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# Injectivity of intuitionistic fuzzy G-modules

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Abstract. In this article, we introduce the notion of injectivity and quasi injectivity of an intuitionistic fuzzy G-module. We also establish a condition on finite dimensional G-modules for which an intuitionistic fuzzy G-module is injective with respect to another intuitionistic fuzzy G-module. We discuss the injectivity of an intuitionistic fuzzy G-module with respect to the quotient intuitionistic fuzzy G-module of another G-module. Some important properties of injectivity of intuitionistic fuzzy G-modules with regards to direct sum of intuitionistic fuzzy G-modules are also studied. A relationship between injectivity and quasi injectivity of intuitionistic fuzzy G-modules is also obtained.

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## 1. Introduction

The concept of injectivity of modules was introduced by Eckmann and Schopf [8]. Banaschewski [5] extended this work of injective and projective modules. After this, many authors tried to generalize the concept of injective modules, for example, Johnson and Wong [14] introduced quasi-injective modules. The notion of M-injective modules was introduced by Azumaya in [4]. Singh [21] introduced the notion of Pseudo injective modules. After the introduction of fuzzy sets by Zadeh [23], the researchers have been carrying out research in various concepts of abstract algebra in fuzzy setting. Rosenfeld [17] was the first one to define the concept of fuzzy subgroups of a group. The literature of various fuzzy algebraic concepts have been growing rapidly. In particular, Nagoita and Ralescu [15] introduced and examined the notion of fuzzy submodule of a module. Zahedi and Ameri [22] studied about the fuzzy projectivity and fuzzy injectivity of modules. Fernadez introduced and studied the notion of fuzzy G-modules in [9] and fuzzy G-modules injectivity in [10].

As an important extension of fuzzy set theory, Atanassov [1, 2, 3] introduced and developed the theory of intuitionistic fuzzy sets. Using the Atanassov′s idea, Biswas [6] established the intuitionistic fuzzification of the concept of subgroup of a group. Later on many mathematicians worked on it and introduced the notion of intuitionistic fuzzy subring, intuitionistic fuzzy submodule etc.(See [11, 12, 13, 16]). The author et al. in [18, 19, 20] have studied intuitionistic fuzzy G-modules ; intuitionistic fuzzy representation, reducibility and complete reducibility of intuitionistic fuzzy G-modules respectively. As a continuation of author's work [18, 19, 20] here the concept of intuitionistic fuzzy G-module injectivity has been introduced and analysed.

## 2. Preliminaries

Here we shall provide some definitions and a few elementary results associated with injectivity of G-modules. Throughout the paper, R and C denote the field of real numbers and field of complex numbers respectively. Unless otherwise stated all G-modules are assumed to be taken over the field K, where K is a subfield of the field of complex numbers and all homomorphisms are G-module homomorphism.

**Definition 2.1** ([7])**.** Let $G$ be a group and let $M$ be a vector space over a field $K$. Then $M$ is called a $G$-module if for every $g \in G$ and $m \in M$, $\exists$ a product(called the action of $G$ on $M$), $gm \in M$ satisfies the following axioms :
(i) $1_G m = m, \forall m \in M$ ($1_G$ being the identity of $G$).
(ii) $(gh)m = g(hm), \forall m \in M, g, h \in G$.
(iii) $g(k_1m_1 + k_2m_2) = k_1(gm_1) + k_2(gm_2), \forall k_1, k_2 \in K; m_1, m_2 \in M$ and $g \in G$.

**Definition 2.2** ([7])**.** Let $G$ be a group and let $M$ be a $G$-module over the field $K$. Let $N$ be a subspace of the vector space $M$ over $K$. Then $N$ is called a $G$-submodule of $M$ if $an_1 + bn_2 \in N$, for all $a, b \in K$ and $n_1, n_2 \in N$.

**Definition 2.3** ([7])**.** Let $M$ and $M^*$ be G-modules. A mapping $f : M \to M^*$ is called a G-module homomorphism if
(i) $f(k_1m_1 + k_2m_2) = k_1f(m_1) + k_2f(m_2)$,
(ii) $f(gm) = gf(m), \forall k_1, k_2 \in K; m, m_1, m_2 \in M$ and $g \in G$.

