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Invited Paper

Bipolar-valued fuzzy subalgebras based on bipolar-valued fuzzy points

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Abstract. The notions of bipolar-valued fuzzy point, contained in relation ($\Subset$) and bipolar quasi-coincident with relation ($\mathcal{Q}$) are introduced. Using these notions, the concept of bipolar-valued fuzzy subalgebra of type (Ω, Θ) where Ω and Θ are any two of $\{\Subset, \mathcal{Q}, \Subset\vee\mathcal{Q}, \Subset\wedge\mathcal{Q}\}$ with $\Omega \neq \Subset\wedge\mathcal{Q}$ is introduced, and related properties are investigated. Conditions for the negative and positive 0-supports to be subalgebras are provided. A characterization of a bipolar-valued fuzzy subalgebra of type ($\Subset$, $\Subset\vee\mathcal{Q}$) is given, and conditions for a bipolar-valued fuzzy set to be a bipolar-valued fuzzy subalgebra of type ($\mathcal{Q}$, $\Subset\vee\mathcal{Q}$) are considered.

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1. Introduction

The notion of bipolar-valued fuzzy set is introduced by Lee [9] as an extension of fuzzy set, and then this notion is applied to BCK/BCI-algebras, CI-algebras, semigroups, Lie algebras and hyper BCK-algebras etc. (see [1], [3], [4], [5], [6], [7], [8], [13], [14]). The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [12], played an important role to generate some different types of fuzzy notions in several algebraic structures.

In this paper, as a generalization of fuzzy points and quasi-coincidence in fuzzy sets, we introduce the notion of bipolar-valued fuzzy point and bipolar quasi-coincidence in bipolar-valued fuzzy sets. Based on these notions, we introduce the concept of bipolar-valued fuzzy subalgebra of type (Ω, Θ) where Ω and Θ are any two of $\{\Subset, \mathcal{Q}, \Subset\vee\mathcal{Q}, \Subset\wedge\mathcal{Q}\}$ with $\Omega \neq \Subset\wedge\mathcal{Q}$, and investigate related properties. We provide conditions for the negative and positive 0-supports to be subalgebras. We discuss a characterization of a bipolar-valued fuzzy subalgebra of type $(\Subset, \Subset\vee\mathcal{Q})$. We consider conditions for a bipolar-valued fuzzy set to be a bipolar-valued fuzzy subalgebra of type $(\mathcal{Q}, \Subset\vee\mathcal{Q})$. Using a a bipolar-valued fuzzy subalgebra of type $(\Subset, \Subset\vee\mathcal{Q})$, we make a a bipolar-valued fuzzy subalgebra.

2. Preliminaries

Let $K(\tau)$ be the class of all algebras with type $\tau = (2, 0)$. By a *BCI-algebra* we mean a system $X := (X, *, 0) \in K(\tau)$ in which the following axioms hold:

(i) $((x * y) * (x * z)) * (z * y) = 0$,
(ii) $(x * (x * y)) * y = 0$,
(iii) $x * x = 0$,
(iv) $x * y = y * x = 0 \implies x = y$,

for all $x, y, z \in X$. If a BCI-algebra X satisfies $0 * x = 0$, for all $x \in X$, then we say that X is a BCK-algebra. We can define a partial ordering $\leq$ by

$$(\forall x, y \in X)\,(x \leq y \iff x * y = 0).$$

In a BCK/BCI-algebra X, the following hold:

(a1) $(\forall x \in X)\ (x * 0 = x)$,
(a2) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,

for all $x, y, z \in X$.

A non-empty subset S of a BCK/BCI-algebra X is called a subalgebra of X, if $x * y \in S$ for all $x, y \in S$. For our convenience, the empty set $\varnothing$ is regarded as a subalgebra of X.

We refer the reader to the books [2] and [11] for further information regarding BCK/BCI-algebras.

Let X be the universe of discourse. A bipolar-valued fuzzy set $f = (X; f_n, f_p)$ in X is an object having the form

$$f = \{(x, f_n(x), f_p(x)) \mid x \in X\},$$

where $f_n : X \to [-1, 0]$ and $f_p : X \to [0, 1]$ are mappings. The positive membership degree $f_p(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set

$$f = \{(x, f_n(x), f_p(x)) \mid x \in X\},$$

and the negative membership degree $f_n(x)$ denotes the satisfaction degree of x to some implicit counter-property of $f = \{(x, f_n(x), f_p(x)) \mid x \in X\}$. If $f_p(x) \neq 0$ and $f_n(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for $f = \{(x, f_n(x), f_p(x)) \mid x \in X\}$. If $f_p(x) = 0$ and $f_n(x) \neq 0$, it is the situation that x does not satisfy the property of $f = \{(x, f_n(x), f_p(x)) \mid x \in X\}$ but somewhat satisfies the counter-property of $f = \{(x, f_n(x), f_p(x)) \mid x \in X\}$. It is possible for an element x to be $f_p(x) \neq 0$ and $f_n(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain (see [10]). For the sake of simplicity, we shall use the symbol $f = (X; f_n, f_p)$ for the bipolar-valued fuzzy set $f = \{(x, f_n(x), f_p(x)) \mid x \in X\}$. Bipolar-valued fuzzy sets and intuitionistic fuzzy sets look similar each other. However, they are different each other (see [10]).

