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Operators on soft inner product spaces II

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Abstract. *In the present paper we have further extended the study of operators on soft inner product spaces. In fact, in this paper, normal operators, unitary operators, isometric operators and square root of positive operators on soft inner product spaces have been introduced and some basic properties of such operators have been investigated.*

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1. Introduction

Molodtsov [25] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties while modeling the problems in engineering, physics, computer science, economics, social sciences, and medical sciences. As he argued that soft sets are more general than fuzzy sets, as a mathematical structure and serves as a better tool for processing uncertainty because of its non-restrictive parametrization and easy applicability to various real life problems. Following his work Maji et al. [21, 22] introduced several operations on soft sets and applied soft sets to decision making problems. Chen et al. [4] presented a new definition of soft set parametrization reduction and some works in this direction have been found in [20, 28, 32]. Soft group was introduced by Aktas and Cagman [1] and soft BCK/BCI – algebras and their applications in ideal theory were investigated by Jun [18, 19]. Ali et al. [2] introduced some new operations on soft sets. Feng et al. [14] worked on soft semirings, soft ideals and idealistic soft semirings. Some works on semigroups and soft ideals over a semi-group are found in [29]. The idea of a soft topological space was first given by Shabir and Naz [30] and subsequently Hazra et al. [17], Pazar et al. [23] and Cagman et al. [3] introduced new definitions of soft topology. Soft topological groups and soft topological soft groups were studied by Nazmul and Samanta [26, 27]. Mappings between soft sets were described by Majumdar and Samanta [24].

Guler et al. [16] studied fixed point theorem on soft G-metric spaces. Feng et al. [15] worked on soft sets combined with fuzzy sets and rough sets. Recently we have introduced soft real sets, soft real numbers, soft complex sets, soft complex numbers in [5, 6]. Two different concepts of soft metric have been presented in [7, 8]. 'Soft linear (vector) space' and 'soft norm' on an absolute 'soft vector space' have been introduced in [9]. An idea of 'soft inner product' has been introduced in [10]. In [11, 12], we have proposed ideas of 'soft linear operator' and 'soft linear functional' on 'soft linear spaces' and 'soft normed linear spaces'. In [12], four fundamental theorems of functional analysis have also been extended in soft set settings.

In fuzzy settings, metric and norm structures are nicely developed based on the theory of interval analysis. But there are some inherent difficulties in the non-comparable order structure of complex numbers on one hand and the lattice ordering of the gradation function of fuzzy sets on the other. So the theory of fuzzy complex numbers is not so well-developed. The notion of fuzzy inner product has a lot of potentials for being a tool for quantum mechanics or in the development of fuzzy operator theory. The reason behind the lack of progress in fuzzy inner products may be due to the fact that the theory of fuzzy complex numbers is not so well-developed. However, in soft set settings, it has been possible to extend the inner product theory nicely. In the present paper an attempt has been made to extend the operator theory on soft inner product spaces. In [13], we introduced notions of self-adjoint soft linear operators and completely continuous soft linear operators on soft inner product spaces and study some of their properties.

In this paper we have further extended the study of operators on soft inner product spaces. In fact, in this paper, normal operators, unitary operators, isometric operators and square root of positive operators on soft inner product spaces have been introduced and some basic properties of such operators have been investigated. In section 2, some preliminary results are given. In section 3, a notion of normal operator on a 'soft inner product space' is given and some properties of such operators are studied. In section 4, unitary operators and isometric operators over soft inner product spaces are introduced and some fundamental properties of such operators are studied. In section 5, square roots of positive operators over soft inner product spaces are introduced and some basic properties of such operators are studied. Section 6 concludes the paper.

2. Preliminaries

Definition 2.1 ([25]). Let U be a universe and E be a set of parameters. Let $\mathcal{P}(U)$ denote the power set of U and A be a non-empty subset of E. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to \mathcal{P}(U)$. In other words, a soft set over U is a parametrized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (F, A).

Let X be an initial universal set and A be the universal parameter set. Then any soft sets (F, E_1), (G, E_2) can be considered as soft sets (F, A), (G, A) with respect to the universal parameter set A, where $F(e) = \varnothing$, if $e \in A - E_1$, $G(e) = \varnothing$, if $e \in A - E_2$.

In our study, throughout this work, we shall consider the soft sets with respect to the universal parameter set A.

Definition 2.2 ([22])**.** (i) A soft set (F, A) over U is said to be an absolute soft set, denoted by $\check{U}$, if for each $\varepsilon \in A$, $F(\varepsilon) = U$.

(ii) A soft set (F, A) over U is said to be a null soft set, denoted by Φ, if for each $\varepsilon \in A$, $F(\varepsilon) = \varnothing$.

Definition 2.3 ([5])**.** Let X be a non-empty set and A be a non-empty parameter set. Then a function $\varepsilon : A \to X$ is said to be a soft element of X. A soft element ε of X is said to belongs to a soft set B of X, which is denoted by $\varepsilon \tilde{\in} B$, if $\varepsilon(e) \in B(e)$, for each $e \in A$.

Thus for a soft set A of X with respect to the index set A, we have $B(e) = \left\{\varepsilon(e), \varepsilon \tilde{\in} B\right\}$, $e \in A$.

It is to be noted that every singleton soft set (a soft set (F, A) for which $F(e)$ is a singleton set, for each $e \in A$) can be identified with a soft element by simply identifying the singleton set with the element that it contains for each $e \in A$.

Definition 2.4 ([5])**.** Let R be the set of all real numbers, $\mathcal{B}(R)$ the collection of all non-empty bounded subsets of R. Let A be taken as the set of parameters. Then a mapping $F : A \to \mathcal{B}(R)$ is called a soft real set. It is denoted by (F, A).

If specifically (F, A) is a singleton soft set, then after identifying (F, A) with the corresponding soft element, it will be called a soft real number.

The set of all soft real numbers is denoted by $\mathcal{R}(A)$ and the set of all non-negative soft real numbers by $\mathcal{R}(A)^*$.

We use notations $\tilde{r}$, $\tilde{s}$, $\tilde{t}$ to denote soft real numbers whereas $\overline{r}$, $\overline{s}$, $\overline{t}$ denotes a particular type of soft real numbers such that $\overline{r}(\lambda) = r$, for each $\lambda \in A$. For example $\overline{0}$ is the soft real number where $\overline{0}(\lambda) = 0$, for each $\lambda \in A$.

Definition 2.5 ([6])**.** Let C be the set of all complex numbers and $\wp(C)$ be the collection of all non-empty bounded subsets of the set of complex numbers. Let A be the set of parameters. Then a mapping

$$F : A \to \wp(C)$$

is called a soft complex set. It is denoted by (F, A).

In particular, If (F, A) is a singleton soft set , then identifying (F, A) with the corresponding soft element, it will be called a soft complex number.

The set of all soft complex numbers is denoted by $\mathcal{C}(A)$.

Let X be a non-empty set. Let $\check{X}$ be the absolute soft set, i.e., $F(\lambda) = X$, for each $\lambda \in A$, where $(F, A) = \check{X}$. Let $\mathcal{S}(\check{X})$ be the collection of all soft sets (F, A) over X for which $F(\lambda) \neq \varnothing$, for each $\lambda \in A$ together with the null soft set Φ.

Let $(F, A)(\neq \Phi) \in \mathcal{S}(\check{X})$. Then the collection of all soft elements of (F, A) is denoted by $SE(F, A)$. For a collection $\mathcal{B}$ of soft elements of $\check{X}$, the soft set generated by $\mathcal{B}$ is denoted by $SS(\mathcal{B})$.

Definition 2.6 ([9])**.** Let V be a vector space over a field K and let A be a parameter set. Let G be a soft set over V. Now G is said to be a soft vector space or soft linear space of V over K, if $G(\lambda)$ is a vector subspace of V, for each $\lambda \in A$.

Definition 2.7 ([9])**.** Let G be a soft vector space of V over K. Then a soft element of G is said to be a soft vector of G.

For example Θ is the null soft vector defined by $\Theta(\lambda) = \theta$ (the null vector of V), for each $\lambda \in A$.

In a similar manner, a soft element of the soft set (K, A) is said to be a soft scalar, K being the scalar field.

Definition 2.8 ([9])**.** Let $\tilde{x}, \tilde{y}$ be soft vectors of G and $\tilde{k}$ be a soft scalar. Then the addition $\tilde{x} + \tilde{y}$ of $\tilde{x}, \tilde{y}$ and scalar multiplication $\tilde{k}.\ \tilde{x}$ of $\tilde{k}$ and $\tilde{x}$ are defined by

$$(\tilde{x} + \tilde{y})(\lambda) = \tilde{x}(\lambda) + \tilde{y}(\lambda), \ (\tilde{k}.\tilde{x})(\lambda) = \tilde{k}(\lambda).\tilde{x}(\lambda), \text{ for each } \lambda \in A.$$

Obviously, $\tilde{x} + \tilde{y}$, $\tilde{k}.\tilde{x}$ are soft vectors of G.

Definition 2.9 ([9])**.** Let $\check{X}$ be the absolute soft vector space, i.e., $\check{X}(\lambda) = X$, for each $\lambda \in A$. Then a mapping $\|.\| : SE(\check{X}) \to \mathcal{R}(A)^*$ is said to be a soft norm on the soft vector space $\check{X}$, if $\|.\|$ satisfies the following conditions:

(N1) $\|\tilde{x}\| \tilde{\geq} \overline{0}$, for each $\tilde{x} \tilde{\in} \check{X}$,

(N2) $\|\tilde{x}\| = \overline{0}$ if and only if $\tilde{x} = \Theta$,

(N3) $\|\widetilde{\alpha}.\tilde{x}\| = |\widetilde{\alpha}| \, \|\tilde{x}\|$ for each $\tilde{x} \tilde{\in} \check{X}$ and for every soft scalar $\widetilde{\alpha}$, where $|\widetilde{\alpha}|$ denotes the modulus of $\widetilde{\alpha}$,

(N4) For each $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$, $\|\tilde{x} + \tilde{y}\| \tilde{\leq} \|\tilde{x}\| + \|\tilde{y}\|$.