**Definition 2.4** ([7])**.** A $G$-module $M$ is said to be injective if for any $G$-module $M^*$ and for any $G$-submodule $N^*$ of $M^*$, every monomorphism from $N^*$ into $M$ can be extended to a homomorphism from $M^*$ into $M$.

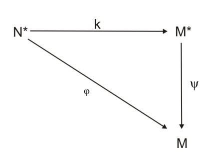


Figure 1. Figure-1

In other words, a G-module $M$ is said to be injective if for all homomorphisms $\varphi : N^* \to M$ and injection $k : N^* \to M^*$, there exists homomorphism $\psi : M^* \to M$ such that $\psi o k = \varphi$ (i.e., injection defined on submodules can be extended to the entire module.i.e., $\psi|_{N*} = \varphi$), that is such that the above diagram commutes.

**Example 2.5** ([10])**.** Let $G = \{1, -1, i, -i\}$ and $M = C$, which is a vector space over $C$. Then $M$ is a $G$ - module with respect to trivial action . Also, we see that no proper subset of $C$ becomes a $G$-module. Let $M^*$ be any other $G$-module. Then following are some prominent cases of $M^*$ :

(i) $M^* = \{0\}$.

(ii) $M^* = C^n (n \geq 1)$ or a $G$-submodule of $C^n$.

(iii) $M^* =$ Space of all functions from any set $S$ into $C$.

(iv) $M^* = C^{m\times n} =$ Space of all $m\times n$ matrices over the field $C$ or a $G$- submodule of $M^*$.

Let $N^*$ be a G-submodule of $M^*$ and $\varphi : N^* \to M$ be a homomorphism.

Case(i): Here $N^* = M^* = \{0\}$. Then $0 = \psi : M^* \to M$ extends the homomorphism $\varphi$.

Case(ii): Since $C^n$ is $n$ dimensional, we have dim$M^* = k \leq n$. Let dim$N^* = m$ and let $\{\alpha_1, \alpha_2, ...., \alpha_m\}$ be a basis of $N^*$ such that $\{\alpha_1, \alpha_2, ...., \alpha_m, \alpha_{m+1}, ..., \alpha_k\}$ be a basis of $M^*$. Then

$$N^* = C\alpha_1 \oplus C\alpha_2 \oplus ...... \oplus C\alpha_m$$

and

$$M^* = C\alpha_1 \oplus C\alpha_2 \oplus .... \oplus C\alpha_m \oplus C\alpha_{m+1} \oplus .... \oplus C\alpha_k.$$

Thus the map $\psi : M^* \to M$ defined by

$$\psi(c_1\alpha_1 + c_2\alpha_2 + ...... + c_k\alpha_k) = \varphi(c_1\alpha_1 + c_2\alpha_2 + ...... + c_m\alpha_m)$$

is a homomorphism which extends $\varphi$.

Case(iii): Here $M^* = M_1 \oplus M_2$, where $M_1$ is the $G$-submodule of $M^*$ consisting of all odd functions and $M_2$ is the $G$-submodule of $M^*$ of all even functions. Then as in Case(ii), there exist a homomorphism $\psi : M^* \to M$ which extends $\varphi$.

Case(iv): Since $C^{m\times n}$ is an $mn$-dimensional vector space over $C$, dim $M^* \leq mn$. Then as in Case(ii), there exist a homomorphism $\psi : M^* \to M$ which extends $\varphi$.

Similarly, for any $G$-module $M^*$ and any $G$-submodule $N^*$ of $M^*$, every homomorphism $\varphi : N^* \to M$ can be extended to a homomorphism $\psi : M^* \to M$. Thus $M$ is injective.

**Definition 2.6** ([7])**.** Let $M$ and $M^*$ be $G$-modules. Then $M$ is $M^*$-injective if for every $G$-submodule $N^*$ of $M^*$, any homomorphism $\varphi : N^* \to M$ can be extended to a homomorphism $\psi : M^* \to M$.