3. Bipolar-valued fuzzy subalgebras

In what follows, let X be a BCK/BCI-algebra unless otherwise specified.

A bipolar-valued fuzzy set $f = (X; f_n, f_p)$ can be represented as more wide version:

$$f : X \times X \to [-1, 0] \times [0, 1],\ (x, y) \mapsto (f_n(x), f_p(y)). \tag{3.1}$$

If we take $x = y$ in (3.1), then it can be written as follows:

$$f : X \to [-1, 0] \times [0, 1],\ x \mapsto (f_n(x), f_p(x)) \tag{3.2}$$

which is originally defined bipolar-valued fuzzy set.

For any $(a, b), (x, y) \in X \times X$, we use the notation $(a, b) \neq (x, y)$, if $a \neq x$ and $b \neq y$.

Definition 3.1 ([7]). A bipolar-valued fuzzy set $f = (X; f_n, f_p)$ is called a bipolar-valued fuzzy subalgebra of X, if it satisfies:

$$\begin{aligned} f_n(x * y) &\leq \max\{f_n(x), f_n(y)\}, \\ f_p(x * y) &\geq \min\{f_p(x), f_p(y)\}, \end{aligned} \tag{3.3}$$

for all $x, y \in X$.

Given a point $(a, b) \in X \times X$, the bipolar-valued fuzzy set $f = (X; f_n, f_p)$ defined by

$$f(x, y) = \begin{cases} (\alpha, \beta) & \text{if } \ (x, y) = (a, b), \\ (0, 0) & \text{if } \ (x, y) \neq (a, b), \end{cases}$$

where $(\alpha, \beta) \in [-1, 0) \times (0, 1]$ is called a bipolar-valued fuzzy point with support (a, b) and value (α, β), and is denoted by $\langle (a, b); (\alpha, \beta) \rangle$. Also denote by a_α and (b, β) we mean an $\mathcal{N}$-point and a fuzzy point, respectively. Note that a bipolar-valued fuzzy point $\langle (a, b); (\alpha, \beta) \rangle$ contains an $\mathcal{N}$-point a_α and a fuzzy point (b, β) simultaneously.

Given a bipolar-valued fuzzy set $f = (X; f_n, f_p)$, we say that a bipolar-valued fuzzy point $\langle (a, b); (\alpha, \beta) \rangle$ is contained in $f := (X, f_n, f_p)$, denoted by $\langle (a, b); (\alpha, \beta) \rangle \Subset (X, f_n, f_p)$, if $a_\alpha \ni f_n$ and $(b, \beta) \in f_p$, that is,

$$f_n(a) \leq \alpha, \quad \text{and} \quad f_p(b) \geq \beta. \tag{3.4}$$

We say that a bipolar-valued fuzzy point $\langle (a, b); (\alpha, \beta) \rangle$ is bipolar quasi-coincident with $f := (X, f_n, f_p)$, denoted by $\langle (a, b); (\alpha, \beta) \rangle\ \mathcal{Q}\ (X, f_n, f_p)$, if $a_\alpha \varrho f_n$ and $(b, \beta)\, q\, f_p$, that is,

$$f_n(a) + \alpha + 1 < 0 \ \text{ and } \ f_p(b) + \beta > 1. \tag{3.5}$$

If $\langle (a, b); (\alpha, \beta) \rangle \Subset (X, f_n, f_p)$ or $\langle (a, b); (\alpha, \beta) \rangle\ \mathcal{Q}\ (X, f_n, f_p)$, we denote it by

$$\langle (a, b); (\alpha, \beta) \rangle \Subset\!\vee \mathcal{Q}\, (X, f_n, f_p).$$

Definition 3.2. A bipolar-valued fuzzy set $f = (X; f_n, f_p)$ is called a bipolar-valued fuzzy subalgebra of type (Ω, Θ) if whenever

$$\langle (x, y); (\alpha_1, \beta_1)\rangle \, \Omega \, (X, f_n, f_p) \text{ and } \langle (a, b); (\alpha_2, \beta_2)\rangle \, \Omega \, (X, f_n, f_p),$$

then $\langle (x * a, y * b); (\max\{\alpha_1, \alpha_2\}, \min\{\beta_1, \beta_2\})\rangle \, \Theta(X, f_n, f_p)$, for all $(x, y), (a, b) \in X \times X$ and (α_1, β_1), $(\alpha_2, \beta_2) \in [-1, 0) \times (0, 1]$, where $\Omega, \Theta \in \{\Subset, \mathcal{Q}, \Subset \vee \mathcal{Q}, \Subset \wedge \mathcal{Q}\}$ with $\Omega \neq \Subset \wedge \mathcal{Q}$.

