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Fuzzy filters of ordered semigroups

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Abstract. In the present paper, first we introduce the notion of (k^*, q)-quasi-coincident. Then, using the idea of (k^*, q)-quasi-coincident of a fuzzy point with a fuzzy set, we also introduce the notions of $(\in, \in \vee (k^*, q_k))$-fuzzy filters and $(\in, \in \vee (k^*, q_k))$-fuzzy bi-filters of an ordered semigroup. Then, we characterize these fuzzy filters. Finally, using the properties of these fuzzy filters, we characterize different classes of ordered semigroups. Also, we characterize some implication operators in terms of these fuzzy filters.

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1. Introduction

The theory of fuzzy sets was introduced by Zadeh[22] in 1965. This theory has provided a useful mathematical tool for describing the behaviour of systems that are too complex or ill-defined to admits precise mathematical analysis by classical methods and tools. Extensive applications of the fuzzy set theory have been found in various field. Since Rosenfeld[18] applied the notion of fuzzy sets to algebra and introduced the notion of fuzzy subgroups, the literature of various fuzzy algebraic concepts has been growing very rapidly. Kuroki initiated the theory of fuzzy semigroups in his paper[15]. The monograph by Mordeson *et al*[16] deals with the theory of fuzzy semigroups and their use in fuzzy codes, fuzzy finite state machines and fuzzy languages. Due to their possibilities of applications, semigroups and related structure are presently extensively investigated in fuzzy settings. Kehayopulu in[10, 12] characterized regular, left regular and right regular ordered semigroups by means of fuzzy left, right and bi-ideals. Murali[17] defined the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set played a vital role in generating different types of fuzzy subgroups. Using these ideas, Bhakat and

Das[1, 2, 3] introduced the concept of (α,β)-fuzzy subgroups by using the 'belong to'($\in$) relation and 'quasi coincident with' (q) relation between a fuzzy point and a fuzzy subgroup and introduced the concept of $(\in,\in\vee q)$-fuzzy subgroup. Davvaz defined $(\in,\in\vee q)$-fuzzy subnearrings and ideals of a near ring in [4]. Kazanci and Yamak[9] studied $(\in,\in\vee q)$-fuzzy bi-ideals of a semigroup. In[7] Jun *et al* studied the generalized fuzzy interior ideals in semigroups. In[19] Shabir *et al* characterized the regular semigroups by $(\in,\in\vee q_k)$-fuzzy ideals. In[13] Khan and Shabir characterized ordered semigroup in terms (α,β)-fuzzy interior ideals. In[14] Khan *et al* studied new types of intuitionistic fuzzy interior ideals of ordered semigroups. Generalizing the concept of the quasi-coincident of a fuzzy point with a fuzzy subset, Jun[8] defined $(\in,\in\vee q_k)$- fuzzy subalgebras in BCK/BCI-algebras, respectively. Following the terminology given by Zadeh, fuzzy sets in an ordered semigroup S were first considered by Kehayopulu and Tsingelis in[10, 12]. As we know, fuzzy filters with special properties of ordered semigroups always play an important role in the study of ordered semigroups structure. They defined fuzzy analogous for several notions, which have prove useful in the theory of ordered semigroups. Moreover[12], they defined fuzzy filter and characterized some basic properties of ordered semigroup in terms of these fuzzy filters. In[6], Davvaz *et al* studied the generalized fuzzy filters in ordered semigroup.

The plan of the article are as follows. In the next section, we define some basic definitions of filter, fuzzy filter and some well known results. In the section 3, first we introduce the notion of (k^*,q)-quasi-coincident and using the idea of (k^*,q)-quasi-coincident of a fuzzy point with a fuzzy set. We also introduce the notions of $(\in,\in\vee(k^*,q_k))$-fuzzy filters and $(\in,\in\vee(k^*,q_k))$-fuzzy bi-filters of an ordered semigroup. Then, we characterize these various fuzzy filters. In the section 4, we define the (k^*,k)-lower parts of fuzzy left(right) filter, filter and bi-filter. Finally, using the properties of these fuzzy filter, in the section 5, we characterize some well known implication operators in terms of left (right) fuzzy filters.

2. Preliminaries

A partially ordered semigroup(briefly ordered semigroup) is a pair $(S,.)$ comprising of a semigroup S and a partial order $\leq$ (on S) that is compatible with the binary operation; i.e. for all $a,b,s\in S$, $a\leq b$ implies that $sa\leq sb$ and $as\leq bs$.

t For any non-empty subset A of an ordered semigroup S, define[10] by

$$(A]=\{a\in S\mid a\leq b \text{ for some } b\in A\}.$$

Suppose A be any non-empty subset of an ordered semigroup S. Then

(i) A is called a subsemigroup of S, if $A^2\subseteq A$.

(ii) A is called a left(resp. right) filter[12] of S, if

(a) for any $b\in S$ and $a\in A$, if $a\leq b$, then $b\in A$,

(b) $a,b\in A\Rightarrow ab\in A$,

(c) $ab\in A\Rightarrow a\in A$(resp. $b\in A$).

(iii) A is called a filter[12] of S, if

(a) for any $b\in S$ and $a\in A$, if $a\leq b$, then $b\in A$,

(b) $a,b\in A\Rightarrow ab\in A$,

(c) $ab\in A\Rightarrow a,b\in A$.

(iv) A is called a bi-filter[6] of S, if

(a) for any $b \in S$ and $a \in A$, if $a \leq b$, then $b \in A$,

(b) $a, b \in A \Rightarrow ab \in A$,

(c) $aba \in A \Rightarrow a \in A$.

Any function f from S to the closed interval $[0, 1]$ is called a fuzzy subset[22] of S. The ordered semigroup S itself is regarded as a fuzzy subset of S by defining $S(x) \equiv 1$ for all $x \in S$.

Let f and g be two fuzzy subsets of S. Then, the inclusion relation $f \subseteq g$ is defined[5] by $f(x) \leq g(x)$ for all $x \in S$. Also

$$(f \cap g)(x) = \min\{f(x), g(x)\}$$

$$(f \cup g)(x) = \max\{f(x), g(x)\}$$

for all $x \in S$.

For any ordered semigroup $(S, ., \leq)$ and $x \in S$, define[5] by

$$A_x = \{(y, z) \in S \times S \mid x \leq yz\}.$$

Then product $f \circ g$ of any fuzzy subsets f and g of S is defined[11] by

$$(f \circ g)(x) = \begin{cases} \bigvee_{(y,z) \in A_x} \min\{f(y), g(z)\}, & A_x \neq \varnothing, \\ 0, & A_x = \varnothing, \end{cases}$$

for all $x \in S$.

Let A be a non-empty subset of S. We denote by f_A, the characteristic function of A, that is the mapping of S into $[0, 1]$ defined by

$$f_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Clearly f_A is a fuzzy subset of S.

Let S be an ordered semigroup and f be a fuzzy subset of S. Then

(i) f is called a fuzzy subsemigroup[10] of S, if $f(xy) \geq \min\{f(x), f(y)\}$ for all $x, y \in S$.

(ii) f is called a fuzzy left(resp. right) filter[12] of S, if

(a) $x \leq y \Rightarrow f(x) \leq f(y)$,

(b) $f(xy) \geq \min\{f(x), f(y)\}$,

(c) $f(xy) \leq f(x)$ (resp. $f(xy) \leq f(y)$),

for all $x, y \in S$.

(iii) f is called a fuzzy filter[12] of S, if

(a) $x \leq y \Rightarrow f(x) \leq f(y)$,

(b) $f(xy) = \min\{f(x), f(y)\}$,

for all $x, y \in S$.

(iv) f is called a fuzzy bi-filter[6] of S, if

(a) $x \leq y \Rightarrow f(x) \leq f(y)$,

(b) $f(xy) \geq \min\{f(x), f(y)\}$,

(c) $f(xyx) \leq f(x)$,

for all $x, y \in S$.

Definition 2.1. [20] Let S be an ordered semigroup, $a \in S$ and $u \in (0, 1]$. An ordered fuzzy point a_u of S is defined by

$$a_u(x) = \begin{cases} u, & \text{if } x \in (a], \\ 0, & \text{if } x \notin (a]. \end{cases}$$

Then a_u is a mapping from S into [0,1]. Thus an ordered fuzzy point of S is a fuzzy subset of S. For any fuzzy subset f of S, we shall also denote $a_u \subseteq f$ by $a_u \in f$ in the sequel. Then $a_u \in f$ if and only if $f(a) \geq u$.

Definition 2.2. [20] Let f be any fuzzy subset of an ordered semigroup S. For any $u \in (0, 1]$, the set

$$U(f; u) = \{x \in S \mid f(x) \geq u\},$$

is called a level subset of f.

