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Similarity and dissimilarity relations in intuitionistic fuzzy matrices using implication operators

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Abstract. In this paper, we study similarity and dissimilarity relations in Intuitionistic Fuzzy Matrix(IFM) using two implication operators and derive a method to get a max-min idempotent, min-max idempotent IFM from an IFM of order n.

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1. Introduction

An $m \times n$ matrix $A = (a_{ij})$ whose entries are in the unit interval $[0, 1]$ is called a Fuzzy Matrix (FM). After the introduction of fuzzy matrix (FM) theory using max-min algebra by Thomason [21], several authors [13, 16] have developed the fuzzy matrix theory and studied about convergence [5, 3], decomposition [6], reduction [7, 8] of a fuzzy matrix (FM). In all the above said study implication operator plays a vital role. Atanassov [1] developed the concept of Intuitionistic Fuzzy

Set (IFS) theory analogous to fuzzy set theory. In [2] Atanassov studied intuitionistic fuzzy index matrix on the basis of index matrix, and extended intuitionistic fuzzy index matrix (EIFIM). Im et al. [10], Jeong and Lee Hong [11], Khan S. K, Pal M and Amiya K. Shyamal [12], Meenakshi AR and Gandhimathi [15] and several others have studied Intuitionistic Fuzzy Matrices (IFM). IFM is very useful in the discussion of intuitionistic fuzzy relation. Pal and Mondal[17] studied the similarity relation, invertibility and eigenvalues of IFM.

Nora and Atanassov [19] used implication operators in IFSs and they have put forth more than hundred and fifty such implication operators and derived some axioms for those implication operators. Sriram and Murugadas [20] used $\leftarrow$ implication operator for IFM, studied the concept of g-inverse and semi-inverse of an IFM which was a generalization of FM studied by Hiroshi Hashimoto [9].

In fuzzy set theory, for any two fuzzy elements $x, y \in [0,1]$, $x \leftarrow y \geq x$ for all x, y. However this is not true in the case of fuzzy matrices (FMs)A, B. That means $A \leftarrow B \ngeq A$ always in the case of implication defined in[9]. Also Hashimoto [4] used $\leftharpoondown$ implication operator to fuzzy sets as well as to FMs, in which case also $x \leftharpoondown y \leq x$ for all $x, y \in [0,1]$, but it is not true that $A \leftharpoondown B \leq A$ for all fuzzy matrices (FMs) A, B. In this paper we extend the new implication operator $\leftharpoondown$ for IFM and discuss the relation with the existing implication operator. Many authors [7, 22, 14] have used $\leftharpoondown$ operator component wise in FM and in most of the cases the results are evident from fuzzy set theoretic point.

In this paper we try to overcome the above problem and we have shown the condition under which $A \leftarrow B \geq A$ and $A \leftharpoondown B \leq A$ are true using $A \leftarrow B$, $A \leftharpoondown B$ defined in [9] for FM, we extend this study to IFM and the relation between the above two implication operators. Also we obtain the unique maximum solution for the Intuitionistic Fuzzy inequality $A \times X \times B \leq C$ under max-min composition and unique minimum solution for $A \diamond X \diamond B \geq C$ under min-max composition. Further, we investigate the similarity relation for IFM using max-min operation and $\leftarrow$, dissimilarity relation for IFM using min-max operation and $\leftharpoondown$. Finally we structured a method to find max-min transitive IFM and min-max c-transitive IFM from any square IFM A.

2. Preliminaries

Throughout this paper $\mathscr{F}_{mn}$ denotes IFMs of order $m \times n$, $\mathscr{F}_n$ denotes IFMs of order $n \times n$, and entries in IFMs are comparable.

Definition 2.1 ([1]). An Intuitionistic Fuzzy Set(IFS) A in E(universal set) is defined as an object of the following form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}$, where $\mu_A(x) : E \to [0,1]$ and $\nu_A(x) : E \to [0,1]$ define the membership and non-membership of the element $x \in E$ respectively and for every $x \in E : 0 \leq \mu_{A(x)} + \nu_A(x) \leq 1$.

For simplicity we consider the pair $\langle x, x'\rangle$ as membership and non-membership function of an IFS with $x + x' \leq 1$.

Definition 2.2 ([1]). For $\langle x, x'\rangle, \langle y, y'\rangle \in IFS$, define
$\langle x, x'\rangle \vee \langle y, y'\rangle = \langle max\{x, y\}, min\{x', y'\}\rangle$,
$\langle x, x'\rangle \wedge \langle y, y'\rangle = \langle min\{x, y\}, max\{x', y'\}\rangle$,
$\langle x, x'\rangle^c = \langle x', x\rangle$.

Definition 2.3 ([1]). Let $X = \{x_1, x_2, ...x_m\}$ be a set of alternatives and $Y = \{y_1, y_2, ...y_n\}$ be the attribute set of each element of X. Then an intuitionistic fuzzy matrix (IFM) is defined by

$$A = (\langle (x_i, y_j), \mu_A(x_i, y_j), \nu_A(x_i, y_j)\rangle),$$

for $i = 1, 2...m$ and $j = 1, 2, ...n$,, where $\mu_A : X \times Y \to$ [0,1] and $\nu_A : X \times Y \to$ [0,1] satisfy the condition $0 \leq \mu_A(x_i, y_j) + \nu_A(x_i, y_j) \leq 1$. For simplicity we denote an intuitionistic fuzzy matrix (IFM) as matrix of pairs $A = (\langle a_{ij}, a'_{ij}\rangle)$ of non negative real numbers satisfying $a_{ij} + a'_{ij} \leq 1$ for all i, j.

For any two elements $A = (\langle a_{ij}, a'_{ij}\rangle), B = (\langle b_{ij}, b'_{ij}\rangle) \in \mathscr{F}_{mn}$ and $C \in \mathscr{F}_{np}$, define : for all $1 \leq i \leq m$ and $1 \leq j \leq n$,

(i) (Component wise addition)
$A \oplus B = (\langle max\{a_{ij}, b_{ij}\}, min\{a'_{ij}, b'_{ij}\}\rangle)$,

(ii) (Component wise multiplication)
$A \odot B = (\langle min\{a_{ij}, b_{ij}\}, max\{a'_{ij}, b'_{ij}\}\rangle)$,

(iii) (The universal matrix)
$J = (\langle 1, 0\rangle)$ is the matrix in which all entries are $\langle 1, 0\rangle$,

(iv) (Identity matrix)

$$I = (\langle \delta_{ij}, \delta'_{ij}\rangle) \text{ , where } \langle \delta_{ij}, \delta'_{ij}\rangle = \begin{cases} \langle 1, 0\rangle & \text{if } i = j \\ \langle 0, 1\rangle & \text{if } i \neq j \end{cases},$$

(v) (The zero matrix)
O is the matrix in which all the entries are $\langle 0, 1\rangle$,

(vi) $A \geq B$, if $a_{ij} \geq b_{ij}$ and $a'_{ij} \leq b'_{ij}$, for all i, j,
$A > B$, if $a_{ij} > b_{ij}$ and $a'_{ij} < b'_{ij}$, for all i, j, in which case A and B are comparable.

(vii) (Complement of A)
$A^c = (\langle a'_{ij}, a_{ij} \rangle)$,

(viii) (Transpose of A)
$A^T = (\langle a_{ji}, a'_{ji} \rangle)$,

(ix) $AC = (\langle max_{k=1}^{n} min\{a_{ik}, c_{kj}\}, min_{k=1}^{n} max\{a'_{ik}, c'_{kj}\} \rangle)$
$= (\langle \sum_{k=1}^{n} a_{ik} c_{kj}, \prod_{k=1}^{n} (a'_{ik} + c'_{kj}) \rangle)$,

(x) $A \leftarrow C = (min_{k=1}^{n} (\langle a_{ik}, a'_{ik} \rangle \leftarrow \langle c_{kj}, c'_{kj} \rangle))$
$= (\bigwedge_{k=1}^{n} (\langle a_{ik}, a'_{ik} \rangle \leftarrow \langle c_{kj}, c'_{kj} \rangle))$,

(xi) For any two comparable elements $\langle x, x' \rangle, \langle y, y' \rangle \in IFS$, define

$$\langle x, x' \rangle \leftarrow \langle y, y' \rangle = \begin{cases} \langle 1, 0 \rangle & \text{if } \langle x, x' \rangle \geq \langle y, y' \rangle \\ \langle x, x' \rangle & \text{if } \langle x, x' \rangle < \langle y, y' \rangle \end{cases}.$$

Definition 2.4 ([1])**.** For $A = (\langle a_{ij}, a'_{ij} \rangle) \in \mathscr{F}_{mn}$ and $C = (\langle c_{ij}, c'_{ij} \rangle) \in \mathscr{F}_{np}$, define

$$A \diamond C = (\bigwedge_{k} (\langle a_{ik}, a'_{ik} \rangle \vee \langle c_{kj}, c'_{kj} \rangle)).$$