The soft vector space $\check{X}$ with a soft norm $\|.\|$ on $\check{X}$ is said to be a soft normed linear space and is denoted by $(\check{X}, \|.\|, A)$ or $(\check{X}, \|.\|)$. (N1), (N2), (N3) and (N4) are said to be soft norm axioms.

Theorem 2.10 ([9, 31])**.** *Every soft norm $\|.\|$ satisfies the condition*
(A) *For $\xi \in X$, and $\lambda \in A$, $\{\|\tilde{x}\|(\lambda) : \tilde{x}(\lambda) = \xi\}$ is a singleton set.*
And hence each soft norm $\|.\|$ can be decomposed into a family of crisp norms $\{\|.\|_\lambda, \ \lambda \in A\}$, where $\|.\|_\lambda : X \to \mathcal{R}^+$ is defined by the following:

$$\text{for each } \xi \in X, \ \|\xi\|_\lambda = \|\tilde{x}\|(\lambda), \text{ with } \tilde{x} \tilde{\in} \check{X} \text{ such that } \tilde{x}(\lambda) = \xi.$$

Definition 2.11 ([10])**.** Let $\check{X}$ be the absolute soft vector space i.e., $\check{X}(\lambda) = X$, for each $\lambda \in A$. Then a mapping $\langle . \rangle : SE(\check{X}) \times SE(\check{X}) \to \mathcal{C}(A)$ is said to be a soft inner product on the soft vector space $\check{X}$, if $\langle . \rangle$ satisfies the following conditions:

(I1) $\langle \tilde{x}, \tilde{x} \rangle \tilde{\geq} \overline{0}$, for each $\tilde{x} \tilde{\in} \check{X}$ and $\langle \tilde{x}, \tilde{x} \rangle = \overline{0}$ if and only if $\tilde{x} = \Theta$,

(I2) $\langle \tilde{x}, \tilde{y} \rangle = \overline{\langle \tilde{y}, \tilde{x} \rangle}$, where bar denotes the complex conjugate of soft complex numbers,

(I3) $\langle \widetilde{\alpha}.\tilde{x}, \tilde{y} \rangle = \widetilde{\alpha}. \langle \tilde{x}, \tilde{y} \rangle$ for each $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$ and for every soft scalar $\widetilde{\alpha}$,

(I4) For each $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$, $\langle \tilde{x} + \tilde{y}, \tilde{z} \rangle = \langle \tilde{x}, \tilde{z} \rangle + \langle \tilde{y}, \tilde{z} \rangle$.

The soft vector space $\check{X}$ with a soft inner product $\langle . \rangle$ on $\check{X}$ is said to be a soft inner product space and is denoted by $(\check{X}, \langle . \rangle, A)$ or $(\check{X}, \langle . \rangle)$. (I1), (I2), (I3) and (I4) are said to be soft inner product axioms.

Example 2.12. Let $X = l_2$. Then X is an inner product space with respect to the inner product $\langle x, y \rangle = \sum_{i=1}^{\infty} \xi_i \overline{\eta_i}$ for $x = \{\xi_i\}$, $y = \{\eta_i\}$ of l_2. Let $\tilde{x}, \tilde{y}$ be soft elements of the absolute soft vector space $\check{X}$. Then $\tilde{x}(\lambda) = \{\xi_i^\lambda\}, \tilde{y}(\lambda) = \{\eta_i^\lambda\}$ are elements of l_2. The mapping $\langle . \rangle : SE(\check{X}) \times SE(\check{X}) \to \mathbb{C}(A)$ defined by $\langle \tilde{x}, \tilde{y} \rangle(\lambda) = \sum_{i=1}^{\infty} \xi_i^\lambda \overline{\eta_i^\lambda} = \langle \tilde{x}(\lambda), \tilde{y}(\lambda) \rangle$, for each $\lambda \in A$, is a soft inner product on the soft vector space $\check{X}$.

We now state the following result which is a modified version of Decomposition Theorem of [10].

Theorem 2.13 ([10])**.** *Every soft inner product* $\langle . \rangle$ *satisfies the condition*
(D) *For* $(\xi, \eta) \in X \times X$ *and* $\lambda \in A$, $\left\{ \langle \tilde{x}, \tilde{y} \rangle (\lambda) : \tilde{x}, \tilde{y} \tilde{\in} \check{X} \text{ such that } \tilde{x}(\lambda) = \xi, \tilde{y}(\lambda) = \eta \right\}$ *is a singleton set.*
Hence every soft inner product can be decomposed into a family of crisp inner products $\{\langle . \rangle_\lambda,\ \lambda \in A\}$*, where for each* $\lambda \in A$, $\langle . \rangle_\lambda : X \times X \to \mathbb{C}$ *is defined by for each* $(\xi, \eta) \in X \times X, \langle \xi, \eta \rangle_\lambda = \langle \tilde{x}, \tilde{y} \rangle (\lambda)$*, with* $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$ *such that* $\tilde{x}(\lambda) = \xi, \tilde{y}(\lambda) = \eta$*.*

Theorem 2.14 ([10])**.** *Let* $(\check{X}, \langle . \rangle, A)$ *be a soft inner product space. Let us define* $\|.\| : \check{X} \to R(A)^*$ *by* $\|\tilde{x}\| = \sqrt{\langle \tilde{x}, \tilde{x} \rangle}$*, for each* $\tilde{x} \tilde{\in} \check{X}$*. Then* $\|.\|$ *is a soft norm on* $\check{X}$*.*

Definition 2.15 ([10])**.** A soft inner product space is said to be complete, if it is complete with respect to the soft metric defined by soft inner product. A complete soft inner product space is said to be a soft Hilbert space.

Definition 2.16 ([11])**.** Let $T : SE(\check{X}) \to SE(\check{Y})$ be an operator. Then T is said to be soft linear, if

(L1) T is additive, i.e., $T(\tilde{x}_1 + \tilde{x}_2) = T(\tilde{x}_1) + T(\tilde{x}_2)$ for every soft elements $\tilde{x}_1, \tilde{x}_2 \tilde{\in} \check{X}$,

(L2) T is homogeneous, i.e., for every soft scalar $\tilde{c}$, $T(\tilde{c}\tilde{x}) = \tilde{c}T(\tilde{x})$, for every soft element $\tilde{x} \tilde{\in} \check{X}$.

The properties (L1) and (L2) can be put in a combined form $T(\tilde{c}_1\tilde{x}_1 + \tilde{c}_2\tilde{x}_2) = \tilde{c}_1 T(\tilde{x}_1) + \tilde{c}_2 T(\tilde{x}_2)$ for every soft elements $\tilde{x}_1, \tilde{x}_2 \tilde{\in} \check{X}$ and every soft scalars $\tilde{c}_1,\ \tilde{c}_2$.

Definition 2.17 ([11])**.** The operator $T : SE(\check{X}) \to SE(\check{Y})$ is said to be continuous at $\tilde{x}_0 \tilde{\in} \check{X}$, if for every sequence $\{\tilde{x}_n\}$ of soft elements of $\check{X}$ with $\tilde{x}_n \to \tilde{x}_0$ as $n \to \infty$, we have $T(\tilde{x}_n) \to T(\tilde{x}_0)$ as $n \to \infty$, i.e., $\|\tilde{x}_n - \tilde{x}_0\| \to \overline{0}$ as $n \to \infty$ implies $\|T(\tilde{x}_n) - T(\tilde{x}_0)\| \to \overline{0}$ as $n \to \infty$.

If T is continuous at each soft element of $\check{X}$, then T is said to be a continuous operator.

Definition 2.18 ([11])**.** Let $T : SE(\check{X}) \to SE(\check{Y})$ be a soft linear operator, where $\check{X}, \check{Y}$ are soft normed linear spaces. The operator T is called bounded, if there exists some positive soft real number $\tilde{M}$ such that for each $\tilde{x} \tilde{\in} \check{X}$, $\|T(\tilde{x})\| \tilde{\leq} \tilde{M} \|\tilde{x}\|$.

Theorem 2.19 ([11])**.** *Let* $T : SE(\check{X}) \to SE(\check{Y})$ *be a soft linear operator, where* $\check{X}, \check{Y}$ *are soft normed linear spaces. Then* T *is bounded if and only if* T *is continuous.*

We now state the following result which is a modified form of the Decomposition Theorem of [11].

Theorem 2.20 ([11])**.** *Every soft linear operator* $T : SE(\check{X}) \to SE(\check{Y})$*, where* $\check{X}, \check{Y}$ *are soft vector spaces, satisfies the condition*
(B) *For* $\xi \in X$*, and* $\lambda \in A$, $\left\{ T(\tilde{x})(\lambda) : \tilde{x} \tilde{\in} \check{X} \text{ such that } \tilde{x}(\lambda) = \xi \right\}$ *is a singleton set.*
And hence every soft linear operator can be decomposed into a family of crisp linear operators $\{T_\lambda,\ \lambda \in A\}$*, where for each* $\lambda \in A$*, the mapping* $T_\lambda : X \to Y$ *is defined by* $T_\lambda(\xi) = T(\tilde{x})(\lambda)$*, for each* $\xi \in X$ *and* $\tilde{x} \tilde{\in} \check{X}$ *with* $\tilde{x}(\lambda) = \xi$*.*

Theorem 2.21 ([11]). *Let $\check{X}, \check{Y}$ be soft normed linear spaces and $T : SE(\check{X}) \to SE(\check{Y})$ be a bounded soft linear operator. Then $\|T(\tilde{x})\| \tilde{\leq} \|T\| \|\tilde{x}\|$,* for all $\tilde{x} \tilde{\in} \check{X}$.