**Remark 2.7.** A $G$-module $M$ is injective if and only if $M$ is $M^*$-injective for every $G$-module $M^*$.

**Example 2.8.** Let $M^* = R^n$, is an $n$-dimensional vector space over $R$.
Let $\{\alpha_1, \alpha_2, ............, \alpha_n\}$ be a basis for $M^*$. Then $M^* = R\alpha_1 \oplus R\alpha_2 \oplus ............ \oplus R\alpha_n$. Let $M = R$ and $G$ be any finite multiplicative subgroup of $R$. Then both $M^*$ and $M$ are $G$-modules. Let $N^*$ be any $G$-submodule of $M^*$ and $\varphi : N^* \to M$ be a homomorphism.

(1) If $N^* = \{0\}$, then $\varphi = 0$. Thus $\psi = 0 : M^* \to M$ extends $\varphi$.
(2) If $N^* = R\alpha_j (1 \le j \le n)$, then $\psi : M^* \to M$ defined by

$$\psi(c_1\alpha_1 + ...... + c_j\alpha_j + ...... + c_n\alpha_n) = \varphi(c_j\alpha_j)$$

is a homomorphism which extends $\varphi$.
(3) $N^* = \oplus_{j=1}^{k} R\alpha_j (k \le n)$, then $\psi : M^* \to M$ defined by

$$\psi(c_1\alpha_1 + ...... + c_j\alpha_j + ...... + c_n\alpha_n) = \varphi(c_1\alpha_1 + ...... + c_k\alpha_k)$$

is a homomorphism which extends $\varphi$. Thus $M$ is $M^*$ -injective.

**Proposition 2.9** ([10])**.** *Let $M = M_1 \oplus M_2$, where $M_1$ and $M_2$ are G-submodules of $M$. Then $M$ is injective if and only if $M_1$ and $M_2$ are both injective.*

*Proof.* Let M be injective. Let $M^*$ be G-module and $N^*$ be any G-submodule of $M^*$. Then the monomorphism $\varphi : N^* \to M$ can be extended to homomorphism $\psi : M^* \to M$ such that $\psi o k = \varphi$, where $k : N^* \to M^*$ is an injection homomorphism. Let $\pi_1 : M \to M_1$ and $\pi_2 : M \to M_2$ be the projection mappings. Then $\psi_1 = \pi_1 o \psi :$

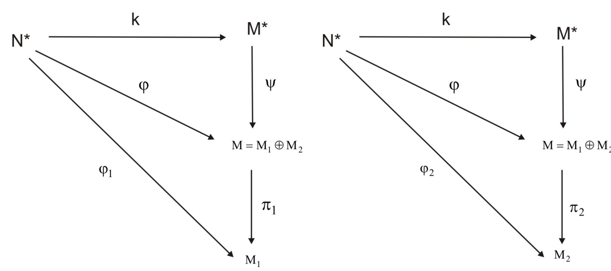


FIGURE 2. Figure-2

$M^* \to M_1$ is an extension of $\varphi_1 = \pi_1 o \varphi : N^* \to M_1$ for
$\psi_1 o k = (\pi_1 o \psi) o k = \pi_1 o (\psi o k) = \pi_1 o \varphi = \varphi_1$ . Thus $M_1$ is injective.

Similarly, $\psi_2 = \pi_2 \circ \psi : M^* \to M_2$ is an extension of $\varphi_2 = \pi_2 \circ \varphi : N^* \to M_2$. So $M_2$ is injective.