Definition 2.5 ([18])**.** Let $R \in \mathscr{F}_n$. Then

(i) If $R \geq I_n$, then R is reflexive.
(ii) If $R^2 \leq R$, then R is transitive.
(iii) If $R \diamond R \geq R$ is c-transitive.
(iv) If $R \diamond R \leq R$ is c-compact.
(v) If $R^2 \geq R$, then R is compact.
(vi) If R is reflexive and transitive, then R is idempotent.
(vii) In R, if all the diagonal entries are $\langle 0, 1 \rangle$, then R is irreflexive.
(viii) R is symmetric if and only if $\langle r_{ij}, r'_{ij} \rangle = \langle r_{ji}, r'_{ji} \rangle$, for all i, j.
(ix) R is antisymmetric if and only if $R \odot R^T \leq I_n$,

where I_n is the unit matrix with all entries in the main diagonal as $\langle 1, 0 \rangle$ and remaining entries with $\langle 0, 1 \rangle$, and $R \odot R^T \leq I_n$ means $\langle r_{ij}, r'_{ij} \rangle \langle r_{ji}, r'_{ji} \rangle = \langle 0, 1 \rangle$ for all $i \neq j$ and $\langle r_{ii}, r'_{ii} \rangle \leq \langle 1, 0 \rangle$ for all i. So if $\langle r_{ij}, r'_{ij} \rangle = \langle 1, 0 \rangle$, then $\langle r_{ji}, r'_{ji} \rangle = \langle 0, 1 \rangle$.

(x) If R is irreflexive and transitive, then R is nilpotent.
(xi) If $\langle r_{ii}, r'_{ii} \rangle \geq \langle r_{ij}, r'_{ij} \rangle$, for all i, j, then R is weakly reflexive.

Proposition 2.6 ([20])**.** *For $A \in \mathscr{F}_{mn}, B \in \mathscr{F}_{mp}$, and $C \in \mathscr{F}_{qn}$, the following inequality hold:*

(1) $B \times (B^T \rightarrow A \leftarrow C^T) \times C \leq A$.

(2) $B \times (B^T \rightarrow A) \leq A$.
(3) $(A \leftarrow C^T) \times C \leq A$.

Lemma 2.7 ([20]). *For $A \in \mathscr{F}_{mn}, A \leftarrow A^T$ is reflexive and transitive.*

Proposition 2.8 ([20]). *For any $A \in \mathscr{F}_{mn}$, the following inequality hold:*
(1) $A \rightarrow A^T \geq I_m$.
(2) $A \leftarrow A^T \geq I_m$.
(3) $(A \rightarrow A^T)A \geq A$.

3. Maximal and minimal solution of intuitionistic fuzzy inequalities

In this section we extend implication operator $\leftharpoondown$ used by Hashimoto [9] for fuzzy matrices to IFS and IFM and discuss the relation between $\leftarrow$ and $\leftharpoondown$.

Lemma 3.1. *For $A, B \in \mathscr{F}_n$, $A \leftarrow B \geq A$ if and only if A is weakly reflexive.*

Proof. Let $A = (\langle a_{ij}, a'_{ij}\rangle)$ and $B = (\langle b_{ij}, b'_{ij}\rangle)$. Then

$$\begin{aligned}(A \leftarrow B) \geq A &\Leftrightarrow \bigwedge_{k=1}^{n} (\langle a_{ik}, a'_{ik}\rangle \leftarrow \langle b_{kj}, b'_{kj}\rangle) \geq \langle a_{ij}, a'_{ij}\rangle\\ &\Leftrightarrow \langle a_{il}, a'_{il}\rangle \leftarrow \langle b_{lj}, b'_{lj}\rangle \geq \langle a_{ij}, a'_{ij}\rangle \text{ for some } l\\ &\Leftrightarrow \langle a_{il}, a'_{il} \geq \langle a_{ij}, a'_{ij}\rangle\\ &\Leftrightarrow \text{in particular, } \langle a_{ll}, a'_{ll}\rangle \geq \langle a_{lj}, a'_{lj}\rangle.\end{aligned}$$

□

Lemma 3.2. *For $A, B \in \mathscr{F}_n, A \leftharpoondown B \leq A$ if and only if $\langle a_{ij}, a'_{ij}\rangle \geq \langle a_{ii}, a'_{ii}\rangle$.*

Proof. Let $A = (\langle a_{ij}, a'_{ij}\rangle)$ and $B = (\langle b_{ij}, b'_{ij}\rangle)$. Then

$$\begin{aligned}(A \leftharpoondown B) \leq A &\Leftrightarrow \bigvee_{k=1}^{n} (\langle a_{ik}, a'_{ik}\rangle \leftharpoondown \langle b_{kj}, b'_{kj}\rangle) \leq \langle a_{ij}, a'_{ij}\rangle\\ &\Leftrightarrow \langle a_{il}, a'_{il}\rangle \leftharpoondown \langle b_{lj}, b'_{lj}\rangle \leq \langle a_{ij}, a'_{ij}\rangle \text{ for some } l\\ &\Leftrightarrow \langle a_{il}, a'_{il} \leq \langle a_{ij}, a'_{ij}\rangle\\ &\Leftrightarrow \text{in particular, } \langle a_{ll}, a'_{ll}\rangle \leq \langle a_{lj}, a'_{lj}\rangle.\end{aligned}$$

□

Definition 3.3. For any two comparable elements $\langle x, x'\rangle, \langle y, y'\rangle \in IFS$, define

$$\langle x, x'\rangle \leftharpoondown \langle y, y'\rangle = \begin{cases}\langle 0, 1\rangle & \text{if } \langle x, x'\rangle \leq \langle y, y'\rangle\\ \langle x, x'\rangle & \text{if } \langle x, x'\rangle > \langle y, y'\rangle\end{cases}$$

and for $A \in \mathscr{F}_{mn}$ and $C \in \mathscr{F}_{np}$, define

$$A \leftharpoondown C = \bigvee_k (\langle a_{ik}, a'_{ik}\rangle \leftharpoondown \langle c_{kj}, c'_{kj}\rangle)$$

and

$$A \rightharpoondown C = \bigvee_k (\langle a_{ik}, a'_{ik}\rangle \rightharpoondown \langle c_{kj}, c'_{kj}\rangle).$$

Lemma 3.4. *For* $\langle a, a'\rangle, \langle b, b'\rangle \in IFS$,

$$(\langle a, a'\rangle \leftarrow \langle b, b'\rangle)^c = (\langle a, a'\rangle^c \leftharpoondown \langle b, b'\rangle^c). \tag{3.1}$$

Proof. Case (i): If $\langle a, a'\rangle \geq \langle b, b'\rangle$, then $\langle a, a'\rangle \leftarrow \langle b, b'\rangle = \langle 1, 0\rangle$. By assumption, $(\langle a, a'\rangle)^c \leq (\langle b, b'\rangle)^c$. Thus

$$(\langle a, a'\rangle \leftarrow \langle b, b'\rangle)^c = \langle 0, 1\rangle. \tag{3.2}$$

So, by Definition 3.3,

$$(\langle a, a'\rangle)^c \leftharpoondown (\langle b, b'\rangle)^c = \langle 0, 1\rangle. \tag{3.3}$$

Hence, by (3.2) and (3.3), (3.1) holds.

Case (ii): If $\langle a, a'\rangle < \langle b, b'\rangle$, then $\langle a, a'\rangle \leftarrow \langle b, b'\rangle = \langle a, a'\rangle$. Thus

$$(\langle a, a'\rangle \leftarrow \langle b, b'\rangle)^c = \langle a, a'\rangle^c = \langle a', a\rangle. \tag{3.4}$$

By assumption, $\langle a, a'\rangle^c > \langle b, b'\rangle^c$. So, by Definition 3.3,

$$\langle a, a'\rangle^c \leftharpoondown \langle b, b'\rangle^c = \langle a', a\rangle. \tag{3.5}$$

Hence, by (3.4) and (3.5), (3.1) holds. □

Lemma 3.5. *For* $\langle a, a'\rangle, \langle b, b'\rangle \in IFS$,

$$(\langle a, a'\rangle \leftharpoondown \langle b, b'\rangle)^c = (\langle a, a'\rangle^c \leftarrow \langle b, b'\rangle^c). \tag{3.6}$$

Proof. Case (i): If $\langle a, a'\rangle \leq \langle b, b'\rangle$, then

$$(\langle a, a'\rangle \leftharpoondown \langle b, b'\rangle)^c = \langle 1, 0\rangle. \tag{3.7}$$

By assumption, $(\langle a, a'\rangle)^c \geq (\langle b, b'\rangle)^c$. Thus

$$(\langle a, a'\rangle^c \leftarrow \langle b, b'\rangle^c) = \langle 1, 0\rangle. \tag{3.8}$$

So, from (3.7) and (3.8), (3.6) holds.