Theorem 2.22. [11] *Let $\check{X}$, $\check{Y}$ be soft normed linear spaces. Let $T : SE(\check{X}) \to SE(\check{Y})$ be a continuous soft linear operator. Then T_λ is continuous on X for each $\lambda \in A$.*

Theorem 2.23 ([12]). *Let $\check{X}$, $\check{Y}$ be soft normed linear spaces. Let $\{T_\lambda; \lambda \in A\}$ be a family of continuous linear operators such that $T_\lambda : X \to Y$ for each λ. Then the soft linear operator $T : SE(\check{X}) \to SE(\check{Y})$ defined by $(T(\tilde{x}))(\lambda) = T_\lambda(\tilde{x}(\lambda))$, for each $\lambda \in A$, is a continuous soft linear operator.*

Theorem 2.24 ([12]). *Let $\check{X}$, $\check{Y}$ be soft normed linear spaces. Let $\{T_\lambda; \lambda \in A\}$ be a family of bounded linear operators such that $T_\lambda : X \to Y$ for each λ. Then the soft linear operator $T : SE(\check{X}) \to SE(\check{Y})$ defined by $(T(\tilde{x}))(\lambda) = T_\lambda(\tilde{x}(\lambda))$, for each $\lambda \in A$, is a bounded soft linear operator.*

Definition 2.25 ([12], Soft linear space of operators). Let $\check{X}, \check{Y}$ be soft normed linear spaces. Consider the set W of all continuous soft linear operators S, T etc. each mapping $SE(\check{X})$ into $SE(\check{Y})$. Then W can be interpreted as to form an absolute soft vector space. We shall denote this absolute soft linear (vector) space by $L(\check{X}, \check{Y})$.

Proposition 2.26 ([12]). *Each element of $SE\left(L\left(\check{X}, \check{Y}\right)\right)$ can be identified uniquely with a member of W, i.e., to a continuous soft linear operator $T : SE(\check{X}) \to SE\left(\check{Y}\right)$.*

Definition 2.27 ([12]). A soft linear functional f is a soft linear operator such that $f : SE(\check{X}) \to K$ where $\check{X}$ is a soft linear space and $K = \mathcal{R}(A)$ if $\check{X}$ is a real soft linear space and $K{=}\mathcal{C}(A)$, if $\check{X}$ is a complex soft linear space.

Theorem 2.28 ([12]). *Let $\check{X}$ be a soft normed linear space. Let $f : SE(\check{X}) \to K$ be a continuous soft linear functional on $\check{X}$. Then f_λ is continuous linear on X for each $\lambda \in A$.*

Theorem 2.29 ([12]). *Let $\check{X}$ be a soft normed linear space. Let $\{f_\lambda; \lambda \in A\}$ be a family of continuous linear functionals such that $f_\lambda : X \to \mathcal{R}$ or $\mathcal{C}$ for each λ. Then the functional $f : SE\left(\check{X}\right) \to K(= \mathcal{R}(A)$ or $\mathcal{C}(A))$ defined by $(f(\tilde{x}))(\lambda) = f_\lambda(\tilde{x}(\lambda))$, for each $\lambda \in A$ and for each $\tilde{x} \tilde{\in} \check{X}$, is a continuous soft linear functional.*

Definition 2.30 ([9]). A sequence of soft elements $\{\tilde{x}_n\}$ in a soft normed linear space $(\check{X}, \|.\|, A)$ is said to be convergent (or strong convergent) and converges to a soft element $\tilde{x}$ if $\|\tilde{x}_n - \tilde{x}\| \to \overline{0}$ as $n \to \infty$. This means for every $\tilde{\varepsilon} \tilde{>} \overline{0}$, chosen arbitrarily, there exists a natural number $N = N(\tilde{\varepsilon})$, such that $\overline{0} \tilde{\leq} \|\tilde{x}_n - \tilde{x}\| \tilde{<} \tilde{\varepsilon}$, whenever $n > N$, i.e., $n > N \Longrightarrow \tilde{x}_n \in B(\tilde{x}, \tilde{\varepsilon})$.

We denote this by $\tilde{x}_n \to \tilde{x}$ as $n \to \infty$ or by $\lim_{n\to\infty} \tilde{x}_n = \tilde{x}$. In this case, $\tilde{x}$ is said to be the limit of the sequence $\tilde{x}_n$ as $n \to \infty$.

Definition 2.31 ([12]). Let $\check{X}$ be a soft normed linear space. Suppose that $\tilde{x}_n$, $\tilde{x}_0 \tilde{\in} \check{X}$. The sequence $\{\tilde{x}_n\}$ of soft elements is said to converge weakly to $\tilde{x}_0$, if for each $f \in \check{X}^*$, $f(\tilde{x}_n) \to f(\tilde{x}_0)$ as $n \to \infty$. We write $\tilde{x}_n \to^{wk} \tilde{x}_0$ and we say that $\tilde{x}_0$ is a *weak limit* of the sequence $\{\tilde{x}_n\}$.

Theorem 2.32 ([12]). *Let $\check{X}$ be a soft normed linear space. Then for any sequence of soft elements in $\check{X}$, strong convergence implies weak convergence.*

Theorem 2.33 ([12]). *If a sequence $\{\tilde{x}_n\}$ of soft elements of $\check{X}$ converges weakly then the sequence of norms $\{\|\tilde{x}_n\|\}$ is bounded.*

Theorem 2.34 ([13]). *Let $\check{X}$, $\check{Y}$ be soft normed linear spaces. Let $T : SE(\check{X}) \to SE(\check{Y})$ be a continuous soft linear operator. If $\{\tilde{x}_n\}$ be a sequence of soft elements of $\check{X}$, then $\tilde{x}_n \to^{wk} \tilde{x}_0$ implies $T(\tilde{x}_n) \to^{wk} T(\tilde{x}_0)$.*

We now state the modified version of Uniform Boundedness Principle Theorem of [13].

Theorem 2.35 ([13], Uniform Boundedness Principle). *Let $\check{X}$ be a soft Banach space and $\check{Y}$ be a soft normed linear space. Let $\{T_i\}$ be a non-empty sequence of continuous soft linear operators such that $T_i : SE(\check{X}) \to SE(\check{Y})$ for each i. If the sequence $\{T_i(\tilde{x})\}$ is bounded in $\check{Y}$ for each $\tilde{x}\tilde{\in}\check{X}$, then $\{\|T_i\|\}$ is a bounded sequence of soft real numbers.*

Theorem 2.36 ([13], Riesz Representation Theorem). *Let $\tilde{H}$ be a soft Hilbert space and f be a soft linear functional on $\tilde{H}$. Then f is continuous on $\tilde{H}$ if and only if there exists a unique soft element $\tilde{y}$ in $\tilde{H}$ such that $f(\tilde{x}) = \langle \tilde{x}, \tilde{y} \rangle$ for each $\tilde{x}\tilde{\in}\tilde{H}$. Also, $\|f\| = \|\tilde{y}\|$.*

Definition 2.37 ([13]). Let $\tilde{H}$ be a soft Hilbert space and let T be a continuous soft linear operator such that $T : SE(\tilde{H}) \to SE(\tilde{H})$ and $\tilde{y}\tilde{\in}\tilde{H}$. We define a functional $f_{\tilde{y}}$ on $\tilde{H}$ by

$$f_{\tilde{y}}(\tilde{x}) = \langle T(\tilde{x}), \tilde{y} \rangle . \tag{2.1}$$

Then $f_{\tilde{y}}$ is a soft linear functional. Moreover, $f_{\tilde{y}}$ is bounded and thus $f_{\tilde{y}}$ is a continuous soft linear functional defined everywhere on $\tilde{H}$ and $\|f_{\tilde{y}}(\tilde{x})\| \tilde{\leq} \|T\| \|\tilde{y}\|$.

By Theorem 2.36, $f_{\tilde{y}}$ has the form

$$f_{\tilde{y}}(\tilde{x}) = \langle \tilde{x}, \tilde{y}^* \rangle \tag{2.2}$$

for each $\tilde{x}\tilde{\in}\tilde{H}$, where $\tilde{y}^*\tilde{\in}\tilde{H}$ is uniquely determined by $f_{\tilde{y}}$. Thus to each $\tilde{y}\tilde{\in}\tilde{H}$, we get a unique $\tilde{y}^*$ satisfying (2.2). So we obtain an operator T^* such that

$$\tilde{y}^* = T^*(\tilde{y}) .$$

This operator $T^* : SE(\tilde{H}) \to SE(\tilde{H})$ is called the *adjoint operator* to T. From (2.1) and (2.2), we see that the operator T and its adjoint operator T^* are connected by the relation

$$\langle T(\tilde{x}), \tilde{y} \rangle = \langle \tilde{x}, T^*(\tilde{y}) \rangle . \tag{2.3}$$

Theorem 2.38 ([13]). *Let $\tilde{H}$ be a soft Hilbert space. Let T be a continuous soft linear operator such that $T : SE(\tilde{H}) \to SE(\tilde{H})$ and T^* be the adjoint operator to T. Then the following properties are satisfied:*

(1) *T^* is unique.*
(2) *T^* is a soft linear operator.*
(3) *T^* is a continuous soft linear operator with $\|T^*\| \tilde{\leq} \|T\|$.*

(4) $T^{**} = T$.
(5) $\|T^*\| = \|T\|$.
(6) $(T_1 T_2)^* = T_2{}^* T_1{}^*$, *where* $T_1, \ \ T_2 : SE(\tilde{H}) \to SE(\tilde{H})$ *are continuous soft linear operators.*
(7) $\|T^*T\| = \|T\|^2$.
(8) $\|TT^*\| = \|T\|^2$.
(9) $(T_1 + T_2)^* = T_1{}^* + T_2{}^*$, *where* $T_1, \ \ T_2 : SE(\tilde{H}) \to SE(\tilde{H})$ *are continuous soft linear operators.*
(10) *If* $\widetilde{\alpha}$ *be any soft scalar, then* $(\widetilde{\alpha}T)^* = \overline{\widetilde{\alpha}}T^*$*; where* $\overline{\widetilde{\alpha}}$ *denote the conjugate of* $\widetilde{\alpha}$.
(11) $T^*T =$ *the null soft linear operator if and only if* T *is the null soft linear operator.*

Proposition 2.39 ([13])**.** *Let* $\tilde{H}$ *be a soft Hilbert space and let* T *be a continuous soft linear operator such that* $T : SE(\tilde{H}) \to SE(\tilde{H})$ *and* T^* *be the adjoint operator to* T*. Then* T^*_λ *defined by* $T^*_\lambda(\tilde{x}(\lambda)) = (T^*(\tilde{x}))(\lambda)$ *is the adjoint operator of* T_λ*, for each* $\lambda \in A$.