Given a bipolar-valued fuzzy set $f = (X; f_n, f_p)$, consider the sets

$$\begin{aligned} N(f; 0) &:= \{x \in X \mid f_n(x) < 0\}, \\ P(f; 0) &:= \{x \in X \mid f_p(x) > 0\} \end{aligned} \tag{3.6}$$

which are called the negative 0-support and the positive 0-support, respectively, of $f = (X; f_n, f_p)$.

Theorem 3.3. *If $f = (X; f_n, f_p)$ is a bipolar-valued fuzzy subalgebra of type $(\Subset, \Subset)$ or $(\Subset, \mathcal{Q})$, then the negative and positive 0-supports of $f = (X; f_n, f_p)$ are subalgebras of X.*

Proof. If $f = (X; f_n, f_p)$ is zero, that is, $f_n(x) = 0$ and $f_p(x) = 0$, for all $x \in X$, then $N(f; 0) = \varnothing$ and $P(f; 0) = \varnothing$ which are subalgebras of X. Suppose that $f = (X; f_n, f_p)$ is nonzero, i.e., $f_n(x) \neq 0$ and $f_p(y) \neq 0$, for all $x, y \in X$. Assume that $f = (X; f_n, f_p)$ is a bipolar-valued fuzzy subalgebra of type $(\Subset, \Subset)$. Let $x, y \in N(f; 0)$ and $a, b \in P(f; 0)$, for $x, y, a, b \in X$. Then $f_n(x) < 0$, $f_n(y) < 0$, $f_p(a) > 0$ and $f_p(b) > 0$. Note that

$$\langle (x, a); (f_n(x), f_p(a)\rangle \Subset (X; f_n, f_p) \text{ and } \langle (y, b); (f_n(y), f_p(b)\rangle \Subset (X; f_n, f_p).$$

If $f_n(x * y) = 0$ or $f_p(a * b) = 0$, then $f_n(x * y) = 0 > \max\{f_n(x), f_n(y)\}$ or $f_p(a * b) = 0 < \min\{f_p(a), f_p(b)\}$. Thus

$$\langle (x * y, a * b); (\max\{f_n(x), f_n(y)\}, \min\{f_p(a), f_p(b)\})\rangle \, \overline{\Subset} \, (X; f_n, f_p),$$

which is a contradiction. So $f_n(x * y) < 0$ and $f_p(a * b) > 0$, i.e., $x * y \in N(f; 0)$ and $a * b \in P(f; 0)$. Hence the negative and positive 0-supports of $f = (X; f_n, f_p)$ are subalgebras of X.

Now suppose that $f = (X; f_n, f_p)$ is a bipolar-valued fuzzy subalgebra of type $(\Subset, \mathcal{Q})$. Let $a, b \in N(f; 0)$ and $x, y \in P(f; 0)$ for $x, y, a, b \in X$. Then $f_n(a) < 0$, $f_n(b) < 0$, $f_p(x) > 0$ and $f_p(y) > 0$. If $f_n(a * b) = 0$ or $f_p(x * y) = 0$, then

$$f_n(a * b) + \max\{f_n(a), f_n(b)\} + 1 = \max\{f_n(a), f_n(b)\} + 1 \geq 0$$

or

$$f_p(x * y) + \min\{f_p(x), f_p(y)\} = \min\{f_p(x), f_p(y)\} \leq 1.$$

Thus

$$\langle (a * b, x * y); (\max\{f_n(a), f_n(b)\}, \min\{f_p(x), f_p(y)\})\rangle \, \overline{\mathcal{Q}} \, (X; f_n, f_p),$$

which is a contradiction because

$$\langle (a, x); (f_n(a), f_p(x)\rangle \Subset (X; f_n, f_p) \text{ and } \langle (b, y); (f_n(b), f_p(y)\rangle \Subset (X; f_n, f_p).$$

So $f_n(a*b) < 0$ and $f_p(x*y) > 0$, i.e., $a*b \in N(f;0)$ and $x*y \in P(f;0)$ for all $a,b,x,y \in X$. Hence the negative and positive 0-supports of $f = (X; f_n, f_p)$ are subalgebras of X. □

Corollary 3.4. *If $f = (X; f_n, f_p)$ is a bipolar-valued fuzzy subalgebra of type $(\Subset, \Subset)$ or $(\Subset, \mathcal{Q})$, then the intersection of negative and positive 0-supports of $f = (X; f_n, f_p)$ is a subalgebra of X.*