3. $(\in, \in \vee(k^*, q_k))$-fuzzy filter of ordered semigroups

In this section, first we define (k^*, q)-quasi-coincident and investigate mainly the properties of $(\in, \in \vee(k^*, q_k))$-fuzzy left (resp. right) filter, and $(\in, \in \vee(k^*, q_k))$-fuzzy bi-filter of an ordered semigroup S.

Definition 3.1. An ordered fuzzy point a_u of an ordered semigroup S, for any $k^* \in (0, 1]$, is said to be (k^*, q)-quasi-coincident with a fuzzy subset f of S, written as $a_u(k^*, q)f$, if

$$f(a) + u > k^*.$$

Let $(S, ., \leq)$ be an ordered semigroup and $0 \leq k < k^* \leq 1$. For an ordered fuzzy point x_u, we denote that
(i) $x_u(k^*, q_k)f$ if $f(x) + u + k > k^*$,
(ii) $x_u \in \vee(k^*, q_k)f$ if $x_u \in f$ or $x_u(k^*, q_k)f$,
(iii) $x_u\overline{\alpha}f$ if $x_u\alpha f$ does not hold for $\alpha \in \{(k^*, q_k), \in \vee(k^*, q_k)\}$.

Definition 3.2. A fuzzy subset f of an ordered semigroup S is called an $(\in, \in \vee(k^*, q_k))$-fuzzy subsemigroup of S, if $x_u \in f$ and $y_v \in f$ imply $(xy)_{\min\{u,v\}} \in \vee(k^*, q_k)f$ for all $u, v \in (0, 1]$ and $x, y \in S$.

Example 3.3. Let $S = \{0, 1, 2, 3, 4\}$ be an ordered semigroup with respect to the order relation $\leq := \{(0, 0), (0, 2), (0, 4), (1, 1), (1, 2), (1, 4), (2, 2), (2, 4), (3, 4), (3, 3), (4, 4)\}$ and the operation $'.'$ defined by the following Cayley table:

.	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	0	2	2	4
3	0	0	2	3	4
4	0	0	2	2	4

Define f on S by $f(0) = f(1) = f(3) = 0.80$, $f(3) = 0.70$ and $f(4) = 0.60$. Then f is an $(\in, \in \vee(k^*, q_k))$-fuzzy subsemigroup of S.

Definition 3.4. A fuzzy subset f of an ordered semigroup S is called an $(\in, \in \vee(k^*, q_k))$-fuzzy left (resp. right) filter of S if for all $u, v \in (0, 1]$ and $x, y \in S$, the following conditions hold:
(i) $x \leq y$, $x_u \in f \Rightarrow y_u \in \vee(k^*, q_k)f$,
(ii) $x_u \in f$, $y_v \in f$ imply $(xy)_{\min\{u,v\}} \in \vee(k^*, q_k)f$,
(iii) $(xy)_u \in f \Rightarrow x_u \in \vee(k^*, q_k)f$ (resp. $y_u \in \vee(k^*, q_k)f$).

Example 3.5. Let $S = \{0, 1, 2, 3, 4, 5\}$ be an ordered semigroup with respect to the order relation $\leq := \{(0, 0), (0, 3), (0, 4), (1, 1), (1, 3), (1, 4), (1, 5), (2, 2), (2, 3), (2, 5), (3, 3), (3, 4), (3, 5), (4, 4), (5, 5)\}$ and the operation $'.'$ defined by the following Cayley table:

.	0	1	2	3	4	5
0	1	2	3	3	3	3
1	2	3	3	3	3	3
2	3	3	3	3	3	3
3	3	3	3	3	3	4
4	4	4	4	4	4	4
5	5	5	5	5	5	5

Define f on S by $f(0) = f(1) = f(2) = f(3) = 0.40$, $f(4) = 0.80$ and $f(5) = 0.50$. Then f is an $(\in, \in \vee(k^*, q_k))$-fuzzy left filter of S.

Definition 3.6. A fuzzy subset f of an ordered semigroup S is called an $(\in, \in \vee(k^*, q_k))$-fuzzy filter of S, if for all $u, v \in (0, 1]$ and $x, y \in S$, the following conditions hold:
(i) $x \leq y$, $x_u \in f \Rightarrow y_u \in \vee(k^*, q_k)f$,
(ii) $x_u \in f$, $y_v \in f$ imply $(xy)_{\min\{u,v\}} \in \vee(k^*, q_k)f$,
(iii) $(xy)_u \in f \Rightarrow x_u \in \vee(k^*, q_k)f$, $y_u \in \vee(k^*, q_k)f)$.

Example 3.7. Let $S = \{0, 1, 2, 3, 4, 5\}$ be an ordered semigroup with respect to the order relation $\leq := \{(0, 0), (0, 3), (0, 4), (1, 1), (1, 5), (2, 2), (2, 5), (3, 3), (3, 4), (3, 5), (4, 4), (5, 5), (5, 4)\}$ and the operation $'.'$ defined by the following Cayley table:

.	0	1	2	3	4	5
0	0	1	1	3	4	5
1	1	1	1	1	1	1
2	1	1	1	1	1	1
3	3	1	1	3	4	5
4	4	5	5	4	4	5
5	5	5	5	5	5	5

Define f on S by $f(0) = 0.70$, $f(1) = f(2) = f(5) = 0.30$, $f(4) = 0.60$ and $k^* = 0.90$, $k = 0.30$. Then f is an $(\in, \in \vee(k^*, q_k))$-fuzzy filter of S.

Definition 3.8. A fuzzy subset f of an ordered semigroup S is called an $(\in, \in \vee(k^*, q_k))$-fuzzy bi-filter of S, if for all $u, v \in (0, 1]$ and $x, y \in S$, the following conditions hold:
(i) $x \leq y$, $x_u \in f \Rightarrow y_u \in \vee(k^*, q_k)f$,
(ii) $x_u \in f$, $y_v \in f$ imply $(xy)_{\min\{u,v\}} \in \vee(k^*, q_k)f$,
(iii) $(xyx)_u \in f \Rightarrow x_u \in \vee(k^*, q_k)f$.

Example 3.9. Let $S = \{0, 1, 2, 3, 4, 5\}$ be an ordered semigroup with respect to the order relation $0 \leq 3$ and $2 \leq 5$ and the operation $'.'$ defined by the following Cayley table:

.	0	1	2	3	4	5
0	0	1	1	3	4	5
1	1	1	1	1	1	1
2	1	1	1	1	1	1
3	3	1	1	3	4	5
4	4	5	5	4	4	5
5	5	5	5	5	5	5

Define f on S by $f(0) = 0.60$, $f(1) = f(2) = f(5) = 0.20$, $f(4) = 0.80$ and $k^* = 0.80$, $k = 0.20$. Then f is an $(\in, \in \vee(k^*, q_k))$-fuzzy bi-filter of S.

Theorem 3.10. *Let S be an ordered semigroup and A be a filter of S. Let f be a fuzzy subset of S defined by*

$$f(x) = \begin{cases} \geq \frac{k^*-k}{2}, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

Then (1) *f is a $((k^*, q), \in \vee(k^*, q_k))$-fuzzy filter of S.*
(2) *f is an $(\in, \in \vee(k^*, q_k))$-fuzzy filter of S.*

Proof. (1) Let $x, y \in S$, $x \leq y$ and $u \in (0, 1]$ be such that $x_u(k^*, q)f$. Then $x \in A$, $f(x) + u > k^*$. Since A is a filter of S and $y \geq x \in A$, we have $y \in A$. Thus $f(y) \geq \frac{k^*-k}{2}$. If $u \leq \frac{k^*-k}{2}$, then $f(y) \geq u$. So $y_u \in f$. If $u > \frac{k^*-k}{2}$, then $f(y) + u > \frac{k^*-k}{2} + \frac{k^*-k}{2} = k^* - k$. Hence $y_u \in \vee(k^*, q_k)f$.

Let $x, y \in S$ and $u, v \in (0, 1]$ be such that $x_u(k^*, q)f$ and $y_v(k^*, q)f$. Then $x, y \in A$, $f(x) + u > k^*$ and $f(y) + v > k^*$. Since A is a filter of S, we have $xy \in A$. Thus $f(xy) \geq \frac{k^*-k}{2}$. If $\min\{u, v\} \leq \frac{k^*-k}{2}$, then $f(xy) \geq \min\{u, v\}$. Thus $(xy)_{\min\{u,v\}} \in f$. Again if $\min\{u, v\} > \frac{k^*-k}{2}$, then $f(xy) + \min\{u, v\} > \frac{k^*-k}{2} + \frac{k^*-k}{2} = k^* - k$. So $(xy)_{\min\{u,v\}}(k^*, q_k)f$. Hence $(xy)_{\min\{u,v\}} \in \vee(k^*, q_k)f$.