Proposition 2.40 ([13])**.** *Let* $\tilde{H}$ *be a soft Hilbert space and let* T *be a continuous soft linear operator such that* $T : SE(\tilde{H}) \to SE(\tilde{H})$*. Let* $\{T^*_\lambda, \lambda \in A\}$ *be a family of adjoint operators of* T_λ*. Then the soft linear operator* T^* *defined by* $(T^*(\tilde{x}))(\lambda) = T^*_\lambda(\tilde{x}(\lambda))$*, for each* $\lambda \in A$*, for each* $\tilde{x} \tilde{\in} \tilde{H}$ *is the adjoint operator of* T.

Definition 2.41 ([13])**.** A continuous soft linear operator $T : SE(\tilde{H}) \to SE(\tilde{H})$ is called self-adjoint soft linear operator or simply self-adjoint, if $T^* = T$.

Theorem 2.42 ([13])**.** (1) *If* $T_1, \ T_2$ *are self-adjoint, so is* $T_1 + \ T_2$.
(2) *If* T *is self-adjoint and* $\widetilde{\alpha}$ *is a soft real number, then* $\widetilde{\alpha}T$ *is self-adjoint.*
(3) *If* T *is any continuous soft linear operator, then* $T^*T, \ TT^*$ *and* $T + T^*$ *are self-adjoint.*
(4) *If* $T_1, \ T_2$ *are self-adjoint, then* T_1T_2 *is self-adjoint if and only if* $T_1T_2 = T_2T_1$.

Theorem 2.43 ([13])**.** *Let* $\tilde{H}$ *be a soft Hilbert space. Let* $T : SE(\tilde{H}) \to SE(\tilde{H})$ *be a self-adjoint soft linear operator. Then* T_λ *is self-adjoint on* $\tilde{H}\left(\lambda\right)$ *for each* $\lambda \in A$.

Theorem 2.44 ([13])**.** *Let* $\tilde{H}$ *be a soft Hilbert space. Let* $\{T_\lambda; \lambda \in A\}$ *be a family of self-adjoint continuous linear operators such that* $T_\lambda : \tilde{H}\left(\lambda\right) \to \tilde{H}\left(\lambda\right)$ *for each* λ*. Then the operator* $T : SE(\tilde{H}) \to SE(\tilde{H})$ *defined by* $T(\tilde{x})\left(\lambda\right) = T_\lambda(\tilde{x}(\lambda))$*, for each* $\lambda \in A$ *and for each* $\tilde{x} \tilde{\in} \tilde{H}$*, is a self-adjoint soft linear operator.*

Theorem 2.45 ([13])**.** *Suppose that* $T \in L(\tilde{H}, \tilde{H})$*. Then* $\langle T(\tilde{x}), \tilde{y}\rangle = \overline{0}$ *for all* $\tilde{x}, \tilde{y} \tilde{\in} \tilde{H}$ *if and only if* $T = O$*, the zero soft linear operator.*

Theorem 2.46 ([13])**.** *Suppose that* $T \in L(\tilde{H}, \tilde{H})$*. Then* $\langle T(\tilde{x}), \tilde{x}\rangle = \overline{0}$ *for all* $\tilde{x} \tilde{\in} \tilde{H}$ *if and only if* $T = O$*, the zero operator.*

Lemma 2.47 ([13])**.** *Suppose that* $T : SE(\tilde{H}) \to SE(\tilde{H})$ *is a self-adjoint soft linear operator. Then*

$$\{|\langle T\left(\tilde{x}\right), \tilde{x}\rangle|\left(\lambda\right) : \tilde{x} \tilde{\in} \tilde{H}, \|\tilde{x}\| = \overline{1}\} = \{|\langle T_\lambda\left(x\right), x\rangle_\lambda| : x \in \tilde{H}\left(\lambda\right), \|x\|_\lambda = 1\}.$$

Theorem 2.48 ([13]). *Suppose that $T : SE(\tilde{H}) \to SE(\tilde{H})$ is a self-adjoint soft linear operator. Then*

$$\|T\|(\lambda) = \sup\{|\langle T(\tilde{x}), \tilde{x}\rangle|(\lambda) : \|\tilde{x}\| = \overline{1}\}, \forall \lambda \in A.$$

Definition 2.49 ([13]). Let $\tilde{H}$ and $\tilde{H}_1$ be soft Hilbert spaces. A soft linear operator $T : SE(\tilde{H}) \to SE(\tilde{H}_1)$ is said to be completely continuous soft linear operator or simply completely continuous, if given any sequence of soft elements $\{\tilde{x}_n\}$ of $\tilde{H}$ such that $\{\|\tilde{x}_n\|\}$ is bounded, the sequence $\{T(\tilde{x}_n)\}$ has a convergent subsequence.

Theorem 2.50 ([13]). *A completely continuous soft linear operator is continuous.*

Theorem 2.51 ([13]). *If S and T are completely continuous, then $S + T$ is completely continuous.*

3. Normal soft linear operators

Definition 3.1. A continuous soft linear operator T mapping a soft Hilbert space $\tilde{H}$ into itself is said to be *normal soft linear operator or simply normal if it commutes with its adjoint, that is, if $TT^* = T^*T$.*

From the fact that $T^{**} = T$, we can say immediately that if T is normal then its adjoint T^* is also normal. The following theorem gives a criteria for a soft linear operator to be normal.

Theorem 3.2. *A continuous soft linear operator T is normal if and only if $\|T^*(\tilde{x})\| = \|T(\tilde{x})\|$ for every $\tilde{x}\tilde{\in}\tilde{H}$.*

Proof. $\|T^*(\tilde{x})\| = \|T(\tilde{x})\|$, for every $\tilde{x}\tilde{\in}\tilde{H}$
if and only if $\|T^*(\tilde{x})\|^2 = \|T(\tilde{x})\|^2$, for every $\tilde{x}\tilde{\in}\tilde{H}$
if and only if $\langle T^*(\tilde{x}), T^*(\tilde{x})\rangle = \langle T(\tilde{x}), T(\tilde{x})\rangle$, for every $\tilde{x}\tilde{\in}\tilde{H}$
if and only if $\langle TT^*(\tilde{x}), \tilde{x}\rangle = \langle T^*T(\tilde{x}), \tilde{x}\rangle$, for every $\tilde{x}\tilde{\in}\tilde{H}$
if and only if $\langle [TT^* - T^*T](\tilde{x}), \tilde{x}\rangle = \overline{0}$, for every $\tilde{x}\tilde{\in}\tilde{H}$
if and only if $TT^* = T^*T$, by Theorem 2.46.
Then $\|T^*(\tilde{x})\| = \|T(\tilde{x})\|$, for each $\tilde{x}\tilde{\in}\tilde{H}$ if and only if T is normal. This proves the theorem. □

The sum and product of two normal soft linear operators are normal under certain restrictions, as the following theorem shows.

Definition 3.3. Two soft linear operators T_1 and T_2 are said to permute, if $T_1T_2 = T_2T_1$, i.e., for every $\tilde{x}$, $T_1T_2(\tilde{x}) = T_2T_1(\tilde{x})$.

Theorem 3.4. *If T_1 and T_2 are normal soft linear operators such that one of them permutes with the adjoint of the other then $T_1 + T_2$ and T_1T_2 are normal.*

Proof. Suppose that T_1 permutes with the adjoint of T_2 i.e., $T_1T_2^* = T_2^*T_1$. Then we see that $(T_1T_2^*)^* = (T_2^*T_1)^*$, i.e., $T_2T_1^* = T_1^*T_2$, i.e., T_2 also permutes with the adjoint of T_1.

To verify that $T_1 + T_2$ is normal, we see that

$$(T_1 + T_2)(T_1 + T_2)^* = (T_1 + T_2)(T_1^* + T_2^*) = T_1T_1^* + T_1T_2^* + T_2T_1^* + T_2T_2^*$$

and

$$(T_1+T_2)^*(T_1+T_2) = (T_1^*+T_2^*)(T_1+T_2) = T_1^*T_1 + T_1^*T_2 + T_2^*T_1 + T_2^*T_2.$$

Under the stated conditions, the right hand sides are equal. Thus T_1+T_2 is normal.