Theorem 3.5. *If $f = (X; f_n, f_p)$ is a bipolar-valued fuzzy subalgebra of type $(\mathcal{Q}, \Subset)$, then the negative and positive 0-supports of $f = (X; f_n, f_p)$ are subalgebras of X.*

Proof. Let $x, y \in N(f;0)$ and $a, b \in P(f;0)$ for $x,y,a,b \in X$. Then $f_n(x) < 0$, $f_n(y) < 0$, $f_p(a) > 0$ and $f_p(b) > 0$. It follows that

$$\langle (x,a); (-1,1)\rangle\, \mathcal{Q}\, (X; f_n, f_p) \text{ and } \langle (y,b); (-1,1)\rangle\, \mathcal{Q}\, (X; f_n, f_p).$$

Since $f = (X; f_n, f_p)$ is a bipolar-valued fuzzy subalgebra of type $(\mathcal{Q}, \Subset)$, we have $\langle (x*y, a*b); (-1,1)\rangle \Subset (X; f_n, f_p)$. If $f_n(x*y) = 0$ or $f_p(a*b) = 0$, then

$$f_n(x*y) = 0 > -1 \text{ or } f_p(a*b) = 0 < 1.$$

Thus

$$\langle (x*y, a*b); (-1,1)\rangle\, \overline{\Subset}\, (X; f_n, f_p),$$

which is a contradiction. So $f_n(x*y) < 0$ and $f_p(a*b) > 0$, that is, $x*y \in N(f;0)$ and $a*b \in P(f;0)$ for all $x,y,a,b \in X$. Consequently, the negative and positive 0-supports of $f = (X; f_n, f_p)$ are subalgebras of X. □

Corollary 3.6. *If $f = (X; f_n, f_p)$ is a bipolar-valued fuzzy subalgebra of type $(\mathcal{Q}, \Subset)$, then the intersection of negative and positive 0-supports of $f = (X; f_n, f_p)$ is a subalgebra of X.*

Theorem 3.7. *If $f = (X; f_n, f_p)$ is a bipolar-valued fuzzy subalgebra of type $(\mathcal{Q}, \mathcal{Q})$, then $f = (X; f_n, f_p)$ is constant on the negative and positive 0-supports of $f = (X; f_n, f_p)$, that is, f_n and f_p are constant on negative 0-support of $f = (X; f_n, f_p)$ and positive 0-support of $f = (X; f_n, f_p)$, respectively.*

Proof. Assume that f_n is not constant on the negative 0-support of $f = (X; f_n, f_p)$. Then there exists $y \in N(f;0)$ such that $\alpha_y = f_n(y) \neq f_n(0) = \alpha_0$. Thus either $\alpha_y < \alpha_0$ or $\alpha_y > \alpha_0$.

Suppose that $\alpha_y > \alpha_0$ and take $\alpha_1, \alpha_2 \in [-1, 0)$ such that $\alpha_2 < -1 - \alpha_y < \alpha_1 < -1 - \alpha_0$. Then $f_n(0) + \alpha_1 + 1 = \alpha_0 + \alpha_1 + 1 < 0$ and $f_n(y) + \alpha_2 + 1 = \alpha_y + \alpha_2 + 1 < 0$. Thus $0_{\alpha_1} \varrho f_n$ and $y_{\alpha_2} \varrho f_n$. Since

$$f_n(y*0) + \max\{\alpha_1, \alpha_2\} + 1 = f_n(y) + \alpha_1 + 1 = \alpha_y + \alpha_1 + 1 > 0,$$

we have $(y*0)_{\max\{\alpha_1,\alpha_2\}} \overline{\varrho} f_n$.

Next suppose that $\alpha_y < \alpha_0$. Then

$$f_n(y) + (-1 - \alpha_0) + 1 = \alpha_y - \alpha_0 < 0.$$

Thus $y_{-1-\alpha_0}\,\varrho f_n$. Since

$$f_n(y*y)+(-1-\alpha_0)+1=f_n(0)-\alpha_0=\alpha_0-\alpha_0=0,$$

we get $(y*y)_{\max\{-1-\alpha_0,-1-\alpha_0\}}\overline{\varrho} f_n$. This is a contradiction. So f_n is constant on the negative 0-support of $f=(X;f_n,f_p)$.

