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# On fuzzy soft $\delta$- open sets and fuzzy soft $\delta$- continuity

Prakash Mukherjee, Ramkrishna Prasad Chakraborty, Choonkil Park

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Abstract. In this paper, we introduce fuzzy soft regularly open sets, fuzzy soft regularly closed sets. Fuzzy soft $\delta$- closure operation are introduced and studied, it is observed that fuzzy soft $\delta$- closure operator on a fuzzy soft topological space $(X, E, \tau)$ satisfies the Kuratowski closure axioms. Fuzzy soft $\delta$- closed sets, fuzzy soft $\delta$- open sets, fuzzy soft $\delta$- *nbd*, fuzzy soft $\delta$- interior are introduced along with fuzzy soft $\delta$- continuous functions and fuzzy soft almost continuous functions. An attempt has made to characterize them in different ways.

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Corresponding Author: Prakash Mukherjee (prakashmukherjee25@gmail.com)

## 1. Introduction

Most of our real life problems in engineering, social and medical science, economics, environment, etc. involve imprecise data and their solutions involve the use of mathematical principles based on uncertainty and imprecision. To handle such uncertainties, Zadeh [18] introduced the concept of fuzzy sets and fuzzy set operations. The analytical part of fuzzy set theory was practically started with the paper of Chang [7] introduced the concept of fuzzy topological space but this theory is associated with an inherent limitation, which is the inadequacy of the parametrization tool associated with this theory as it was mentioned by Molodtsov in [11]. In 1999, Molodtsov [11] introduced the concept of soft set theory which is free from the above problems and started to develop the basics of the corresponding theory as a new approach for modelling uncertainties. Shabir and Naz [15] studied the topological structures of soft sets. In recent times, many researchers have contributed a lot towards fuzzification of soft set theory. In 2001, Maji, Roy and Biswas [10] introduced fuzzy soft set which is a combination of fuzzy set and soft set. Researchers got interest to improve this theory and tried to apply it into different physical problems. Aygunoglu and Aygun [3] applied fuzzy soft sets on group theory. Kharal and Ahmad [9] defined the notion of a mapping on classes of fuzzy soft sets, which is of fundamental importance in fuzzy soft set theory; they have also studied the properties of fuzzy soft images and fuzzy soft inverse images of fuzzy soft sets. Tanay and Burc Kandemir [16] introduced topological structure of fuzzy soft set and gave an introductory theoretical base to carry further study on this concept. The study, was pursued by some others (see [1, 2, 4, 5, 6, 8, 12, 13, 14, 17], for instance). The objective of this paper is divided into three parts. In the first part, we introduce fuzzy soft regularly open and fuzzy soft regularly closed sets and investigate some of their properties. In the second part, $\delta$- open and $\delta$- closed sets on fuzzy soft topological spaces are introduced and studied. Furthermore, fuzzy soft $\delta$- closure operation is introduced and studied and observed that fuzzy soft $\delta$- closure operator on a fuzzy soft topological space $(X, E, \tau)$ satisfies the Kuratowski closure axioms. Finally, an attempt has made to introduce fuzzy soft $\delta$- continuous mapping together with fuzzy soft almost continuous mapping and characterize them in different ways.

## 2. Preliminaries

Throughout this paper $X$ denotes the initial universe, $E$ denotes the set of all possible parameters for $X$. $P(X)$ denotes the power set of $X$, $I^X$ denotes the set of all fuzzy sets on $X$ and $I$ stands for $[0, 1]$.

**Definition 2.1** ([18]). A fuzzy set $A$ in $X$ is defined by a membership function $\mu_A : X \to [0, 1]$ whose value $\mu_A(x)$ represents the "grade of membership" of $x$ in $A$ for $x \in X$.

If $A$, $B \in I^X$ then from [18] we have the following:
(i) $A \leq B \Leftrightarrow \mu_A(x) \leq \mu_B(x),\ \forall\ x \in X$.
(ii) $A = B \Leftrightarrow \mu_A(x) = \mu_B(x),\ \forall\ x \in X$.
(iii) $C = A \vee B \Leftrightarrow \mu_C(x) = \max(\mu_A(x), \mu_B(x)),\ \forall\ x \in X$.
(iv) $D = A \wedge B \Leftrightarrow \mu_D(x) = \min(\mu_A(x), \mu_B(x)),\ \forall\ x \in X$.
(v) $E = A^c \Leftrightarrow \mu_E(x) = 1 - \mu_A(x),\ \forall\ x \in X$.

**Definition 2.2** ([11]). Let $A \subseteq E$. A pair $(F, A)$ is called a soft set over $X$, where $F$ is a mapping from $A$ into $P(X)$, i.e., $F : A \to P(X)$.

In other words, a soft set is a parameterized family of subsets of the set $X$. For $e \in A$, $F(e)$ may be considered as the set of $e-$approximate elements of the soft set $(F, A)$.

**Definition 2.3** ([10]). Let $A \subseteq E$. A pair $(f, A)$ is called a fuzzy soft set over $X$, where $f : A \to I^X$ is a function, i.e., for each $a \in A$, $f(a) = f_a : X \to [0, 1]$ is a fuzzy set on $X$.

We will use $FS(X, E)$ to denote the family of all fuzzy soft sets over $X$.

Modification in the above definition was done by Roy and Samanta [14],which goes as follows.

**Definition 2.4** ([14]). Let $A \subseteq E$. A fuzzy soft set $f_A$ over $X$ is a mapping from the parameter set $E$ to $I^X$, i.e., $f_A : E \to I^X$, where $f_A(e) \neq 0_X$ if $e \in A \subseteq E$ and $f_A(e) = 0_X$ if $e \notin A$,where $0_X$ denotes the empty fuzzy set on $X$.

**Definition 2.5** ([14]). The fuzzy soft set $f_\phi \tilde{\in} FS(X,E)$ is called null fuzzy soft set, denoted by $\tilde{0}_E$, if for all $e \in E$, $f_A(e) = 0_X$.

**Definition 2.6** ([14]). Let $f_E \tilde{\in} FS(X,E)$. The fuzzy soft set $f_E$ is called universal fuzzy soft set, denoted by $\tilde{1}_E$, if for all $e \in E$, $f_E(e) = 1_X$, where $1_X(x) = 1$ for all $x \in X$.

**Definition 2.7** ( [10]). Let $A, B \subseteq E$ and $f_A, g_B \tilde{\in} FS(X,E)$. We say that $f_A$ is a fuzzy soft subset of $g_B$ and write $f_A \tilde{\subseteq} g_B$ if and only if

(1) $A \subseteq B$,

(2) For every $e \in E$, $f_A(e) \leq g_B(e)$.

**Definition 2.8** ([10]). Let $f_A, g_B \tilde{\in} FS(X,E)$. Then $f_A$ and $g_B$ are said to be equal, denoted by $f_A = g_B$, if $f_A \tilde{\subseteq} g_B$ and $g_B \tilde{\subseteq} f_A$.

**Definition 2.9** ([10]). Let $f_A, g_B \tilde{\in} FS(X,E)$. Then the union of $f_A$ and $g_B$, denoted by $h_C = f_A \sqcup g_B$, where $C = A \cup B$ and $h_C$ is defined by

$$h_C = \begin{cases} f_A(e) & \text{if e} \in A - B, \\ g_B(e) & \text{if e} \in B - A, \\ f_A(e) \vee g_B(e) & \text{if e} \in A \cap B. \end{cases}$$

**Definition 2.10** ([3]). Let $f_A, g_B \tilde{\in} FS(X,E)$. Then the intersection of $f_A$ and $g_B$ is also a fuzzy soft set $h_C$, defined by $h_C(e) = f_A(e) \wedge g_B(e)$ for all $e \in E$, where $C = A \cap B$. Here we write $h_C = f_A \sqcap g_B$.