To see that T_1T_2 is normal, we have

$$(T_1T_2)(T_1T_2)^* = T_1T_2T_2^*T_1^* = T_1T_2^*T_2T_1^* = T_2^*T_1T_1^*T_2 = T_2^*T_1^*T_1T_2 = (T_1T_2)^*(T_1T_2).$$

This proves the theorem. □

Theorem 3.5. *If T is a normal soft linear operator, then $\|T^2\| = \|T\|^2$.*

Proof. We first observe that if $\|T_1(\tilde{x})\| = \|T_2(\tilde{x})\|$ for each $\tilde{x}\tilde{\in}\tilde{H}$, then $\|T_1\| = \|T_2\|$.

By Theorem 3.2, $\|T^2(\tilde{x})\| = \|TT(\tilde{x})\| = \|T^*T(\tilde{x})\|$ for every $\tilde{x}\tilde{\in}\tilde{H}$. Then

$$\|T^2\| = \|T^*T\|. \tag{3.1}$$

From Theorem 2.38, we have

$$\|T^*T\| = \|T\|^2. \tag{3.2}$$

Combining (3.1) and (3.2), we get $\|T^2\| = \|T\|^2$ and this proves the theorem. □

Theorem 3.6. *If T is any continuous soft linear operator from a soft Hilbert space $\tilde{H}$ into itself, then T can be expressed uniquely in the form $T = T_1 + \bar{i}T_2$ where T_1 and T_2 are self-adjoint soft linear operators(Here $\bar{i}$ is the soft complex number such that $\bar{i}(\lambda) = i$, for each $\lambda \in A$).*

Proof. We define T_1 and T_2 by the formulas $T_1 = \overline{(\frac{1}{2})}(T+T^*)$, $T_2 = \overline{(\frac{1}{2i})}(T-T^*)$. Then $T_1^* = \overline{(\frac{1}{2})}(T^*+T) = T_1$ and $T_2^* = -\overline{(\frac{1}{2i})}(T^*-T) = \overline{(\frac{1}{2i})}(T-T^*) = T_2$. Thus T_1 and T_2 are self-adjoint soft linear operators and $T = T_1 + \bar{i}T_2$.

If $T = T_3 + \bar{i}T_4$, where T_3 and T_4 are self-adjoint soft linear operators, then $T^* = (T_3+\bar{i}T_4)^* = T_3 - \bar{i}T_4$. Thus $T + T^* = \bar{2}T_3$ and $T - T^* = \overline{2i}T_4$. So $T_1 = T_3$ and $T_2 = T_4$. Hence T is expressed uniquely. □

Definition 3.7. The self-adjoint soft linear operators T_1 and T_2 are called the real part and the imaginary part of the operator T, respectively.

Theorem 3.8. *If T is a continuous soft linear operator from a soft Hilbert space $\tilde{H}$ into itself, then T is normal if and only if its real and imaginary parts permute.*

Proof. Let $T = T_1+\bar{i}T_2$, where T_1 and T_2 are self-adjoint soft linear operators. Then

$T^* = T_1 - \bar{i}T_2$,

$TT^* = (T_1+\bar{i}T_2)(T_1-\bar{i}T_2) = T_1^2 + T_2^2 + \bar{i}(T_2T_1 - T_1T_2)$

and

$T^*T = (T_1-\bar{i}T_2)(T_1+\bar{i}T_2) = T_1^2 + T_2^2 + \bar{i}(T_1T_2 - T_2T_1)$.

Suppose $T_1T_2 = T_2T_1$. Then it is clear that $TT^* = T^*T$, i.e., T is normal.

Conversely, suppose T is normal, i.e., $TT^* = T^*T$. Then $T_1T_2 - T_2T_1 = T_2T_1 - T_1T_2$. Thus $\bar{2}T_1T_2 = \bar{2}T_2T_1$. So $T_1T_2 = T_2T_1$. This proves the theorem. □

Theorem 3.9. *Let T be normal. Then $T(\tilde{x}) = \tilde{\lambda}\tilde{x}$ if only if $T^*(\tilde{x}) = \bar{\tilde{\lambda}}\tilde{x}$, for $\tilde{x}\tilde{\in}\tilde{H}$ and for any soft scalar $\tilde{\lambda}$.*

Proof. We consider the operator $T - \tilde{\lambda}I$, where I being the identity soft linear operator. Then we have

$$(T - \tilde{\lambda}I)(T - \tilde{\lambda}I)^* = (T - \tilde{\lambda}I)(T^* - \bar{\tilde{\lambda}}I) = TT^* - \bar{\tilde{\lambda}}T - \tilde{\lambda}T^* + |\tilde{\lambda}|^2 I$$

and

$$(T - \tilde{\lambda}I)^*(T - \tilde{\lambda}I) = (T^* - \bar{\tilde{\lambda}}I)(T - \tilde{\lambda}I) = T^*T - \tilde{\lambda}T^* - \bar{\tilde{\lambda}}T + |\tilde{\lambda}|^2 I.$$

Since T is normal, $TT^* = T^*T$. Thus it follows that $T - \tilde{\lambda}I$ is normal.
By Theorem 3.2, $\|(T - \tilde{\lambda}I)(\tilde{x})\| = \|(T - \tilde{\lambda}I)^*(\tilde{x})\|$ for every $\tilde{x}\tilde{\in}\tilde{H}$. So

$$\|T(\tilde{x}) - \tilde{\lambda}(\tilde{x})\| = \|(T^*(\tilde{x}) - \bar{\tilde{\lambda}}(\tilde{x})\| \text{ for every } \tilde{x}\tilde{\in}\tilde{H}.$$

Hence, $T(\tilde{x}) = \tilde{\lambda}(\tilde{x})$ if and only if $T^*(\tilde{x}) = \bar{\tilde{\lambda}}(\tilde{x})$. This proves the theorem. □

Theorem 3.10. (1) *Every self-adjoint operator is normal.*

(2) *If T is normal and $\tilde{\lambda}$ is a soft scalar, then $\tilde{\lambda}T$ is normal.*

(3) *If T_n is a sequence of normal operators that converges (in the norm of $L(\tilde{H}, \tilde{H})$) to T, then T is normal.*

Proof. Proofs of (1) and (2) are obvious. We prove only (3).

(3). Suppose T_n is a sequence of normal operators that converges (in the norm of $L(\tilde{H}, \tilde{H})$) to T. Then $\|T_n - T\| \to \bar{0}$ as $n \to \infty$.

On one hand, $\|T_n^* - T^*\| = \|(T_n - T)^*\| = \|T_n - T\|$. then $\|T_n^* - T^*\| \to \bar{0}$ as $n \to \infty$. Thus,

$$\begin{aligned}\|T_nT_n^* - TT^*\| &\tilde{\leq} \|T_nT_n^* - T_nT^*\| + \|T_nT^* - TT^*\| \\ &\tilde{\leq} \|T_n\|\|T_n^* - T^*\| + \|T^*\|\|T_n - T\| \to \bar{0} \text{ as } n \to \infty,\end{aligned}$$

because $\|T_n - T\| \to \bar{0}$. So $\|T_n\|$ is bounded. Hence, $T_nT_n^* \to TT^*$.

Similarly, $T_n^*T_n \to T^*T$. Now,

$$\begin{aligned}\|TT^* - T^*T\| &\tilde{\leq} \|TT^* - T_nT_n^*\| + \|T_nT_n^* - T_n^*T_n\| + \|T_n^*T_n - T^*T\| \\ &= \|TT^* - T_nT_n^*\| + \|T_n^*T_n - T^*T\| \to \bar{0} \text{ as } n \to \infty.\end{aligned}$$

Then, $TT^* = T^*T$. Thus T is normal. This proves the theorem. □

4. Unitary and isometric soft linear operators

Definition 4.1. A continuous soft linear operator $T : SE(\tilde{H}) \to SE(\tilde{H})$ is said to be unitary, if it satisfies the condition $TT^* = T^*T = I$, where I is the identity soft linear operator.

Proposition 4.2. *The unitary operators on $\tilde{H}$ are those operators whose inverses are equal to their adjoints.*

Proof. Let T be unitary. Suppose $T(\tilde{x_1}) = T(\tilde{x_2})$. Then operating both sides by the operator T^*, we get $T^*T(\tilde{x_1}) = T^*T(\tilde{x_2})$ and $TT^* = T^*T = I$. Thus $\tilde{x_1} = \tilde{x_2}$. So T is injective.

Also for $\tilde{y}\tilde{\in}\tilde{H}$, $T(T^*(\tilde{y})) = TT^*(\tilde{y}) = I(\tilde{y}) = \tilde{y}$. Then T is surjective. Thus T is bijective. So the result holds. □

Every self-adjoint soft linear operator is normal. Also, it is evident that every unitary operator is normal. We now exhibit an example to show that a normal operator need not be self-adjoint or unitary.

Example 4.3. Let $I : SE(\tilde{H}) \to SE(\tilde{H})$ be the identity soft linear operator. Let $T = \overline{2i}I$. Then $T^* = (\overline{2i}I)^* = -\overline{2i}I$ and also $T^{-1} = -\overline{(\frac{1}{2})i}I$. Thus, $TT^* = T^*T = \overline{4}I, T^* \neq T^{-1}$. So T is normal which is neither self-adjoint nor unitary.

Theorem 4.4. *If $T \in L(\tilde{H}, \tilde{H})$, then the following conditions are equivalent to one another:*
(1) $T^*T = I$.
(2) $\langle T(\tilde{x}), T(\tilde{y})\rangle = \langle \tilde{x}, \tilde{y}\rangle$, *for each* $\tilde{x}, \tilde{y} \tilde{\in} \tilde{H}$.
(3) $\|T(\tilde{x})\| = \|\tilde{x}\|$, *for each* $\tilde{x} \tilde{\in} \tilde{H}$.

