Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 12, No. 6, (December 2016), pp. 825–834
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

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Weakly fuzzy pre-I-open sets and decomposition of fuzzy continuity

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Received 22 January 2016; Revised 5 May 2016; Accepted 7 June 2016

Abstract. In this paper, we introduce the notion of weakly fuzzy pre-$\mathcal{I}$-open set, fuzzy strong t-$\mathcal{I}$-set, fuzzy $\mathcal{I}_\beta$-set, fuzzy strong $B_\mathcal{I}$-set and fuzzy $S\mathcal{I}_\beta$-set in fuzzy ideal topological spaces, and investigated several characterizations and some properties of these sets. Moreover, we obtained a new decompositions of fuzzy continuity via fuzzy idealization.

2010 AMS Classification: 54A40

Keywords: Weakly fuzzy pre-$\mathcal{I}$-open, Weakly fuzzy pre-$\mathcal{I}$-continuous, Fuzzy strong t-$\mathcal{I}$-set, Fuzzy $\mathcal{I}_\beta$-set, Fuzzy strong $B_\mathcal{I}$-set, Fuzzy $S\mathcal{I}_\beta$-set.

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1. Introduction

The fundamental concept of fuzzy set was introduced by Zadeh [17] and an alternative definition of fuzzy topology was given by Lowen (1976). The notions of the sets and functions in fuzzy topological spaces are used extensively in many engineering problems, computational topology for geometric design, computer-aided geometric design, engineering design research and mathematical sciences. Currently, Shihong Du, et al. are working to fuzzify the 9-intersection Egenhofer model for describing topological relations in Geographic Information System. In general topology by introducing the notion of ideal, Kuratowski [7], Vaidyanathaswamy [14] and several other authors carried out such analyses. In 1997, Mahmoud [8] and Sarkar [12] independently presented some of the ideal concepts in the fuzzy trend and studied many other properties.

In this paper, we introduce the notion of weakly fuzzy pre-$\mathcal{I}$-open set, fuzzy strong t-$\mathcal{I}$-set, fuzzy $\mathcal{I}_\beta$-set, fuzzy strong $B_\mathcal{I}$-set and fuzzy $S\mathcal{I}_\beta$-set and obtain several characterizations and some properties of these sets. Finally we obtained a new decompositions of fuzzy continuity by using weakly fuzzy pre-$\mathcal{I}$-continuity and fuzzy strong $B_\mathcal{I}$-continuity.

2. Preliminaries

Throughout this paper, X represents a nonempty fuzzy set and fuzzy subset A of X, denoted by $A \leq X$, is characterized by a membership function in the sense of Zadeh [17]. The basic fuzzy sets are the empty set, the whole set and the class of all fuzzy subsets of X which will be denoted by $0, 1$ and $\mathcal{I}^X$, respectively. By (X, τ), we mean a fuzzy topological space in Chang's sense. A fuzzy point in X with support $x \in X$ and value $\alpha(0 < \alpha \leq 1)$ is denoted by x_α. For a fuzzy subset A of X, Cl(A), Int(A) and $1 - A$ will, respectively, denote the closure, interior and complement of A. A nonempty collection $\mathcal{I}$ of fuzzy subsets of X is called a fuzzy ideal [12] if and only if

1. $B \in \mathcal{I}$ and $A \leq B$, then $A \in \mathcal{I}$ (heredity),
2. if $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \vee B \in \mathcal{I}$ (finite additivity).

A fuzzy ideal topological space, denoted by $(X, \tau, \mathcal{I})$, means a fuzzy topological space with a fuzzy ideal $\mathcal{I}$ and fuzzy topology τ.

For $(X, \tau, \mathcal{I})$, the fuzzy local function of $A \leq X$ with respect to τ and $\mathcal{I}$ is denoted by $A^*(\tau, \mathcal{I})$ (briefly A^*) and is defined as $A^*(\tau, \mathcal{I}) = \vee\{x \in X : A \wedge U \notin \mathcal{I}$ for every $U \in \tau(x)\}$. While A^* is the union of the fuzzy points x such that if $U \in \tau(x)$ and $E \in \mathcal{I}$, then there is at least one $y \in X$ for which $U(y) + A(y) - 1 > E(y)$. Fuzzy closure operator of a fuzzy set in$(X, \tau, \mathcal{I})$is defined as $Cl^*(A) = A \vee A^*$. In $(X, \tau, \mathcal{I})$, the collection $\tau^*(\mathcal{I})$ means an extension of fuzzy topological space than τ via fuzzy ideal which is constructed by considering the class $\beta = \{U - E : U \in \tau, E \in \mathcal{I}\}$as a base [12]. This topology of fuzzy sets is considered as generalization of the ordinary one. First, we shall recall some definitions used in the sequel.

