i}), (b_{1i}, b_{2i})\Big) < 1.$

Consider the sequence $\{q_1 = q_{10}, q_{11}, q_{12}, \cdots, q_{1n} = p_1\}$
and the sequence $\{a_{11}, a_{12}, a_{13}, \cdots, a_{1n}\} \in \Sigma_1$
and $\{b_{11}, b_{12}, b_{13}, \cdots, b_{1n}\} \in X_1$ such that $\forall i = 1, 2, \cdots, n$,
$\mu_{A_1}(q_{1i-1}, a_{1i}, q_{1i}, b_{1i}) > 0$ and $\nu_{A_1}(q_{1i-1}, a_{1i}, q_{1i}, b_{1i}) < 1$.
Then $\mathscr{A}_1$ is intuitionistic connected.

Consider the sequence $\{q_2 = q_{20}, q_{21}, q_{22}, \cdots, q_{2n} = p_2\}$
and the sequence $\{a_{21}, a_{22}, a_{23}, \cdots, a_{2n}\} \in \Sigma_2$
and $\{b_{21}, b_{22}, b_{23}, \cdots, b_{2n}\} \in X_2$ such that $\forall i = 1, 2, \cdots, n$,
$\mu_{A_2}(q_{2i-1}, a_{2i}, q_{2i}, b_{2i}) > 0$ and $\nu_{A_2}(q_{2i-1}, a_{2i}, q_{2i}, b_{2i}) < 1$.
Then $\mathscr{A}_2$ is intuitionistic connected. □

Theorem 3.31. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be ifm, $i = 1, 2$. Then intuitionistic restricted direct product of ifm $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic connected then $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic connected.*

Proof. Let $\times$ be intuitionistic restricted direct product. Suppose $\mathscr{A}_1 \times \mathscr{A}_2$ are intuitionistic connected. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$. Then
$\exists$ a sequence of states $\{(q_{10}, q_{20}), (q_{11}, q_{21}), \cdots, (q_{1n}, q_{2n})\} \in Q_1 \times Q_2$
and the sequence $\{a_1, a_2, a_3, \cdots, a_n\} \in \Sigma$
and $\{b_1, b_2, b_3, \cdots, b_n\} \in X$ such that $\forall i = 1, 2, \cdots, n$,
either $\mu_{A_1} \times \mu_{A_2}\Big((q_{1i-1}, q_{2i-1}), a_i, (q_{1i}, q_{2i}), b_i\Big) > 0$
or $\mu_{A_1} \times \mu_{A_2}\Big((q_{1i}, q_{2i}), a_i, (q_{1i-1}, q_{2i-1}), b_i\Big) > 0$
and either $\nu_{A_1} \times \nu_{A_2}\Big((q_{1i-1}, q_{2i-1}), a_i, (q_{1i}, q_{2i}), b_i\Big) < 1$
or $\nu_{A_1} \times \nu_{A_2}\Big((q_{1i}, q_{2i}), a_i, (q_{1i-1}, q_{2i-1}), b_i\Big) < 1$.

Without the loss of generality, suppose

$$(\mu_{A_1} \times \mu_{A_2})\Big((q_{1i-1}, q_{2i-1}), a_i, (q_{1i}, q_{2i}), b_i\Big) > 0$$

and

$$(\nu_{A_1} \times \nu_{A_2})\Big((q_{1i-1}, q_{2i-1}), a_i, (q_{1i}, q_{2i}), b_i\Big) < 1.$$

Consider the sequence $\{q_1 = q_{10}, q_{11}, q_{12}, \cdots, q_{1n} = p_1\}$
and the sequence $\{a_1, a_2, a_3, \cdots, a_n\} \in \Sigma$
and $\{b_1, b_2, b_3, \cdots, b_n\} \in X$ such that $\forall i = 1, 2, \cdots, n$,
$\mu_{A_1}(q_{1i-1}, a_i, q_{1i}, b_i) > 0$, $\nu_{A_1}(q_{1i-1}, a_i, q_{1i}, b_i) < 1$.
Then $\mathscr{A}_1$ is intuitionistic connected.