Proof. Suppose that (1) is true. Since $\langle T^*T(\tilde{x}), \tilde{y}\rangle = \langle T(\tilde{x}), T(\tilde{y})\rangle$, for each $\tilde{x}, \tilde{y} \tilde{\in} \tilde{H}$, $\langle T(\tilde{x}), T(\tilde{y})\rangle = \langle \tilde{x}, \tilde{y}\rangle$, for each $\tilde{x}, \tilde{y} \tilde{\in} \tilde{H}$. Then (2) holds.

Suppose that (2) is true. Then by making $\tilde{y} = \tilde{x}$, we obtain
$\langle T(\tilde{x}), T(\tilde{x})\rangle = \langle \tilde{x}, \tilde{x}\rangle$ or $\|T(\tilde{x})\|^2 = \|\tilde{x}\|^2$ or $\|T(\tilde{x})\| = \|\tilde{x}\|$.
Thus (3) holds.

Now suppose that (3) is true. Then $\|T(\tilde{x})\| = \|\tilde{x}\|$ or $\|T(\tilde{x})\|^2 = \|\tilde{x}\|^2$ or $\langle T(\tilde{x}), T(\tilde{x})\rangle = \langle \tilde{x}, \tilde{x}\rangle$, i.e., $\langle T^*T(\tilde{x}), \tilde{x}\rangle = \langle \tilde{x}, \tilde{x}\rangle$, i.e., $\langle (T^*T - I)(\tilde{x}), \tilde{x}\rangle = \bar{0}$.
By Theorem 2.46, $T^*T = I$. Thus (1) holds. This proves the theorem. □

Theorem 4.5. *A continuous soft linear operator $T : SE(\tilde{H}) \to SE(\tilde{H})$ is unitary if and only if T is an isomorphism.*

Proof. Suppose T is unitary. Then T is injective and surjective, by Proposition 4.2 and $T^*T = I$. Thus by Theorem 4.4, $\|T(\tilde{x})\| = \|\tilde{x}\|$. So T is an isomorphism.

Conversely, suppose T is an isomorphism. Then T^{-1} exists and $\|T(\tilde{x})\| = \|\tilde{x}\|$. Thus by Theorem 4.4, $T^*T = I$. It follows that $(T^*T)T^{-1} = IT^{-1}$ or $T^* = T^{-1}$ or $TT^* = I$. So $TT^* = T^*T = I$. Hence T is unitary. □

Definition 4.6. A continuous soft linear operator $T : SE(\tilde{H}) \to SE(\tilde{H})$ is called isometric, if $\|T(\tilde{x})\| = \|\tilde{x}\|$, for each $\tilde{x} \tilde{\in} \tilde{H}$.

Theorem 4.4 shows immediately that a unitary soft linear operator is isometric. But the converse is not true as shown by the following example.

Example 4.7. Consider the soft Hilbert space $\tilde{H}$ as defined in Example 2.12. Then a soft element of the soft Hilbert space is of the form $\tilde{x}$, where $\tilde{x}(\lambda) = \{\xi_1^\lambda, \xi_2^\lambda, \xi_3^\lambda, ...\}, \lambda \in A$; $\{\xi_1^\lambda, \xi_2^\lambda, \xi_3^\lambda, ...\}$ being an element of the Hilbert space l_2. Let us define $T : SE(\tilde{H}) \to SE(\tilde{H})$ by $T(\tilde{x}) = \tilde{y}$, where $\tilde{y}(\lambda) = \{0, \xi_1^\lambda, \xi_2^\lambda, ...\}$, for each $\lambda \in A$. Then $\|T(\tilde{x})\| = \|\tilde{x}\|$, for each $\tilde{x} \tilde{\in} \tilde{H}$.

But T is not unitary, because T does not map $SE(\tilde{H})$ onto itself.

5. Square roots of positive operators

Definition 5.1. Let $\tilde{H}$ be a soft Hilbert space and $T : SE(\tilde{H}) \to SE(\tilde{H})$ be a self-adjoint soft linear operator. The operator T is called *positive*, if $\langle T(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \bar{0}$, for each $\tilde{x} \tilde{\in} \tilde{H}$. In notation, we write $T \tilde{\geq} O$.

Let T_1 and T_2 be two self-adjoint soft linear operators on $\tilde{H}$. If for each $\tilde{x} \tilde{\in} \tilde{H}$,
$\langle T_1(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \langle T_2(\tilde{x}), \tilde{x}\rangle$, i.e., $\langle (T_1 - T_2)(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \bar{0}$, then T_1 is called greater than T_2 or T_2 is said to be smaller than T_1. In notation, $T_1 \tilde{\geq} T_2$ or $T_2 \tilde{\leq} T_1$.

If T is a self-adjoint soft linear operator on $\tilde{H}$, then
$\langle T^2(\tilde{x}), \tilde{x}\rangle = \langle T(T(\tilde{x})), \tilde{x}\rangle = \langle T(\tilde{x}), T^*(\tilde{x})\rangle = \langle T(\tilde{x}), T(\tilde{x})\rangle = \|T(\tilde{x})\|^2 \tilde{\geq} \overline{0}$.
Thus $T^2 \tilde{\geq} O$.

If T is any continuous soft linear operator on $\tilde{H}$, then the operators TT^* and T^*T are self adjoint. We have also
$\langle TT^*(\tilde{x}), \tilde{x}\rangle = \langle T^*(\tilde{x}), T^*(\tilde{x})\rangle = \|T^*(\tilde{x})\|^2 \tilde{\geq} \overline{0}$
and
$\langle T^*T(\tilde{x}), \tilde{x}\rangle = \langle T(\tilde{x}), T(\tilde{x})\rangle = \|T(\tilde{x})\|^2 \tilde{\geq} \overline{0}$.
Thus, the operators TT^* and T^*T are always positive.

Theorem 5.2. *The product of two positive permutable soft linear operators S and T is positive.*

Proof. We consider the following two cases:

(Case 1): Let $\|S\|(\lambda) \neq 0$, for any $\lambda \in A$. By theorem 2.42 (4), ST is self-adjoint. We construct a sequence of operators $\{S_n\}$, by the definition

$$(5.1)\quad S_1 = \frac{S}{\|S\|},\ S_2 = S_1 - {S_1}^2,\ S_3 = S_2 - {S_2}^2,\ \cdots,\ S_{n+1} = S_n - S_n^2,\ \cdots \text{and } O \tilde{\leq} S_n \tilde{\leq} I$$

for $n = 1, 2, \cdots$.
Since S is self-adjoint, it is clear that each S_n is self-adjoint.

Let $n = 1$. Then $\langle S_1(\tilde{x}), \tilde{x}\rangle = \frac{1}{\|S\|}\langle S(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \overline{0}$. Thus $S_1 \tilde{\geq} O$ and

$$\langle (I - S_1)(\tilde{x}), \tilde{x}\rangle = \langle \tilde{x}, \tilde{x}\rangle - \langle S_1(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \overline{0},$$

because $\langle S_1(\tilde{x}), \tilde{x}\rangle \tilde{\leq} \|S_1(\tilde{x})\|\|\tilde{x}\| \tilde{\leq} \|S_1\|\|\tilde{x}\|^2 = \|\tilde{x}\|^2 = \langle \tilde{x}, \tilde{x}\rangle$. This shows that $S_1 \tilde{\leq} I$. So, (5.1) is true for $n = 1$.

Let (5.1) be true for $n = k$. Then
$\langle S_k^2(I - S_k)(\tilde{x}), \tilde{x}\rangle = \langle S_k(I - S_k)(\tilde{x}), S_k^*(\tilde{x})\rangle = \langle (I - S_k)S_k(\tilde{x}), S_k(\tilde{x})\rangle \tilde{\geq} \overline{0}$,
because $(I - S_k)$ is positive. Thus, $S_k^2(I - S_k) \tilde{\geq} O$. Similarly, it follows that $S_k(I - S_k)^2 \tilde{\geq} O$. As the sum of two positive operators is clearly positive, it follows that

$$S_{k+1} = S_k^2(I - S_k) + S_k(I - S_k)^2 \tilde{\geq} O.$$

Also, $I - S_{k+1} = (I - S_k) + S_k^2 \tilde{\geq} O$. So, $O \tilde{\leq} S_{k+1} \tilde{\leq} I$, i.e., (5.1) is true for $n = k+1$. Hence (5.1) is true.

Now $S_1 = S_1^2 + S_2 = S_1^2 + S_2^2 + S_3 = \cdots = S_1^2 + S_2^2 + S_3 + \cdots + S_n^2 + S_{n+1}$ and this implies that

$$(5.2)\qquad \sum_{k=1}^{n} S_k^2 = S_1 - S_{n+1} \tilde{\leq} S_1.$$

Then, $S_1 - \sum_{k=1}^{n} S_k^2 \tilde{\geq} O$ or $\langle (S_1 - \sum_{k=1}^{n} S_k^2)(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \overline{0}$ or $\langle S_1(\tilde{x}), \tilde{x}\rangle - \sum_{k=1}^{n}\langle S_k^2(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \overline{0}$ or $\langle S_1(\tilde{x}), \tilde{x}\rangle - \sum_{k=1}^{n}\langle S_k(\tilde{x}), S_k(\tilde{x})\rangle \tilde{\geq} \overline{0}$, because S_k is self-adjoint. Thus we see that
$\sum_{k=1}^{n}\langle S_k(\tilde{x}), S_k(\tilde{x})\rangle \tilde{\leq} \langle S_1(\tilde{x}), \tilde{x}\rangle$,
whatever n may be. This gives that the series $\sum_{k=1}^{n}\langle S_k(\tilde{x}), S_k(\tilde{x})\rangle = \sum_{k=1}^{n}\|S_k(\tilde{x})\|^2$ is convergent. So $\|S_k(\tilde{x})\| \to \overline{0}$ as $k \to \infty$. Hence from (5.2),
$(\sum_{k=1}^{n} S_k^2)(\tilde{x}) = S_1(\tilde{x}) - S_{n+1}(\tilde{x}) \to S_1(\tilde{x})$ as $n \to \infty$.