Let $x, y \in S$ and $u \in (0, 1]$ be such that $(xy)_u(k^*, q)f$. Since A is a filter of S, we have $xy \in A$. Then $x, y \in A$. Thus $f(x) \geq \frac{k^*-k}{2}$ and $f(y) \geq \frac{k^*-k}{2}$. If $u \leq \frac{k^*-k}{2}$, then $f(x) \geq u$, $f(y) \geq u$ and Thus $(x)_u \in f$, $(y)_u \in f$. Again if $u > \frac{k^*-k}{2}$, then

$$f(x) + u > \frac{k^*-k}{2} + \frac{k^*-k}{2} = k^* - k$$

and

$$f(y) + u > \frac{k^*-k}{2} + \frac{k^*-k}{2} = k^* - k.$$

So $(x)_u(k^*, q_k)f$ and $(y)_u(k^*, q_k)f$. Hence $x_u \in \vee(k^*, q_k)f$ and $y_u \in \vee(k^*, q_k)f$.

(2) Let $x, y \in S$, $x \leq y$ and $u \in (0, 1]$ be such that $x_u \in f$. Then $x \in A$ and $f(x) \geq u$. Since A is a filter of S and $y \geq x \in A$, we have $y \in A$. Thus $f(y) \geq \frac{k^*-k}{2}$. If $u \leq \frac{k^*-k}{2}$, then $f(y) \geq u$. So we have $y_u \in f$. If $u > \frac{k^*-k}{2}$, then $f(y) + u > \frac{k^*-k}{2} + \frac{k^*-k}{2} = k^* - k$. Thus $y_u(k^*, q_k)f$. Hence $y_u \in \vee(k^*, q_k)f$.

Let $x, y \in S$ and $u, v \in (0, 1]$ be such that $x_u \in f$ and $y_v \in f$. Then $x, y \in A$, $f(x) \geq u$ and $f(y) \geq v$. Since A is a filter of S, we have $xy \in A$. Thus $f(xy) \geq \frac{k^*-k}{2}$. If $\min\{u, v\} \leq \frac{k^*-k}{2}$, then $f(xy) \geq \min\{u, v\}$. Thus $(xy)_{\min\{u,v\}} \in f$.

Again if $\min\{u,v\} > \frac{k^*-k}{2}$, then $f(xy) + \min\{u,v\} > \frac{k^*-k}{2} + \frac{k^*-k}{2} = k^* - k$. So $(xy)_{\min\{u,v\}}(k^*, q_k)f$. Hence $(xy)_{\min\{u,v\}} \in \vee(k^*, q_k)f$.

Let $x, y \in S$ and $u \in (0,1]$ be such that $(xy)_u \in f$. Since A is a filter of S, we have $xy \in A$. Then $x, y \in A$. Thus $f(x) \geq \frac{k^*-k}{2}$ and $f(y) \geq \frac{k^*-k}{2}$. If $u \leq \frac{k^*-k}{2}$, then $f(x) \geq u$, $f(y) \geq u$. So $(x)_u \in f$, $(y)_u \in f$. Again if $u > \frac{k^*-k}{2}$, then

$$f(x) + u > \tfrac{k^*-k}{2} + \tfrac{k^*-k}{2} = k^* - k$$

and

$$f(y) + u > \tfrac{k^*-k}{2} + \tfrac{k^*-k}{2} = k^* - k.$$

Hence $(x)_u(k^*, q_k)f$ and $(y)_u(k^*, q_k)f$. Therefore $x_u \in \vee(k^*, q_k)f$, $y_u \in \vee(k^*, q_k)f$. □

Theorem 3.11. *Let f be a fuzzy subset of an ordered semigroup S. Then f is an $(\in, \in \vee(k^*, q_k))$-fuzzy left filter of S if and only if*

(1) $x \leq y \Rightarrow \min\{f(x), \frac{k^*-k}{2}\} \leq f(y)$,

(2) $f(xy) \geq \min\{f(x), f(y), \frac{k^*-k}{2}\}$,

(3) $f(x) \geq \min\{f(xy), \frac{k^*-k}{2}\}$,

for all $x, y \in S$.

Proof. Let f be an $(\in, \in \vee(k^*, q_k))$-fuzzy left filter of S and $x, y \in S$.

Suppose to the contrary that $f(y) < \min\{f(x), \frac{k^*-k}{2}\}$ for some $x, y \in S$. Choose $u \in (0, \frac{k^*-k}{2})$ such that $f(y) < u \leq \min\{f(x), \frac{k^*-k}{2}\}$. Then $x_u \in f$, but $(y)_u \overline{\in \vee(k^*, q_k)} f$. This is a contradiction. Thus $f(y) \geq \min\{f(x), \frac{k^*-k}{2}\}$.

Suppose to the contrary that $f(xy) < \min\{f(x), f(y), \frac{k^*-k}{2}\}$ for some $x, y \in S$. Choose $u \in (0, \frac{k^*-k}{2})$ such that $f(xy) < u \leq \min\{f(x), f(y), \frac{k^*-k}{2}\}$. Then $x_u \in f$ and $y_u \in f$, but $(xy)_u \overline{\in \vee(k^*, q_k)} f$, which is a contradiction. Thus $f(xy) \geq \min\{f(x), f(y), \frac{k^*-k}{2}\}$.

Again suppose to the contrary that $f(x) < \min\{f(xy), \frac{k^*-k}{2}\}$ for some $x, y \in S$. Choose $u \in (0, \frac{k^*-k}{2})$ such that $f(x) < u \leq \min\{f(xy), \frac{k^*-k}{2}\}$. Then $(xy)_u \in f$, but $(x)_u \overline{\in \vee(k^*, q_k)} f$. Thus we get a contradiction. So $f(x) \geq \min\{f(xy), \frac{k^*-k}{2}\}$.

Conversely, suppose that conditions (1), (2) and (3) hold.

Let $f(y) \geq \min\{f(x), \frac{k^*-k}{2}\}$ for all $x, y \in S$ and let $x_u \in f$ $(u \in (0,1])$. Then $f(x) \geq u$. Thus $f(y) \geq \min\{f(x), \frac{k^*-k}{2}\} \geq \min\{u, \frac{k^*-k}{2}\}$. Now, if $u \leq \frac{k^*-k}{2}$, then $f(y) \geq u$. Thus $y_u \in f$. Again, if $u > \frac{k^*-k}{2}$, then $f(y) \geq \frac{k^*-k}{2}$. Thus

$$f(y) + u > \frac{k^* - k}{2} + \frac{k^* - k}{2} = k^* - k,$$

which implies that $y_u(k^*, q_k)f$. So $y_u \in \vee(k^*, q_k)f$.

suppose that $f(xy) \geq \min\{f(x), f(y), \frac{k^*-k}{2}\}$ for all $x, y \in S$. Let $x_u \in f$ and $y_v \in f$ for all $u, v \in (0,1]$. Then $f(x) \geq u$ and $f(y) \geq v$. Thus

$$f(xy) \geq \min\{f(x), f(y), \frac{k^* - k}{2}\} \geq \min\{u, v, \frac{k^* - k}{2}\}.$$

If $\min\{u,v\} \leq \frac{k^*-k}{2}$, then $f(xy) \geq \min\{u,v\}$ imply that $(xy)_{\min\{u,v\}} \in f$. If $\min\{u,v\} > \frac{k^*-k}{2}$, then $f(xy) \geq \frac{k^*-k}{2}$. Thus

$$f(xy) + \min\{u,v\} > \frac{k^*-k}{2} + \frac{k^*-k}{2} = k^* - k,$$

which implies that $(xy)_{\min\{u,v\}}(k^*, q_k)f$. So $(xy)_{\min\{u,v\}} \in \vee(k^*, q_k)f$.

Suppose next that $f(x) \geq \min\{f(xy), \frac{k^*-k}{2}\}$ for all $x, y \in S$. Let $(xy)_u \in f$ for all $u \in (0,1]$. Then $f(xy) \geq u$. Thus

$$f(x) \geq \min\{f(xy), \frac{k^*-k}{2}\} \geq \min\{u, \frac{k^*-k}{2}\}.$$

Now if $u \leq \frac{k^*-k}{2}$, then $f(x) \geq u$. Thus $x_u \in f$. If $u > \frac{k^*-k}{2}$, then $f(x) \geq \frac{k^*-k}{2}$. So

$$f(x) + u > \frac{k^*-k}{2} + \frac{k^*-k}{2} = k^* - k,$$

which implies that $x_u(k^*, q_k)f$. Hence $x_u \in \vee(k^*, q_k)f$. Therefore, f is an $(\in, \in \vee(k^*, q_k))$-fuzzy left filter of S. □

Dually we may prove the following.