Definition 2.1. A subset A of a fuzzy topological space (X, τ) is said to be:
(i) fuzzy α-open [3] if $A \leq Int(Cl(Int(A)))$.
(ii) fuzzy pre-open [3] if $A \leq Int(Cl(A))$.
(iii) fuzzy semi-open [1] if $A \leq Cl(Int(A))$.
(iv) fuzzy β-open [9, 13] if $A \leq Cl(Int(Cl(A)))$.

Definition 2.2. A subset A of a fuzzy ideal topological space $(X, \tau, \mathcal{I})$ is said to be:
(i) fuzzy α-$\mathcal{I}$-open [16] if $A \leq Int(Cl^*(Int(A)))$.
(ii) fuzzy pre-$\mathcal{I}$-open [10] if $A \leq Int(Cl^*(A))$.
(iii) fuzzy semi-$\mathcal{I}$-open [4] if $A \leq Cl^*(Int(A))$.
(iv) fuzzy β-$\mathcal{I}$-open [16] if $A \leq Cl(Int(Cl^*(A)))$.
(v) weakly fuzzy semi-$\mathcal{I}$-open [2] if $A \leq Cl^*(Int(Cl(A)))$.

Definition 2.3 ([15])**.** Let A be a subset of a fuzzy topological space (X, τ). The complement of a fuzzy semi-open set is said to be fuzzy semi-closed.

The intersection of all fuzzy semi-closed sets containing A is called the fuzzy semi-closure of A and is denoted by $sCl(A)$.

The fuzzy semi-interior of A, denoted by $sInt(A)$, is defined by the union of all fuzzy semi-open sets contained in A.

The following lemma is well-known.

Lemma 2.4 ([15])**.** *For a subset A of a fuzzy topological space (X, τ), the following properties hold:*

(1) $sCl(A) = A \vee Int(Cl(A))$.
(2) $sCl(A) = Int(Cl(A))$, *if* A *is fuzzy open.*

For the local function, the following lemma is basic and useful in the sequel.

Lemma 2.5 ([6]). *Let* $(X, \tau, \mathcal{I})$ *be a fuzzy ideal topological space and* A, B *be subsets of* X. *Then*
(1) *if* $A \leq B$, *then* $A^* \leq B^*$,
(2) $A^* = Cl(A^*) \leq Cl(A)$,
(3) *if* $U \in \tau$, *then* $U \cap A^* \leq (U \vee A)^*$.

3. Weakly fuzzy pre-$\mathcal{I}$-open sets

Definition 3.1. A subset A of a fuzzy ideal topological space $(X, \tau, \mathcal{I})$ is said to be weakly fuzzy pre-$\mathcal{I}$-open if $A \leq sCl(Int(Cl^*(A)))$.

Proposition 3.2. *For a subset of a fuzzy ideal topological space, the following properties hold:*
(1) *Every fuzzy pre-$\mathcal{I}$-open set is weakly fuzzy pre-$\mathcal{I}$-open,*
(2) *Every weakly fuzzy pre-$\mathcal{I}$-open set is fuzzy β-$\mathcal{I}$-open,*
(3) *Every fuzzy α-open set is weakly fuzzy pre-$\mathcal{I}$-open,*
(4) *Every weakly fuzzy pre-$\mathcal{I}$-open set is fuzzy pre-open.*