On the other hand, by continuity of T,

$T(\sum_{k=1}^n S_k^2)(\tilde{x}) \to TS_1(\tilde{x})$ or $\sum_{k=1}^n T(S_k^2)(\tilde{x}) \to TS_1(\tilde{x})$.
Then

$\langle T(\sum_{k=1}^n S_k^2)(\tilde{x}), \tilde{x}\rangle \to \langle TS_1(\tilde{x}), \tilde{x}\rangle$ or $\sum_{k=1}^n \langle TS_k^2(\tilde{x}), \tilde{x}\rangle \to \langle TS_1(\tilde{x}), \tilde{x}\rangle$.

Now T is permutable with S. So T is permutable with S_k. Hence we obtain

$$\begin{aligned}\langle ST(\tilde{x}), \tilde{x}\rangle &= \|S\|\langle TS_1(\tilde{x}), \tilde{x}\rangle = \|S\|\ lim\ \sum_{k=1}^n \langle TS_k^2(\tilde{x}), \tilde{x}\rangle \\ &= \|S\|\ lim\ \sum_{k=1}^n \langle S_k^2 T(\tilde{x}), \tilde{x}\rangle = \|S\|\ lim\ \sum_{k=1}^n \langle S_k T(\tilde{x}), S_k^*(\tilde{x})\rangle \\ &= \|S\|\ lim\ \sum_{k=1}^n \langle TS_k(\tilde{x}), S_k(\tilde{x})\rangle \tilde{\geq} \overline{0},\end{aligned}$$

because T is positive. This shows that ST is a positive operator.

(Case 2): Let $\|S\|(\lambda) = 0$, for some $\lambda \in A$. Let $D \subset A$ be such that $\|S\|(\lambda) = 0$, for each $\lambda \in D$ and $\|S\|(\mu) \neq 0$, for any $\mu \in (A - D)$. Then $\|S_\lambda\|_\lambda = \|S\|(\lambda) = 0$, for each $\lambda \in D$, i.e., $S_\lambda = 0$, for each $\lambda \in D$.

Let us define a soft linear operator W such that $W(\lambda) = I_\lambda$, for each $\lambda \in D$ and $W(\mu) = S(\mu)$, for each $\mu \in (A - D)$. Then $\|W\|(\lambda) \neq 0$, for any $\lambda \in A$. Thus by (Case 1), the operator WT is positive, i.e., $\langle WT(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \overline{0}$, for each $\tilde{x} \tilde{\in} \tilde{H}$, i.e., $\langle WT(\tilde{x}), \tilde{x}\rangle(\lambda) \geq 0$, i.e., $\langle W_\lambda T_\lambda(\tilde{x}(\lambda)), \tilde{x}(\lambda)\rangle_\lambda \geq 0$, for each $\tilde{x} \tilde{\in} \tilde{H}$ and for each $\lambda \in A$.

Now, for each $\tilde{x} \tilde{\in} \tilde{H}$, we have,

$$\langle ST(\tilde{x}), \tilde{x}\rangle(\mu) = \langle S_\mu T_\mu(\tilde{x}(\mu)), \tilde{x}(\mu)\rangle_\mu = \langle W_\mu T_\mu(\tilde{x}(\mu)), \tilde{x}(\mu)\rangle_\mu \geq 0,$$

for each $\mu \in (A - D)$ (Since $W_\mu = W(\mu) = S(\mu) = S_\mu$, for each $\mu \in (A - D)$).
And $\langle ST(\tilde{x}), \tilde{x}\rangle(\lambda) = \langle S_\lambda T_\lambda(\tilde{x}(\lambda)), \tilde{x}(\lambda)\rangle_\lambda = 0$, for each $\lambda \in D$.
So, $\langle ST(\tilde{x}), \tilde{x}\rangle(\lambda) = \langle S_\lambda T_\lambda(\tilde{x}(\lambda)), \tilde{x}(\lambda)\rangle_\lambda \geq 0$, for each $\lambda \in A$, i.e., $\langle ST(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \overline{0}$, i.e., ST is positive.

Hence in either case, the product of two permutable positive soft linear operators is positive. □

Definition 5.3. A sequence $\{T_n\}$ of self-adjoint soft linear operators on a soft Hilbert space $\tilde{H}$ is called monotone increasing, if $T_1 \tilde{\leq} T_2 \tilde{\leq} T_3 \tilde{\leq} \cdots$
and monotone decreasing, if $T_1 \tilde{\geq} T_2 \tilde{\geq} T_3 \tilde{\geq} \cdots$.

Theorem 5.4. *Let $\{T_n\}$ be a sequence of self-adjoint soft linear operators on a soft Hilbert space $\tilde{H}$ such that*

$$T_1 \tilde{\leq} T_2 \tilde{\leq} T_3 \tilde{\leq} \cdots \tilde{\leq} S,$$

where S is a self-adjoint soft linear operator on $\tilde{H}$. Suppose, further that any T_j permutes with S and with every T_m. Then $\{T_n\}$ is strongly convergent and the limit operator is soft linear, bounded and self-adjoint and satisfies $T \tilde{\leq} S$. An analogous result holds for monotone decreasing sequence.

Proof. Let $C_n = S - T_n$. Then

(i) C_n is self-adjoint soft linear operator, because S and T_n are so,

(ii) C_n is a positive operator because $T_n \tilde{\leq} S$,

(iii) since T_j permutes with S and with every T_m, we have $C_n C_m = C_m C_n$, so that C_n's are permutable operators,

(iv) if $n > m$, then $C_m - C_n = T_n - T_m \tilde{\geq} O$, so that C_n is monotonically decreasing sequence.

For $n > m$, the operator $C_m - C_n$ is positive and also C_n and C_m are positive. Thus by Theorem 5.2, their products $C_m(C_m - C_n)$ and $C_n(C_m - C_n)$ are also positive. So,

$\langle (C_m^2 - C_m C_n)(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \overline{0}$ or $\langle C_m^2(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \langle C_m C_n(\tilde{x}), \tilde{x}\rangle$

and

$\langle (C_n C_m - C_n^2)(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \overline{0}$ or $\langle C_n C_m(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \langle C_n^2(\tilde{x}), \tilde{x}\rangle$.

These together imply

$$\langle C_m^2(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \langle C_m C_n(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \langle C_n^2(\tilde{x}), \tilde{x}\rangle. \tag{5.3}$$

Now, C_n^2 is positive. Then $\langle C_n^2(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \overline{0}$. From Equation (5.3), we obtain for $n > m$, $\langle C_n^2(\tilde{x}), \tilde{x}\rangle \tilde{\leq} \langle C_m^2(\tilde{x}), \tilde{x}\rangle$. Thus, the sequence $\{\langle C_n^2(\tilde{x}), \tilde{x}\rangle\}$ is monotonically decreasing sequence of non-negative soft real numbers and so has a limit. Form (5.3), it follows that the sequence

$\{\langle C_m C_n(\tilde{x}), \tilde{x}\rangle\}$ converges to the same limit as $m,\ n \to \infty$.

Since for any self-adjoint soft linear operator T, $\|T(\tilde{x})\|^2 = \langle T^2(\tilde{x}), \tilde{x}\rangle$. Then we have

$$\begin{aligned}
&\|C_m(\tilde{x}) - C_n(\tilde{x})\|^2\\
&= \|(C_m - C_n)(\tilde{x})\|^2\\
&= \langle (C_m - C_n)^2(\tilde{x}), \tilde{x}\rangle\\
&= \langle (C_m^2 - 2C_m C_n + C_n^2)(\tilde{x}), \tilde{x}\rangle\\
&= \langle C_m^2(\tilde{x}), \tilde{x}\rangle - 2\langle C_m C_n(\tilde{x}), \tilde{x}\rangle + \langle C_n^2(\tilde{x}), \tilde{x}\rangle \to \overline{0} \text{ as } m,\ n \to \infty.
\end{aligned}$$

Thus, $\{C_n(\tilde{x})\}$ is a Cauchy sequence and therefore convergent. Because $C_n = S - T_n$, i.e., $C_n(\tilde{x}) = S(\tilde{x}) - T_n(\tilde{x})$, this implies that $\{T_n(\tilde{x})\}$ converges to $T(\tilde{x})$, say for arbitrary $\tilde{x} \tilde{\in} \tilde{H}$, i.e., $T(\tilde{x}) = \lim T_n(\tilde{x})$ for $\tilde{x} \tilde{\in} \tilde{H}$. This defines a soft linear operator $T : SE(\tilde{H}) \to SE(\tilde{H})$ which is clearly soft linear. Uniform boundedness principle (Theorem 2.35) implies that T is bounded. Also T is self-adjoint, because T_n is self-adjoint and the soft inner product is continuous.

Now, for each n, $\langle (S - T_n)(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \overline{0}$ or $\langle S(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \langle T_n(\tilde{x}), \tilde{x}\rangle$.

Since $T_n(\tilde{x}) \to T(\tilde{x})$, we have $\langle S(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \langle T(\tilde{x}), \tilde{x}\rangle$, i.e., $\langle (S - T)(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \overline{0}$ or $T \tilde{\leq} S$.