Case (ii): If $\langle a, a'\rangle > \langle b, b'\rangle$, then $\langle a, a'\rangle^c < \langle b, b'\rangle^c$. Thus

$$(\langle a, a'\rangle \leftharpoondown \langle b, b'\rangle)^c = \langle a, a'\rangle^c \tag{3.9}$$

and

$$\langle a, a'\rangle^c \leftarrow \langle b, b'\rangle^c = \langle a, a'\rangle^c. \tag{3.10}$$

So, by (3.9) and (3.10), (3.6) holds. □

Lemma 3.6. *For* $\langle a, a'\rangle, \langle b, b'\rangle \in IFS$,

$$(\langle b, b'\rangle \rightarrow \langle a, a'\rangle)^c = (\langle b, b'\rangle^c \rightharpoondown \langle a, a'\rangle^c). \tag{3.11}$$

Proof. Case (i): If $\langle a, a'\rangle \geq \langle b, b'\rangle$, then

$$(\langle b, b'\rangle \rightarrow \langle a, a'\rangle)^c = (\langle 1, 0\rangle)^c. \tag{3.12}$$

By assumption, $(\langle a, a'\rangle)^c \leq (\langle b, b'\rangle)^c$. Thus

$$(\langle b, b'\rangle^c \rightharpoondown \langle a, a'\rangle)^c = \langle 0, 1\rangle = \langle 1, 0\rangle^c. \tag{3.13}$$

So, by (3.12) and (3.13), (3.11) holds.

Case (ii): If $\langle a, a'\rangle < \langle b, b'\rangle$, then

$$(\langle b, b'\rangle \rightarrow \langle a, a'\rangle)^c = \langle a, a'\rangle^c \tag{3.14}$$

and

$$\langle b, b'\rangle^c \rightharpoondown \langle a, a'\rangle^c = \langle a, a'\rangle^c. \tag{3.15}$$

Thus, by (3.14) and (3.15), (3.11) holds. □

In the same manner, we prove the following Lemma.

Lemma 3.7. *For* $\langle a, a'\rangle, \langle b, b'\rangle \in IFS$,

$$(\langle b, b'\rangle \rightharpoondown \langle a, a'\rangle)^c = \langle b, b'\rangle^c \rightarrow \langle a, a'\rangle^c.$$

Lemma 3.8. *For* $\langle a, a'\rangle, \langle b, b'\rangle, \langle c, c'\rangle \in IFS$,

$$\langle a, a'\rangle \wedge \langle b, b'\rangle \leq \langle c, c'\rangle \Leftrightarrow \langle a, a'\rangle \leq \langle c, c'\rangle \leftarrow \langle b, b'\rangle.$$

Proof. Case(i): If $\langle a, a'\rangle \leq \langle b, b'\rangle$, then

$$\begin{aligned} &\langle a, a'\rangle \wedge \langle b, b'\rangle = \langle a, a'\rangle \leq \langle c, c'\rangle \\ \Leftrightarrow\ &\langle a, a'\rangle \leq \langle c, c'\rangle \leftarrow \langle b, b'\rangle. \end{aligned}$$

Case (ii): If $\langle a, a'\rangle > \langle b, b'\rangle$, then

$$\begin{aligned} &\langle a, a'\rangle \wedge \langle b, b'\rangle = \langle b, b'\rangle \leq \langle c, c'\rangle \\ \Leftrightarrow\ &\langle c, c'\rangle \leftarrow \langle b, b'\rangle = \langle 1, 0\rangle \\ \Leftrightarrow\ &\langle a, a'\rangle \leq \langle c, c'\rangle \leftarrow \langle b, b'\rangle. \end{aligned}$$

□

Lemma 3.9. *For* $\langle a, a'\rangle, \langle b, b'\rangle, \langle c, c'\rangle \in IFS$,

$$\langle a, a'\rangle \vee \langle b, b'\rangle \geq \langle c, c'\rangle \Leftrightarrow \langle a, a'\rangle \geq \langle c, c'\rangle \leftharpoondown \langle b, b'\rangle.$$

Proof. Case (i): If $\langle a, a'\rangle \geq \langle b, b'\rangle$, then

$$\langle a, a'\rangle \vee \langle b, b'\rangle = \langle a, a'\rangle \geq \langle c, c'\rangle \Leftrightarrow \langle a, a'\rangle \geq \langle c, c'\rangle \leftharpoondown \langle b, b'\rangle.$$

Case (ii): If $\langle a, a'\rangle < \langle b, b'\rangle$, then

$$\begin{aligned} &\langle a, a' \rangle \vee \langle b, b' \rangle = \langle b, b' \rangle \\ \Leftrightarrow\ &\langle c, c' \rangle \leftharpoondown \langle b, b' \rangle = \langle 0, 1 \rangle \\ \Leftrightarrow\ &\langle a, a' \rangle \geq \langle c, c' \rangle \leftharpoondown \langle b, b' \rangle. \end{aligned}$$

□

Now we extend this result to $IFMs$.

Proposition 3.10. *For $A \in \mathscr{F}_{mn}, B \in \mathscr{F}_{nm}$, the following results hold:*
(1) $(A \leftarrow B)^c = A^c \leftharpoondown B^c$.
(2) $(A \leftharpoondown B)^c = A^c \leftarrow B^c$.
(3) $(A \rightarrow B)^c = A^c \rightharpoondown B^c$.
(4) $(A \rightharpoondown B)^c = A^c \rightarrow B^c$.

Proof. The proof is evident from Lemma 3.4, Lemma 3.5, Lemma 3.6 and Lemma 3.7 □

Lemma 3.11. *For $A \in \mathscr{F}_{mn}, B \in \mathscr{F}_{nm}$,*
(1) $(A \leftarrow B)^T = B^T \rightarrow A^T$.
(2) $(A \leftharpoondown B)^T = B^T \rightharpoondown A^T$.

Proof. (1) Let $A = (\langle a_{ij}, a'_{ij} \rangle),\ B = (\langle b_{ij}, b'_{ij} \rangle)$. Then

$$\langle a_{ik}, a'_{ik} \rangle \leftarrow \langle b_{kj}, b'_{kj} \rangle = \begin{cases} \langle 1, 0 \rangle & \text{if } \langle a_{ik}, a'_{ik} \rangle \geq \langle b_{kj}, b'_{kj} \rangle, \\ \langle a_{ik}, a'_{ik} \rangle & \text{if } \langle a_{ik}, a'_{ik} \rangle < \langle b_{kj}, b'_{kj} \rangle. \end{cases}$$

Thus,

$$(\langle a_{ik}, a'_{ik} \rangle \leftarrow \langle b_{kj}, b'_{kj} \rangle)^T = \begin{cases} \langle 1, 0 \rangle & \text{if } \langle a_{ki}, a'_{ki} \rangle \geq \langle b_{jk}, b'_{jk} \rangle. \\ \langle a_{ki}, a'_{ki} \rangle & \text{if } \langle a_{ki}, a'_{ki} \rangle < \langle b_{jk}, b'_{jk} \rangle. \end{cases} \tag{3.16}$$

Now,

$$\langle b_{jk}, b'_{jk} \rangle \rightarrow \langle a_{ki}, a'_{ki} \rangle = \begin{cases} \langle 1, 0 \rangle & \text{if } \langle a_{ki}, a'_{ki} \rangle \geq \langle b_{jk}, b'_{jk} \rangle, \\ \langle a_{ki}, a'_{ki} \rangle & \text{if } \langle a_{ki}, a'_{ki} \rangle < \langle b_{jk}, b'_{jk} \rangle. \end{cases} \tag{3.17}$$

So, by (3.16) and (3.17), (1) holds.
(2) Clearly,

$$\langle a_{ik}, a'_{ik} \rangle \leftharpoondown \langle b_{kj}, b'_{kj} \rangle = \begin{cases} \langle 0, 1 \rangle & \text{if } \langle a_{ik}, a'_{ik} \rangle \leq \langle b_{kj}, b'_{kj} \rangle, \\ \langle a_{ik}, a'_{ik} \rangle & \text{if } \langle a_{ik}, a'_{ik} \rangle > \langle b_{kj}, b'_{kj} \rangle. \end{cases}$$

Then

$$(\langle a_{ik}, a'_{ik} \rangle \leftharpoondown \langle b_{kj}, b'_{kj} \rangle)^T = \begin{cases} \langle 0, 1 \rangle & \text{if } \langle a_{ki}, a'_{ki} \rangle \leq \langle b_{jk}, b'_{jk} \rangle, \\ \langle a_{ki}, a'_{ki} \rangle & \text{if } \langle a_{ki}, a'_{ki} \rangle > \langle b_{jk}, b'_{jk} \rangle. \end{cases} \tag{3.18}$$

and

$$(3.19)\quad \langle b_{jk}, b'_{jk}\rangle \rightharpoondown \langle a_{ki}, a'_{ki}\rangle = \begin{cases} \langle 0,1\rangle & \text{if } \langle a_{ki}, a'_{ki}\rangle \leq \langle b_{jk}, b'_{jk}\rangle, \\ \langle a_{ki}, a'_{ki}\rangle & \text{if } \langle a_{ki}, a'_{ki}\rangle > \langle b_{jk}, b'_{jk}\rangle. \end{cases}$$

Thus, by (3.18) and (3.19), (2) holds. □

Lemma 3.12. *For $A \in \mathscr{F}_{mn}, B \in \mathscr{F}_{np}$ and $C \in \mathscr{F}_{mp}$,*
(1) $A \times B \leq C \Leftrightarrow A \leq C \leftarrow B^T \Leftrightarrow B \leq A^T \rightarrow C$.
(2) $A \diamond B \geq C \Leftrightarrow A \geq C \leftharpoondown B^T \Leftrightarrow B \geq A^T \rightharpoondown C$.