Suppose that f_p is not constant on the positive 0-support of $f=(X;f_n,f_p)$. Then there exists $b\in X$ such that $\beta_b=f_p(b)\neq f_p(0)=\beta_0$. For the case $\beta_b<\beta_0$, if we choose $\beta_1,\beta_2\in(0,1]$ such that $1-\beta_0<\beta_1<1-\beta_b<\beta_2$, then

$$f_p(0)+\beta_1=\beta_0+\beta_1>1 \text{ and } f_p(b)+\beta_2=\beta_b+\beta_2>1,$$

i.e, $(0,\beta_1)\;q\,f_p$ and $(b,\beta_2)\;q\,f_p$. Since

$$f_p(b*0)+\min\{\beta_1,\beta_2\}=f_p(b)+\beta_1=\beta_b+\beta_1<1,$$

it follows that $(b*0,\min\{\beta_1,\beta_2\})\,\overline{q}\,f_p$, which is a contradiction. If $\beta_b>\beta_0$, then $f_p(b)+(1-\beta_0)=\beta_b+1-\beta_0>1$. Thus $(b,1-\beta_0)\,q\,f$. Since

$$f_p(b*b)+(1-\beta_0)=f_p(0)+1-\beta_0=\beta_0+1-\beta_0=1,$$

we have $(b*b,\min\{1-\beta_0,1-\beta_0\})\,\overline{q}\,f_p$. This is a contradiction. So f_p is constant on the positive 0-support of $f=(X;f_n,f_p)$. Consequently, $f=(X;f_n,f_p)$ is constant on the negative and positive 0-supports of $f=(X;f_n,f_p)$. □

Theorem 3.8. *A bipolar-valued fuzzy set $f=(X;f_n,f_p)$ is a bipolar-valued fuzzy subalgebra of type $(\Subset,\Subset\vee\mathcal{Q})$ if and only if the following conditions are valid.*

$$(3.7)\qquad (\forall x,y\in X)\left(\begin{array}{l} f_n(x*y)\leq\max\{f_n(x),f_n(y),-0.5\}\\ f_p(x*y)\geq\min\{f_p(x),f_p(y),0.5\}\end{array}\right).$$

Proof. Suppose that $f=(X;f_n,f_p)$ is a bipolar-valued fuzzy subalgebra of type $(\Subset,\Subset\vee\mathcal{Q})$. For any $x,y\in X$, assume that $\max\{f_n(x),f_n(y)\}>-0.5$. If $f_n(a*b)>\max\{f_n(a),f_n(b)\}$ for some $a,b\in X$, then there exists $\alpha\in[-1,0)$ such that

$$f_n(a*b)>\alpha\geq\max\{f_n(a),f_n(b)\}.$$

Thus $a_\alpha\ni f_n$ and $b_\alpha\ni f_n$, but $(a*b)_{\max\{\alpha,\alpha\}}\overline{\ni\vee\varrho}f_n$, a contradiction. So $f_n(x*y)\leq\max\{f_n(x),f_n(y)\}$, whenever $\max\{f_n(x),f_n(y)\}>-0.5$, for all $x,y\in X$.

Now suppose that $\max\{f_n(x),f_n(y)\}\leq-0.5$. Then $x_{-0.5}\ni f_n$ and $y_{-0.5}\ni f_n$ which imply that $(x*y)_{\max\{-0.5,-0.5\}}\ni\vee\varrho\,f_n$. Thus $f_n(x*y)\leq-0.5$. Otherwise,

$$f_n(x*y)-0.5+1>-0.5-0.5+1=0, \text{ i.e., } (x*y)_{-0.5}\,\overline{\varrho}f_n.$$

This is a contradiction. Consequently, $f_n(x*y)\leq\max\{f_n(x),f_n(y),-0.5\}$, for all $x,y\in X$. If $\min\{f_p(x),f_p(y)\}<0.5$, then $f_p(x*y)\geq\min\{f_p(x),f_p(y)\}$. For, suppose that $f_p(x*y)<\min\{f_p(x),f_p(y)\}\triangleq\beta$. Then $(x,\beta)\in f_p$ and $(y,\beta)\in f_p$, but $(x*y,\beta)=(x*y,\min\{\beta,\beta\})\,\overline{\in\vee q}\,f_p$, a contradiction. Thus

$$f_p(x*y)\geq\min\{f_p(x),f_p(y)\}, \text{ whenever } \min\{f_p(x),f_p(y)\}<0.5.$$

Now assume that

$$\min\{f_p(x),f_p(y)\}\geq 0.5.$$

Then $(x,0.5)\in f_p$ and $(y,0.5)\in f_p$, which imply that

$$(x*y,0.5)=(x*y,\min\{0.5,0.5\})\in\vee\,q f_p.$$

Thus $f_p(x*y) \geq 0.5$. Otherwise, $f_p(x*y)+0.5 < 0.5+0.5 = 1$, a contradiction. So, $f_p(x*y) \geq \min\{f_p(x), f_p(y), 0.5\}$, for all $x, y \in X$. Hence (3.7) is valid.