Proof. Let $(X, \tau, \mathcal{I})$ be a fuzzy ideal topological space and A be a subset of X.
(1) Let A be a fuzzy pre-$\mathcal{I}$-open set. Then we have

$$A \leq Int(Cl^*(A)) \leq sCl(Int(Cl^*(A))).$$

This shows that A is a weakly fuzzy pre-$\mathcal{I}$-open set.
(2) Let A be a weakly fuzzy pre-$\mathcal{I}$-open set. Since $sCl(A) \leq Cl(A)$, we have

$$A \leq sCl(Int(Cl^*(A))) \leq Cl(Int(Cl^*(A))).$$

This shows that A is a fuzzy β-$\mathcal{I}$-open set.
(3) Let A be a fuzzy α-open set. Then, by Lemma 2.4, we have

$$\begin{aligned} A &\leq Int(Cl(Int(A))) \leq Int(Cl(Int(A \vee A^*))) \\ &= Int(Cl(Int(Cl^*(A)))) = sCl(Int(Cl^*(A))). \end{aligned}$$

This shows that A is a weakly fuzzy pre-$\mathcal{I}$-open set.
(4) Let A be a weakly fuzzy pre-$\mathcal{I}$-open set. Then, by Lemma 2.4, we have

$$\begin{aligned} A &\leq sCl(Int(Cl^*(A))) \leq sCl(Int(Cl(A))) \\ &= Int(Cl(Int(Cl(A)))) = Int(Cl(A)). \end{aligned}$$

This shows that A is a fuzzy pre-open set. □

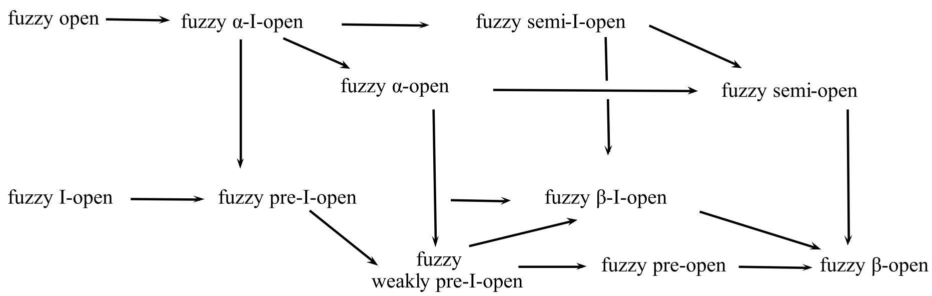
Lemma 3.3. *In a fuzzy ideal topological space* $(X, \tau, \mathcal{I})$, *every fuzzy semi-$\mathcal{I}$-open set is fuzzy β-$\mathcal{I}$-open.*

Proof. Let A be a fuzzy semi-$\mathcal{I}$-open set. Then we have

$$A \leq Cl^*(Int(A)) \leq Cl(Int(A)) \leq Cl(Int(Cl^*(A))).$$

This shows that A is fuzzy β-$\mathcal{I}$-open. □

Remark 3.4. For several sets defined above, by Remark 2.1 of [5], Proposition 3.2 and Lemma 3.3, we have the following diagram. The converse of Proposition 3.2 are not true as shown by the following examples.



Example 3.5. Let $X = \{a, b, c\}$ and A, B and C be the fuzzy subsets of X defined as follows:
$A(a) = 0.6,\ A(b) = 0.7,\ A(c) = 0.5,$
$B(a) = 0.4,\ B(b) = 0.3,\ B(c) = 0.2,$
$C(a) = 0.5,\ C(b) = 0.7,\ C(c) = 0.4.$
We put $\tau = \{0, C, 1\}$. If we take $\mathcal{I} = \rho(X)$, then A is weakly fuzzy pre-$\mathcal{I}$-open, but A is not fuzzy pre-$\mathcal{I}$-open. And also A is not fuzzy semi-$\mathcal{I}$-open.

Example 3.6. Let $X = \{a, b, c\}$ and A, B and C be the fuzzy subsets of X defined as follows:
$A(a) = 0.5,\ A(b) = 0.3,\ A(c) = 0.6,$
$B(a) = 0.3,\ B(b) = 0.2,\ B(c) = 0.4,$
$C(a) = 0.4,\ C(b) = 0.6,\ C(c) = 0.7.$
We put $\tau = \{0, B, C, 1\}$. If we take $\mathcal{I} = \rho(X)$, then A is fuzzy β-$\mathcal{I}$-open which is not weakly fuzzy pre-$\mathcal{I}$-open.

Example 3.7. Let $X = \{a, b, c\}$ and A, B be the fuzzy subsets of X defined as follows:
$A(a) = 0.5,\ A(b) = 0.7,\ A(c) = 0.8,$
$B(a) = 0.4,\ B(b) = 0.5,\ B(c) = 0.3.$
We put $\tau = \{0, B, 1\}$. If we take $\mathcal{I} = \{0\}$, then A is weakly fuzzy pre-$\mathcal{I}$-open which is not fuzzy semi-open; hence not fuzzy α-open.