This proves the theorem. □

Definition 5.5. Let $T : SE(\tilde{H}) \to SE(\tilde{H})$ be a positive soft linear operator on a soft Hilbert space $\tilde{H}$. Then a self-adjoint soft linear operator S is called a square root of T, if $S^2 = T$. If, in addition $S \tilde{\geq} O$, then S is called a positive square root of T and is denoted by $S = T^{\frac{1}{2}}$.

Theorem 5.6. *The positive square root S of an arbitrary positive self-adjoint soft linear operator T exists and is unique. It is permutable with each operator permutable with T.*

Proof. We consider the following two cases.

(Case 1): Let $\|T\|(\lambda) \neq 0$, for each $\lambda \in A$.

We can further assume that $T \tilde{\leq} I$, where I is the identity operator. Because if not, we can start with the operator T_1, where $T_1 = \frac{T}{\|T\|}$, then $\|T_1\| = \overline{1}$. By Schwarz inequality, $\langle T_1(\tilde{x}), \tilde{x}\rangle \tilde{\leq} \|T_1\| \|\tilde{x}\|^2 = \|\tilde{x}\|^2 = \langle \tilde{x}, \tilde{x}\rangle$, i.e., $\langle I(\tilde{x}) - T_1(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \overline{0}$ or $\langle (I - T_1)(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \overline{0}$. Then $T_1 \tilde{\leq} I$.

We now construct a sequence of operators by

$S_0 = O$,

$S_1 = S_0 + \frac{1}{2}(T - S_0^2) = \frac{1}{2}T$,

$S_2 = S_1 + \frac{1}{2}(T - S_1^2)$,

$\dots \quad \dots \quad \dots$

$$S_{n+1} = S_n + \frac{1}{2}(T - S_n^2) \tag{5.4}$$

and so on. Because T is self-adjoint and square of a self-adjoint soft linear operator is self-adjoint, it follows that S_n are self-adjoint. It also follows that all S_n are permutable with every operator permutable with T. In particular, we have $S_nT = TS_n$ and $S_mT = TS_m$ and thus $S_nS_m = S_mS_n$.

Now $\frac{1}{2}(I - S_n)^2 + \frac{1}{2}(I - T) = I - [S_n + \frac{1}{2}(T - S_n^2)] = I - S_{n+1}$ and so

$$S_n \tilde{\leq} I, \quad \forall n. \tag{5.5}$$

Further using (5.4), we obtain that

$$S_{n+1} - S_n = \frac{1}{2}[(I - S_{n-1}) + (I - S_n)](S_n - S_{n-1}). \tag{5.6}$$

We now show that $S_n \tilde{\leq} S_{n+1}$ for each n. Equality (5.6) shows that $S_{n+1} - S_n \tilde{\geq} O$ if $S_n - S_{n-1} \tilde{\geq} O$. But $S_1 = \frac{1}{2}T \tilde{\geq} O = S_0$. Then $S_n \tilde{\leq} S_{n+1}$, for each n. Thus, we obtain a monotonically increasing sequence $\{S_n\}$ of self-adjoint operators

$$S_0 \tilde{\leq} S_1 \tilde{\leq} S_2 \tilde{\leq} \cdots \tilde{\leq} S_n \tilde{\leq} \cdots \tilde{\leq} I.$$

By Theorem 5.4, this sequence converges to an operator S which is self-adjoint.

We now show that S is positive. We have $S_1 = \frac{1}{2}T$ is positive and because the sequence $\{S_n\}$ is monotone increasing, each S_n is positive. Then $\langle S_n(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \bar{0}$ for each n. Passing to the limit, $\langle S(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \bar{0}$. Thus S is positive. Letting $n \to \infty$ in (5.4), we get $S = S + \frac{1}{2}(T - S^2)$, i.e., $S^2 = T$.

(Case 2): Let $\|T\|(\lambda) = 0$, for some $\lambda \in A$. Let $D \subset A$ be such that $\|T\|(\lambda) = 0$, for each $\lambda \in D$ and $\|T\|(\mu) \neq 0$, for any $\mu \in (A - D)$. Then $\|T_\lambda\|_\lambda = \|T\|(\lambda) = 0$, for each $\lambda \in D$, i.e., $T_\lambda = 0$, for each $\lambda \in D$. Let us define a soft linear operator W such that $W(\lambda) = I_\lambda$, for each $\lambda \in D$ and $W(\mu) = T(\mu)$, for each $\mu \in (A - D)$. Then $\|W\|(\lambda) \neq 0$, for any $\lambda \in A$. Thus by (Case 1), there exists a positive soft linear operator V such that $V^2 = W$.

Let us consider the soft linear operator S defined by $S(\mu) = V(\mu)$, for each $\mu \in (A - D)$ and $S(\lambda) = 0$, for all $\lambda \in D$. Then $S^2 = T$. Let $\tilde{x} \tilde{\in} \tilde{H}$ and let $\mu \in (A - D)$. Since $V_\mu = V(\mu) = S(\mu) = S_\mu$,

$$\langle S(\tilde{x}), \tilde{x}\rangle(\mu) = \langle S_\mu(\tilde{x}(\mu)), \tilde{x}(\mu)\rangle_\mu = \langle V_\mu(\tilde{x}(\mu)), \tilde{x}(\mu)\rangle_\mu \geq 0.$$

Furthermore, $\langle S(\tilde{x}), \tilde{x}\rangle(\lambda) = \langle S_\lambda(\tilde{x}(\lambda)), \tilde{x}(\lambda)\rangle_\lambda = 0$, for each $\lambda \in D$.
Then, $\langle S(\tilde{x}), \tilde{x}\rangle(\lambda) = \langle S_\lambda(\tilde{x}(\lambda)), \tilde{x}(\lambda)\rangle_\lambda \geq 0$, for each $\lambda \in A$. Thus $\langle S(\tilde{x}), \tilde{x}\rangle \tilde{\geq} \bar{0}$. So S is positive and hence S is a positive square root of T.

Therefore in either case, the existence of a positive square root S of the operator T is obtained.

Now, S_n is permutable with every operator permutable with T. Let the operator C permute with T, then $S_nC = CS_n$ i.e., $S_nC(\tilde{x}) = CS_n(\tilde{x})$, for each $\tilde{x} \tilde{\in} \tilde{H}$. Taking limit, $SC(\tilde{x}) = CS(\tilde{x})$, for each $\tilde{x} \tilde{\in} \tilde{H}$, i.e., $SC = CS$. So, S is permutable with C, i.e., S is permutable with every operator permutable with T.

We now prove the uniqueness. Let S_1 be another positive square root of T. Since S_1 permutes with T, by the preceding observation, $S_1S = SS_1$.

If $\tilde{x}\tilde{\in}\tilde{H}$ and $\tilde{y} = (S - S_1)\tilde{x}$, then

$\langle S(\tilde{y}), \tilde{y}\rangle + \langle S_1(\tilde{y}), \tilde{y}\rangle = \langle (S + S_1)(\tilde{y}), \tilde{y}\rangle = \langle (S^2 - S_1^2)(\tilde{x}), \tilde{y}\rangle = \langle (T - T)(\tilde{x}), \tilde{y}\rangle = \overline{0}$.

Since both S and S_1 are positive, it follows that $\langle S(\tilde{y}), \tilde{y}\rangle = \overline{0}$ and $\langle S_1(\tilde{y}), \tilde{y}\rangle = \overline{0}$. Because S is positive, by what we have already proved, there exists a self-adjoint soft linear operator C such that $S = C^2$. Thus,

$$\begin{aligned}\|C(\tilde{y})\|^2 &= \langle C(\tilde{y}), C(\tilde{y})\rangle = \langle \tilde{y}, C^*C(\tilde{y})\rangle \\ &= \langle \tilde{y}, C^2(\tilde{y})\rangle = \langle \tilde{y}, S(\tilde{y})\rangle = \overline{0}, \text{ i.e., } C(\tilde{y}) = \Theta.\end{aligned}$$

So $S(\tilde{y}) = C^2(\tilde{y}) = C(C(\tilde{y})) = \Theta$. Similarly, $S_1(\tilde{y}) = \Theta$. Hence for $\tilde{x}\tilde{\in}\tilde{H}$,

$\|S(\tilde{x}) - S_1(\tilde{x})\|^2 = \langle (S - S_1)^2(\tilde{x}), \tilde{x}\rangle = \langle (S - S_1)(\tilde{y}), \tilde{x}\rangle = \langle S(\tilde{y}), \tilde{x}\rangle - \langle S_1(\tilde{y}), \tilde{x}\rangle = \overline{0}$.

Therefore, $S(\tilde{x}) = S_1(\tilde{x})$, for each $\tilde{x}\tilde{\in}\tilde{H}$, i.e., $S = S_1$. This proves the uniqueness of the square root. □

6. Conclusions

The concept of operator plays a very important role in many aspects of linear algebra and functional analysis. In the dynamics of quantum theory, one must study operators on infinite dimensional Hilbert spaces. On the other hand, the usual uncertainty principle of Heisenberg ultimates generalized uncertainty principle, this has been motivated by string theory and non-commutative geometry. In string quantum gravity regime space-time points are determined in a fuzzy manner. Thus Hilbert spaces and operators on Hilbert spaces involving the uncertainties need to be developed. In this regard it is to be noted that the study of operator theory on fuzzy inner product spaces is limited since in fuzzy setting complex valued inner product space is not so developed. In soft set settings it has been possible to develop the concept of soft inner product nicely. A concept of operator theory on soft inner product spaces is also introduced in [13]. In this paper we have further extended the operator theory on soft inner product spaces. This concept can be extended to spectral theory of bounded self-adjoint operators and unbounded linear operators on soft Hilbert spaces. A generalization of quantum mechanics can be done by using the generalized unbounded linear operators on soft inner product spaces. There is an ample scope for further research on operators on soft inner product spaces.

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