**Definition 2.11** ([10]). Let $f_A \tilde{\in} FS(X,E)$. Then the complement of $f_A$, denoted by ${f_A}^c$, is a fuzzy soft set defined by ${f_A}^c(e) = 1_X \setminus f_A(e), \forall e \in E$.
Let us call ${f_A}^c$ to be the fuzzy soft complement of $f_A$ in $FS(X,E)$.

Clearly, $({f_A}^c)^c = f_A$, $(\tilde{1}_E)^c = \tilde{0}_E$ and $(\tilde{0}_E)^c = \tilde{1}_E$.

**Theorem 2.12** ([1]). *Let $f_A, g_B \tilde{\in} FS(X,E)$. Then the following holds:*

*(1)* $(f_A \sqcup g_B)^c = {f_A}^c \sqcap {g_B}^c$,

*(2)* $(f_A \sqcap g_B)^c = {f_A}^c \sqcup {g_B}^c$.

**Definition 2.13.** [2] A fuzzy soft set $f_A \tilde{\in} FS(X,E)$ is called fuzzy soft point if $A = \{e\} \subseteq E$ and $f_A(e)$ is a fuzzy point in $X$, i.e., there exists $x \in X$ such that $f_A(e)(x) = \alpha (0 < \alpha \leq 1)$ and $f_A(e)(y) = 0$ for all $y \in X - \{x\}$. We denote by $e_x^\alpha$.

**Definition 2.14** ([2]). Let $f_A, g_B \tilde{\in} FS(X,E)$. Then $f_A$ is said to be soft quasi-coincident with $g_B$, denoted by $f_A \tilde{q} g_B$, if there exist $e \in E$ and $x \in X$ such that $f_A(e)(x) + g_B(e)(x) > 1$.

If $f_A$ is not soft quasi-coincident with $g_B$, then we write $f_A \bar{\tilde{q}} g_B$.

**Definition 2.15** ([9]). Let $FS(X,E)$ and $FS(Y,K)$ be families of all fuzzy soft sets over $X$ and $Y$, respectively. Let $u : X \to Y$ and $p : E \to K$ be two functions. Then $f_{up}$ is called a fuzzy soft mapping from $FS(X,E)$ to $FS(Y,K)$ and denoted by $f_{up} : FS(X,E) \to FS(Y,K)$ .
(1) Let $f_A \tilde{\in} FS(X,E)$. Then the image of $f_A$ under the fuzzy soft mapping $f_{up}$ is the fuzzy soft set over $Y$ defined by $f_{up}(f_A)$, where

$$f_{up}(f_A)(k)(y) = \begin{cases} \bigvee\limits_{x\in u^{-1}(y)} (\bigvee\limits_{e\in p^{-1}(k)\cap A} f_A(e))(x) & \text{if } \mathrm{u^{-1}(y)} \neq \phi \text{ and if } \mathrm{p^{-1}(k)} \cap \mathrm{A} \neq \phi \\ 0_Y & \text{otherwise.} \end{cases}$$

(2) Let $g_B \tilde{\in} FS(X,E)$ . Then the pre-image of $g_B$ under the fuzzy soft mapping $f_{up}$ is the fuzzy soft set over $X$ defined by ${f_{up}}^{-1}(g_B)$, where

$${f_{up}}^{-1}(g_B)(e)(x) = \begin{cases} g_B(p(e))(u(x)) & \text{for } \mathrm{p(e)} \in \mathrm{B} \\ 0_X & \text{otherwise.} \end{cases}$$

If $u$ and $p$ are injective then the fuzzy soft mapping $f_{up}$ is said to be injective. If $u$ and $p$ are surjective then the fuzzy soft mapping $f_{up}$ is said to be surjective. The fuzzy soft mapping $f_{up}$ is called constant if $u$ and $p$ are constant.

**Theorem 2.16** ([9]). *Let $f_A \tilde{\in} FS(X,E)$, $\{f_{A_i}\}_{i\in j} \tilde{\subseteq} FS(X,E)$ and $g_B \tilde{\in} FS(Y,K)$, $\{g_{B_i}\}_{i\in j} \tilde{\subseteq} FS(Y,K)$, where $J$ is an index set.*
*(1) $f_{up}(\bigwedge\limits_{i\in j} f_{A_i}) = \bigwedge\limits_{i\in j} f_{up}(f_{A_i})$, if $f_{up}$ is injective.*
*(2) $f_{up}(\tilde{1}_E) = \tilde{1}_K$, if $f_{up}$ is surjective.*
*(3) ${f_{up}}^{-1}({f_A}^c) = ({f_{up}}^{-1}(f_A))^c$.*

**Definition 2.17** ([16]). A fuzzy soft topological space is a triple $(X,E,\tau)$ where $X$ is a non-empty set and $\tau$ is a family of fuzzy soft sets over $(X,E)$ satisfying the following properties:

(1) $\tilde{0}_E, \tilde{1}_E \tilde{\in} \tau$;
(2) If $f_A, g_B \tilde{\in} \tau$, then $f_A \sqcap g_B \tilde{\in} \tau$;
(3) If $f_{A_\alpha} \tilde{\in} \tau$ for all $\alpha \in \Lambda$, an index set, then $\bigsqcup\limits_{\alpha\in\Lambda} f_{A_\alpha} \tilde{\in} \tau$.

Then $\tau$ is called a fuzzy soft topology over $(X,E)$. Also each member of $\tau$ is called a fuzzy soft open set in $(X,E,\tau)$.

$g_B$ is called fuzzy soft closed in $(X,E,\tau)$ if $(g_B)^c \tilde{\in} \tau$.

**Definition 2.18** ([17]). Let $(X,E,\tau)$ be a fuzzy soft topological space and $f_A \tilde{\in} FS(X,E)$. The fuzzy soft closure of $f_A$, denoted by $\overline{f_A}$, is the intersection of all fuzzy soft closed supersets of $f_A$.

Clearly, $\overline{f_A}$ is the smallest fuzzy soft closed set over $(X,E)$ which contains $f_A$.

**Definition 2.19** ([16]). Let $(X,E,\tau)$ be a fuzzy soft topological space and $f_A \tilde{\in} FS(X,E)$. The fuzzy soft interior of $f_A$, denoted by ${f_A}^0$, is the union of all fuzzy soft open subsets of $f_A$.

Clearly, ${f_A}^0$ is the largest fuzzy soft open set over $(X,E)$ which contained in $f_A$.

**Theorem 2.20** ([8]). *Let $(X, E, \tau)$ be a fuzzy soft topological space and $f_A \tilde{\in} FS(X, E)$. Then*
(1) $({f_A}^0)^c = \overline{({f_A}^c)}$,
(2) $(\overline{f_A})^c = ({f_A}^c)^0$.

**Definition 2.21** ([2]). Let $(X, E, \tau)$ be a fuzzy soft topological space. Then $f_A \tilde{\in} FS(X, E)$ is called fuzzy soft $q$-neighbourhood(briefly, fuzzy soft *q-nbd*) of $g_B$ if and only if there exists a fuzzy soft open set $h_C$ in $\tau$ such that $g_B \tilde{q} h_C \tilde{\subseteq} f_A$.

If, in addition, $f_A$ is fuzzy soft open, then $f_A$ is called a fuzzy soft open *q-nbd* of $g_B$.

**Definition 2.22** ([6]). Let $(X, E, \tau)$ be a fuzzy soft topological space and $f_A \tilde{\in} FS(X, E)$. Then $f_A$ is called

(1) a fuzzy soft semi-open set in $(X, E, \tau)$ if and only if there exist a $g_B \tilde{\in} \tau$ such that $g_B \tilde{\subseteq} f_A \tilde{\subseteq} \overline{g_B}$;

(2) a fuzzy soft semi-closed set in $(X, E, \tau)$ if and only if there is a fuzzy soft closed set $h_C$ in $(X, E, \tau)$ such that ${h_C}^0 \tilde{\subseteq} f_A \tilde{\subseteq} h_C$.