Conversely, let $f = (X; f_n, f_p)$ be a bipolar-valued fuzzy set satisfying the condition (3.7). Let $x, y \in X$ and $\alpha_1, \alpha_2 \in [-1, 0)$ be such that $x_{\alpha_1} \ni f_n$ and $y_{\alpha_2} \ni f_n$. If $f_n(x*y) \leq \max\{\alpha_1, \alpha_2\}$, then $(x*y)_{\max\{\alpha_1,\alpha_2\}} \ni f_n$. Suppose that $f_n(x*y) > \max\{\alpha_1, \alpha_2\}$. Then $\max\{f_n(x), f_n(y)\} \leq -0.5$. Otherwise, we have

$$f_n(x*y) \leq \max\{f_n(x), f_n(y), -0.5\} = \max\{f_n(x), f_n(y)\} \leq \max\{\alpha_1, \alpha_2\},$$

a contradiction. It follows that

$$\begin{aligned} f_n(x*y) + \max\{\alpha_1, \alpha_2\} + 1 &< 2f_n(x*y) + 1 \\ &\leq 2\max\{f_n(x), f_n(y), -0.5\} + 1 = 0. \end{aligned}$$

Thus $(x*y)_{\max\{\alpha_1,\alpha_2\}} \varrho f_n$. So, $(x*y)_{\max\{\alpha_1,\alpha_2\}} \ni\vee\varrho f_n$.

Let $a, b \in X$ and $\beta_1, \beta_2 \in (0, 1]$ be such that $(a, \beta_1) \in f_p$ and $(b, \beta_2) \in f_p$. Then $f_p(a) \geq \beta_1$ and $f_p(b) \geq \beta_2$. If $f_p(a*b) < \min\{\beta_1, \beta_2\}$, then $\min\{f_p(a), f_p(b)\} \geq 0.5$. Otherwise, we get

$$f_p(a*b) \geq \min\{f_p(a), f_p(b), 0.5\} \geq \min\{f_p(a), f_p(b)\} \geq \min\{\beta_1, \beta_2\},$$

which is a contradiction. Thus

$$f_p(a*b) + \min\{\beta_1, \beta_2\} > 2f_p(a*b) \geq 2\min\{f_p(a), f_p(b), 0.5\} = 1.$$

So $(a*b, \min\{\beta_1, \beta_2\})\, q\, f_p$. Hence, $f = (X; f_n, f_p)$ is a bipolar-valued fuzzy subalgebra of type $(\Subset, \Subset\vee\mathcal{Q})$. □

We provide conditions for a bipolar-valued fuzzy set to be a bipolar-valued fuzzy subalgebra of type $(\mathcal{Q}, \Subset\vee\mathcal{Q})$.

Theorem 3.9. *Let A be a subalgebra of X and let $f = (X; f_n, f_p)$ be a bipolar-valued fuzzy set such that*

$$(3.8) \qquad (\forall x \in X)\left(\begin{array}{l} x \in A \ \Rightarrow\ f_n(x) \leq -0.5,\ f_p(x) \geq 0.5, \\ x \notin A \ \Rightarrow\ f_n(x) = 0,\ f_p(x) = 0 \end{array}\right).$$

Then $f = (X; f_n, f_p)$ is a bipolar-valued fuzzy subalgebra of type $(\mathcal{Q}, \Subset\vee\mathcal{Q})$.

Proof. Let $x, y, a, b \in X$ and $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in [-1, 0) \times (0, 1]$ be such that

$$\langle (x, y); (\alpha_1, \beta_1)\rangle\, \mathcal{Q}\, (X, f_n, f_p) \text{ and } \langle (a, b); (\alpha_2, \beta_2)\rangle\, \mathcal{Q}\, (X, f_n, f_p).$$

Then $x_{\alpha_1} \varrho f_n$, $(y, \beta_1)\, q\, f_p$, $a_{\alpha_2} \varrho f_n$ and $(b, \beta_2)\, q\, f_p$, i.e.,

$$f_n(x) + \alpha_1 + 1 < 0,\ f_p(y) + \beta_1 > 1,\ f_n(a) + \alpha_2 + 1 < 0 \text{ and } f_p(b) + \beta_2 > 1.$$

If $x \notin A$ or $a \notin A$, then $f_n(x) = 0$ or $f_n(a) = 0$. Thus $\alpha_1 + 1 < 0$ or $\alpha_2 + 1 < 0$. This is impossible, and so $x \in A$ and $a \in A$. Since A is a subalgebra of X, it follows that $x*a \in A$. Also, we get $y*b \in A$ because if it is impossible, then $y \notin A$ or $b \notin A$. Thus $f_p(y) = 0$ or $f_p(b) = 0$, which imply that $\beta_1 > 1$ or $\beta_2 > 1$. This is a contradiction. If $\max\{\alpha_1, \alpha_2\} < -0.5$, then $f_n(x*a) + \max\{\alpha_1, \alpha_2\} + 1 < 0$, i.e., $(x*a)_{\max\{\alpha_1,\alpha_2\}} \varrho f_n$. If $\max\{\alpha_1, \alpha_2\} \geq -0.5$, then $f_n(x*a) \leq -0.5 \leq \max\{\alpha_1, \alpha_2\}$, i.e., $(x*a)_{\max\{\alpha_1,\alpha_2\}} \ni f_n$. Thus