Theorem 3.12. *Let f be a fuzzy subset of an ordered semigroup S. Then f is an $(\in, \in \vee(k^*, q_k))$-fuzzy right filter of S if and only if*

(1) $x \leq y \Rightarrow \min\{f(x), \frac{k^*-k}{2}\} \leq f(y)$,

(2) $f(xy) \geq \min\{f(x), f(y), \frac{k^*-k}{2}\}$,

(3) $f(y) \geq \min\{f(xy), \frac{k^*-k}{2}\}$,

for all $x, y \in S$.

It is clear that every fuzzy left filter of an ordered semigroup S is an $(\in, \in \vee(k^*, q_k))$-fuzzy left filter of S, but the converse is not necessarily true (see the following example).

Example 3.13. Let $S = \{0, 1, 2, 3\}$ be an ordered semigroup with respect to the order relation $0 \leq 3$ and the operation $'.'$ defined by the following Cayley table:

.	0	1	2	3
0	0	1	1	3
1	1	1	1	1
2	1	1	1	1
3	3	1	1	3

Define f on S by $f(0) = 0.5$, $f(1) = f(2) = 0.40$ and $f(3) = 0.60$. Then f is an $(\in, \in \vee(k^*, q_k))$-fuzzy left filter of S. But, since $f(0) = 0.50 < f(03) = f(3) = 0.60$; that is f is not a fuzzy left filter of S.

Theorem 3.14. *Let f be a fuzzy subset of an ordered semigroup S. Then f is an $(\in, \in \vee(k^*, q_k))$-fuzzy filter of S if and only if*

(1) $x \leq y \Rightarrow \min\{f(x), \frac{k^*-k}{2}\} \leq f(y)$,

(2) $f(xy) \geq \min\{f(x), f(y), \frac{k^*-k}{2}\}$,

(3) $\min\{f(xy), \frac{k^*-k}{2}\} \leq \min\{f(x), f(y)\}$,

for all $x, y \in S$.

Proof. The proof of part (1) and (2) follows from Theorem 3.11.

Now for part (3), suppose to the contrary that $\min\{f(xy), \frac{k^*-k}{2}\} > \min\{f(x), f(y)\}$. Choose $u \in (0, \frac{k^*-k}{2})$ such that

$$\min\{f(xy), \frac{k^*-k}{2}\} \geq u > \min\{f(x), f(y)\}.$$

If $f(xy) < \frac{k^*-k}{2}$, then $f(xy) \geq u > \min\{f(x), f(y)\}$, which implies that $(xy)_u \in f$ but $x_u \notin f$ and $y_u \notin f$, this is a contradiction. Thus $f(xy) \leq \min\{f(x), f(y)\}$. If $f(xy) \geq \frac{k^*-k}{2}$, then $\min\{f(x), f(y)\} < \frac{k^*-k}{2}$ which implies that $(xy)_{\frac{k^*-k}{2}}(k^*, q_k)f$ but $x_{\frac{k^*-k}{2}}\overline{(k^*, q_k)}f$ and $y_{\frac{k^*-k}{2}}\overline{(k^*, q_k)}f$, again a contradiction. So $\min\{f(x), f(y)\} \geq \frac{k^*-k}{2}$. Hence $\min\{f(xy), \frac{k^*-k}{2}\} \leq \min\{f(x), f(y)\}$.

Conversely, Suppose the condition (1), (2) and (3) hold. The proof of first two condition follows from the converse part of Theorem 3.11. Now, we have to show that $(xy)_u \in f \Rightarrow x_u \in \vee(k^*, q_k)f$ and $y_u \in \vee(k^*, q_k)f)$. Suppose that $\min\{f(xy), \frac{k^*-k}{2}\} \leq \min\{f(x), f(y)\}$ for all $x, y \in S$. Let $(xy)_u \in f$ for all $u \in (0, 1]$. Then $f(xy) \geq u$. Thus

$$\min\{f(x), f(y)\} \geq \min\{f(xy), \frac{k^*-k}{2}\} \geq \min\{u, \frac{k^*-k}{2}\}.$$

Now if $u \leq \frac{k^*-k}{2}$, then $\min\{f(x), f(y)\} \geq u$. Thus $x_u, y_u \in f$. If $u > \frac{k^*-k}{2}$, then $\min\{f(x), f(y)\} \geq \frac{k^*-k}{2}$. So

$$f(x) + u > \tfrac{k^*-k}{2} + \tfrac{k^*-k}{2} = k^* - k$$

and

$$f(y) + u > \tfrac{k^*-k}{2} + \tfrac{k^*-k}{2} = k^* - k,$$

which implies that $x_u, y_u(k^*, q_k)f$. Hence $x_u, y_u \in \vee(k^*, q_k)f$. Therefore, f is an $(\in, \in \vee(k^*, q_k))$-fuzzy filter of S. □

Theorem 3.15. *Let f be a fuzzy subset of an ordered semigroup S. Then f is an $(\in, \in \vee(k^*, q_k))$-fuzzy bi-filter of S if and only if*

(1) $x \leq y \Rightarrow \min\{f(x), \frac{k^*-k}{2}\} \leq f(y)$,

(2) $f(xy) \geq \min\{f(x), f(y), \frac{k^*-k}{2}\}$,

(3) $f(x) \geq \min\{f(xyx), \frac{k^*-k}{2}\}$,

for all $x, y \in S$.

Proof. The proof of part (1) and (2) follows from Theorem 3.11.

Now for part (3), suppose to the contrary that $f(x) < \min\{f(xyx), \frac{k^*-k}{2}\}$ for some $x, y \in S$. Choose $u \in (0, \frac{k^*-k}{2})$ such that $f(x) < u \leq \min\{f(xyx), \frac{k^*-k}{2}\}$. Then $(xyx)_u \in f$, but $x_u \overline{\in \vee(k^*, q_k)} f$. Thus we get a contradiction. So $f(x) \geq \min\{f(xyx), \frac{k^*-k}{2}\}$.

Conversely, Suppose the condition (1), (2) and (3) hold. The proof of first two condition follows from the converse part of Theorem 3.11. Now, we have to show that $(xyx)_u \in f \Rightarrow x_u \in \vee(k^*, q_k)f$. Suppose next that $f(x) \geq \min\{f(xyx), \frac{k^*-k}{2}\}$ for all $x, y \in S$. Let $(xyx)_u \in f$ for all $u \in (0, 1]$. Then $f(xyx) \geq u$. Thus

$f(x) \geq \min\{f(xyx), \frac{k^*-k}{2}\} \geq \min\{u, \frac{k^*-k}{2}\}$. Now if $u \leq \frac{k^*-k}{2}$, then $f(x) \geq u$. Thus $x_u \in f$. If $u > \frac{k^*-k}{2}$, then $f(x) \geq \frac{k^*-k}{2}$. So

$$f(x) + u > \frac{k^* - k}{2} + \frac{k^* - k}{2} = k^* - k,$$

which implies that $x_u(k^*, q_k)f$. Hence $x_u \in \vee(k^*, q_k)f$. Therefore, f is an $(\in, \in \vee(k^*, q_k))$-fuzzy bi-filter of S. □

Theorem 3.16. *If f is a nonzero $(\in, \in \vee(k^*, q_k))$-fuzzy bi-filter of S, then the set $f_0 = \{x \in S \mid f(x) > 0\}$ is a bi-filter of S.*

Proof. Let f be an $(\in, \in \vee(k^*, q_k))$-fuzzy bi-filter of S. Let $x, y \in S$, $x \leq y$ and $x \in f_0$. Then $f(x) > 0$. Since f is an $(\in, \in \vee(k^*, q_k))$-fuzzy bi-filter of S and $f(x) > 0$, we have $f(y) \geq \min\{f(x), \frac{k^*-k}{2}\} > 0$. Thus $f(y) > 0$, i.e., $y \in f_0$. Let $x, y \in f_0$. Then $f(x) > 0$ and $f(y) > 0$. Thus $f(xy) \geq \min\{f(x), f(y), \frac{k^*-k}{2}\} > 0$. So $xy \in f_0$. Let $xyx \in f_0$. Then $f(xyx) > 0$. Thus $f(x) \geq \min\{f(xyx), \frac{k^*-k}{2}\} > 0$. So $x \in f_0$. Hence f_0 is a bi-filter of S. □

Dually we may show that the following.

Theorem 3.17. *If f is a nonzero $(\in, \in \vee(k^*, q_k))$-fuzzy filter of S, then the set $f_0 = \{x \in S \mid f(x) > 0\}$ is a filter of S.*

Theorem 3.18. *Let f be a fuzzy subset of S. Then f is an $(\in, \in \vee(k^*, q_k))$-fuzzy filter of S if and only if $U(f; u)(\neq \varnothing)$ $(u \in (0, \frac{k^*-k}{2}])$ is a filter of S.*

Proof. Suppose that f is an $(\in, \in \vee(k^*, q_k))$-fuzzy filter of S.