**Definition 2.23** ([4]). Let $(X, E, \tau_1)$ and $(X, K, \tau_2)$ be two fuzzy soft topological spaces. A fuzzy soft function $f_{up} : (X, E, \tau_1) \to (X, K, \tau_2)$ is called a fuzzy soft continuous if ${f_{up}}^{-1}(g_B) \tilde{\in} \tau_1$ for all $g_B \tilde{\in} \tau_2$.

## 3. Fuzzy soft regularly open and regularly closed sets

In this section, we introduce fuzzy soft regularly open and fuzzy soft regularly closed sets and investigate some of their properties.

**Definition 3.1.** Let $(X, E, \tau)$ be a fuzzy soft topological space and $f_A \tilde{\in} FS(X, E)$. Then $f_A$ is called a fuzzy soft regularly open set in $(X, E, \tau)$ if and only if $(\overline{f_A})^0 = f_A$.

**Definition 3.2.** Let $(X, E, \tau)$ be a fuzzy soft topological space and $f_A \tilde{\in} FS(X, E)$. Then $f_A$ is called a fuzzy soft regularly closed set in $(X, E, \tau)$ if and only if $\overline{{f_A}^0} = f_A$.

The following remark follows from Theorem 2.20.

**Remark 3.3.** A fuzzy soft set $f_A$ in a fuzzy soft topological space $(X, E, \tau)$ is fuzzy soft regularly open if and only if ${f_A}^c$ is fuzzy soft regularly closed.

**Remark 3.4.** $\tilde{0}_E$ and $\tilde{1}_E$ are always fuzzy soft regularly open and fuzzy soft regularly closed sets.

**Remark 3.5.** It is clear from definition that in a fuzzy soft topological space $(X, E, \tau)$, every fuzzy soft regularly open set is fuzzy soft open set, but the converse is not true, which follows from the following example.

**Example 3.6.** Let $X = \{a, b, c\}$, and $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2\} \subseteq E$, $B = \{e_1, e_2, e_3\} \subseteq E$.
Let us consider the following fuzzy soft sets over $(X, E)$.

$$f_A = \{f(e_1) = \{a/0.5, b/0.2, c/0\}, f(e_2) = \{a/0.7, b/0.6, c/0.3\}, f(e_3) = \{a/0, b/0, c/0\}, f(e_4) = \{a/0, b/0, c/0\}\},$$

$$g_B = \{g(e_1) = \{a/0.5, b/0.3, c/0\}, g(e_2) = \{a/0.7, b/0.8, c/0.5\},$$

$g(e_3) = \{a/0.4, b/0.9, c/0.8\}, g(e_4) = \{a/0, b/0, c/0\}\}$.

Let us consider the fuzzy soft topology $\tau = \{\tilde{0}_E, \tilde{1}_E, f_A, g_B\}$ over $(X, E)$.

Now, ${f_A}^c = \{f^c(e_1) = \{a/0.5, b/0.8, c/1\}, f^c(e_2) = \{a/0.3, b/0.4, c/0.7\}, f^c(e_3) = \{a/1, b/1, c/1\}, f^c(e_4) = \{a/1, b/1, c/1\}\}$ and

${g_B}^c = \{g^c(e_1) = \{a/0.5, b/0.7, c/1\}, g^c(e_2) = \{a/0.3, b/0.2, c/0.5\}, g^c(e_3) = \{a/0.6, b/0.1, c/0.2\}, g^c(e_4) = \{a/1, b/1, c/1\}\}$.

Clearly, ${f_A}^c$ and ${g_B}^c$ are fuzzy soft closed sets.

Then the fuzzy soft closure of $f_A$ is the intersection of all fuzzy soft closed sets containing $f_A$.

That is, $\overline{f_A} = \tilde{1}_E$.

The fuzzy soft interior of $\overline{f_A}$ is the union of all fuzzy soft open sets contained in $\overline{f_A}$.

That is, $(\overline{f_A})^0 = (\tilde{1}_E)^0 = \tilde{1}_E$.

Hence $f_A$ is open but not a fuzzy soft regularly open set.

**Theorem 3.7.** *(a) The intersection of two fuzzy soft regular open sets is a fuzzy soft regular open set.*

*(b) The union of two fuzzy soft regular closed sets is a fuzzy soft regular closed set.*

*Proof.* (a) Let $f_A$ and $g_B$ be any two fuzzy soft regular open sets of a fuzzy soft topological space $(X, E, \tau)$. Then it is clear that $(\overline{(f_A \sqcap g_B)})^0 \tilde{\subseteq} (\overline{f_A})^0 = f_A$ and $(\overline{(f_A \sqcap g_B)})^0 \tilde{\subseteq} (\overline{g_B})^0 = g_B$.

(b) The proof is obvious. □

**Theorem 3.8.** *(a) The closure of a fuzzy soft open set is a fuzzy soft regular closed set.*

*(b) The interior of a fuzzy soft closed set is a fuzzy soft regular open set.*

*Proof.* (a) Let $f_A$ be a fuzzy soft open set of a fuzzy soft topological space $(X, E, \tau)$. Clearly, $(\overline{f_A})^0 \tilde{\subseteq} \overline{f_A}$ implies that $\overline{(\overline{f_A})^0} \tilde{\subseteq} \overline{f_A}$. Now, the fact that $f_A$ is fuzzy soft open implies that $f_A \tilde{\subseteq} (\overline{f_A})^0$ and $\overline{f_A} \tilde{\subseteq} \overline{(\overline{f_A})^0}$.

Thus $\overline{f_A}$ is a fuzzy soft regular closed set.

(b) The proof is obvious. □

**Remark 3.9.** The union (intersection) of any two fuzzy soft regular open (closed) sets need not be a fuzzy soft regular open (closed) set.

**Example 3.10.** Let $X = \{a, b\}$, $E = \{e_1, e_2, e_3\}$, $A = \{e_1, e_2\} \subseteq E$, $B = \{e_1, e_3\} \subseteq E$, $C = \{e_2, e_3\} \subseteq E$, $D = \{e_1\} \subseteq E$, $F = \{e_2\} \subseteq E$, $G = \{e_3\} \subseteq E$.

Let us consider the following fuzzy soft sets over $(X, E)$.

$f_A = \{f(e_1) = \{a/0.5, b/0.6\}, f(e_2) = \{a/0.7, b/0.4\}, f(e_3) = \{a/0, b/0\}\}$,

$g_B = \{g(e_1) = \{a/0.5, b/0.6\}, g(e_2) = \{a/0, b/0\}, g(e_3) = \{a/0.4, b/0.7\}\}$,

$h_C = \{h(e_1) = \{a/0, b/0\}, h(e_2) = \{a/0.7, b/0.4\}, h(e_3) = \{a/0.4, b/0.7\}\}$,

$i_E = \{i(e_1) = \{a/0.5, b/0.6\}, i(e_2) = \{a/0.7, b/0.4\}, i(e_3) = \{a/0.4, b/0.7\}\}$,

$j_D = \{j(e_1) = \{a/0.5, b/0.6\}, j(e_2) = \{a/0, b/0\}, j(e_3) = \{a/0, b/0\}\}$,

$k_F = \{k(e_1) = \{a/0, b/0\}, k(e_2) = \{a/0.7, b/0.4\}, k(e_3) = \{a/0, b/0\}\},$

$u_G = \{u(e_1) = \{a/0, b/0\}, u(e_2) = \{a/0, b/0\}, u(e_3) = \{a/0.4, b/0.7\}\}.$

Let us consider the fuzzy soft topology $\tau = \{\tilde{0}_E, \tilde{1}_E, f_A, g_B, h_C, i_E, j_D, k_F, u_G\}$ over $(X, E)$.