Example 3.8. Let $X = \{a, b, c\}$ and A, B and C be the fuzzy subsets of X defined as follows:
$A(a) = 0.6,\ A(b) = 0.4,\ A(c) = 0.5,$
$B(a) = 0,\ B(b) = 0,\ B(c) = 0.5,$
$C(a) = 0.7,\ C(b) = 0.6,\ C(c) = 0.7.$
We put $\tau = \{0, A, B, 1\}$. If we take $\mathcal{I} = \rho(X)$, then A is fuzzy pre-open which is not weakly fuzzy pre-$\mathcal{I}$-open.

Example 3.9. Let $X = \{a, b, c\}$ and A and B be the fuzzy subsets of X defined as follows:

$A(a) = 0.5$, $A(b) = 0.4$, $A(c) = 0.6$,
$B(a) = 0.2$, $B(b) = 0.3$, $B(c) = 0.4$.
We put $\tau = \{0, B, 1\}$. If we take $\mathcal{I} = \rho(X)$, then A is fuzzy semi-$\mathcal{I}$-open which is not weakly fuzzy pre-$\mathcal{I}$-open.

Lemma 3.10. *For subsets A and B of a fuzzy ideal topological space $(X, \tau, \mathcal{I})$, the following properties hold:*
(1) $U \wedge Cl^*(A) \leq Cl^*(U \wedge A)$, *if* $U \in \tau$.
(2) $Int(Cl(A)) \wedge Int(Cl(B)) = Int(Cl(A \wedge B))$, *if either A or B is fuzzy semi-open.*

Proof. (1) By Lemma 2.5, we have $U \wedge Cl^*(A) = U \wedge (A \vee A^*) = (U \wedge A) \vee (U \wedge A^*) \leq (U \wedge A) \vee (U \wedge A)^* = Cl^*(U \wedge A)$.
(2) The proof is from Lemma 3.5 in [11]. □

Theorem 3.11. *Let $(X, \tau, \mathcal{I})$, be a fuzzy ideal topological space. Let A, U and A_α $(\alpha \in \Delta)$ be subsets of X. Then*
(1) *If A_α is weakly fuzzy pre-$\mathcal{I}$-open for each $\alpha \in \Delta$, then $\bigvee_{\alpha \in \Delta} A_\alpha$ is weakly fuzzy pre-$\mathcal{I}$-open.*
(2) *If U is fuzzy α-open and A is weakly fuzzy pre-$\mathcal{I}$-open, then $U \wedge A$ is weakly fuzzy pre-$\mathcal{I}$-open.*

Proof. (1) Since A_α is weakly fuzzy pre-$\mathcal{I}$-open for each $\alpha \in \Delta$, we have

$$A_\alpha \leq sCl(Int(Cl^*(A_\alpha))) \leq sCl(Int(Cl^*(\textstyle\bigvee_{\alpha \in \Delta} A_\alpha))) \text{ for each } \alpha \in \Delta.$$

Then

$$\textstyle\bigvee_{\alpha \in \Delta} A_\alpha \leq sCl(Int(Cl^*(\bigvee_{\alpha \in \Delta} A_\alpha))).$$

This shows that $\bigvee_{\alpha \in \Delta} A_\alpha$ is weakly fuzzy pre-$\mathcal{I}$-open.
(2) By Lemmas 2.4 and 3.10, we have

$$\begin{aligned} A \wedge U &\leq sCl(Int(Cl^*(A))) \wedge Int(Cl(Int(U))) \\ &= Int(Cl(Int(Cl^*(A)))) \wedge Int(Cl(Int(U))) \\ &= Int(Cl(Int(Cl^*(A)) \wedge Int(U))) \\ &= sCl(Int(Cl^*(A) \wedge Int(U))) \\ &\leq sCl(Int(Cl^*(A \wedge Int(U)))) \\ &\leq sCl(Int(Cl^*(A \wedge U))). \end{aligned}$$

This shows that $A \wedge U$ is weakly fuzzy pre-$\mathcal{I}$-open. □

Remark 3.12. The finite intersection of weakly fuzzy pre-$\mathcal{I}$-open sets need not be weakly fuzzy pre-$\mathcal{I}$-open in general as the following examples shows.