Let $x, y \in S$ be such that $y \geq x \in U(f; u)$, where $u \in (0, \frac{k^*-k}{2}]$. Then $f(x) \geq u$. By Theorem 3.14,

$$f(y) \geq \min\{f(x), \frac{k^* - k}{2}\} \geq \min\{u, \frac{k^* - k}{2}\} = u.$$

Thus $y \in U(f; u)$.

Let $x, y \in U(f; u)$. Then $f(x) \geq u$ and $f(y) \geq u$. Thus, by Theorem 3.14,

$$f(xy) \geq \min\{f(x), f(y), \frac{k^* - k}{2}\} \geq \min\{u, \frac{k^* - k}{2}\} = u.$$

So $f(xy) \geq u$. Hence $xy \in U(f; u)$.

Let $xy \in U(f; u)$. Then $f(xy) \geq u$. By Theorem 3.14,

$$\min\{f(x), f(y)\} \geq \min\{f(xy), \frac{k^* - k}{2}\} \geq \min\{u, \frac{k^* - k}{2}\} = u.$$

Thus $f(x) \geq u$ and $f(y) \geq u$. So $x, y \in U(f; u)$. Hence $U(f; u)$ is a filter of S.

Conversely, suppose that $U(f; u)(\neq \varnothing)$ is a filter of S for all $u \in (0, \frac{k^*-k}{2}]$. Take any $x, y \in S$ with $x \leq y$ and suppose to the contrary that $f(y) < \min\{f(x), \frac{k^*-k}{2}\}$. Then $f(y) < u \leq \min\{f(x), \frac{k^*-k}{2}\}$, for some $u \in (0, \frac{k^*-k}{2}]$. This implies that $x \in U(f; u)$, but $y \notin U(f; u)$. This is a contradiction. Thus $f(y) \geq \min\{f(x), \frac{k^*-k}{2}\}$ for all $x, y \in S$ with $x \leq y$. Again suppose to the contrary that $f(xy) < \min\{f(x), f(y), \frac{k^*-k}{2}\}$ for some $x, y \in S$. Then there exist $u \in (0, \frac{k^*-k}{2}]$ such that $f(xy) < u \leq \min\{f(x), f(y), \frac{k^*-k}{2}\}$. This implies that $x_u \in U(f; u)$ and

$y_u \in U(f;u)$, but $(xy)_u \notin U(f;u)$. We get a contradiction. Therefore $f(xy) \geq \min\{f(x), f(y), \frac{k^*-k}{2}\}$ for all $x, y \in S$.

Again, also suppose to the contrary that $\min\{f(x), f(y)\} < \min\{f(xy), \frac{k^*-k}{2}\}$ for some $x, y \in S$. Then there exist $u \in (0, \frac{k^*-k}{2}]$ such that

$$\min\{f(x), f(y)\} < u \leq \min\{f(xy), \frac{k^*-k}{2}\}.$$

This implies that $(xy)_u \in U(f;u)$, but $x_u, y_u \notin U(f;u)$. We get a contradiction. Thus $\min\{f(x), f(y)\} \geq \min\{f(x), \frac{k^*-k}{2}\}$ for all $x, y \in S$. So, by Theorem 3.14, f is an $(\in, \in \vee(k^*, q_k))$-fuzzy filter of S. □

Theorem 3.19. *Let S be an ordered semigroup and f be an $(\in, \in \vee(k^*, q_k))$-fuzzy bi-filter of S such that $f(x) < \frac{k^*-k}{2}$ for all $x \in S$. Then f is an $(\in, \in)$-fuzzy bi-filter of S.*

Proof. Let f be an $(\in, \in \vee(k^*, q_k))$-fuzzy bi-filter of S and $x, y \in S$, $x \leq y$. Then, $x_u \in f$ for some $u \in (0, 1]$. Thus $f(x) \geq u$ and we have $f(y) \geq \min\{f(x), \frac{k^*-k}{2}\} \geq \{u, \frac{k^*-k}{2}\} = u$. So $y_u \in f$.

Let $x, y \in S$ and $u, v \in (0, 1]$ such that $x_u \in f$ and $y_v \in f$. Then $f(x) \geq u$ and $f(y) \geq v$. Thus

$$f(xy) \geq \min\{f(x), f(y), \frac{k^*-k}{2}\} \geq \min\{u, v, \frac{k^*-k}{2}\} = \min\{u, v\}.$$

So $f(xy) \geq \min\{u, v\}$. Hence $xy \in f$.

Let $x, y \in S$ and $u \in (0, 1]$ such that $(xyx)_u \in f$. Then $f(xyx) \geq u$. Thus

$$f(x) \geq \min\{f(xyx), \frac{k^*-k}{2}\} \geq \min\{u, \frac{k^*-k}{2}\} = u.$$

So $f(x) \geq u$. Hence $x \in f$. Therefore f is an $(\in, \in)$-fuzzy bi-filter of S. □

Definition 3.20. Let f be any fuzzy subset of an ordered semigroup S. The set

$$[f]_u = \{x \in S \mid x_u \in \vee(k^*, q_k) f\},$$

where $u \in (0, 1]$, is called an $(\in \vee(k^*, q_k))$-level subset of f.

Theorem 3.21. *Let S be an ordered semigroup and f be a fuzzy subset of S. Then f is an $(\in, \in \vee(k^*, q_k))$-fuzzy filter of S if and only if the $(\in \vee(k^*, q_k))$-level subset $[f]_u$ of f is a filter of S for all $u \in (0, 1]$.*

Proof. Suppose f is an $(\in, \in \vee(k^*, q_k))$-fuzzy filter of S. To show that $[f_u]$ is a filter of S, take any $x \in [f]_u$ and $x \leq y$. As $x \in [f]_u$, we have $x_u \in \vee(k^*, q_k) f$, that is, $f(x) \geq u$ or $f(x) + u + k > k^*$. Now, by Theorem 3.14, $f(y) \geq f(x) \geq u$ or $f(y) \geq f(x) \geq k^* - u - k$, that is, $y_u \in \vee(k^*, q_k) f$. So $y \in [f]_u$.

Next, take any $x, y \in [f]_u$ for $u \in (0, 1]$. Then $x_u \in \vee(k^*, q_k) f$ and $y_u \in \vee(k^*, q_k) f$, that is, $f(x) \geq u$ or $f(x) + u + k > k^*$ and $f(y) \geq u$ or $f(y) + u + k > k^*$. Since f is an $(\in, \in \vee(k^*, q_k))$-fuzzy filter of S, by Theorem 3.14,

$$f(xy) \geq \min\{f(x), f(y), \frac{k^*-k}{2}\}.$$

Case(i): Let $f(x) \geq u$ and $f(y) \geq u$. If $u > \frac{k^*-k}{2}$, then

$$f(xy) \geq \min\{f(x), f(y), \frac{k^*-k}{2}\} \geq \min\{u, u, \frac{k^*-k}{2}\} = \frac{k^*-k}{2},$$

and thus $(xy)_u(k^*, q_k)f$. If $u \leq \frac{k^*-k}{2}$, then

$$f(xy) \geq \min\{f(x), f(y), \frac{k^*-k}{2}\} \geq \min\{u, u, \frac{k^*-k}{2}\} = u.$$

So $(xy)_u \in f$. Hence $(xy)_u \in \vee(k^*, q_k)f$.

Case(ii): Let $f(x) \geq u$ and $f(x) + u + k > k^*$. If $u > \frac{k^*-k}{2}$, then

$$\begin{aligned} f(xy) &\geq \min\{f(x), f(y), \tfrac{k^*-k}{2}\} = \min\{f(y), \tfrac{k^*-k}{2}\} \\ &> \min\{(k^* - u - k), \tfrac{k^*-k}{2}\} = k^* - u - k, \end{aligned}$$

that is, $f(xy) + u + k > k^*$. Thus $(xy)_u(k^*, q_k)f$. If $u \leq \frac{k^*-k}{2}$, then

$$f(xy) \geq \min\{f(x), f(y), \frac{k^*-k}{2}\} \geq \min\{u, (k^* - u - k), \frac{k^*-k}{2}\} = u.$$

So $(xy)_u \in f$. Hence $(xy)_u \in \vee(k^*, q_k)f$.

Case(iii): Let $f(x) + u + k > k^*$ and $f(y) \geq u$. The proof in this case is similar to the proof of Case(ii).

Case(iv): Let $f(x) + u + k > k^*$ and $f(y) + u + k > k^*$. The proof in this case is similar to the proof of Case(ii) and (iii).