Now, ${f_A}^c = \{f^c(e_1) = \{a/0.5, b/0.4\}, f^c(e_2) = \{a/0.3, b/0.6\}, f^c(e_3) = \{a/1, b/1\}\},$

${g_B}^c = \{g^c(e_1) = \{a/0.5, b/0.4\}, g^c(e_2) = \{a/1, b/1\}, g^c(e_3) = \{a/0.6, b/0.3\}\},$

${h_C}^c = \{h^c(e_1) = \{a/1, b/1\}, h^c(e_2) = \{a/0.3, b/0.6\}, h^c(e_3) = \{a/0.6, b/0.3\}\},$

${i_E}^c = \{i^c(e_1) = \{a/0.5, b/0.4\}, i^c(e_2) = \{a/0.3, b/0.6\}, i^c(e_3) = \{a/0.6, b/0.3\}\},$

${j_D}^c = \{j^c(e_1) = \{a/0.5, b/0.4\}, j^c(e_2) = \{a/1, b/1\}, j^c(e_3) = \{a/1, b/1\}\},$

${k_F}^c = \{k^c(e_1) = \{a/1, b/1\}, k^c(e_2) = \{a/0.3, b/0.6\}, k^c(e_3) = \{a/1, b/1\}\},$

${u_G}^c = \{u^c(e_1) = \{a/1, b/1\}, u^c(e_2) = \{a/1, b/1\}, u^c(e_3) = \{a/0.6, b/0.3\}\}.$

Clearly, $f_A$ and $g_B$ are fuzzy soft regularly open sets.

Since, $f_A \sqcup g_B = i_E$, the fuzzy soft closure of $i_E$ is the intersection of all fuzzy soft closed set containing $i_E$.

That is, $\overline{i_E} = \tilde{1}_E$, $(\overline{i_E})^0 = (\tilde{1}_E)^0 = \tilde{1}_E$.

Hence, $i_E$ is not a fuzzy soft regularly open set.

Thus the union of any two fuzzy soft regularly open sets need not be fuzzy soft regularly open.

**Remark 3.11.** In a fuzzy soft topological space $(X, E, \tau)$, the collection of all fuzzy soft regularly open sets forms a base for some topology $\tau_s$ on $(X, E)$.

**Definition 3.12.** In a fuzzy soft topological space $(X, E, \tau)$, if $\tau_s$ coincides with $\tau$, then $\tau$ is said to be a fuzzy soft semi regularization topology.

## 4. Fuzzy soft $\delta$-open and fuzzy soft $\delta$-closed sets

In this section, we define fuzzy soft $\delta$-closure, fuzzy soft $\delta$-open and fuzzy soft $\delta$-closed sets and investigate their related properties. Further, we point out the inter-relationship between fuzzy soft regular open set and fuzzy soft $\delta$-open set.

**Definition 4.1.** A fuzzy soft point $e_x^\alpha$ is called a fuzzy soft $\delta$-cluster point of a fuzzy soft set $f_A$ in an fuzzy soft topological space $(X, E, \tau)$ if and only if every fuzzy soft regularly open *q-nbd* of $e_x^\alpha$ is soft quasi-coincident with $f_A$.

The union of all fuzzy soft $\delta$-cluster points of $f_A$ is called the fuzzy soft $\delta$-closure of $f_A$ and is denoted by $fs\delta cl(f_A)$.

**Remark 4.2.** Let $(X, E, \tau)$ be a fuzzy soft topological space and $f_A \tilde{\in} FS(X, E)$. Then $fs\delta cl(f_A) = \sqcap\{g_B \tilde{\in} FS(X, E)/f_A \tilde{\subseteq} g_B, g_B = \overline{{g_B}^0}\}$.

**Proposition 4.3.** *Let* $(X, E, \tau)$ *be a fuzzy soft topological space and* $f_A \tilde{\in} FS(X, E)$. *Then* $\overline{f_A} \tilde{\subseteq} fs\delta cl(f_A)$.

*Proof.* The proof is obvious. □

**Definition 4.4.** A fuzzy soft set $f_A$ of a fuzzy soft topological space $(X, E, \tau)$ is said to be fuzzy soft $\delta$-closed if and only if $f_A = fs\delta cl(f_A)$. The complement of a fuzzy soft $\delta$-closed set is said to be fuzzy soft $\delta$-open.

**Proposition 4.5.** *Every fuzzy soft $\delta$-closed set is fuzzy soft closed in a fuzzy soft topological space $(X, E, \tau)$.*

*Proof.* Let $f_A$ be a fuzzy soft $\delta$-closed set in $(X, E, \tau)$. Then $\overline{f_A} \tilde{\subseteq} fs\delta cl(f_A)$ and $f_A = fs\delta cl(f_A)$ imply that $\overline{f_A} = f_A$, so that $f_A$ is fuzzy soft closed. □

**Corollary 4.6.** *Every fuzzy soft $\delta$-open set is fuzzy soft open in a fuzzy soft topological space $(X, E, \tau)$.*

**Remark 4.7.** A fuzzy soft open set may not be fuzzy soft $\delta$-open in a fuzzy soft topological space $(X, E, \tau)$, which follows from the following example.

**Example 4.8.** Let $X = \{a, b\}$, $E = \{e_1, e_2\}$.
Let us consider the following fuzzy soft sets over $(X, E)$.

$f_E = \{f(e_1) = \{a/0.1, b/0.1\}, f(e_2) = \{a/0.1, b/0.2\}\}$,

$g_E = \{g(e_1) = \{a/0.2, b/0.2\}, g(e_2) = \{a/0.1, b/0.2\}\}$,

$h_E = \{h(e_1) = \{a/0.2, b/0.7\}, h(e_2) = \{a/0.2, b/0.7\}\}$,

$i_E = \{i(e_1) = \{a/0.9, b/0.9\}, i(e_2) = \{a/0.7, b/0.7\}\}$,

$j_E = \{j(e_1) = \{a/0.9, b/1\}, j(e_2) = \{a/0.7, b/0.9\}\}$.

Let us consider the fuzzy soft topology $\tau = \{\tilde{0}_E, \tilde{1}_E, f_E, g_E, h_E, i_E, j_E\}$ over $(X, E)$.
Now, ${f_E}^c = \{f^c(e_1) = \{a/0.9, b/0.9\}, f^c(e_2) = \{a/0.9, b/0.8\}\}$,

${g_E}^c = \{g^c(e_1) = \{a/0.8, b/0.8\}, g^c(e_2) = \{a/0.9, b/0.8\}\}$,

${h_E}^c = \{h^c(e_1) = \{a/0.8, b/0.3\}, h^c(e_2) = \{a/0.8, b/0.3\}\}$,

${i_E}^c = \{i^c(e_1) = \{a/0.1, b/0.1\}, i^c(e_2) = \{a/0.3, b/0.3\}\}$,

${j_E}^c = \{j^c(e_1) = \{a/0.1, b/0\}, j^c(e_2) = \{a/0.3, b/0.1\}\}$.

Clearly, ${f_E}^c, {g_E}^c, {h_E}^c, {i_E}^c$ and ${j_E}^c$ are fuzzy soft closed sets.

Obviously, $f_E, g_E, h_E, i_E$ are fuzzy soft regular open sets.

Again, $\overline{j_E} = \tilde{1}_E$ and hence $j_E$ is not a fuzzy soft regular open set.
Thus $fs\delta cl({j_E}^c) = {i_E}^c$.

So ${j_E}^c$ is not fuzzy soft $\delta$-closed and hence $j_E$ is not fuzzy soft $\delta$-open but $j_E$ is fuzzy soft open.

**Proposition 4.9.** *A fuzzy soft regularly open set is fuzzy soft $\delta$-open in a fuzzy soft topological space $(X, E, \tau)$.*

*Proof.* The proof follows from the definitions. □

**Remark 4.10.** A fuzzy soft $\delta$-open set need not be fuzzy soft regular open in a fuzzy soft topological space $(X, E, \tau)$.