Proof. (1) $A \times B \leq C$
$\Leftrightarrow (\bigvee_k(\langle a_{ik}, a'_{ik}\rangle) \wedge \langle b_{kj}, b'_{kj}\rangle) \leq \langle c_{ij}, c'_{ij}\rangle)$ for all i, j
$\Leftrightarrow \langle a_{ik}, a'_{ik}\rangle \leq \langle c_{ij}, c'_{ij}\rangle \leftarrow \langle b_{kj}, b'_{kj}\rangle$ for all i, j, k, by Lemma 3.8
$\Leftrightarrow \langle a_{ik}, a'_{ik}\rangle \leq \bigwedge_j(\langle c_{ij}, c'_{ij}\rangle \leftarrow \langle b_{kj}, b'_{kj}\rangle)$ for all i, k
$\Leftrightarrow (\langle a_{ij}, a'_{ij}\rangle) \leq \bigwedge_k\langle c_{ik}, c'_{ik}\rangle) \leftarrow \langle b_{jk}, b'_{jk}\rangle)$ for all i, j, k
$\Leftrightarrow A \leq C \leftarrow B^T$.

On one hand,
$A \times B \leq C \Leftrightarrow B^T \times A^T \leq C^T \Leftrightarrow B^T \leq C^T \leftarrow A \Leftrightarrow B \leq (C^T \leftarrow A)^T$.
By Lemma 3.11 (1), $(C^T \leftarrow A)^T = A^T \rightarrow C$. Then

$$A \times B \leq C \Leftrightarrow B \leq A^T \rightarrow C.$$

(2) $A \diamond B \geq C$
$\Leftrightarrow (\bigwedge_k(\langle a_{ik}, a'_{ik}\rangle) \vee \langle b_{kj}, b'_{kj}\rangle) \geq \langle c_{ij}, c'_{ij}\rangle)$ for all i, j
$\Leftrightarrow \langle a_{ik}, a'_{ik}\rangle) \vee \langle b_{kj}, b'_{kj}\rangle \geq \langle c_{ij}, c'_{ij}\rangle$ for all i, j, k.
$\Leftrightarrow \langle a_{ik}, a'_{ik}\rangle \geq \langle c_{ij}, c'_{ij}\rangle \leftharpoondown \langle b_{kj}, b'_{kj}\rangle$ for all i, j, k
$\Leftrightarrow \langle a_{ik}, a'_{ik}\rangle \geq \bigvee_j(\langle c_{ij}, c'_{ij}\rangle \leftharpoondown \langle b_{kj}, b'_{kj}\rangle)$ for all i, k
$\Leftrightarrow \langle a_{ij}, a'_{ij}\rangle \geq \bigvee_k(\langle c_{ik}, c'_{ik}\rangle \leftharpoondown \langle b_{jk}, b'_{jk}\rangle)$ for all i, j.
Then $A \diamond B \geq C \Leftrightarrow A \geq C \leftharpoondown B^T \Leftrightarrow B \geq A^T \rightharpoondown C$ is obvious. □

Proposition 3.13. *For $A \in \mathscr{F}_{mn}, B \in \mathscr{F}_{pn}$,*
(1) $(A \leftharpoondown B^T) \diamond B \geq A$.
(2) $B \diamond (B^T \rightharpoondown A) \geq A$, *for* $B \in \mathscr{F}_{mp}$.

Proof. (1) Since $A \leftharpoondown B^T \geq A \leftharpoondown B^T$, by Lemma 3.12(2), (1) holds.
(2) Put $D = B^T \rightharpoondown A$. Then, by Lemma 3.12(2),

$$B \diamond D \geq A \Leftrightarrow D \geq B^T \rightharpoondown A \Leftrightarrow B^T \rightharpoondown A \geq B^T \rightharpoondown A.$$

Thus $B \diamond (B^T \rightharpoondown A) \geq A$. □

Remark 3.14. Consider the inequality $A \times X \times B \leq C$ where $A \in \mathscr{F}_{mn}, B \in \mathscr{F}_{pq}$ and $C \in \mathscr{F}_{mq}$ with unknown $X \in \mathscr{F}_{np}$.

By Lemma 3.12(1),

$$X \leq (A^T \rightarrow C) \leftarrow B^T. \tag{3.20}$$

By Proposition 2.6,

$$A \times (A^T \rightarrow C \leftarrow B^T) \times B \leq C. \tag{3.21}$$

From (3.20) and (3.21), $(A^T \rightarrow C \leftarrow B^T)$ is the maximum solution of $A \times X \times B \leq C$.

Remark 3.15. Consider the inequality $A \diamond X \diamond B \geq C$ with suitable size and unknown X. Then, by Lemma 3.12(2),

$$X \geq A^T \rightharpoondown C \leftharpoondown B^T. \tag{3.22}$$

By Proposition 3.13,

$$A \diamond \left(A^T \rightharpoondown C \leftharpoondown B^T\right) \diamond B \geq A \diamond \left(A^T \rightharpoondown C\right) \geq C. \tag{3.23}$$

From (3.22) and (3.23), $A^T \rightharpoondown C \leftharpoondown B^T$ is the minimum solution of $A \diamond X \diamond B \geq C$.

4. Intuitionistic fuzzy similarity relation (IFSR) and intuitionistic fuzzy dissimilarity relation(IFDR) using $\leftarrow$ and $\leftharpoondown$.

This section provides a method to obtain IFSR and IFDR from any IFM. Using this IFSR and IFDR we shall obtain max-min idempotent $(R \times R = R)$ and min-max idempotent $(R \diamond R = R)$ for any IFM.

Proposition 4.1. *For $A \in \mathscr{F}_{mn}$, $B \in \mathscr{F}_{fnp}$,*
(1) $A \leftarrow (A^T \leftarrow B) \geq B$.
(2) $A \leftharpoondown (A^T \leftharpoondown B) \leq B$.
(3) $(B \rightarrow A^T) \rightarrow A \geq B$, *for* $B \in \mathscr{F}_{pn}$.
(4) $(B \rightharpoondown A^T) \rightharpoondown A \leq B$, *for* $B \in \mathscr{F}_{pn}$.

Proof. (1) Since $(B^T \rightarrow A) \leq (B^T \rightarrow A)$,

$$(A^T \leftarrow B)^T = B^T \rightarrow (A^T)^T = B^T \rightarrow A \leq B^T \rightarrow A.$$

Since $AB \leq C \Leftrightarrow A \leq C \leftarrow B^T \Leftrightarrow B \leq A^T \rightarrow C)$, using Lemma 3.12(1),

$$B(A^T \leftarrow B)^T \leq A$$

and

$$B \leq A \leftarrow ((A^T \leftarrow B)^T)^T = A \leftarrow (A^T \leftarrow B).$$

(2) Since $B^T \rightharpoondown A \geq B^T \rightharpoondown A$ or $(A^T \leftharpoondown B)^T \geq B^T \rightharpoondown A$, using Lemma 3.12(2), $B \diamond (A^T \leftharpoondown B)^T \geq A$ and $B \geq A \leftharpoondown (A^T \leftharpoondown B)$.

(3) Since $A \leftarrow B^T \leq A \leftarrow B^T$ or $(B \to A^T)^T \leq A \leftarrow B^T$,

$(B \to A^T)^T B \leq A$ and $B \leq (B \to A^T) \to A$.

(4) Since $A \leftharpoondown B^T \geq A \leftharpoondown B^T$ or $(B \rightharpoondown A^T)^T \geq A \leftharpoondown B^T$, by Lemma 3.12,

$(B \rightharpoondown A^T)^T \diamond B \geq A$ and $B \geq (B \rightharpoondown A^T) \rightharpoondown A$. □

Proposition 4.2. *For $A \in \mathscr{F}_{mn}$, $B \in \mathscr{F}_{pn}$,*

(1) $((A \leftarrow B^T)B) \leftarrow B^T = A \leftarrow B^T$.

(2) $((A \leftharpoondown B^T) \diamond B) \leftharpoondown B^T = A \leftharpoondown B^T$.

(3) $B^T \to (B(B^T \to A)) = B^T \to A$, *for* $B \in \mathscr{F}_{mp}$.

(4) $B^T \rightharpoondown (B \diamond (B^T \rightharpoondown A)) = B^T \rightharpoondown A$, *for* $B \in \mathscr{F}_{mp}$.