Then, in any case, we have $(xy)_u \in \vee(k^*, q_k)f$. Thus $xy \in [f]_u$.

Finally, we may show that $x_u, y_u \in \vee(k^*, q_k)f$ on the lines similar to the above proof. So $x, y \in [f]_u$.

Conversely, Let f be a fuzzy subset of S and $u \in (0, 1]$ be such that $[f]_u$ is a filter of S. Suppose to the contrary that $f(y) < \min\{f(x), \frac{k^*-k}{2}\}$ for some $x, y \in S$. Then there exist $u \in (0, \frac{k^*-k}{2}]$ such that $f(y) < u \leq \min\{f(x), \frac{k^*-k}{2}\}$. Now $x \in [f]_u$, which implies that $y \in [f]_u$. Thus $f(y) \geq u$ or $f(y) + u + k > k^*$, which is not possible. Therefore $f(y) \geq \min\{f(x), \frac{k^*-k}{2}\}$ for all $x, y \in S$. If possible, let $f(xy) < \min\{f(x), f(y), \frac{k^*-k}{2}\}$ for some $x, y \in S$. Then there exist $u \in (0, \frac{k^*-k}{2}]$ such that $f(xy) < u \leq \min\{f(x), f(y), \frac{k^*-k}{2}\}$. Now $x, y \in [f]_u$, which implies that $xy \in [f]_u$. Thus $f(xy) \geq u$ or $f(xy) + u + k > k^*$, which is not possible. So $f(xy) \geq \min\{f(x), f(y), \frac{k^*-k}{2}\}$ for all $x, y \in S$. In a similar way, we may show that $\min\{f(x), f(y)\} \geq \min\{f(xy), \frac{k^*-k}{2}\}$ for all $x, y \in S$. Hence f is an $(\in, \in \vee(k^*, q_k))$-fuzzy filter of S. □

4. (k^*, k)-lower parts of $(\in, \in \vee(k^*, q_k))$-fuzzy filters

In this section we investigate properties of (k^*, k)-lower parts of $(\in, \in \vee(k^*, q_k))$-fuzzy filters of ordered semigroup.

Definition 4.1. Let f be a fuzzy subset of an ordered semigroup S. Then we define the (k^*, k)-lower part $\underline{f_k^{k^*}}$ of f as follows:

$$\underline{f_k^{k^*}}(x) = \min\left\{f(x), \frac{k^*-k}{2}\right\}$$

for all $x \in S$ and $0 \leq k < k^* \leq 1$.

Clearly, $\underline{f_k^{k^*}}$ is a fuzzy subset of S.

For any non-empty subset A of S and fuzzy subset f of S, $(\underline{f_A})_k^{k^*}$, the (k^*, k)-lower part of the characteristic function f_A, will be denoted by $(\underline{f_k^{k^*}})_A$ in the sequel.

Definition 4.2. Let f and g be any fuzzy subsets of S. Define $f(\cap)_k^{k^*} g$, $f(\cup)_k^{k^*} g$ and $f(\circ)_k^{k^*} g$ as follows:

$$\begin{aligned}(f(\cap)_k^{k^*} g)(x) &= \min\{(f \cap g)(x), \tfrac{k^*-k}{2}\},\\ (f(\cup)_k^{k^*} g)(x) &= \min\{(f \cup g)(x), \tfrac{k^*-k}{2}\},\\ (f(\circ)_k^{k^*} g)(x) &= \min\{(f \circ g)(x), \tfrac{k^*-k}{2}\},\end{aligned}$$

for all $x \in S$ and $0 \leq k < k^* \leq 1$.
Then, clearly, $f(\cap)_k^{k^*} g$, $f(\cup)_k^{k^*} g$ and $f(\circ)_k^{k^*} g$ are all fuzzy subsets of S.

The following Lemma easily follows.

Lemma 4.3. *Let f and g be any fuzzy subsets of an ordered semigroup S. Then*
(1) $(\underline{f_k^{k^*}})_k^{k^*} = \underline{f_k^{k^*}}$ *and* $\underline{f_k^{k^*}} \subseteq f$.
(1) *If $f \subseteq g$, and $h \in F(S)$, then $f(\circ)_k^{k^*} h \subseteq g(\circ)_k^{k^*} h$ and $h(\circ)_k^{k^*} f \subseteq h(\circ)_k^{k^*} g$.*
(2) $f(\cap)_k^{k^*} g = \underline{f_k^{k^*}} \cap \underline{g_k^{k^*}}$.
(3) $f(\cup)_k^{k^*} g = \underline{f_k^{k^*}} \cup \underline{g_k^{k^*}}$.
(4) $f(\circ)_k^{k^*} g = \underline{f_k^{k^*}} \circ \underline{g_k^{k^*}}$.

In Example 3.13, we have shown that the $(\in, \in \vee(k^*, q_k))$-fuzzy left filter of an ordered semigroup S is not necessarily a fuzzy left filter of S. However, in the following theorem, we show that, if f is an $(\in, \in \vee(k^*, q_k))$-fuzzy left filter of S, then the (k^*, k)-lower part $\underline{f_k^{k^*}}$ of f is a fuzzy left filter of S.

Theorem 4.4. *If f is an $(\in, \in \vee(k^*, q_k))$-fuzzy left filter of an ordered semigroup S, then the (k^*, k)-lower part $\underline{f_k^{k^*}}$ of f is a fuzzy left filter of S.*

Proof. Let f be an $(\in, \in \vee(k^*, q_k))$-fuzzy left filter of an ordered semigroup S and $x, y \in S$. Suppose $x \leq y$. Since f is an $(\in, \in \vee(k^*, q_k))$-fuzzy left filter of an ordered semigroup S, $f(y) \geq \min\{f(x), \frac{k^*-k}{2}\}$. Thus

$$\underline{f_k^{k^*}}(y) = \min\{f(y), \frac{k^*-k}{2}\} \geq \min\{f(x), \frac{k^*-k}{2}\} = \underline{f_k^{k^*}}(x).$$

Let $x, y \in S$. Then $f(xy) \geq \min\{f(x), f(y), \frac{k^*-k}{2}\}$. Thus we have

$$\begin{aligned}\underline{f_k^{k^*}}(xy) &= \min\{f(xy), \tfrac{k^*-k}{2}\}\\ &\geq \min\{f(x), f(y), \tfrac{k^*-k}{2}\}\\ &= \min\{\underline{f_k^{k^*}}(x), \underline{f_k^{k^*}}(y)\}.\end{aligned}$$

On one hand, $f(x) \geq \min\{f(xy), \frac{k^*-k}{2}\}$. So

$$\begin{aligned}\underline{f_k^{k^*}}(x) &= \min\{f(x), \tfrac{k^*-k}{2}\}\\ &\geq \min\{f(xy), \tfrac{k^*-k}{2}\} = \underline{f_k^{k^*}}(xy).\end{aligned}$$

Hence, $\underline{f_k^{k^*}}$ is a fuzzy left filter of S. □

Theorem 4.5. *Let S be an ordered semigroup and A non-empty subset of S. Then A is a left filter of S if and only if the characteristic function $(\underline{f_k^{k^*}})_A$ of A is an $(\in, \in \vee(k^*, q_k))$-fuzzy left filter of S.*

Proof. Let $x, y \in S$, $x \leq y$.

If $x \in A$, then $(\underline{f_k^{k^*}})_A(x) = \frac{k^*-k}{2}$. Since $x \in A$ and $x \leq y$, we have $y \in A$. Then $(\underline{f_k^{k^*}})_A(y) = \frac{k^*-k}{2}$. Thus $(\underline{f_k^{k^*}})_A(y) \geq (\underline{f_k^{k^*}})_A(x)$.