Example 3.13. Let $X = \{a, b, c\}$ and A, B, C and D be the fuzzy subsets of X defined as follows:
$A(a) = 0.3$, $A(b) = 0.5$, $A(c) = 0.6$,
$B(a) = 0.5$, $B(b) = 0.7$, $B(c) = 0.6$,
$C(a) = 0.3$, $C(b) = 0.5$, $C(c) = 0.8$,
$D(a) = 0.6$, $D(b) = 0.5$, $D(c) = 0.4$.
We put $\tau = \{0, D, 1\}$ and $A = B \cap C$. If we take $\mathcal{I} = \{0\}$, then B and C are weakly fuzzy pre-$\mathcal{I}$-open but A is not weakly fuzzy pre-$\mathcal{I}$-open.

Definition 3.14. A subset A of a fuzzy ideal topological space $(X, \tau, \mathcal{I})$, is said to be weakly fuzzy pre-$\mathcal{I}$-closed if its complement is weakly fuzzy pre-$\mathcal{I}$-open.

Lemma 3.15. [6] *Let $(X, \tau, \mathcal{I})$ be a fuzzy ideal topological space and $A \leq S \leq X$. Then $A^*(\tau_{|S}, \mathcal{I}_{|S}) = A^*(\tau, \mathcal{I}) \wedge S$ holds.*

Theorem 3.16. *Let $(X, \tau, \mathcal{I})$ be a fuzzy ideal topological space and $A \leq U \in \tau$. Then, A is weakly fuzzy pre-$\mathcal{I}$-open in $(X, \tau, \mathcal{I})$ if and only if A is weakly fuzzy pre-$\mathcal{I}$-open in $(U, \tau_{|U}, \mathcal{I}_{|U})$.*

Proof. Necessity: Let A be weakly fuzzy pre-$\mathcal{I}$-open in $(X, \tau, \mathcal{I})$. Then we have

$$A \leq sCl(Int(Cl^*(A))) = Int(Cl(Int(Cl^*(A)))).$$

Thus

$$\begin{aligned} A = U \wedge A &\leq U \wedge Int(Cl(Int(Cl^*(A)))) = Int(U \wedge Cl(Int(Cl^*(A)))) \\ &\leq Int_U(U \wedge Cl(U \wedge Int(Cl^*(A)))) = Int_U(Cl_U(U \wedge Int(Cl^*(A)))) \\ &= sCl_U(Int(U \wedge Cl^*(A))) \leq sCl_U(Int_U(U \wedge Cl^*(A))) \\ &\leq sCl_U(Int_U(Cl^*_U(A))). \end{aligned}$$

This shows that A is weakly fuzzy pre-$\mathcal{I}$-open in $(U, \tau_{|_U}, \mathcal{I}_{|_U})$.

Sufficiency: Let A be weakly fuzzy pre-$\mathcal{I}$-open in $(U, \tau_{|_U}, \mathcal{I}_{|_U})$. Then we have

$$\begin{aligned} A &\leq sCl_U(Int_U(Cl^*_U(A))) = sCl_U(Int_U(Cl^*(A) \wedge U)) \\ &= sCl_U(Int(Cl^*(A) \wedge U)) = Int_U(Cl_U(Int(Cl^*(A) \wedge U))) \\ &= Int(U \wedge Cl(Int(Cl^*(A) \wedge U))) \leq Int(Cl(Int(Cl^*(A)))) \\ &= sCl(Int(Cl^*(A))). \end{aligned}$$

This shows that A is weakly fuzzy pre-$\mathcal{I}$-open in $(X, \tau, \mathcal{I})$. □

Corollary 3.17. *Let$(X, \tau, \mathcal{I})$be a fuzzy ideal topological space. If $U \in \tau$ and A is weakly fuzzy pre-$\mathcal{I}$-open, then $U \wedge A$ is weakly fuzzy pre-$\mathcal{I}$-open in $(U, \tau_{|U}, \mathcal{I}_{|U})$.*