Consider the sequence $\{q_2 = q_{20}, q_{21}, q_{22}, \cdots, q_{2n} = p_2\}$
and the sequence $\{a_1, a_2, a_3, \cdots, a_n\} \in \Sigma$
and $\{b_1, b_2, b_3, \cdots, b_n\} \in X$ such that $\forall i = 1, 2, \cdots, n$,
$\mu_{A_2}(q_{2i-1}, a_i, q_{2i}, b_i) > 0$ and $\nu_{A_2}(q_{2i-1}, a_i, q_{2i}, b_i) < 1$.
Then $\mathscr{A}_2$ is intuitionistic connected. □

Theorem 3.32. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be ifm, $i = 1, 2$. Then intuitionistic cascade product of ifm $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic connected then $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic connected.*

Proof. Suppose $\times$ is intuitionistic cascade product. Suppose $\mathscr{A}_1 \times \mathscr{A}_2$ are intuitionistic connected. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$. Then

$\exists$ a sequence of states $\{(q_{10}, q_{20}), (q_{11}, q_{21}), \cdots, (q_{1n}, q_{2n})\} \in Q_1 \times Q_2$
and the sequence $\{x_{21}, x_{22}, x_{23}, \cdots, x_{2n}\} \in \Sigma_2$
and $\{y_{21}, y_{22}, y_{23}, \cdots, y_{2n}\} \in X_2$ $\forall i = 1, 2, \cdots, n$,

$$\text{either } (\mu_{A_1} \times \mu_{A_2})\Big((q_{1i-1}, q_{2i-1}), x_{2i}, (q_{1i}, q_{2i}), y_{2i}\Big) > 0$$
$$\text{or } (\mu_{A_1} \times \mu_{A_2})\Big((q_{1i}, q_{2i}), x_{2i}, (q_{1i-1}, q_{2i-1}), y_{2i}\Big) > 0$$
$$\text{and either } (\nu_{A_1} \times \nu_{A_2})\Big((q_{1i-1}, q_{2i-1}), x_{2i}, (q_{1i}, q_{2i}), y_{2i}\Big) < 1$$
$$\text{or } (\nu_{A_1} \times \nu_{A_2})\Big((q_{1i}, q_{2i}), x_{2i}, (q_{1i-1}, q_{2i-1}), y_{2i}\Big) < 1.$$

Without loss of generality, suppose

$$(\mu_{A_1} \times \mu_{A_2})\Big((q_{1i-1}, q_{2i-1}), x_{2i}, (q_{1i}, q_{2i}), y_{2i}\Big) > 0$$
$$\text{and } (\nu_{A_1} \times \nu_{A_2})\Big((q_{1i-1}, q_{2i-1}), x_{2i}, (q_{1i}, q_{2i}), y_{2i}\Big) < 1.$$

Consider the sequence $\{q_1 = q_{10}, q_{11}, q_{12}, \cdots, q_{1n} = p_1\}$
and the sequence $\{\omega_x(q_{21}) = x_{11}(\text{say}), \omega_x(q_{22}) = x_{12}, \cdots, \omega_x(q_{2n}) = x_{1n}\} \in \Sigma_1$
and $\{\omega_y(q_{21}) = y_{11}, \omega_y(q_{22}) = y_{12}, \cdots, \omega_y(q_{2n}) = y_{1n}\} \in X_1$ such that $\forall i = 1, 2, \cdots, n$, $\mu_{A_1}(q_{1i-1}, x_{1i}, q_{1i}, y_{1i}) > 0$ and $\nu_{A_1}(q_{1i-1}, x_{1i}, q_{1i}, y_{1i}) < 1$. Then $\mathscr{A}_1$ is intuitionistic connected.

Now consider the sequence $\{q_2 = q_{20}, q_{21}, q_{22}, \cdots, q_{2n} = p_2\}$
and the sequence $\{x_{21}, x_{22}, x_{23}, \cdots, x_{2n}\} \in \Sigma_2$
and $\{y_{21}, y_{22}, y_{23}, \cdots, y_{2n}\} \in X_2$ such that $\forall i = 1, 2, \cdots, n$,

$$\mu_{A_2}(q_{2i-1}, x_{2i}, q_{2i}, y_{2i}) > 0 \text{ and } \nu_{A_2}(q_{2i-1}, x_{2i}, q_{2i}, y_{2i}) < 1.$$

Then $\mathscr{A}_2$ is intuitionistic connected. □

Theorem 3.33. *Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be ifm, $i = 1, 2$. Then intuitionistic wreath product of ifm $\mathscr{A}_1 \times \mathscr{A}_2$ is intuitionistic connected then $\mathscr{A}_1$ and $\mathscr{A}_2$ are intuitionistic connected.*

Proof. Suppose $\times$ is intuitionistic wreath product. Suppose $\mathscr{A}_1 \times \mathscr{A}_2$ are intuitionistic connected. Let $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$. Then