If $x \notin A$, then $(\underline{f_k^{k^*}})_A(x) = 0$. Since $y \in S$, we have $(\underline{f_k^{k^*}})_A(y) \geq 0$. Thus $(\underline{f_k^{k^*}})_A(x) = 0 \leq (\underline{f_k^{k^*}})_A(y)$. Since A be any left filter of S and $x, y \in S$. Then

$$(\underline{f_k^{k^*}})_A(xy) \geq \min\{(\underline{f_k^{k^*}})_A(x), (\underline{f_k^{k^*}})_A(y)\}.$$

If $x, y \in A$, then $xy \in A$. Thus

$$(\underline{f_k^{k^*}})_A(xy) = \frac{k^*-k}{2} \geq \min\{(\underline{f_k^{k^*}})_A(x), (\underline{f_k^{k^*}})_A(y)\}.$$

If $y \in A$,$x \notin A$, then $(\underline{f_k^{k^*}})_A(x) = 0$. Since $xy \in S$, we have $(\underline{f_k^{k^*}})_A(xy) \geq 0$. Thus

$$(\underline{f_k^{k^*}})_A(xy) \geq \min\{(\underline{f_k^{k^*}})_A(x), (\underline{f_k^{k^*}})_A(y)\}.$$

Similarly, we may show that for the case $x \in A$, $y \notin A$ and $y \notin A$, $x \notin A$. So

$$(\underline{f_k^{k^*}})_A(xy) \geq \min\{(\underline{f_k^{k^*}})_A(x), (\underline{f_k^{k^*}})_A(y)\}.$$

Let A be any left filter of S and $x, y \in S$. Then $(\underline{f_k^{k^*}})_A(x) \geq (\underline{f_k^{k^*}})_A(xy)$. Indeed, if $xy \in A$, then $x \in A$. Thus

$$(\underline{f_k^{k^*}})_A(x) = \frac{k^*-k}{2} \geq (\underline{f_k^{k^*}})_A(xy).$$

Conversely, let $x \in A$ and $x \leq y$. Then $y \in A$. It is enough to show that $(\underline{f_k^{k^*}})_A(y) = \frac{k^*-k}{2}$. Since $x \in A$, $(\underline{f_k^{k^*}})_A(x) = \frac{k^*-k}{2}$. Since $(\underline{f_k^{k^*}})_A$ is an $(\in, \in \vee(k^*, q_k))$-fuzzy left filter of S and $x \leq y$, $(\underline{f_k^{k^*}})_A(y) \geq (\underline{f_k^{k^*}})_A(x) = \frac{k^*-k}{2}$. Since $y \in S$, we have $(\underline{f_k^{k^*}})_A(y) \leq \frac{k^*-k}{2}$. Thus $(\underline{f_k^{k^*}})_A(y) = \frac{k^*-k}{2}$. So $y \in A$.

Let $x, y \in A$ and $(\underline{f_k^{k^*}})_A$ be an $(\in, \in \vee(k^*, q_k))$-fuzzy left filter of S. Then

$$(\underline{f_k^{k^*}})_A(xy) \geq \min\{(\underline{f_k^{k^*}})_A(x), (\underline{f_k^{k^*}})_A(y)\} = \frac{k^*-k}{2} > 0.$$

Thus $(\underline{f_k^{k^*}})_A(xy) = \frac{k^*-k}{2}$, that is, $xy \in A$.

Let $xy \in A$ and $(\underline{f_k^{k^*}})_A$ be an $(\in, \in \vee(k^*, q_k))$-fuzzy left filter of S. Then

$$(\underline{f_k^{k^*}})_A(x) \geq (\underline{f_k^{k^*}})_A(xy) = \frac{k^*-k}{2} > 0.$$

So $(\underline{f_k^{k^*}})_A(x) = \frac{k^*-k}{2}$, that is, $x \in A$. Hence A is a left filter of S. □

Theorems 4.6-4.8 may be proved on the similar lines.

Theorem 4.6. *Let S be an ordered semigroup and A non-empty subset of S. Then A is a right filter of S if and only if the characteristic function $(\underline{f_k^{k^*}})_A$ of A is an $(\in, \in \vee(k^*, q_k))$-fuzzy right filter of S.*

Theorem 4.7. *Let S be an ordered semigroup and A non-empty subset of S. Then A is a filter of S if and only if the characteristic function $(\underline{f_k^{k^*}})_A$ of A is an $(\in, \in \vee(k^*, q_k))$-fuzzy filter of S.*

Theorem 4.8. *Let S be an ordered semigroup and A non-empty subset of S. Then A is a bi-filter of S if and only if the characteristic function $(\underline{f_k^{k^*}})_A$ of A is an $(\in, \in \vee(k^*, q_k))$-fuzzy bi-filter of S.*

5. Implications based fuzzy logic

The implication operator $(\rightarrow)$ plays a significant role in the classical two-valued logic. Firstly, from the classical implication one can obtain all other basic logical connectives of the binary logic, viz., the binary operators − and $(\wedge)$, or $(\vee)$− and the unary negation operator $(\neg)$. Secondly, the implication operator holds the center stage in the inference mechanisms of any logic, like modus ponens, modus tollens, hypothetical syllogism in classical logic. A fuzzy implication is a generalization of the classical one to fuzzy logic, much the same way as a t-norm and a t-conorm are generalizations of the classical conjunction and disjunction, respectively. In this section we give the basic definitions and examples of fuzzy implications. Next, we characterize some main properties of implication operators in terms of fuzzy filters.

Definition 5.1. [21] A function $I : [0, 1]^2 \rightarrow [0, 1]$ is called a fuzzy implication if it satisfies, for all $x, x_1, x_2, y, y_1, y_2 \in [0, 1]$, the following conditions:

if $x_1 \leq x_2$, then $I(x_1, y) \geq I(x_2, y)$, i.e., $I(\cdot, y)$ is decreasing , $\quad (I1)$

if $y_1 \leq y_2$, then $I(x, y_1) \leq I(x, y_2)$, i.e., $I(x, \cdot)$ is increasing , $\quad (I2)$

$$I(0, 0) = 1, \quad (I3)$$
$$I(1, 1) = 1, \quad (I4)$$
$$I(1, 0) = 0. \quad (I5)$$

The set of all fuzzy implications will be denoted by FI.

The property $(I1)$ is the left antitonicity of the function I that gives a fuzzy implication its unique flavor. It captures the idea that a decrease in the truth value of the antecedent increases its efficacy to state more about the truth value of its consequent. In fact, $(I1)$ strongly encourages the truth-functional realization of an implication, since the truth values of both the antecedent and the consequent independently affect the overall truth value of the statement itself. The axiom $(I2)$ captures the right isotonicity of the overall truth value as a direct function of the consequent. Axioms $(I3) \rightarrow (I5)$ comes from the basic properties of the classical implication. Therefore, we can also use the alternative name ÔÇÿmonotonic implicationsÔÇÖ as the characteristic property of the family FI.

Examples of basic fuzzy implications

(1) $I_{KD} < I_{RC} < I_{LK} < I_{WB}$.
(2) $I_{RS} < I_{GD} < I_{GG} < I_{LK} < I_{WB}$.
(3) $I_{YG} < I_{RC} < I_{LK} < I_{WB}$.
(4) $I_{KD} < I_{FD} < I_{LK} < I_{WB}$.
(5) $I_{RS} < I_{GD} < I_{FD} < I_{LK} < I_{WB}$.

Name	Year	Formula
Lukasiewicz	1923	$I_{LK}(x,y) = \min(1, 1-x+y)$.
Gödel	1932	$I_{GD}(x,y) = \begin{cases} 1, & \text{if } x \leq y; \\ y, & \text{otherwise.} \end{cases}$
Reichenbach	1935	$I_{RC}(x,y) = 1 - x + xy$
Kleene-Dienes	1938	$I_{KD}(x,y) = max(1-x, y)$
Goguen	1969	$I_{GG}(x,y) = \begin{cases} 1, & \text{if } x \leq y; \\ \frac{y}{x}, & \text{otherwise.} \end{cases}$
Weber	1983	$I_{WB}(x,y) = \begin{cases} 1, & \text{if } x < 1; \\ y, & \text{othewise.} \end{cases}$
Fodor	1993	$I_{FD}(x,y) = \begin{cases} 1, & \text{if } x \leq y; \\ max(1-x, y), & \text{otherwise.} \end{cases}$

Definition 5.2. Let I be an implication operator and f be a fuzzy subset of S. Then f is a t-implication based fuzzy left (resp. right) filter of S if and only if

(i) $x \leq y \Rightarrow I(f(x), f(y)) \geq t$,

(ii) $I(\min\{f(x), f(y)\}, f(xy)) \geq t$,

(iii) $I(f(xy), f(x)) \geq t$ (resp. $I(f(xy), f(y)) \geq t$),

for all $x, y \in S$ and $t \in (0, 1]$.

Theorem 5.3. *Let f be any fuzzy subset of S and $I = I_{FD}$. Then f is a $(\frac{k^*-k}{2})$-implication based fuzzy left filter of S if and only if*

(1) $x \leq y \Rightarrow \min\{f(y), \frac{k^*-k}{2}\} \geq f(x)$,

(2) $\min\{f(xy), \frac{k^*-k}{2}\} \geq \min\{f(x), f(y)\}$,

(3) $\min\{f(x), \frac{k^*-k}{2}\} \geq f(xy)$,

for all $x, y \in S$.