Proof. (1) Clearly, $(A \leftarrow B^T)B \leq (A \leftarrow B^T)B$. Then, by Lemma 3.12(1), $(A \leftarrow B^T) \leq ((A \leftarrow B^T)B) \leftarrow B^T$. Thus, by Proposition 2.6,

$$(A \leftarrow B^T) \leq ((A \leftarrow B^T)B) \leftarrow B^T \leq A \leftarrow B^T.$$

So $((A \leftarrow B^T)B) \leftarrow B^T = A \leftarrow B^T$.

(2) Clearly, $((A \leftharpoondown B^T) \diamond B) \geq (A \leftharpoondown B^T) \diamond B$. Then

$$(A \leftharpoondown B^T) \geq ((A \leftharpoondown B^T) \diamond B) \leftharpoondown B^T \geq A \leftharpoondown B^T.$$

Then, by Lemma 3.12(i), and Proposition 3.13(1),

$$((A \leftharpoondown B) \diamond B) \leftharpoondown B^T = A \leftharpoondown B^T.$$

(3) Clearly, $B(B^T \to A) \leq B(B^T \to A)$. Then

$$(B^T \to A)) \leq B^T \to B(B^T \to A) \leq B^T \to A.$$

By Proposition 2.6, $(B(B^T \to A) \leq A$. Thus

$$B^T \to (B(B^T \to A)) = B^T \to A.$$

(4) Clearly, $B \diamond (B^T \rightharpoondown A) \geq B \diamond (B^T \rightharpoondown A)$. Then, by Proposition 3.13, $B^T \rightharpoondown A \geq B^T \rightharpoondown (B \diamond (B^T \rightharpoondown A)) \geq B^T \rightharpoondown A$. Thus

$$B^T \rightharpoondown (B \diamond (B^T \rightharpoondown A)) = B^T \rightharpoondown A.$$

□

Proposition 4.3. (1) *For $A \in \mathscr{F}_{mn}$, $B \in \mathscr{F}_{pn}$ and $C \in \mathscr{F}_{qn}$,*

(a) $(A \leftarrow B^T) \times (B \leftarrow C^T) \times C \leq A$.

(b) $(A \leftharpoondown B^T) \diamond (B \leftharpoondown C^T) \diamond C \geq A$.

(2) *For $A \in \mathscr{F}_{mn}, B \in \mathscr{F}_{mp}$ and $C \in \mathscr{F}_{mq}$,*

(a) $C \times (C^T \to B) \times (B^T \to A) \leq A$.

(b) $C \diamond (C^T \rightharpoondown B) \diamond (B^T \rightharpoondown A) \geq A$.

Proof. (1) (a) By proposition 2.6, $(A \leftarrow B^T)B \leq A$, $(B \leftarrow C^T)C \leq B$. Then $(A \leftarrow B^T)(B \leftarrow C^T)C \leq (A \leftarrow B^T)B \leq A$. Thus (a) holds.

(b) By Proposition 2.6, $(A \leftharpoondown B^T) \diamond B \geq A \Rightarrow (B \leftharpoondown C^T) \diamond C \geq B$. Then $(A \leftharpoondown B^T) \diamond (B \leftharpoondown C^T) \diamond C \geq (A \leftharpoondown B^T)B \geq A$. Thus (b) holds.

(2) (a) By proposition 2.6, $B \times (B^T \rightarrow A) \leq A$, $C \times (C^T \rightarrow B) \leq B$. Then $C \times (C^T \rightarrow B) \times (B^T \rightarrow A) \leq B(B^T \rightarrow A) \leq A$. Thus (a) holds.

(b) By proposition 3.13, $B \diamond (B^T \rightharpoondown A) \geq A$, $C \diamond (C^T \rightharpoondown B) \geq B$. Then $C \diamond (C^T \rightharpoondown B) \diamond (B^T \rightharpoondown A) \geq B \diamond (B^T \rightharpoondown A) \geq A$. Thus (b) holds. □

Proposition 4.4. (1) *For* $A \in \mathscr{F}_{mn}$, $B \in \mathscr{F}_{pn}$ *and* $F \in \mathscr{F}_{nq}$,

(a) $(A \leftarrow B^T)(B \leftarrow F) \leq A \leftarrow F$.

(b) $(A \leftharpoondown B^T) \diamond (B \leftharpoondown F) \geq A \leftharpoondown F$.

(2) *For* $A \in \mathscr{F}_{mn}$, $B \in \mathscr{F}_{mp}$ *and* $F \in \mathscr{F}_{qm}$,

(a) $(F \rightarrow B)(B^T \rightarrow A) \leq F \rightarrow A$.

(b) $(F \rightharpoondown B) \diamond (B^T \rightharpoondown A) \geq F \rightharpoondown A$.

Proof. (1) (a) By Proposition 4.3, $(A \leftarrow B^T)(B \leftarrow F^T)F \leq A$. Then, by Lemma 3.12, $(A \leftarrow B^T)(B \leftarrow F^T) \leq A \leftarrow F^T$.

(b) Clearly, $(A \leftharpoondown B^T) \diamond (B \leftharpoondown F^T) \diamond F \geq A$. Then (b) holds.

(2) (a) By Proposition 4.3, $F(F^T \rightarrow B)(B^T \rightarrow A) \leq A$. Then (a) holds.

(b) Clearly, $F \diamond (F^T \rightharpoondown B) \diamond (B^T \rightharpoondown A) \geq A$. Then (b) holds. □

Proposition 4.5. (1) *For* $A \in \mathscr{F}_{mn}$, $B \in \mathscr{F}_{pn}$ *and* $F \in \mathscr{F}_{n}$,

(a) $(A \leftarrow B^T)(B \wedge (B \leftarrow F)) \leq A \wedge (A \leftarrow F)$.

(b) $(A \leftharpoondown B^T) \diamond (B \vee (B \leftharpoondown F)) \geq A \vee (A \leftharpoondown F)$.

(2) *For* $A \in \mathscr{F}_{mn}$, $B \in \mathscr{F}_{mp}$ *and* $F \in \mathscr{F}_{m}$,

(a) $(B \wedge (F \rightarrow B)) \times (B^T \rightarrow A) \leq A \wedge (F \rightarrow A)$.

(b) $(B \vee (F \rightharpoondown B)) \diamond (B^T \rightharpoondown A) \geq A \vee (F \rightharpoondown A)$.

Proof. (1)(a) It is clear that $B \wedge (B \leftarrow F) \leq B$, $B \wedge (B \leftarrow F) \leq B \leftarrow F$. Then

$$(A \leftarrow B^T)(B \wedge (B \leftarrow F)) \leq (A \leftarrow B^T)B \leq A. \tag{4.1}$$

By Proposition 4.4,

$$A \leftarrow B^T)(B \wedge (B \leftarrow F)) \leq (A \leftarrow B^T)(B \leftarrow F) \leq A \leftarrow F. \tag{4.2}$$

From (4.1) and (4.2),

$$(A \leftarrow B^T)(B \wedge (B \leftarrow F)) \leq A \wedge (A \leftarrow F).$$

(b) Clearly, $B \vee (B \leftharpoondown F) \geq B$ and $B \vee (B \leftharpoondown F) \geq B \leftharpoondown F$. By Proposition 3.13,

$$(A \leftharpoondown B^T) \diamond (B \vee (B \leftharpoondown F)) \geq (A \leftharpoondown B^T) \diamond B \geq A.$$

By Proposition 4.4(1b),

$$(A \leftharpoondown B^T) \diamond (B \vee (B \leftharpoondown F)) \geq (A \leftharpoondown B^T) \diamond (B \leftharpoondown F) \geq A \leftharpoondown F.$$

Thus (1b) holds.

Similarly, we can prove (2a) and (2b), using this similar argument and proposition 4.4(2a) and 4.4(2b). □

Proposition 4.6. (1) *For* $A \in \mathscr{F}_{mn}, B \in \mathscr{F}_{pn}, C \in \mathscr{F}_{pn}, D \in \mathscr{F}_{mn}$, $E \in \mathscr{F}_{pn}$, *and* $F \in \mathscr{F}_{nn}$,

(a) *If* $A \leq D, C \leq B, E \leq B$, *then*

$$(A \leftarrow B^T) \times (C \wedge (E \leftarrow F)) \leq D \wedge (A \leftarrow F).$$

(b) *If* $A \geq D$, $C \geq B$, $E \geq B$, *then*

$$(A \leftharpoondown B^T) \diamond (C \vee (E \leftharpoondown F)) \geq D \vee (A \leftharpoondown F).$$

(2) *For* $A \in \mathscr{F}_{mn}, B \in \mathscr{F}_{np}, C \in \mathscr{F}_{np}, D \in \mathscr{F}_{mn}, E \in \mathscr{F}_{mp}$, *and* $F \in \mathscr{F}_{nm}$,

(a) *If* $A \leq D$, $C \leq B$, $E \leq B$, *then*

$$(C \wedge (F \rightarrow E)) \times (B^T \rightarrow A) \leq D \wedge (F \rightarrow A).$$