Conversely, suppose both $M_1$ and $M_2$ are injective. Let $M^*$ be a G-module and $N^*$ be any G-submodule of $M^*$ and let $\varphi : N^* \to M$ be a homomorphism. Let $\pi_1$ and $\pi_2$ be the projections of $M_1$ and $M_2$ on M respectively.
Then $\varphi_1 = \pi_1 o \varphi : N^* \to M_1$ and $\varphi_2 = \pi_2 o \varphi : N^* \to M_2$. Since both $M_1$ and

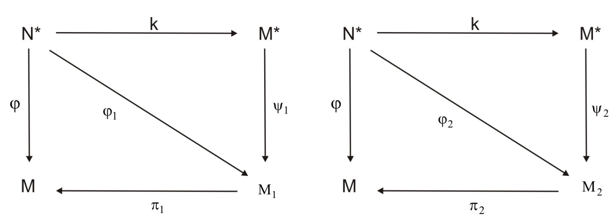


FIGURE 3. Figure-3

$M_2$ are injective, the mappings $\psi_1$ and $\psi_2$ can be extended to homomorphisms $\psi_1 : M^* \to M_1$ and $\psi_2 : M^* \to M_2$ respectively such that $\psi_1 o k = \varphi_1$ and $\psi_2 o k = \varphi_2$.

Define $\psi : M^* \to M$ by $\psi(m) = \psi_1(m) + \psi_2(m)$ , $\forall\ m \in M^*$.
Then $\psi$ is a homomorphism. Also for any $m \in M$, we have

$$\begin{aligned}(\psi ok)(m) &= \psi(k(m)) = \psi_1(k(m)) + \psi_2(k(m))\\ &= (\psi_1 ok)(m) + (\psi_2 ok)(m) = \varphi_1(m) + \varphi_2(m)\\ &= (\varphi_1 + \varphi_2)(m) = \varphi(m).\end{aligned}$$

Thus $\psi ok = \varphi$. So $\psi$ is an extention of $\varphi$. Hence $M$ is injective. □

**Corollary 2.10.** *$M = \oplus_{i=1}^{n} M_i$ is injective if and only if $M_i$ is injective, for every $i$ =1,2,....,n.*

**Proposition 2.11** ([10])**.** *Let M and $M^*$ be G-modules such that M is $M^*$-injective. If $N^*$ is a G-submodule of $M^*$, then M is $N^*$-injective and M is $M^*/N^*$-injective.*

*Proof.* Since $N^* \subseteq M^*$ and $M$ is $M^*$-injective, it is obvious that $M$ is $N^*$-injective. Let $X^*/N^*$ be a G-submodule of $M^*/N^*$ and $\varphi : X^*/N^* \to M$ be a homomorphism.

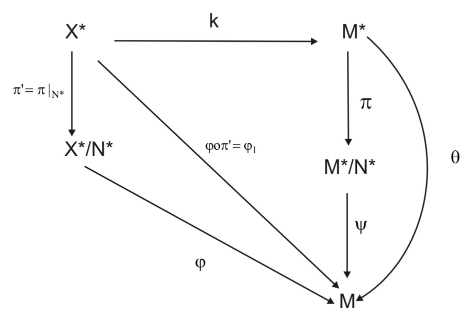


FIGURE 4. Figure-4

Let $\pi : M^* \to M^*/N^*$ be the canonical map, which is given by $\pi(x) = x + N^*$, $\forall x \in M^*$ and $\pi' = \pi|_{X^*} : X^* \to M^*/N^*$. Then $\varphi_1 = \varphi o \pi' : X^* \to M$. Since $M$ is $M^*$-injective, $\exists$ an extension $\theta : M^* \to M$ of $\varphi_1$. Thus, we have

$$\theta(N^*) = \varphi_1(N^*) = (\varphi o \pi')(N^*) = \varphi(\pi'(N^*)) = \varphi(0) = 0.$$

So Ker $\pi$ is a G-submodule of Ker $\theta$. Hence $\exists'$s a map $\psi : M^*/N^* \to M$ such that $\psi o \pi = \theta$. Also for any $x \in X^*$, we have

$$\psi(x + N^*) = \psi(\pi(x)) = (\psi o \pi)(x) = \theta(x) = (\varphi o \pi')(x) = (\varphi(\pi'(x)) = \varphi(x + N^*).$$

Therefore $\psi$ extends $\varphi$ and thus $M$ is $M^*/N^*$-injective. □

**Definition 2.12** ([4])**.** A G-module M is called quasi-injective, or self-injective, when it is M- injective.

For example, injective modules and semisimple (or completely reducible) modules are quasi-injective and direct summands of quasi-injective modules are also quasi-injective.