$$(x*a)_{\max\{\alpha_1,\alpha_2\}} \ni\vee\varrho f_n.$$

Also, if $\min\{\beta_1, \beta_2\} > 0.5$, then $f_p(y * b) + \min\{\beta_1, \beta_2\} > 1$. So

$$(y * b, \min\{\beta_1, \beta_2\}) \, q \, f_p.$$

If $\min\{\beta_1, \beta_2\} \leq 0.5$, then $f_p(y * b) \geq 0.5 \geq \min\{\beta_1, \beta_2\}$, i.e.,

$$(y * b, \min\{\beta_1, \beta_2\}) \in f_p.$$

Hence $(y * b, \min\{\beta_1, \beta_2\}) \in\!\vee\, q f_p$. Therefore

$$\langle (x * a, y * b); (\max\{\alpha_1, \alpha_2\}, \min\{\beta_1, \beta_2\}) \rangle \Subset\!\vee\, \mathcal{Q}\, (X; f_n, f_p),$$

and consequently $f = (X; f_n, f_p)$ is a bipolar-valued fuzzy subalgebra of type ($\mathcal{Q}$, $\Subset\!\vee\, \mathcal{Q}$). □

Theorem 3.10. *Let $f = (X; f_n, f_p)$ be a bipolar-valued fuzzy subalgebra of type ($\mathcal{Q}$, $\Subset\!\vee\, \mathcal{Q}$). If $f = (X; f_n, f_p)$ is nonconstant on the negative and positive 0-supports of $f = (X; f_n, f_p)$, then $f_n(x) \leq -0.5$ and $f_p(x) \geq 0.5$ for some $x \in X$. In particular, $f_n(0) \leq -0.5$ and $f_p(0) \geq 0.5$.*

Proof. Assume that $f_n(x) > -0.5$, for all $x \in X$. Since f_n is not constant on the negative 0-support of $f = (X; f_n, f_p)$, there exists $x \in N(f; 0)$ such that $\alpha_x = f_n(x) \neq f_n(0) = \alpha_0$. Then either $\alpha_0 < \alpha_x$ or $\alpha_0 > \alpha_x$. For the case $\alpha_0 < \alpha_x$, choose $r < -0.5$ such that $\alpha_0 + r + 1 < 0 < \alpha_x + r + 1$. Then $0_r \varrho f_n$. Since $x_{-1} \varrho f_n$, it follows that $x_r = (x * 0)_{\max\{r,-1\}} \ni\!\vee \varrho f_n$. But, $f_n(x) > -0.5 > r$ implies that $x_r \overline{\ni} f_n$ Also, $f_n(x) + r + 1 = \alpha_x + r + 1 > 0$ implies that $x_r \overline{\varrho} f_n$. This is a contradiction.

Now, if $\alpha_0 > \alpha_x$ then we can take $r < -0.5$ such that $\alpha_x + r + 1 < 0 < \alpha_0 + r + 1$. Then $x_r \varrho f_n$, and $f_n(x * x) = f_n(0) = \alpha_0 > r = \max\{r, r\}$ induces that $(x * x)_{\max\{r,r\}} \overline{\ni} f_n$. Since

$$f_n(x * x) + \max\{r, r\} + 1 = f_n(0) + r + 1 = \alpha_0 + r + 1 > 0,$$

$(x * x)_{\max\{r,r\}} \overline{\varrho} f_n$. Thus$(x * x)_{\max\{r,r\}} \overline{\ni\!\vee \varrho} f_n$, which is a contradiction. So $f_n(x) \leq -0.5$ for some $x \in X$.

Now, assume that $f_p(x) < 0.5$, for all $x \in X$. Since f_p is not constant on the positive 0-support of $f = (X; f_n, f_p)$, there exists $a \in P(f; 0)$ such that $\beta_a \triangleq f_p(a) \neq f_p(0) \triangleq \beta_0$. For the case $\beta_0 < \beta_a$, choose $\delta > 0.5$ such that $\beta_0 + \delta < 1 < \beta_a + \delta$. It follows that $(a, \delta) \, q \, f_p$, $f_p(a * a) = f_p(0) = \beta_0 < \delta = \min\{\delta, \delta\}$ and $f_p(a * a) + \min\{\delta, \delta\} = f_p(0) + \delta = \beta_0 + \delta < 1$ so that $(a * a, \min\{\delta, \delta\}) \, \overline{\in\!\vee\, q} \, f_p$. This is a contradiction. If $\beta_0 > \beta_a$, we can take $\delta > 0.5$ such that $\beta_a + \delta < 1 < \beta_0 + \delta$. Then $(0, \delta) \, q \, f_p$ and $(a, 1) \, q \, f_p$, but