Proof. Since $U \in \tau$ and A is weakly fuzzy pre-$\mathcal{I}$-open, by Theorem 3.11, $U \wedge A$ is weakly fuzzy pre-$\mathcal{I}$-open in$(X, \tau, \mathcal{I})$. Since $U \in \tau$, by Theorem 3.16, $U \wedge A$ is weakly fuzzy pre-$\mathcal{I}$-open in $(U, \tau_{|U}, \mathcal{I}_{|U})$. □

4. Fuzzy strong $B_{\mathcal{I}}$-Sets

Definition 4.1. A subset A of a fuzzy ideal topological space$(X, \tau, \mathcal{I})$is called

(i) a fuzzy t-$\mathcal{I}$-set [10] if $Int(Cl^*(A)) = Int(A)$,

(ii) a fuzzy strong t-$\mathcal{I}$-set if $sCl(Int(Cl^*(A))) = Int(A)$,

(iii) a fuzzy $\mathcal{I}_\beta$-set if $Cl(Int(Cl^*(A))) = Int(A)$.

Definition 4.2. A subset A of a fuzzy ideal topological space$(X, \tau, \mathcal{I})$is called

(i) fuzzy $B_{\mathcal{I}}$-set [10] if $A = U \wedge V$, where $U \in \tau$ and V is a fuzzy t-$\mathcal{I}$-set,

(ii) a fuzzy strong $B_{\mathcal{I}}$-set if $A = U \wedge V$, where $U \in \tau$ and V is a fuzzy strong t-$\mathcal{I}$-set,

(iii) a fuzzy $S\mathcal{I}_\beta$-set if $A = U \wedge V$, where $U \in \tau$ and V is a fuzzy $\mathcal{I}_\beta$-set.

Proposition 4.3. *Let$(X, \tau, \mathcal{I})$be a fuzzy ideal topological space and A be a subset of X. Then the following properties hold.*

(1) *If A is a fuzzy $\mathcal{I}_\beta$-set, then A is a fuzzy strong t-$\mathcal{I}$-set.*

(2) *If A is a fuzzy strong t-$\mathcal{I}$-set, then A is a fuzzy t-$\mathcal{I}$-set.*

Proof. (1) Let A be a fuzzy $\mathcal{I}_\beta$-set. Then we have

$$sCl(Int(Cl^*(A))) \leq Cl(Int(Cl^*(A))) = Int(A)$$

and

$$Int(A) \leq Int(Cl^*(A)) \leq sCl(Int(Cl^*(A))).$$

Thus $Int(A) = sCl(Int(Cl^*(A)))$.

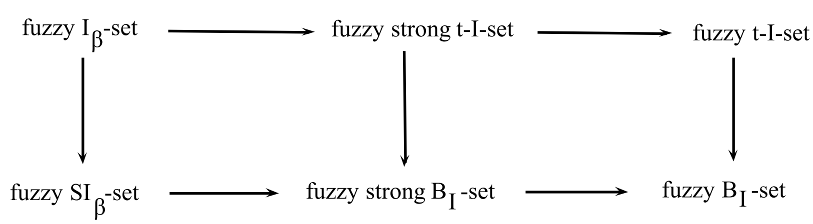
(2) Let A be a fuzzy strong t-$\mathcal{I}$-set. Then we have

$$Int(Cl^*(A)) \leq sCl(Int(Cl^*(A))) = Int(A) \leq Int(Cl^*(A)).$$

Thus $Int(Cl^*(A)) = Int(A)$. This shows that A is a fuzzy t-$\mathcal{I}$-set. □

Proposition 4.4. *Let$(X, \tau, \mathcal{I})$be a fuzzy ideal topological space. For a subset A of X, the following properties hold.*
(1) *If A is a fuzzy $\mathcal{I}_\beta$-set, then A is a fuzzy $S\mathcal{I}_\beta$-set.*
(2) *If A is a fuzzy strong t-$\mathcal{I}$-set, then A is a fuzzy strong $B_\mathcal{I}$-set.*
(3) *If A is a fuzzy $S\mathcal{I}_\beta$-set, then A is a fuzzy strong $B_\mathcal{I}$-set.*
(4) *If A is a fuzzy strong $B_\mathcal{I}$-set, then A is a fuzzy $B_\mathcal{I}$-set.*

By Proposition 3.1 of [5] and Propositions 4.3 and 4.4, we have the following diagram:



Remark 4.5. The converse implications in the above diagram need not be true as the following example shows.