**Example 2.13.** Let S = $\{1, \omega, \omega^2\}$, where $\omega$ is a complex cube root of unity and $G = S_3$, the symmetric group of degree three. Let $M = \{\alpha + \beta\omega + \gamma\omega^2 : \alpha, \beta, \gamma \in R\}$. Then M is a vector space over R spanned by S. For each $x \in G$, define $T_x : M \to M$ by

$$T_x(\alpha + \beta\omega + \gamma\omega^2) = \alpha x(1) + \beta x(\omega) + \gamma x(\omega^2).$$

Then $T_x$ is an isomorphism of M onto itself.

Also the map $T : G \to GL(M)$ defined by $T(x) = T_x$, $\forall\, x \in G$, is a representation of G. Thus M is a G-module. Also the only G-submodules of M are M and $\{0\}$.

We will show that M is M-injective. Let N be any G-submodule of M. Then $N = \{0\}$ or N = M. Let $\varphi : N \to M$ be any homomorphism.

Case(i): Suppose$N = \{0\}$. Then the map $\psi : M \to M$ defined by $\psi(x) = 0\ \forall\ x \in M$ extends $\varphi$.

Case(ii): Suppose N = M. Then $\varphi$ is a homomorphism from M into itself. Thus $\psi = \varphi$ is the required extension.

So, in both cases, $\varphi : N \to M$ can be extend to a homomorphism $\psi : M \to M$. Hence M is M-injective. Therefore M is quasi-injective.

**Proposition 2.14.** *Any finite dimensional quasi- injective G- module is the direct sum of quasi injective G-submodules.*

*Proof.* Let M be a finite dimensional G-module. Then M is completely reducible. Thus M is a direct sum of irreducible G-submodules. Let $M = M_1 \oplus M_2 \oplus .........\oplus M_n$, where $M_i (1 \leq i \leq n)$ be irreducible G-submodules of M. If M is quasi-injective, then by Propositions 2.9 and 2.11, $M_i\ '$s are quasi-injective. Thus M is direct sum of quasi-injective G-submodules. □

**Definition 2.15** ([4])**.** Let $M$ and $M^*$-injective be G-modules. Then $M$ and $M^*$ are relatively injective if $M$ is $M^*$- injective and $M^*$ is $M$-injective.

**Example 2.16.** Let $G = \{1, -1\}$, $M = Q\sqrt{3}$ and $M^* = Q\sqrt{5}$ are vector spaces over $Q$. Also both $M$ and $M^*$ are G-modules. It can be easily proved that $M$ is $M^*$-injective and $M^*$ is $M$-injective and hence $M$ and $M^*$ are relatively injective.

## 3. Intuitionistic fuzzy G-module injectivity

In this section we extend the notion of injectivitity of G-modules to injectivity of intuitionistic fuzzy G-modules. Here homomorphism means G-homomorphism.

**Definition 3.1** ([18])**.** Let G be a group and let M be a G-module over K, which is a subfield of C. Then an intuitionistic fuzzy G-module on M is an intuitionistic fuzzy set A = $(\mu_A, \nu_A)$ such that following conditions are satisfied :

(i) $\mu_A(ax + by) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(ax + by) \leq \nu_A(x) \vee \nu_A(y), \forall a, b \in$K and $x, y \in$M.

(ii) $\mu_A(gm) \geq \mu_A(m)$ and $\nu_A(gm) \leq \nu_A(m), \forall g \in$ G; $m \in$M.

**Remark 3.2.** If A is an intuitionistic fuzzy G-module of a G-module M, then

$$\mu_A(0) \geq \mu_A(x) \text{ and } \nu_A(0) \leq \nu_A(x),\ \forall x \in M.$$

**Example 3.3** ([18])**.** Let G = $\{1, -1\}$, M = $R^n$ over R. Then M is a G-module. Define an intuitionistic fuzzy set A = $(\mu_A, \nu_A)$ on M by

$$\mu_A(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0.5 & \text{if } x \neq 0 \end{cases} ; \nu_A(x) = \begin{cases} 0 & \text{if } x = 0 \\ 0.25 & \text{if } x \neq 0, \end{cases}$$

where $x = (x_1, x_2, ....., x_n) \in R^n$. Then A is an intuitionistic fuzzy G-module on M.