(b) *If* $A \geq D$, $C \geq B$, $E \geq B$, *then*

$$(C \vee (F \rightharpoondown E)) \diamond (B^T \rightharpoondown A) \geq D \vee (F \rightharpoondown A).$$

Proof. (1) (a) Since $(C \wedge (E \leftarrow F)) \leq C \leq B$ and $E \leq B$,

$$(C \wedge (E \leftarrow F)) \leq (E \leftarrow F) \leq B \leftarrow F.$$

Then

$$\begin{aligned}(A \leftarrow B^T) \times (C \wedge (E \leftarrow F)) &\leq (A \leftarrow B^T)C \\ &\leq (A \leftarrow B^T)B \\ &\leq A \leq D.\end{aligned}$$

Thus

$$(A \leftarrow B^T) \times (C \wedge (E \leftarrow F)) \leq D. \tag{4.3}$$

Similarly, by Proposition 4.4),

$$\begin{aligned}(A \leftarrow B^T) \times (C \wedge (E \leftarrow F)) &\leq (A \leftarrow B^T) \times (E \leftarrow F)\\ &\leq (A \leftarrow B^T) \times (B \leftarrow F)\\ &\leq A \leftarrow F.\end{aligned}$$

So

$$(A \leftarrow B^T) \times (C \wedge (E \leftarrow F)) \leq A \leftarrow F. \tag{4.4}$$

Hence, from (4,3) and (4.4),

$$(A \leftarrow B^T)(C \wedge (E \leftarrow F)) \leq D \wedge (A \leftarrow F).$$

(b) Clearly, $C \vee (E \leftharpoondown F) \geq C$ and $C \vee (E \leftharpoondown F) \geq E \leftharpoondown F$. Also, $C \vee (E \leftharpoondown F) \geq B$ and $C \vee (E \leftharpoondown F) \geq B \leftharpoondown F$.
Then

$$(A \leftharpoondown B^T) \diamond (C \vee (E \leftharpoondown F)) \geq (A \leftharpoondown B^T) \diamond B \geq A \geq D.$$

Thus

$$(A \leftharpoondown B^T) \diamond (C \vee (E \leftharpoondown F)) \geq D. \tag{4.5}$$

Using Proposition 4.4,

$$\begin{aligned}(A \leftharpoondown B^T) \diamond (C \vee (E \leftharpoondown F)) &\geq (A \leftharpoondown B^T) \diamond (E \leftharpoondown F)\\ &\geq (A \leftharpoondown B^T) \diamond (B \leftharpoondown F)\\ &\geq A \leftharpoondown F.\end{aligned}$$

So

$$(A \leftharpoondown B^T) \diamond (C \vee (E \leftharpoondown F)) \geq A \leftharpoondown F. \tag{4.6}$$

Hence, from (4.5) and (4.6), we get

$$(A \leftharpoondown B^T) \diamond (C \vee (E \leftharpoondown F)) \geq D \vee (A \leftharpoondown F).$$

By similar argument, we can prove (2a) and (2b). □

Theorem 4.7. (1) *For* $A \in \mathscr{F}_{mn}$, $B \in \mathscr{F}_{pn}$, $C \in \mathscr{F}_{pn}$, $D \in \mathscr{F}_{mn}$, $E \in \mathscr{F}_{pn}$, $F \in \mathscr{F}_{nn}$ *and* $C \in \mathscr{F}_{mp}$ *and* $G \in \mathscr{F}_{mp}$,

(a) *If* $A \leq D$, $C \leq B$, $E \leq B$, *then*

$$(G \wedge (A \leftarrow B^T)) \times (C \wedge (E \leftarrow F)) \leq D \wedge (A \leftarrow F).$$

(b) *If* $A \geq D$, $C \geq B$, $E \geq B$,

$$(G \vee (A \leftharpoondown B^T)) \diamond (C \vee (E \leftharpoondown F)) \geq D \vee (A \leftharpoondown F).$$

(2) *For* $A \in \mathscr{F}_{mn}$, $B \in \mathscr{F}_{mp}$, $C \in \mathscr{F}_{mp}$, $D \in \mathscr{F}_{mn}$, $E \in \mathscr{F}_{np}$, $F \in \mathscr{F}_{mm}$ *and* $G \in \mathscr{F}_{pn}$,

(a) *If* $A \leq D$, $C \leq B$, $E \leq B$, *then*

$$(C \wedge (F \rightarrow E)) \times (G \wedge (B^T \rightarrow A)) \leq D \wedge (F \rightarrow A).$$

(b) *If* $A \geq D$, $C \geq B$, $E \geq B$, *then*

$$(C \vee (F \rightharpoondown E)) \diamond (G \vee (B^T \rightharpoondown A)) \geq D \vee (F \rightharpoondown A).$$

Proof. (1) (a) By Proposition 4.6(1a),

$$G \wedge (A \leftarrow B^T) \leq G \text{ and } G \wedge (A \leftarrow B^T) \leq A \leftarrow B^T.$$

Then

$$\begin{aligned}(G \wedge (A \leftarrow B^T))(C \wedge (E \leftarrow F^T)) &\leq (A \leftarrow B^T)(C \wedge (E \leftarrow F)) \\ &\leq D \wedge (A \leftarrow F).\end{aligned}$$

(b) By Proposition 4.6 (1b),

$$\begin{aligned}(G \vee (A \leftharpoondown B^T)) \diamond (C \vee (E \leftharpoondown F^T)) &\geq (A \leftharpoondown B^T) \diamond (C \vee (E \leftharpoondown F) \\ &\geq D \wedge (A \leftharpoondown F)).\end{aligned}$$

(2) (a)

$$\begin{aligned}(C \wedge (F \rightarrow E)) \times (G \wedge (B^T \rightarrow A)) &\leq (C \wedge (F \rightarrow E)) \times (BV \rightarrow A) \\ &\leq D \wedge (F \rightarrow A).\end{aligned}$$

(b) By Proposition 4.6 (2b),

$$\begin{aligned}(C \vee (F \rightharpoondown E)) \diamond (G \vee (B^T \rightharpoondown A)) &\geq (C \vee (F \leftharpoondown E^T)) \diamond (B^T \rightharpoondown A)) \\ &\geq D \vee (F \rightharpoondown A).\end{aligned}$$

□

Using the above Theorem 4.7, we obtain the following proposition.

Proposition 4.8. *For* $A, B, C \in \mathscr{F}_n$,

(1) (a) *If* $A \leq C \leq B$, *then*

$$(C \wedge (A \leftarrow B^T)) \times (C \wedge (A \leftarrow B^T) \leq C \wedge (A \leftarrow B^T).$$

(b) *If* $A \geq C \geq B$, *then*

$$(C \vee (A \leftharpoondown B^T)) \diamond (C \vee (A \leftharpoondown B^T) \geq C \vee (A \leftharpoondown B^T).$$

(2) (a) *If* $A \leq C \leq B$, *then*

$$(C \wedge (B^T \rightarrow A)) \times (C \wedge (B^T \rightarrow A)) \leq C \wedge (B^T \rightarrow A).$$

(b) *If* $A \geq C \geq B$, *then*

$$(C \vee (B^T \rightharpoondown A)) \diamond (C \vee (B^T \rightharpoondown A)) \geq C \vee (B^T \rightharpoondown A).$$

This proposition shows that $C \wedge (A \leftarrow B^T)$ and $C \wedge (B^T \rightarrow A)$ are transitive and $C \vee (B^T \rightharpoondown A)$ and $C \vee (A \leftharpoondown B^T)$ are c-transitive.

Remark 4.9. In Proposition 4.8, if we replace $CandB$ by A, we get $A \wedge (A \leftarrow A^T)$ and $A \wedge (A^T \rightarrow A)$ are transitive, and $A \vee (A^T \rightharpoondown A)$ and $A \vee (A \leftharpoondown A^T)$ are c-transitive.

Theorem 4.10. (1) *For $A \in \mathscr{F}_{mn}$, $B \in \mathscr{F}_{pn}$, $C \in \mathscr{F}_{pn}$ and $F \in \mathscr{F}_{nq}$,*
(a) *If $C \leq B$, then $(A \leftarrow B^T)(C \leftarrow F) \leq A \leftarrow F$.*
(b) *If $C \geq B$, then $(A \leftharpoondown B^T) \diamond (C \leftharpoondown F) \geq A \leftharpoondown F$.*
(2) *For any IFM $A \in \mathscr{F}_{mn}$, $B \in \mathscr{F}_{mp}$, $C \in \mathscr{F}_{mp}$ and $F \in \mathscr{F}_{qm}$,*
(a) *If $C \leq B$, then $(F \rightarrow C)(B^T \rightarrow A) \leq F \rightarrow A$.*
(b) *If $C \geq B$, then $(F \leftharpoondown C) \diamond (B^T \leftharpoondown A) \geq A \leftharpoondown F$.*

Proof. (1) (a) By Proposition 4.4(1a),

$$(A \leftarrow B^T)(C \leftarrow F) \leq (A \leftarrow B^T)(B \leftarrow F) \leq A \leftarrow F.$$

(b) By Proposition 4.4(1b),

$$(A \leftharpoondown B^T) \diamond (C \leftharpoondown F) \geq (A \leftharpoondown B^T) \diamond (B \leftharpoondown F) \geq A \leftarrow F.$$

Similarly, we can prove (2a) and (2b). □

The following Proposition is evident from the Theorem 4.10.