**Example 4.11.** It follows from Example 3.10 that $f_A, g_B, h_C, j_D, k_F, u_G$ are fuzzy soft regularly open but $i_E$ is not a fuzzy soft regularly open set. Clearly, ${f_A}^c, {g_B}^c, {h_C}^c, {j_D}^c, {k_F}^c, {u_G}^c$ are fuzzy soft regularly closed sets. Now, the fuzzy soft $\delta$-closure of ${i_E}^c$ is the intersection of all fuzzy soft regularly closed sets containing ${i_E}^c$. That is, $fs\delta cl({i_E}^c) = {i_E}^c$. Hence $i_E$ is a fuzzy soft $\delta$-open set.

**Theorem 4.12.** *In a fuzzy soft topological space $(X, E, \tau)$, $\overline{f_A} = fs\delta cl(f_A)$ for every fuzzy soft semi-open set $f_A$ in $(X, E, \tau)$.*

*Proof.* It suffices to show that $fs\delta cl(f_A)\tilde{\sqsubseteq}\overline{f_A}$.
Let $e_x^\alpha$ be a fuzzy soft point in $(X, E, \tau)$ such that $e_x^\alpha\tilde{\in} fs\delta cl(f_A)$ but $e_x^\alpha\tilde{\notin}\overline{f_A}$. Then there is a fuzzy soft $q$-*nbd* $g_B$ of $e_x^\alpha$ such that $g_B\overline{\tilde{q}}f_A$. We then have $g_B\tilde{\sqsubseteq}(\tilde{1}_E \setminus f_A)$, and $(\overline{g_B})^0\tilde{\sqsubseteq}(\overline{\tilde{1}_E \setminus f_A})^0 = \tilde{1}_E \setminus \overline{({f_A}^0)}\tilde{\sqsubseteq}\tilde{1}_E \setminus f_A$ (since $f_A$ is fuzzy soft semi-open). Thus $(\overline{g_B})^0\overline{\tilde{q}}f_A$ and consequently, $e_x^\alpha\tilde{\notin} fs\delta cl(f_A)$, which is a contradiction. □

**Corollary 4.13.** *For a fuzzy soft open set $f_A$ in $(X, E, \tau)$, $\overline{f_A} = fs\delta cl(f_A)$.*

**Theorem 4.14.** *Let $(X, E, \tau)$ be a fuzzy soft topological space and $f_A, g_B\tilde{\in} FS(X, E)$. Then*
*(1) $fs\delta cl(\tilde{0}_E) = \tilde{0}_E$ and $fs\delta cl(\tilde{1}_E) = \tilde{1}_E$;*
*(2) If $f_A\tilde{\sqsubseteq}g_B$, then $fs\delta cl(f_A)\tilde{\sqsubseteq}fs\delta cl(g_B)$;*
*(3) $fs\delta cl(fs\delta cl(f_A)) = fs\delta cl(f_A)$;*
*(4) the finite union of fuzzy soft $\delta$-closed sets is also fuzzy soft $\delta$-closed.*

*Proof.* (1), (2) and (3) are obviously true.

(4) Let $f_A$ and $g_B$ be any two fuzzy soft $\delta$-closed sets. Then $f_A = fs\delta cl(f_A)$ and $g_B = fs\delta cl(g_B)$. Clearly, $(f_A \sqcup g_B)\tilde{\sqsubseteq}fs\delta cl(f_A \sqcup g_B)$. Now we will show that $fs\delta cl(f_A \sqcup g_B)\tilde{\sqsubseteq}(f_A \sqcup g_B)$. Let $e_x^\alpha$ be a fuzzy soft point. Suppose that $e_x^\alpha\tilde{\in} fs\delta cl(f_A \sqcup g_B)$. Then for any fuzzy soft regular open $q$-*nbd* $h_C$ of $e_x^\alpha$, $h_C\tilde{q}(f_A \sqcup g_B)$. Thus $h_C\tilde{q}f_A$ or $h_C\tilde{q}g_B$. Hence $e_x^\alpha\tilde{\in}(fs\delta cl(f_A) \sqcup fs\delta cl(g_B))$. That is, $e_x^\alpha\tilde{\in}(f_A \sqcup g_B)$. □

**Remark 4.15.** The fuzzy soft $\delta$-closure operation on a fuzzy soft topological space $(X, E, \tau)$ satisfies the Kuratowski closure axioms. So there exists one and only one topology on $(X, E)$.

**Definition 4.16.** A fuzzy soft set $f_A$ in an fuzzy soft topological space $(X, E, \tau)$ is said to be a fuzzy soft $\delta$-*nbd* of a fuzzy soft point $e_x^\alpha$ if and only if there exists a fuzzy soft regularly open $q$-*nbd* $g_B$ of $e_x^\alpha$ such that $g_B\overline{\tilde{q}}{f_A}^c$.

**Definition 4.17.** For a fuzzy soft set $f_A$ in an fuzzy soft topological space $(X, E, \tau)$, we define fuzzy soft $\delta$-interior of $f_A$ (denoted by $fs\delta int(f_A)$) is defined by
$fs\delta int(f_A) = \tilde{1}_E - fs\delta cl(\tilde{1}_E - f_A)$.

**Remark 4.18.** It is easy to show that a fuzzy soft set $f_A$ in the fuzzy soft topological space $(X, E, \tau)$ is fuzzy soft $\delta$-open if and only if $f_A = fs\delta int(f_A)$.

**Lemma 4.19.** *For any fuzzy soft set $f_A$ in $(X, E, \tau)$ and any fuzzy soft point $e_x^\alpha$ in $(X, E, \tau)$, $e_x^\alpha\tilde{q}fs\delta int(f_A)$ and so there exists a fuzzy soft open $q$-nbd $g_B$ of $e_x^\alpha$ such that $(\overline{g_B})^0\tilde{\sqsubseteq}f_A$.*

*Proof.* Let $e_x^\alpha \tilde{q} fs\delta int(f_A) \Rightarrow e_x^\alpha \tilde{\notin} (\tilde{1}_E - fs\delta int(f_A)) = fs\delta cl(\tilde{1}_E - f_A)$. so there exists a fuzzy soft open *q-nbd* $g_B$ of $e_x^\alpha$ such that $(\overline{g_B})^0 \tilde{q} (\tilde{1}_E - f_A) \Rightarrow (\overline{g_B})^0 \tilde{\subseteq} f_A$. □

**Remark 4.20.** The set of all fuzzy soft $\delta$-open sets in a fuzzy soft topological space $(X, E, \tau)$ forms a topology, called the fuzzy soft $\delta$-topology and denoted by $\tau_\delta$.

**Remark 4.21.** Fuzzy soft regularly open set $\Rightarrow$ fuzzy soft $\delta$-open set $\Rightarrow$ fuzzy soft open set, but the converse in no case is true.

**Remark 4.22.** It can be observed that for any fuzzy soft topological space $(X, E, \tau)$, the topologies $\tau$, $\tau_s$ and $\tau_\delta$ are different and moreover $\tau_s \tilde{\subseteq} \tau_\delta \tilde{\subseteq} \tau$.

It is clear that in a semi regularization space the above three topologies coincide, since $\tau_s = \tau$ in that case.

## 5. Fuzzy soft $\delta$-continuous mapping and fuzzy soft almost continuous mapping

In this section, we introduce the concept of fuzzy soft $\delta$-continuous mapping and fuzzy soft almost continuous mapping. Some basic properties of above two types of functions are investigated.

**Definition 5.1.** A function $f_{up} : (X, E, \tau_1) \to (Y, K, \tau_2)$ is said to be a fuzzy soft $\delta$-continuous if and only if for each fuzzy soft point $e_x^\alpha$ in $FS(X, E)$ and for any fuzzy soft regularly open *q-nbd* $g_B$ of $f_{up}(e_x^\alpha)$ in $FS(Y, K)$, there exists a fuzzy soft regularly open *q-nbd* $f_A$ of $e_x^\alpha$ such that $f_{up}(f_A) \tilde{\subseteq} g_B$.