**Proposition 3.4.** *Let $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy G-module on $M$. Then for each $r \in [0,1]$ the intuitionistic fuzzy set $A_r = (\mu_{A_r}, \nu_{A_r})$ defined by*

$$\mu_{A_r}(x) = \mu_A(x) \wedge r \text{ and } \nu_{A_r}(x) = \nu_A(x) \vee (1-r), \forall\, x \in M$$

*is an intuitionistic fuzzy G-module on $M$.*

*Proof.* Let $x, y \in M, a, b \in K$. Then

$$\begin{aligned} \mu_{A_r}(ax+by) &= \mu_A(ax+by) \wedge r \\ &\geq \mu_A(x) \wedge \mu_A(y) \wedge r \\ &= (\mu_A(x) \wedge r) \wedge (\mu_A(y) \wedge r) \\ &= \mu_{A_r}(x) \wedge \mu_{A_r}(y). \end{aligned}$$

Similarly, $\nu_{A_r}(ax+by) \leq \nu_{A_r}(x) \vee \nu_{A_r}(y)$.
For any $g \in G, x \in M$, we have

$$\begin{aligned} \mu_{A_r}(gx) &= \mu_A(gx) \wedge r \\ &\geq \mu_A(x) \wedge r \\ &= \mu_{A_r}(x). \end{aligned}$$

Similarly, $\nu_{A_r}(gx) \leq \nu_{A_r}(x)$.
Thus $A_r$ is an intuitionistic fuzzy G-module on M. □

**Proposition 3.5** ([18]). *Let $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy G-module of a G-module M and let N be a G-submodule of M. Then the restriction of A on N is an intuitionistic fuzzy set denoted by $A|_N = (\mu_{A|_N}, \nu_{A|_N})$ and is defined by*

$$\mu_{A|_N}(x) = \mu_A(x) \text{ and } \nu_{A|_N}(x) = \nu_A(x), \forall\, x \in N,$$

*is an intuitionistic fuzzy G-module on N.*

**Proposition 3.6.** *Let $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy G-module of a G-module M and let N be a G-submodule of M. Then the intuitionistic fuzzy set $A_N = (\mu_{A|_N}, \nu_{A|_N})$ of $M/N$ defined by*

$$\mu_{A_N}(x+N) = \mu_A(x) \text{ and } \nu_{A_N}(x+N) = \nu_A(x), \forall\, x \in M,$$

*is an intuitionistic fuzzy G-module on $M/N$.*

*Proof.* For $x+N, y+N \in M/N, g \in G$ and scalar $a, b \in K$, we have

$$\begin{aligned} \mu_{A_N}\{a(x+N)+b(y+N)\} &= \mu_{A_N}\{(ax+by)+N\} \\ &= \mu_A(ax+by) \\ &\geq \mu_A(x) \wedge \mu_A(y) \\ &\geq \mu_{A_N}(x+N) \wedge \mu_{A_N}(y+N). \end{aligned}$$

Similarly, $\nu_{A_N}\{a(x+N)+b(y+N)\} \leq \nu_{A_N}(x+N) \vee \nu_{A_N}(y+N)$ and

$$\begin{aligned} \mu_{A_N}[g(x+N)] &= \mu_{A_N}(gx+N) \\ &= \mu_A(gx) \\ &\geq \mu_A(x) \\ &= \mu_{A_N}(x+N). \end{aligned}$$

Similarly, $\nu_{A_N}[g(x+N)] \leq \nu_{A_N}(x+N)$.
Thus $A_N$ is an intuitionistic fuzzy G-module on $M/N$. □

**Proposition 3.7** ([18]). *Let M be a G-module and let A be an intuitionistic fuzzy set on M, then A is an intuitionistic fuzzy G-module on M if and only if either* $C_{(\alpha,\beta)}(A) = \varnothing$ *or* $C_{(\alpha,\beta)}(A)$*, for all* $\alpha, \beta \in [0,1]$ *such that* $\alpha + \beta \leq 1$*, is a G-submodule of M, where* $C_{(\alpha,\beta)}(A) = \{x \in M : \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta\}$.