$$(a * 0, \min\{1, \delta\}) = (a, \delta) \, \overline{\in\!\vee\, q} \, f_p$$

since $f_p(a) < 0.5 < \delta$ and $f_p(a) + \delta = \beta_a + \delta < 1$. This is a contradiction, and hence $f_p(x) \geq 0.5$ for some $x \in X$. We now prove that $f_n(0) \leq -0.5$ and $f_p(0) \geq 0.5$. Assume that $f_n(0) \triangleq \alpha_0 > -0.5$ or $f_p(0) \triangleq \beta_0 < 0.5$. Note that there exist $x, a \in X$ such that $f_n(x) \triangleq \alpha_x \leq -0.5$ and $f_p(a) \triangleq \beta_a \geq 0.5$. It follows that $\alpha_x < \alpha_0$ and $\beta_0 < \beta_a$. Choose $(\alpha_1, \beta_1) \in [-1, 0) \times (0, 1]$ such that

$$\alpha_1 < \alpha_0 \quad \text{and} \quad \alpha_x + \alpha_1 + 1 < 0 < \alpha_0 + \alpha_1 + 1$$

and

$$\beta_1 > \beta_0 \quad \text{and} \quad \beta_0 + \beta_1 < 1 < \beta_a + \beta_1.$$

Then $f_n(x)+\alpha_1+1=\alpha_x+\alpha_1+1<0$ and $f_p(a)+\beta_1=\beta_a+\beta_1>1$. Thus $x_{\alpha_1}\varrho f_n$ or $(a,\beta_1)\, q\, f_p$, i.e., $\langle(x,a);(\alpha_1,\beta_1)\rangle\, \mathcal{Q}\,(X;f_n,f_p)$. Now we have

$$f_n(x*x)+\max\{\alpha_1,\alpha_1\}+1=f_n(0)+\alpha_1+1=\alpha_0+\alpha_1+1>0$$

and

$$f_p(a*a)+\min\{\beta_1,\beta_1\}=f_p(0)+\beta_1=\beta_0+\beta_1<1.$$

Also, we get

$$f_n(x*x)=f_n(0)=\alpha_0>\alpha_1=\max\{\alpha_1,\alpha_1\}$$

and

$$f_p(a*a)=f_p(0)=\beta_0<\beta_1=\min\{\beta_1,\beta_1\}.$$

So $(x*x)_{\max\{\alpha_1,\alpha_1\}}\,\overline{\ni\vee\varrho}\, f_n$ and $(a*a,\min\{\beta_1,\beta_1\})\,\overline{\in\vee q}\, f_p$. Hence

$$\langle(x*x,a*a);(\max\{\alpha_1,\alpha_1\},\min\{\beta_1,\beta_1\})\rangle\,\overline{\Subset\vee\mathcal{Q}}\,(X;f_n,f_p).$$

This is a contradiction. Therefore $f_n(0)\leq -0.5$ and $f_p(0)\geq 0.5$. □

Theorem 3.11. *Given a bipolar-valued fuzzy set $f=(X;f_n,f_p)$, let $f^*=(X;f_n^*,f_p^*)$ be a bipolar-valued fuzzy set in which $f_n^*(x)=\max\{f_n(x),-0.5\}$ and $f_p^*(x)=\min\{f_p(x),0.5\}$, for all $x\in X$. If $f=(X;f_n,f_p)$ is a bipolar-valued fuzzy subalgebra of X of type $(\Subset,\ \Subset\vee\mathcal{Q})$, then $f^*=(X;f_n^*,f_p^*)$ is a bipolar-valued fuzzy subalgebra of X.*

Proof. Let $f=(X;f_n,f_p)$ be a bipolar-valued fuzzy subalgebra of X of type $(\Subset,$ $\Subset\vee\mathcal{Q})$. For any $x,y\in X$, we have

$$\begin{aligned}f_n^*(x*y)&=\max\{f_n(x*y),-0.5\}\\&\leq\max\{\max\{f_n(x),f_n(y),-0.5\},-0.5\}\\&=\max\{\max\{f_n(x),-0.5\},\max\{f_n(y),-0.5\}\}\\&=\max\{f_n^*(x),f_n^*(y)\}\end{aligned}$$

and

$$\begin{aligned}f_p^*(x*y)&=\min\{f_p(x*y),0.5\}\\&\geq\min\{\min\{f_p(x),f_p(y),0.5\},0.5\}\\&=\min\{\min\{f_p(x),0.5\},\min\{f_p(y),0.5\}\}\\&=\min\{f_p^*(x),f_p^*(y)\}.\end{aligned}$$

Then $f^*=(X;f_n^*,f_p^*)$ is a bipolar-valued fuzzy subalgebra of X. □

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