**Definition 5.2.** Let $f_{up} : (X, E, \tau_1) \to (Y, K, \tau_2)$ be a fuzzy soft mapping. Then
(i) $f_{up}$ is said to be a fuzzy soft $\delta$-open mapping if for each fuzzy soft $\delta$-open set $f_A$ in $(X, E, \tau_1)$, $f_{up}(f_A)$ is fuzzy soft $\delta$-open in $(Y, K, \tau_2)$,
(ii) $f_{up}$ is said to be a fuzzy soft $\delta$-closed mapping if for each fuzzy soft $\delta$-closed set $g_B$ in $(X, E, \tau_1)$, $f_{up}(g_B)$ is fuzzy soft $\delta$-closed in $(Y, K, \tau_2)$.

**Theorem 5.3.** *Let $f_{up} : (X, E, \tau_1) \to (Y, K, \tau_2)$ be a fuzzy soft mapping. Then the following are equivalent:*
*(1) $f_{up}$ is fuzzy soft $\delta$-continuous.*
*(2) $f_{up}(fs\delta cl(f_A)) \tilde{\subseteq} fs\delta cl(f_{up}(f_A))$, for each fuzzy soft set $f_A$ in $FS(X, E)$.*
*(3) $fs\delta cl({f_{up}}^{-1}(f_A)) \tilde{\subseteq} {f_{up}}^{-1}(fs\delta cl(f_A))$, for each fuzzy soft set $f_A$ in $FS(Y, K)$.*
*(4) For each fuzzy soft $\delta$-closed set $f_A$ in $(Y, K, \tau_2)$, ${f_{up}}^{-1}(f_A)$ is a fuzzy soft $\delta$-closed set in $(X, E, \tau_1)$.*
*(5) For each fuzzy soft $\delta$-open set $f_A$ in $(Y, K, \tau_2)$, ${f_{up}}^{-1}(f_A)$ is a fuzzy soft $\delta$-open set in $(X, E, \tau_1)$.*
*(6) For each fuzzy soft regularly open set $f_A$ in $(Y, K, \tau_2)$, ${f_{up}}^{-1}(f_A)$ is a fuzzy soft $\delta$-open set in $(X, E, \tau_1)$.*

*Proof.* (1) $\Rightarrow$ (2) Let $e_x^\alpha \tilde{\in} fs\delta cl(f_A)$ and $g_B$ be any fuzzy soft regularly open *q-nbd* of $f_{up}(e_x^\alpha)$. Then there exists a fuzzy soft regularly open *q-nbd* $h_C$ of $e_x^\alpha$ such that $f_{up}(h_C) \tilde{\subseteq} g_B$. Since $e_x^\alpha \tilde{\in} fs\delta cl(f_A)$, we have $h_C \tilde{q} f_A$. Then $f_{up}(h_C) \tilde{q} f_{up}(f_A)$. Thus $g_B \tilde{q} f_{up}(f_A)$ and hence $f_{up}(e_x^\alpha) \tilde{\in} fs\delta cl(f_{up}(f_A))$. So $f_{up}(fs\delta cl(f_A)) \tilde{\subseteq} fs\delta cl(f_{up}(f_A))$.

(2) $\Rightarrow$ (3) By (2), $fs\delta cl({f_{up}}^{-1}(f_A)) \tilde{\subseteq} fs\delta cl(f_{up}({f_{up}}^{-1}(f_A))) \tilde{\subseteq} fs\delta cl(f_A)$.
Hence $fs\delta cl({f_{up}}^{-1}(f_A)) \tilde{\subseteq} {f_{up}}^{-1}(fs\delta cl(f_A))$.

(3) ⇒ (4) Let $f_A$ be a fuzzy soft $\delta$-closed set in $(Y, K, \tau_2)$. Then $fs\delta cl(f_A) = f_A$ and by (3), $fs\delta cl({f_{up}}^{-1}(f_A))\tilde{\subseteq}{f_{up}}^{-1}(fs\delta cl(f_A)) = {f_{up}}^{-1}(f_A)$. Hence ${f_{up}}^{-1}(f_A)$ is fuzzy soft $\delta$-closed in $(X, E, \tau_1)$.

(4) ⇒ (5) Let $f_A$ be a fuzzy soft $\delta$-open set in $(Y, K, \tau_2)$. Then ${f_A}^c$ is a fuzzy soft $\delta$-closed set in $(Y, K, \tau_2)$. By (4), ${f_{up}}^{-1}({f_A}^c)$ is fuzzy soft $\delta$-closed in $(X, E, \tau_1)$. Since ${f_{up}}^{-1}({f_A}^c) = \tilde{1}_E - {f_{up}}^{-1}(f_A)$, ${f_{up}}^{-1}(f_A)$ is a fuzzy soft $\delta$-open set in $(X, E, \tau_1)$.

(5) ⇒ (6) It follows from Proposition 4.9 and (5). □

Note: The following two examples show that the concepts of fuzzy soft continuity and fuzzy soft $\delta$-continuity are independent.

**Example 5.4.** Let $X = \{a, b\}$, $E = \{e_1, e_2\}$.
Let us consider the following fuzzy soft sets over $(X, E)$.

$f_E = \{f(e_1) = \{a/0.5, b/0.3\}, f(e_2) = \{a/0.6, b/0.4\}\}$,

$g_E = \{g(e_1) = \{a/0.6, b/0.7\}, g(e_2) = \{a/0.2, b/0.7\}\}$.

Let us consider the fuzzy soft topology $\tau_1 = \{\tilde{0}_E, \tilde{1}_E, f_E\}$, $\tau_2 = \{\tilde{0}_E, \tilde{1}_E, g_E\}$ over $(X, E)$.

We define the fuzzy soft mapping $f_{up} : (X, E, \tau_1) \to (X, E, \tau_2)$, where $u : X \to X$ and $p : E \to E$ are mappings, defined by $u(a) = a, u(b) = b, p(e_1) = e_1, p(e_2) = e_2$. Obviously, the fuzzy soft $\delta$-open set in $(X, E, \tau_2)$ is $\tilde{0}_E, \tilde{1}_E$, ${f_{up}}^{-1}(\tilde{0}_E) = \tilde{0}_E \tilde{\in} (X, E, \tau_1)$ and ${f_{up}}^{-1}(\tilde{1}_E) = \tilde{1}_E \tilde{\in} (X, E, \tau_1)$. Thus $f_{up} : (X, E, \tau_1) \to (X, E, \tau_2)$ is fuzzy soft $\delta$-continuous; but not fuzzy soft continuous, since ${f_{up}}^{-1}(g_E) = g_E \tilde{\notin} (X, E, \tau_1)$.

**Example 5.5.** Let $X = \{a, b, c\}$, and $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2, e_4\} \subseteq E$, $B = \{e_1, e_2\} \subseteq E$, $C = \{e_2, e_4\} \subseteq E$.
Let us consider the following fuzzy soft sets over $(X, E)$.

$f_A = \{f(e_1) = \{a/0.6, b/0.1, c/0\}, f(e_2) = \{a/0.4, b/0.7, c/0.5\}, f(e_3) = \{a/0, b/0, c/0\}, f(e_4) = \{a/0.5, b/0.3, c/0.8\}\}$,

$g_B = \{g(e_1) = \{a/0.3, b/0.1, c/0\}, g(e_2) = \{a/0.4, b/0.2, c/0.4\}, g(e_3) = \{a/0, b/0, c/0\}, g(e_4) = \{a/0, b/0, c/0\}\}$,

$h_C = \{h(e_1) = \{a/0, b/0, c/0\}, h(e_2) = \{a/0.8, b/0.3, c/0.5\}, h(e_3) = \{a/0, b/0, c/0\}, h(e_4) = \{a/0.8, b/0.3, c/0.5\}\}$.