Example 4.6. Let $X = \{a, b, c\}$ and A, B and C be the fuzzy subsets of X defined as follows:

$A(a) = 0.6,\ A(b) = 0.5,\ A(c) = 0.6,$
$B(a) = 0.4,\ B(b) = 0.3,\ B(c) = 0.2,$
$C(a) = 0.6,\ C(b) = 0.7,\ C(c) = 0.4.$

We put $\tau = \{0, B, 1\}$.If we take $\mathcal{I} = \rho(X)$, then A is fuzzy strong t-$\mathcal{I}$-set which is not fuzzy $S\mathcal{I}_\beta$-set.

Example 4.7. Let $X = \{a, b, c\}$ and A, B and C be the fuzzy subsets of X defined as follows:

$A(a) = 0.3,\ A(b) = 0.6,\ A(c) = 0.5,$
$B(a) = 0.2,\ B(b) = 0.5,\ B(c) = 0.4,$
$C(a) = 0.6,\ C(b) = 0.7,\ C(c) = 0.5.$

We put $\tau = \{0, A, B, 1\}$.If we take $\mathcal{I} = \rho(X)$, then A is fuzzy t-$\mathcal{I}$-set which is not fuzzy strong $B_\mathcal{I}$-set.

Example 4.8. Let $X = \{a, b, c\}$ and A, B and C be the fuzzy subsets of X defined as follows:

$A(a) = 0.3,\ A(b) = 0.4,\ A(c) = 0.3,$
$B(a) = 0.3,\ B(b) = 0.5,\ B(c) = 0.4,$
$C(a) = 0.7,\ C(b) = 0.6,\ C(c) = 0.7.$

We put $\tau = \{0, A, C, 1\}$.If we take $\mathcal{I} = \{0\}$, then A is fuzzy $S\mathcal{I}_\beta$-set which is not fuzzy t-$\mathcal{I}$-set.

Proposition 4.9. *$t(X, \tau, \mathcal{I})$be a fuzzy ideal topological space. For a subset A of $(X, \tau, \mathcal{I})$, the following properties are equivalent:*
(1) *A is fuzzy open,*
(2) *A is weakly fuzzy pre-$\mathcal{I}$-open and a fuzzy strong $B_\mathcal{I}$-set,*
(3) *A is fuzzy β-$\mathcal{I}$-open and a fuzzy $S\mathcal{I}_\beta$-set.*

Proof. The implications (1) ⇒ (2) and (1) ⇒ (3) are obvious, since$X$ is a fuzzy strong t-$\mathcal{I}$-set and a fuzzy $\mathcal{I}_\beta$-set.

(2) ⇒ (1): By the weak fuzzy pre-$\mathcal{I}$-openness of A,

$$A \leq sCl(Int(Cl^*(A))) = sCl(Int(Cl^*(U \wedge V))),$$

where $A = U \wedge V$, $U \in \tau$ and V is a fuzzy strong t-$\mathcal{I}$-set. Then
$A \leq U \wedge A \leq U \wedge (sCl(Int(Cl^*(U))) \wedge sCl(Int(Cl^*(V)))) = U \wedge Int(V) = Int(A).$
This shows that A is fuzzy open.

(3) ⇒ (1): Let A be a fuzzy β-$\mathcal{I}$-open and a fuzzy $S\mathcal{I}_\beta$-set. Let $A = U \wedge V$, where $U \in \tau$ and V is a fuzzy $\mathcal{I}_\beta$-set. Since A is a fuzzy β-$\mathcal{I}$-open, $A \leq Cl(Int(Cl^*(A)))$ and $A = U \wedge V \leq U$, we have

$$\begin{aligned} A &\leq U \wedge A \leq U \wedge Cl(Int(Cl^*(A))) \\ &= U \wedge Cl(Int(Cl^*(U \wedge V))) \\ &\leq U \wedge Cl(Int(Cl^*(U))) \wedge Cl(Int(Cl^*(V))) \\ &= U \wedge Int(V) = Int(A). \end{aligned}$$

Thus A is fuzzy open. □

Remark 4.10. The notion of weak fuzzy pre-$\mathcal{I}$-openness (resp. fuzzy β-$\mathcal{I}$-openness) is independent of that of fuzzy strong $B_\mathcal{I}$-set (resp. fuzzy $S\mathcal{I}_\beta$-set) as shown by the following example.