$\exists$ a sequence of states $\{(q_{10}, q_{20}), (q_{11}, q_{21}), \cdots, (q_{1n}, q_{2n})\} \in Q_1 \times Q_2$
and the sequence $\{(f_1, x_{21}), (f_2, x_{22}), (f_3, x_{23}), \cdots, (f_n, x_{2n})\} \in \Sigma_1^{Q_1} \times \Sigma_2$
and $\{(g_1, y_{21}), (g_2, y_{22}), (g_3, y_{23}), \cdots, (g_n, y_{2n})\} \in X_1^{Q_2} \times X_2, \forall i = 1, 2, \cdots, n$

$$\text{either } (\mu_{A_1} \times \mu_{A_2})\Big((q_{1i-1}, q_{2i-1}), (f_i, x_{2i}), (q_{1i}, q_{2i}), (g_i, y_{2i})\Big) > 0$$
$$\text{or } (\mu_{A_1} \times \mu_{A_2})\Big((q_{1i}, q_{2i}), (f_i, x_{2i}), (q_{1i-1}, q_{2i-1}), (g_i, y_{2i})\Big) > 0$$
$$\text{and either } (\nu_{A_1} \times \nu_{A_2})\Big((q_{1i-1}, q_{2i-1}), (f_i, x_{2i}), (q_{1i}, q_{2i}), (g_i, y_{2i})\Big) < 1$$
$$\text{or } (\nu_{A_1} \times \nu_{A_2})\Big((q_{1i}, q_{2i}), (f_i, x_{2i}), (q_{1i-1}, q_{2i-1}), (g_i, y_{2i})\Big) < 1.$$

Without the loss of generality, suppose

$(\mu_{A_1} \times \mu_{A_2})\Big((q_{1i-1}, q_{2i-1}), (f_i, x_{2i}), (q_{1i}, q_{2i}), (g_i, y_{2i})\Big) > 0$
and $(\nu_{A_1} \times \nu_{A_2})\Big((q_{1i-1}, q_{2i-1}), (f_i, x_{2i}), (q_{1i}, q_{2i}), (g_i, y_{2i})\Big) < 1.$

Consider the sequence $\{q_1 = q_{10}, q_{11}, q_{12}, \cdots, q_{1n} = p_1\}$
and the sequence $\{f_1(q_{21}) = x_{11}$ (say),$f_2(q_{22}) = x_{12}, \cdots, f_n(q_{2n}) = x_{1n}\} \in \Sigma_1$
and $\{g_1(q_{21}) = y_{11}, g_2(q_{22}) = y_{12}, \cdots, g_n(q_{2n}) = y_{1n}\} \in X_1$ such that $\forall i = 1, 2, \cdots, n$,
$\mu_{A_1}(q_{1i-1}, x_{1i}, q_{1i}, y_{1i}) > 0$ and $\nu_{A_1}(q_{1i-1}, x_{1i}, q_{1i}, y_{1i}) < 1$.
Then $\mathscr{A}_1$ is intuitionistic connected.

Now consider the sequence $\{q_2 = q_{20}, q_{22}, q_{23}, \cdots, q_{2n} = p_2\}$
and the sequence $\{x_{21}, x_{22}, x_{23}, \cdots, x_{2n}\} \in \Sigma_2$
and $\{y_{21}, y_{22}, y_{23}, \cdots, y_{2n}\} \in X_2$ such that $\forall i = 1, 2, \cdots, n$,
$\mu_{A_2}(q_{2i-1}, x_{2i}, q_{2i}, y_{2i}) > 0$ and $\nu_{A_2}(q_{2i-1}, x_{2i}, q_{2i}, y_{2i}) < 1$.
Then $\mathscr{A}_2$ is intuitionistic connected. □

Note 3.34. The converse of Theorems 3.31 to 3.33 is true when individual ifm's are strongly connected.

Remark 3.35. Let $\mathscr{A}_i = (Q_i, \Sigma_i, X_i, A_i)$ be ifm, $i = 1, 2$. By theorems 3.29 to 3.33, ifm $\mathscr{A}_1 \times \mathscr{A}_2$ is strongly connected if and only if ifm $\mathscr{A}_1$ and $\mathscr{A}_2$ are strongly connected, where $\times$ is intuitionistic Cartesian product, intuitionistic full direct product, intuitionistic restricted direct product, intuitionistic cascade product and intuitionistic wreath product.

4. Conclusions

In this paper the results of fuzzy Mealy machine are successfully extended for intuitionistic fuzzy Mealy machines. We introduced various kinds of products of intuitionistic fuzzy Mealy machines such as intuitionistic cartesian product, intuitionistic full direct product, intuitionistic restricted direct product, intuitionistic cascade product and intuitionistic wreath product. Using these products it has been shown that intuitionistic fuzzy Mealy machines preserves some structural properties viz. cyclic, intuitionistic retrievable, connected and strongly connected.

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