Let us consider the fuzzy soft topology $\tau_1 = \{\tilde{0}_E, \tilde{1}_E, h_C\}$, $\tau_2 = \{\tilde{0}_E, \tilde{1}_E, f_A, g_B\}$ over $(X, E)$.

We define the fuzzy soft mapping $f_{up} : (X, E, \tau_1) \to (X, E, \tau_2)$, where $u : X \to X$ and $p : E \to E$ are mappings, defined by $u(a) = c, u(b) = b, u(c) = a, p(e_1) = p(e_3) = e_3, p(e_2) = e_4, p(e_4) = e_4$.

Clearly, $f_A$ and $g_B$ are fuzzy soft regularly open sets in $(X, E, \tau_2)$ but $h_C$ is not a fuzzy soft regularly open set in $(X, E, \tau_1)$. It is easy to show that $f_{up} : (X, E, \tau_1) \to (X, E, \tau_2)$ is fuzzy soft continuous but not fuzzy soft $\delta$-continuous, since $f_A$ is a fuzzy soft $\delta$-open set in $(X, E, \tau_2)$ but ${f_{up}}^{-1}(f_A) = h_C$ is not a fuzzy soft $\delta$-open set in $(X, E, \tau_1)$.

**Corollary 5.6.** *If $f_{up} : (X, E, \tau_1) \to (Y, K, \tau_2)$ is a fuzzy soft $\delta$-continuous mapping. Then for each fuzzy soft open set $g_B$ in $(Y, K, \tau_2)$, $fs\delta cl({f_{up}}^{-1}(g_B))\tilde{\subseteq}{f_{up}}^{-1}(\overline{g_B})$.*

*Proof.* Since $g_B$ is fuzzy soft open in $(Y, K, \tau_2)$, $\overline{g_B} = fs\delta cl(g_B)$. By (3), of the above Theorem 5.3, $fs\delta cl({f_{up}}^{-1}(g_B)) \tilde{\subseteq} {f_{up}}^{-1}(fs\delta cl(g_B)) = {f_{up}}^{-1}(\overline{g_B})$. □

**Theorem 5.7.** *A function $f_{up} : (X, E, \tau_1) \to (Y, K, \tau_2)$ is fuzzy soft $\delta$-continuous if and only if for each fuzzy soft point $e_x^\alpha$ in $FS(X, E)$ and each fuzzy soft $\delta - nbd$ $g_B$ of $f_{up}(e_x^\alpha)$, ${f_{up}}^{-1}(f_A)$ is a fuzzy soft $\delta - nbd$ of $e_x^\alpha$.*

*Proof.* Let $e_x^\alpha \tilde{\in} FS(X, E)$ and $g_B$ be a fuzzy soft $\delta - nbd$ of $f_{up}(e_x^\alpha)$. Then there exists a fuzzy soft regularly open *q-nbd* $h_C$ of $f_{up}(e_x^\alpha)$ such that $h_C \tilde{\subseteq} g_B$. Since $f_{up}$ is fuzzy soft $\delta$-continuous, there exists a fuzzy soft regularly open *q-nbd* $j_A$ of $e_x^\alpha$ such that $f_{up}(j_A) \tilde{\subseteq} h_C$, and hence $j_A \tilde{\subseteq} {f_{up}}^{-1}(f_{up}(j_A)) \tilde{\subseteq} {f_{up}}^{-1}(h_C)$. Therefore, since ${f_{up}}^{-1}(h_C) \tilde{\subseteq} {f_{up}}^{-1}(g_B)$, ${f_{up}}^{-1}(g_B)$ is a fuzzy soft $\delta - nbd$ of $e_x^\alpha$.

Conversely, let $e_x^\alpha \tilde{\in} FS(X, E)$ and $g_B$ be a fuzzy soft regularly open *q-nbd* of $f_{up}(e_x^\alpha)$. Then $g_B$ is a fuzzy soft $\delta - nbd$ of $f_{up}(e_x^\alpha)$. By hypothesis, ${f_{up}}^{-1}(g_B)$ is a fuzzy soft $\delta - nbd$ of $e_x^\alpha$. Therefore there exists a fuzzy soft regularly open *q-nbd* $j_A$ of $e_x^\alpha$ such that $j_A \tilde{\subseteq} {f_{up}}^{-1}(g_B)$ and hence $f_{up}(j_A) \tilde{\subseteq} f_{up}({f_{up}}^{-1}(g_B)) \tilde{\subseteq} g_B$. Hence $f_{up}$ is fuzzy soft $\delta$-continuous. □

**Definition 5.8.** A function $f_{up} : (X, E, \tau_1) \to (Y, K, \tau_2)$ is said to be fuzzy soft almost continuous if and only if, for each fuzzy soft point $e_x^\alpha$ in $FS(X, E)$ and for any fuzzy soft regularly open *q-nbd* $g_B$ of $f_{up}(e_x^\alpha)$, there exists a fuzzy soft open *q-nbd* $f_A$ of $e_x^\alpha$ such that $f_{up}(f_A) \tilde{\subseteq} g_B$.

**Theorem 5.9.** *Let $f_{up} : (X, E, \tau_1) \to (Y, K, \tau_2)$ be a fuzzy soft mapping. Then the following are equivalent:*
*(1) $f_{up}$ is fuzzy soft almost continuous.*
*(2) $f_{up}(\overline{f_A}) \tilde{\subseteq} fs\delta cl(f_{up}(f_A))$ for each fuzzy soft set $f_A$ in $FS(X, E)$.*
*(3) $\overline{{f_{up}}^{-1}(g_B)} \tilde{\subseteq} {f_{up}}^{-1}(fs\delta cl(g_B))$ for each fuzzy soft set $g_B$ in $FS(Y, K)$.*
*(4) For each fuzzy soft $\delta$-closed set $g_B$ in $(Y, K, \tau_2)$, ${f_{up}}^{-1}(g_B)$ is a fuzzy soft closed set in $(X, E, \tau_1)$.*
*(5) For each fuzzy soft $\delta$-open set $g_B$ in $(Y, K, \tau_2)$, ${f_{up}}^{-1}(g_B)$ is a fuzzy soft open set in $(X, E, \tau_1)$.*
*(6) For each fuzzy soft regularly open set $g_B$ in $(Y, K, \tau_2)$, ${f_{up}}^{-1}(g_B)$ is a fuzzy soft open set in $(X, E, \tau_1)$.*
*(7) For each fuzzy soft point $e_x^\alpha$ in $FS(X, E)$ and each fuzzy soft $\delta - nbd$ $g_B$ of $f_{up}(e_x^\alpha)$, ${f_{up}}^{-1}(g_B)$ is a fuzzy soft q-nbd of $e_x^\alpha$.*

*Proof.* (1) $\Rightarrow$ (2) Let $e_x^\alpha \tilde{\in} \overline{f_A}$ and $g_B$ be any fuzzy soft regularly open *q-nbd* of $f_{up}(e_x^\alpha)$. Since $f_{up}$ is fuzzy soft almost continuous, there exists a fuzzy soft open set $h_C$ of $e_x^\alpha$ such that $f_{up}(h_C) \tilde{\subseteq} g_B$. Since $e_x^\alpha \tilde{\in} \overline{f_A}$, we have $h_C \tilde{q} f_A$. Then $f_{up}(h_C) \tilde{q} f_{up}(f_A)$. Thus $g_B \tilde{q} f_{up}(f_A)$ and hence $f_{up}(e_x^\alpha) \tilde{\in} fs\delta cl(f_{up}(f_A)) \Rightarrow e_x^\alpha \tilde{\in} {f_{up}}^{-1}(fs\delta cl(f_{up}(f_A)))$. Thus $\overline{f_A} \tilde{\subseteq} {f_{up}}^{-1}(fs\delta cl(f_{up}(f_A)))$ and so $f_{up}(\overline{f_A}) \tilde{\subseteq} fs\delta cl(f_{up}(f_A))$.