Proposition 4.11. (1) *For $A \in \mathscr{F}_{mn}$, $B \in \mathscr{F}_{mn}$ and $F \in \mathscr{F}_{nq}$,*
(a) *If $A \leq B$, then $(A \leftarrow B^T)(A \leftarrow F) \leq A \leftarrow F$.*
(b) *If $A \geq B$, then $(A \leftharpoondown B^T) \diamond (A \leftharpoondown F) \geq A \leftharpoondown F$.*
(2) *For any IFMs $A \in \mathscr{F}_{mn}$, $B \in \mathscr{F}_{mn}$ and $F \in \mathscr{F}_{qm}$,*
(a) *If $A \leq B$, then $(F \rightarrow A)(B^T \rightarrow A) \leq F \rightarrow A$.*
(b) *If $A \geq B$, then $(F \rightharpoondown A) \diamond (B^T \rightharpoondown A) \geq F \rightharpoondown A$.*

Remark 4.12. If we replace F by B^T in Proposition 4.11, we get
(1) (a) If $A \leq B$, then $(A \leftarrow B^T)(A \leftarrow B^T) \leq A \leftarrow B^T$.
(b) If $A \geq B$, then $(A \leftharpoondown B^T) \diamond (A \leftharpoondown B^T) \geq A \leftharpoondown B^T$.
(2) (a) If $A \leq B$, then $(B^T \rightarrow A)(B^T \rightarrow A) \leq B^T \rightarrow A$.
(b) If $A \geq B$, then $(B^T \rightharpoondown A) \diamond (B^T \rightharpoondown A) \geq B^T \rightharpoondown A$.
Similarly,
(a) If $A \geq B$, then $(A \leftarrow B^T)(A \leftarrow B^T) \geq A \leftarrow B^T$.
(b) If $A \leq B$, then $(A \leftharpoondown B^T) \diamond (A \leftharpoondown B^T) \leq A \leftharpoondown B^T$.
(c) If $A \geq B$, then $(B^T \rightarrow A)(B^T \rightarrow A) \geq B^T \rightarrow A$.
(d) If $A \leq B$, then $(B^T \rightharpoondown A) \diamond (B^T \rightharpoondown A) \leq B^T \rightharpoondown A$.

Remark 4.13. In the above Remark 4.12, if $A = B$, we get
(1) (a) $(A \leftarrow A^T)(A \leftarrow A^T) = A \leftarrow A^T$.
(b) $(A \leftharpoondown A^T) \diamond (A \leftharpoondown A^T) = A \leftharpoondown A^T$.
(2) (a) $(A^T \rightarrow A)(A^T \rightarrow A) = A^T \rightarrow A$.
(b) $(A^T \rightharpoondown A) \diamond (A^T \rightharpoondown A) = A^T \rightharpoondown A$.

Remark 4.14. (1) In Proposition 4.8 (1a), if we replace C by $(A \leftarrow A^T)^T$, then

$$((A \leftarrow A^T)^T \wedge (A \leftarrow A^T) \times ((A \leftarrow A^T)^T \wedge (A \leftarrow A^T))$$
$$\leq (A \leftarrow A^T)^T \wedge (A \leftarrow A^T)).$$

(2) In Proposition 4.8 (1b), if we replace C by $(A \leftharpoondown A^T)^T$ and B by A, then

$$((A \leftharpoondown A^T)^T \vee (A \leftharpoondown A^T) \diamond ((A \leftharpoondown A^T)^T \vee (A \leftharpoondown A^T))$$
$$\geq (A \leftharpoondown A^T)^T \vee (A \leftharpoondown A^T).$$

Let R be an $n \times n$ IFM. If $R \times R \leq R$(transitive), $R^T = R$ (symmetric) and $I \leq R$(reflexive), then R represents a similarity relation or an intuitionistic fuzzy equivalence relation. Clearly

$(A \leftarrow A^T) \wedge (A \leftarrow A^T)^T \geq I$,
$((A \leftarrow A^T) \wedge (A \leftarrow A^T)^T)^T = (A \leftarrow A^T) \wedge (A \leftarrow A^T)^T$
and
$((A \leftarrow A^T) \wedge (A \leftarrow A^T)^T)((A \leftarrow A^T) \wedge (A \leftarrow A^T)^T)$
$= ((A \leftarrow A^T) \wedge (A \leftarrow A^T)^T)$.

Thus $((A \leftarrow A^T) \wedge (A \leftarrow A^T)^T)$ represents a similarity relation.
Also $(A^T \rightarrow A) \wedge (A^T \rightarrow A)^T$ represents a similarity relation.

If $R \diamond R \geq R$ (c-transitive), $R^T = R$ and $I^c \geq R$, then R represents a dissimilarity relation. That is, if $R = (\langle r_{ij}, r'_{ij} \rangle)$ is a dissimilarity matrix, then

$\langle r_{ik} \vee r_{kj}, r'_{ik} \wedge r'_{kj} \rangle \geq \langle r_{ij}, r'_{ij} \rangle, \langle r_{ij}, r'_{ij} \rangle = \langle r_{ji}, r'_{ji} \rangle$
and
$\langle r_{ii}, r'_{ii} \rangle = \langle 0, 1 \rangle$ for $i, j, k = 1, 2, ..., n$.

Thus $(A \leftharpoondown A^T) \vee (A \leftharpoondown A^T)^T$ and $(A^T \rightharpoondown A) \vee (A^T \rightharpoondown A)^T$ represents dissimilarity relations.

Proposition 4.15. (1) *For* $A \in \mathscr{F}_{mn}, F \in \mathscr{F}_{nq}$,
(a) $(A \leftarrow A^T)(A \leftarrow F) = A \leftarrow F$.
(b) $(A \leftharpoondown A^T) \diamond (A \leftharpoondown F) = A \leftharpoondown F$.
(2) *For* $A \in \mathscr{F}_{mn}, F \in \mathscr{F}_{qm}$,
(a) $(F \rightarrow A)(A^T \rightarrow A) = F \rightarrow A$.

(b) $(F \leftharpoondown A) \diamond (A^T \leftharpoondown A) = F \leftharpoondown A$.

Proof. In Proposition 4.11(1a), if we set $B = A$, then

$$(A \leftarrow A^T)(A \leftarrow F) \leq A \leftarrow F.$$

Since $A \leftarrow A^T$ is reflexive, $A \leftarrow A^T \geq I$ $(A \leftarrow A^T)(A \leftarrow F) \geq A \leftarrow F$. Thus $(A \leftarrow A^T)(A \leftarrow F) = A \leftarrow F$.

Similarly, setting in Proposition 4.11(1b), $B = A$. Then we have

$$(A \leftharpoondown A^T) \diamond (A \leftharpoondown F) \geq A \leftharpoondown F.$$

Since $I^c \geq A \leftharpoondown A^T$, $(A \leftharpoondown F) \geq (A \leftharpoondown A^T) \diamond (A \leftharpoondown F)$. So (1b) holds

By similar argument, we can prove (2a) and (2b). □

Proposition 4.16. *For* $A \in \mathscr{F}_{mn}$*,*

(1) (a) $(A \leftarrow A^T)A = A$.

(b) $(A \leftharpoondown A^T) \diamond A = A$.

(2) (a) $A(A^T \rightarrow A) = A$.

(b) $A \diamond (A^T \rightharpoondown A) = A$.

Proof. (1) (a) Clearly, $A \leftarrow A^T \geq I$. Then $(A \leftarrow A^T)A \geq A$. Thus, By Proposition 2.6, $(A \leftarrow A^T)A \leq A$. So $(A \leftarrow A^T)A = A$.

(b) Clearly, $A \leftharpoondown A^T \leq I^c$ and $(A \leftharpoondown A^T) \diamond A \leq A$. By Proposition 3.13, $(A \leftharpoondown A^T) \diamond A \geq A$. Thus $(A \leftharpoondown A^T) \diamond A = A$.

Similarly, we can prove (2a) and (2b). □

The following Proposition is obvious from Proposition 4.16.

Proposition 4.17. *For* $A \in \mathscr{F}_{mn}$*,*

(1) $(A \leftarrow A^T)A(A^T \rightarrow A) = A$.

(2) $(A \leftharpoondown A^T) \diamond A \diamond (A^T \rightharpoondown A) = A$.

Proposition 4.18. *For* $A \in \mathscr{F}_{mn}$*,*

(1) (a) $A \leftarrow (A^T \rightarrow A)^T = A \leftarrow (A^T \leftarrow A) = A$.