Example 4.11. Let $X = \{a, b, c\}$ and A, B and C be the fuzzy subsets of X defined as follows:

$A(a) = 0.4,\ A(b) = 0.5,\ A(c) = 0.6,$
$B(a) = 0.5,\ B(b) = 0.6,\ B(c) = 0.6,$
$C(a) = 0.4,\ C(b) = 0.5,\ C(c) = 0.7.$

We put $\tau = \{0, C, 1\}$. If we take $\mathcal{I} = \rho(X)$, then A is fuzzy strong $B_\mathcal{I}$-set which is not weakly fuzzy pre-$\mathcal{I}$-open.
If we take $\mathcal{I} = \{0\}$, in the same topology, then A is weakly fuzzy pre-$\mathcal{I}$-open set which is not fuzzy strong $B_\mathcal{I}$-set.

Example 4.12. (1) Let $X = \{a, b, c\}$ and A, B and C be the fuzzy subsets of X defined as follows:

$A(a) = 0.7,\ A(b) = 0.5,\ A(c) = 0.6,$
$B(a) = 0.7,\ B(b) = 0.6,\ B(c) = 0.8,$

$C(a) = 0.8$, $C(b) = 0.5$, $C(c) = 0.6$.
We put $\tau = \{0, B, 1\}$. If we take $\mathcal{I} = \rho(X)$, then A is fuzzy $S\mathcal{I}_\beta$-set which is not fuzzy β-$\mathcal{I}$-open.

(2) In Example 4.6, we have A is fuzzy β-$\mathcal{I}$-open set, but it is not fuzzy $S\mathcal{I}_\beta$-set.

5. Decomposition of fuzzy continuity

Definition 5.1. *A* function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be weakly fuzzy pre-$\mathcal{I}$-continuous if for every $V \in \sigma, f^{-1}(V)$ is a weakly fuzzy pre-$\mathcal{I}$-open set of $(X, \tau, \mathcal{I})$.

Definition 5.2. *A* function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be fuzzy β-$\mathcal{I}$-continuous [16] (respectively, fuzzy $B_{\mathcal{I}}$-continuous [10]) if for every $V \in \sigma, f^{-1}(V)$ is a fuzzy β-$\mathcal{I}$-open (respectively, a fuzzy $B_{\mathcal{I}}$-set) in $(X, \tau, \mathcal{I})$.

Definition 5.3. *A* function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be fuzzy strongly $B_{\mathcal{I}}$-continuous (resp. fuzzy $S\mathcal{I}_\beta$-continuous) if for every $V \in \sigma, f^{-1}(V)$ is a fuzzy strong $B_{\mathcal{I}}$-set (resp. fuzzy $S\mathcal{I}_\beta$-set) of $(X, \tau, \mathcal{I})$.

Proposition 5.4. *If a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is fuzzy strongly $B_{\mathcal{I}}$-continuous (resp. fuzzy $S\mathcal{I}_\beta$-continuous), then f is fuzzy $B_{\mathcal{I}}$-continuous (resp. fuzzy strongly $B_{\mathcal{I}}$-continuous.)*

Proof. This follows from Proposition 4.4. □

Theorem 5.5. *Let$(X, \tau, \mathcal{I})$be a fuzzy ideal topological space. For a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following properties are equivalent:*
(1) *f is fuzzy continuous,*
(2) *f is weakly fuzzy pre-$\mathcal{I}$-continuous and fuzzy strongly $B_{\mathcal{I}}$-continuous.*

Proof. This is an immediate consequence of Proposition 4.9. □

Theorem 5.6. *Let$(X, \tau, \mathcal{I})$be a fuzzy ideal topological space. For a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following properties are equivalent:*
(1) *f is fuzzy continuous,*
(2) *f is fuzzy β-$\mathcal{I}$-continuous and fuzzy $S\mathcal{I}_\beta$-continuous.*

Proof. This is an immediate consequence of Proposition 4.9. □

6. Conclusions

Decomposition of fuzzy continuity has been recently of major interest among general topologists. In section 5, we obtained new decompositions of fuzzy continuity by using weakly fuzzy pre-$\mathcal{I}$-open sets and fuzzy strong $B_{\mathcal{I}}$-sets via fuzzy idealization.

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