(2) $\Rightarrow$ (3) By (2), $f_{up}(\overline{{f_{up}}^{-1}(g_B)}) \tilde{\subseteq} fs\delta cl[f_{up}({f_{up}}^{-1}(g_B))]$. So $f_{up}(\overline{{f_{up}}^{-1}(g_B)}) \tilde{\subseteq} fs\delta cl(g_B)$. Then $\overline{{f_{up}}^{-1}(g_B)} \tilde{\subseteq} {f_{up}}^{-1}(fs\delta cl(g_B))$ (since is a surjection).

(3) $\Rightarrow$ (4) Let $g_B$ be a fuzzy soft $\delta$-closed set in $(Y, K, \tau_2)$. Then $fs\delta cl(g_B) = g_B$. By (3), $\overline{{f_{up}}^{-1}(g_B)} \tilde{\subseteq} {f_{up}}^{-1}(fs\delta cl(g_B)) = {f_{up}}^{-1}(g_B)$. Thus ${f_{up}}^{-1}(g_B)$ is fuzzy soft closed in $(X, E, \tau_1)$.

(4) $\Rightarrow$ (5) Let $g_B$ be a fuzzy soft $\delta$-open set in $(Y, K, \tau_2)$. Then ${g_B}^c$ is a fuzzy soft $\delta$- closed set in $(Y, K, \tau_2)$. By (4), ${f_{up}}^{-1}({g_B}^c)$ is fuzzy soft closed in $(X, E, \tau_1)$. Since ${f_{up}}^{-1}({g_B}^c) = \tilde{1}_E - {f_{up}}^{-1}(g_B)$, ${f_{up}}^{-1}(g_B)$ is a fuzzy soft open set in $(X, E, \tau_1)$.

(5) $\Rightarrow$ (6) It follows from Proposition 4.9 and (5).

(1) $\Rightarrow$ (7) Let $e_x^\alpha$ be a fuzzy soft point in $FS(X, E)$ and $g_B$ be any fuzzy soft $\delta - nbd$ of $f_{up}(e_x^\alpha)$. Then there exists a fuzzy soft regularly open *q-nbd* $h_C$ of $f_{up}(e_x^\alpha)$ such that $h_C \bar{\tilde{q}} {g_B}^c$. Since $f_{up}$ is fuzzy soft almost continuous, there exists a fuzzy soft open *q-nbd* $f_A$ of $e_x^\alpha$ such that $f_{up}(f_A) \tilde{\subseteq} h_C \tilde{\subseteq} g_B$, and so $f_A \tilde{\subseteq} {f_{up}}^{-1}(g_B)$ and hence ${f_{up}}^{-1}(g_B)$ is a fuzzy soft *q-nbd* of $e_x^\alpha$.

(7) $\Rightarrow$ (1) Let $e_x^\alpha$ be a fuzzy soft point in $FS(X, E)$ and $g_B$ be any fuzzy soft regularly open *q-nbd* of $f_{up}(e_x^\alpha)$ in $(Y, K, \tau_2)$. Then $g_B$ is a fuzzy soft $\delta - nbd$ of $f_{up}(e_x^\alpha)$. By (7), ${f_{up}}^{-1}(g_B)$ is a fuzzy soft *q-nbd* of $e_x^\alpha$. Hence there exists a fuzzy soft open set $f_A$ in $(X, E, \tau_1)$ such that $e_x^\alpha \tilde{q} f_A \tilde{\subseteq} {f_{up}}^{-1}(g_B)$ so that $f_{up}(f_A) \tilde{\subseteq} g_B$. Thus $f_{up}$ is fuzzy soft almost continuous. □

**Remark 5.10.** A fuzzy soft $\delta$-continuous mapping is fuzzy soft almost continuous but converse is not true.

**Example 5.11.** It follows from Example 5.5 that $f_{up}$ is fuzzy soft almost continuous but not fuzzy soft $\delta$-continuous.

**Remark 5.12.** Fuzzy soft continuity implies fuzzy soft almost continuity but not conversely, which follows from Example 5.4.

## 6. Conclusion

Topology is an important area of mathematics with many applications in the domain of computer science and physical sciences. Fuzzy soft topology [16] is a relatively new and promising domain which can lead to the development of new mathematical models and innovative approaches that will significantly contribute to the solution of complex problems in natural sciences.

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## References

[1] B. Ahmad and A. Kharal, On fuzzy soft sets, Adv. Fuzzy Syst. 2009(2009) Article ID 586507 6 pages.
[2] S. Atmaca and İ. Zorlutuna, On fuzzy soft topological spaces, Ann. Fuzzy Math. Inform. 5 (2) (2013) 377–386.
[3] A. Aygünoglu and H. Aygün, Introduction to fuzzy soft groups, Comput. Math. Appl. 58 (2009) 1279–1286.
[4] B. Bora, On fuzzy soft continuous mapping, Int. J. Basic Sci. Social Sci. 1 (2012) 50–64.
[5] R. P. Chakraborty and P. Mukherjee, On generalised fuzzy soft topological spaces, Afr. J. Math. Comput. Sci. Res. 8 (1) (2015) 1–11.
[6] R. P. Chakraborty, P. Mukherjee and P. K. Gain, A note on fuzzy soft semi open sets and fuzzy soft semi continuous functions, J. Fuzzy Math. 22 (2014) 973–989.
[7] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.
[8] P. K. Gain, P. Mukherjee and R. P. Chakraborty, M. Pal, On some structural properties of fuzzy soft topological spaces, Int. J. Fuzzy Math. Archive 1 (2013) 1–15.
[9] A. Kharal and B. Ahmad, Mappings on fuzzy soft classes, Adv. Fuzzy Syst. 2009 (2009) 1–6, Article ID 407890 6 pages.
[10] P. K. Maji, R. Biswas and A. R. Roy, Fuzzy soft sets, J. Fuzzy Math. 9 (2001) 589–602.
[11] D. Molodtsov, Soft set theory-First results, Comput. Math. Appl. 37 (1999) 19–31.
[12] P. Mukherjee, R. P. Chakraborty and C. Park, Fuzzy soft $\theta-$ closure operator in fuzzy soft topological spaces, Electronic Journal of Mathematical Analysis and Applications 3 (2015) 227–236.
[13] P. Mukherjee and C. Park, On fuzzy soft bitopological spaces, Mathematics and Computer Sciences Journal 10 (2015) 1–8.
[14] S. Roy and T. K. Samanta, A note on fuzzy soft topological spaces, Ann. Fuzzy Math. Inform. 3 (2) (2012) 305–311.
[15] M. Shabir and M. Naz, On soft topological spaces, Comput. Math. Appl. 61 (2011) 1786–1799.
[16] B. Tanay and M. Burc Kandemir, Topological structure of fuzzy soft sets, Comput. Math. Appl. 61 (2011) 2952–2957.
[17] B. P. Varol and H. Aygün, Fuzzy soft topology, Hacettepe J. Math. Stat. 41 (2012) 407–419.
[18] L. A. Zadeh, Fuzzy sets, Inform and Control 8 (1965) 338–353.

Prakash Mukherjee (prakashmukherjee25@gmail.com; pmukherjee@hotmail.com)
Department of mathematics, Hijli College, Hijli, Kharagpur, Paschim Medinipur, postal code 721306, West Bengal, India

Ramkrishna Prasad Chakraborty (ramda2004@rediffmail.com)
Department of mathemathics, Hijli College, Hijli, Kharagpur, Paschim Medinipur, postal code 721306, West Bengal, India

Choonkil Park (baak@hanyan.ac.kr)
Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Republic of Korea