(b) $A \leftharpoondown (A^T \rightharpoondown A)^T = A \leftharpoondown (A^T \leftharpoondown A) = A$.

(2) (a) $(A \leftarrow A^T) \rightarrow A = (A \rightarrow A^T) \rightarrow A = A$.

(b) $(A \leftharpoondown A^T)^T \rightharpoondown A = (A \rightharpoondown A^T) \rightharpoondown A = A$.

Proof. By Proposition 4.1(1), $A \leftarrow (A^T \leftarrow A) \geq A$. Since $(A^T \leftarrow A) \geq I$, $A \leftarrow (A^T \leftarrow A) \geq A \leftarrow I = A$. Thus $A \leftarrow (A^T \leftarrow A) = A$.

(b) By Proposition 4.1(2), $A \leftharpoondown (A^T \leftharpoondown A) \leq A$. Since $I^c \geq A^T \leftharpoondown A$,

$$A \leftharpoondown I^c \geq A \leftharpoondown (A^T \leftharpoondown A) \text{ and } A \geq A \leftharpoondown (A^T \leftharpoondown A).$$

Thus $A \leftharpoondown (A^T \leftharpoondown A) = A$. □

Remark 4.19. By Proposition 4.16 and 4.18, we have

$$(A \leftarrow A^T)A = A \leftarrow (A^T \leftarrow A) \text{ and } A(A^T \rightarrow A) = (A \rightarrow A^T) \rightarrow A.$$

Lemma 4.20. *For $A, B \in \mathscr{F}_n$, $(A \wedge B)^2 \leq A^2 \wedge B^2$.*

Proof. $A \wedge B = (\langle (a_{ij} \wedge b_{ij}), (a'_{ij} \vee b'_{ij}) \rangle)$. Then

$$(A \wedge B)^2$$
$$= (\langle \bigvee_k ((a_{ik} \wedge b_{ik}) \wedge (a_{kj} \wedge b_{kj})), \bigwedge_k ((a'_{ik} \vee b'_{ik}) \vee (a'_{kj} \vee b'_{kj}) \rangle).$$

Without loss of generality, assume $AandB$ are 2×2 matrices.
Consider

$$\bigvee_k ((a_{ik} \wedge b_{ik}) \wedge (a_{kj} \wedge b_{kj}))$$
$$= [(a_{i1} \wedge a_{1j}) \wedge (b_{i1} \wedge b_{1j})] \vee [(a_{i2} \wedge a_{2j}) \wedge (b_{i2} \wedge b_{2j})]$$
$$= [(a_{i1} \wedge a_{1j}) \vee [(a_{i2} \wedge a_{2j}) \wedge (b_{i2} \wedge b_{2j})]] \wedge [(b_{i1} \wedge b_{1j}) \vee [(a_{i2} \wedge a_{2j}) \wedge (b_{i2} \wedge b_{2j})]]$$
$$= [(a_{i1} \wedge a_{1j}) \vee (a_{i2} \wedge a_{2j})] \wedge [(a_{i1} \wedge a_{1j}) \vee (b_{i2} \wedge b_{2j})] \wedge [(b_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge a_{2j})] \wedge [(b_{i1} \wedge b_{1j}) \vee (b_{i2} \wedge b_{2j})]$$
$$\leq [(a_{i1} \wedge a_{1j}) \vee (a_{i2} \wedge a_{2j})] \wedge [(b_{i1} \wedge b_{1j}) \vee (b_{i2} \wedge b_{2j})]$$
$$= (\bigvee_k (a_{ik} \wedge a_{kj})) \wedge (\bigvee_k (b_{ik} \wedge b_{kj})).$$

Similarly,

$$\bigwedge_k [(a'_{ik} \vee b'_{ik}) \vee (a'_{kj} \vee b'_{kj})] \geq (\bigwedge_k (a'_{ik} \vee a'_{kj})) \vee (\bigwedge_k (b'_{ik} \vee b'_{kj})).$$

Thus

$$(A \wedge B)^2$$
$$\leq ((\bigvee_k (a_{ik} \wedge a_{kj})) \wedge (\bigvee_k (b_{ik} \wedge b_{kj})), (\bigwedge_k (a'_{ik} \vee a'_{kj})) \vee (\bigwedge_k (b'_{ik} \vee b'_{kj}))$$
$$= A^2 \wedge B^2.$$

□

In dual fashion we can prove the following Lemma.

Lemma 4.21. *For $A, B \in \mathscr{F}_n, (A \vee B) \diamond (A \vee B) \geq (A \diamond A) \vee (B \diamond B)$.*

Proposition 4.22. (1) *For $A \in \mathscr{F}_n, B \in \mathscr{F}_{mn}$,*

(a) $R = A \wedge (A \leftarrow A^T) \wedge (B^T \rightarrow B) \Rightarrow R \times R \leq R$.

(b) $R = A \vee (A \leftharpoondown A^T) \vee (B^T \rightharpoondown B) \Rightarrow R \diamond R \geq R$..

(1) *For any IFMs $A \in \mathscr{F}_n, B \in \mathscr{F}_{nm}$,*

(a) $R = A \wedge (A \rightarrow A^T) \wedge (B^T \leftarrow B) \Rightarrow R \times R \leq R$.

(b) $R = A \vee (A \rightharpoondown A^T) \vee (B^T \leftharpoondown B) \Rightarrow R \diamond R \geq R$.

Proof. (1) (a) Since $(A \wedge (A \leftarrow A^T)$ and $(B^T \to B)$ are transitive,

$$\begin{aligned} R \times R &= ((A \wedge (A \leftarrow A^T)) \wedge (B^T \to B))^2 \\ &\leq (A \wedge (A \leftarrow A^T))^2 \wedge (B^T \to B)^2 \\ &\leq (A \wedge (A \leftarrow A^T)) \wedge (B^T \to B) \\ &= R. \end{aligned}$$

Similarly, we can prove the other results. □

In the above Proposition, if we replace B by A, we get the following.

Proposition 4.23. *For any IFMs $A \in \mathscr{F}_n$,*
(1) $R = A \wedge (A \leftarrow A^T) \wedge (A^T \to A) \Rightarrow R \times R \leq R$.
(2) $R = A \vee (A \leftharpoondown A^T) \vee (A^T \rightharpoondown A) \Rightarrow R \diamond R \geq R$.

Lemma 4.24. *If $R \in \mathscr{F}_n$ is transitive and symmetric, then it is weakly reflexive.*

Proof. Since R is transitive, $R^2 \leq R$. Then $\langle r_{ik} \wedge r_{ki}, r'_{ik} \vee r'_{ki} \rangle \leq \langle r_{ii}, r'_{ii} \rangle$. Using the symmetricity condition of R, we get $\langle r_{ik}, r'_{ik} \rangle \leq \langle r_{ii}, r'_{ii} \rangle$. Thus the result holds. □

Lemma 4.25. *If $R \in \mathscr{F}_n$ is transitive and weakly reflexive, then R is compact.*

Proof. Since R is weakly reflexive, $\langle r_{ik}, r'_{ik} \rangle \leq \langle r_{ii}, r'_{ii} \rangle$. Then

$$\langle r_{ii}, r'_{ii} \rangle \wedge \langle r_{ik}, r'_{ik} \rangle \geq \langle r_{ik}, r'_{ik} \rangle.$$

This is true for all i, k. thus $R^2 \geq R$. □

Lemma 4.26. *If $R \in \mathscr{F}_n$ is c-transitive and symmetric, then $R \diamond R = R$.*

Proof. If R is c-transitive, then $\langle r_{ik}, r'_{ik} \rangle \vee \langle r_{kj}, r'_{kj} \rangle \geq \langle r_{ij}, r'_{ij} \rangle$.
In particular, $\langle r_{ik}, r'_{ik} \rangle \vee \langle r_{ki}, r'_{ki} \rangle \geq \langle r_{ii}, r'_{ii} \rangle$. By symmetry of R, $\langle r_{ik}, r'_{ik} \rangle \geq \langle r_{ii}, r'_{ii} \rangle$. Thus $\langle r_{ii}, r'_{ii} \rangle \vee \langle r_{ik}, r'_{ik} \rangle \leq \langle r_{ik}, r'_{ik} \rangle$. So $R \diamond R \leq R$. Hence $R \diamond R = R$. □

From the above Lemmas the following proposition is trivial.

Proposition 4.27. *For any symmetric $A \in \mathscr{F}_n$,*
(1) $R = A \wedge (A \leftarrow A^T) \wedge (A^T \to A) \Rightarrow R \times R = R$.
(2) $R = A \vee (A \leftharpoondown A^T) \vee (A^T \rightharpoondown A) \Rightarrow R \diamond R = R$.

The Proposition 4.27 shows that one can derive a max-min idempotent IFM from any symmetric IFM and min-max idempotent IFM from a symmetric IFM. If we replace replace A by $A^c \wedge (A^T)^c$ in Proposition

4.27(1) and A by $A^c \vee (A^T)^c$ in Proposition 4.27 we get idempotent and min-max idempotent IFM for any IFM.

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