Proof. Suppose f is a $(\frac{k^*-k}{2})$-implication based fuzzy left filter of S. Let $x, y \in S$ with $x \leq y$. By Definition 5.2(i), we have

$$I_{FD}(f(x), f(y)) = 1 \text{ or } \max(1 - f(x), f(y)) \geq \tfrac{k^*-k}{2}.$$

Then $f(y) \geq f(x)$ or $f(x) \leq \frac{k^*-k}{2}$, $f(y) \geq \frac{k^*-k}{2}$. Thus $\min\{f(y), \frac{k^*-k}{2}\} \geq f(x)$.

Next we have to show that the (2) is hold. Again, by Definition 5.2(ii), we have

$$I_{FD}\{\min(f(x), f(y)), f(xy)\} = 1 \text{ or } \max(1 - \min\{f(x), f(y)\}, f(xy)) \geq \tfrac{k^*-k}{2}.$$

Then

$$f(xy) \geq \min\{f(x), f(y)\} \text{ or } \min\{f(x), f(y)\} \leq \tfrac{k^*-k}{2},\ f(xy) \geq \tfrac{k^*-k}{2}.$$

Thus $\min\{f(xy), \frac{k^*-k}{2}\} \geq \min\{f(x), f(y)\}$.

Now, we have to show that condition (3) is hold. By Definition 5.2(iii), we have

$$I_{FD}(f(xy), f(x)) = 1 \text{ or } \max(1 - f(xy), f(x)) \geq \tfrac{k^*-k}{2}.$$

Thus $f(x) \geq f(xy)$ or $f(xy) \leq \frac{k^*-k}{2}$, $f(x) \geq \frac{k^*-k}{2}$. So $\min\{f(x), \frac{k^*-k}{2}\} \geq f(xy)$.

Conversely suppose that conditions (1), (2) and (3) hold.

If $f(y) \leq \frac{k^*-k}{2}$, then $f(y) \geq f(x)$. Thus $I_{FD}(f(x), f(y)) = 1 \geq \frac{k^*-k}{2}$.

If $f(y) \geq \frac{k^*-k}{2}$, then $f(x) \leq \frac{k^*-k}{2}$. Thus $\max\{1 - f(x), f(y)\} \geq \frac{k^*-k}{2}$.

So

$$I_{FD}(f(x), f(y)) = \begin{cases} 1 \geq \frac{k^*-k}{2}, & \text{if } f(x) \leq f(y), \\ \max(1 - f(x), f(y)) \geq \frac{k^*-k}{2}, & \text{otherwise.} \end{cases}$$

If $f(xy) \leq \frac{k^*-k}{2}$, then $f(xy) \geq \min\{f(x), f(y)\}$. Thus

$$I_{FD}(\min\{f(x), f(y)\}, f(xy)) = 1 \geq \frac{k^* - k}{2}.$$

If $f(xy) \geq \frac{k^*-k}{2}$, then $\min\{f(x), f(y)\} \leq \frac{k^*-k}{2}$. Thus

$$\max\{1 - \min\{f(x), f(y)\}, f(xy)\} \geq \frac{k^* - k}{2}.$$

So

$$I_{FD}(\min\{f(x), f(y)\}, f(xy)) = \begin{cases} 1 \geq \frac{k^*-k}{2}, & \text{if } \min\{f(x), f(y)\} \leq f(xy), \\ \max(1 - \min\{f(x), f(y)\}, f(xy)) \geq \frac{k^*-k}{2}, & \text{otherwise.} \end{cases}$$

If $f(x) \leq \frac{k^*-k}{2}$, then $f(x) \geq f(xy)$. Thus $I_{FD}(f(xy), f(x)) = 1 \geq \frac{k^*-k}{2}$.
If $f(x) \geq \frac{k^*-k}{2}$, then $f(xy) \leq \frac{k^*-k}{2}$. Thus $\max\{1 - f(xy), f(x)\} \geq \frac{k^*-k}{2}$.
Hence

$$I_{FD}(f(xy), f(x)) = \begin{cases} 1 \geq \frac{k^*-k}{2}, & \text{if } f(xy) \leq f(x), \\ \max(1 - f(xy), f(x)) \geq \frac{k^*-k}{2}, & \text{otherwise.} \end{cases}$$

Therefore, f is a $(\frac{k^*-k}{2})$-implication based fuzzy left filter of S. □

Dually we may prove the following.

Theorem 5.4. *Let f be any fuzzy subset of S and $I = I_{FD}$. Then f is a $(\frac{k^*-k}{2})$-implication based fuzzy right filter of S if and only if*
(1) $x \leq y \Rightarrow \min\{f(y), \frac{k^*-k}{2}\} \geq f(x)$,
(2) $\min\{f(xy), \frac{k^*-k}{2}\} \geq \min\{f(x), f(y)\}$,
(3) $\min\{f(y), \frac{k^*-k}{2}\} \geq f(xy)$,
for all $x, y \in S$.

Theorem 5.5. *Let f be any fuzzy subset of S and $I = I_{GD}$. Then f is a $(\frac{k^*-k}{2})$-implication based fuzzy left filter of S if and only if f is an $(\in, \in \vee(k^*, q_k))$-fuzzy left filter of S.*

Proof. Let f be a $(\frac{k^*-k}{2})$-implication based fuzzy left filter of S. Let $x, y \in S$ be such that $x \leq y$. By Definition 5.2(i), we have $f(y) \geq f(x)$ or $f(x) > f(y) \geq \frac{k^*-k}{2}$. Then $f(y) \geq \min\{f(x), \frac{k^*-k}{2}\}$. By Definition 5.2(ii), we have

$$f(xy) \geq \min\{f(x), f(y)\} \text{ or } \min\{f(x), f(y)\} > f(xy) \geq \tfrac{k^*-k}{2}.$$

Thus $f(xy) \geq \min\{f(x), f(y), \frac{k^*-k}{2}\}$. By Definition 5.2(iii), we have

$$f(x) \geq f(xy) \text{ or } f(xy) > f(x) \geq \tfrac{k^*-k}{2}.$$

So $f(x) \geq \min\{f(xy), \frac{k^*-k}{2}\}$. Hence f is an $(\in, \in \vee(k^*, q_k))$-fuzzy left filter of S.

Conversely, suppose that f is an $(\in, \in \vee(k^*, q_k))$-fuzzy left filter of S. Let $x, y \in S$ be such that $x \leq y$. By Theorem 3.7(1), we get

$$I_{GD}(f(x), f(y)) = \begin{cases} 1 \geq \frac{k^*-k}{2}, & \text{if } \min\{f(x), \frac{k^*-k}{2}\} = f(x), \\ f(y) \geq \frac{k^*-k}{2}, & \text{if } \min\{f(x), \frac{k^*-k}{2}\} = \frac{k^*-k}{2}. \end{cases}$$

Thus $I_{GD}(f(x), f(y)) \geq \frac{k^*-k}{2}$. Again, by Theorem 3.7(2), we have

$$I_{GD}(\min(f(x), f(y)), f(xy)) = \begin{cases} 1 \geq \frac{k^*-k}{2}, & \text{if } \min\{f(x), f(y), \frac{k^*-k}{2}\} = \min\{f(x), f(y)\}, \\ f(xy) \geq \frac{k^*-k}{2}, & \text{if } \min\{f(x), f(y), \frac{k^*-k}{2}\} = \frac{k^*-k}{2}. \end{cases}$$

So $I_{GD}(\min(f(x), f(y)), f(xy)) \geq \frac{k^*-k}{2}$. By using Theorem 3.7(3), we get

$$I_{GD}(f(xy), f(x)) = \begin{cases} 1 \geq \frac{k^*-k}{2}, & \text{if } \min\{f(xy), \frac{k^*-k}{2}\} = f(xy), \\ f(x) \geq \frac{k^*-k}{2}, & \text{if } \min\{f(xy), \frac{k^*-k}{2}\} = \frac{k^*-k}{2}. \end{cases}$$

So $I_{GD}(f(xy), f(x))\frac{k^*-k}{2}$. Hence f is a $(\frac{k^*-k}{2})$-implication based fuzzy right filter of S. □

Dually we may prove the following.

Theorem 5.6. *Let f be any fuzzy subset of S and $I = I_{GD}$. Then f is a $(\frac{k^*-k}{2})$-implication based fuzzy right filter of S if and only if f is an $(\in, \in \vee(k^*, q_k))$-fuzzy right filter of S.*

6. Conclusion

As we know, fuzzy filters of an ordered semigroup with special properties always play an important role in the study of ordered semigroup structure. In this paper we have introduced the concepts of $(\in, \in \vee(k^*, q_k))$-fuzzy filters and $(\in, \in \vee(k^*, q_k))$-fuzzy bi-filters of an ordered semigroup and investigated their related properties. We hope that the research along this direction can be continued, and in fact, some results in this paper have already constituted a platform for further discussion concerning the future development of ordered semigroups. Hopefully, some new results in these topics can be obtained in the forthcoming paper.

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