**Theorem 3.8** ([20]). *Consider a maximal chain of submodules of G-module M*

$$M_0 \subset M_1 \subset M_2 \subset ........ \subset M_n = M \ ,$$

*where* $\subset$ *denotes proper inclusion. Then there exists an intuitionistic fuzzy G-module A of M given by*

$$\mu_A(x) = \begin{cases} \alpha_0 & \text{if } x \in M_0 \\ \alpha_1 & \text{if } x \in M_1 \backslash M_0 \\ \alpha_2 & \text{if } x \in M_2 \backslash M_1 \\ ........... & \\ \alpha_n & \text{if } x \in M_n \backslash M_{n-1} \end{cases} ; \nu_A(x) = \begin{cases} \beta_0 & \text{if } x \in M_0 \\ \beta_1 & \text{if } x \in M_1 \backslash M_0 \\ \beta_2 & \text{if } x \in M_2 \backslash M_1 \\ ........... & \\ \beta_n & \text{if } x \in M_n \backslash M_{n-1}, \end{cases}$$

*where* $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq ...... \geq \alpha_n$ *and* $\beta_0 \leq \beta_1 \leq \beta_2 \leq ...... \leq \beta_n$ *;* $\alpha_i, \beta_i \in [0,1]$ *such that* $\alpha_i + \beta_i \leq 1$*,* $\forall\ i = 0, 1, 2, ..., n$.

**Remark 3.9** ([20]). The converse of the above theorem is also true i.e., any intuitionistic fuzzy G-module A of a G-module M can be expressed in the above form.

**Definition 3.10** ([20]). If $A = (\mu_A, \nu_A)$ and B $= (\mu_B, \nu_B)$ be intuitionistic fuzzy G-module of G-module $M$ and $M^*$ respectively. A function $f : M \to M^*$ is said to be a function from $A$ to $B$ if $\mu_B o f = \mu_A$ and $\nu_B o f = \nu_A$. Further, if $f$ is a G-module

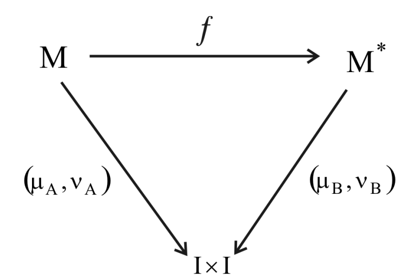


FIGURE 5. Figure-5

homomorphism or G-epimorphism or G-isomorphism, from $M$ to $M^*$, then $f$ is said to be intuitionistic fuzzy G-module homomorphism or G-epimorphism or G-isomorphism from A to B.

Suppose $M$ and $M^*$ be G-modules and let $M$ be $M^*$-injective. Then for every monomorphism $\varphi : N^* \to M$ and injection $k : N^* \to M^*$, there exists homomorphism $\psi : M^* \to M$ such that $\psi o k = \varphi$ , where $N^*$ is a G-submodule of G-module $M^*$. In other words the map $\varphi$ extends to $\psi$, i.e., $\psi|_{N^*} = \varphi$.

If $A = (\mu_B, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy G-modules of $M$ and $M^*$ respectively and $(\mu_{B|_{N^*}}, \nu_{B|_{N^*}})$ be the intuitionistic fuzzy G-module of $N^*$. Then A is said to be $B$-injective if the following diagram is commutative. That is,

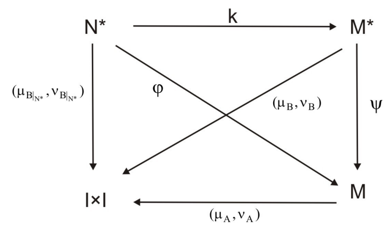


FIGURE 6. Figure-6

$\mu_{B|_{N^*}} = \mu_B ok$ and $\nu_{B|_{N^*}} = \nu_B ok$ ; $\mu_B = \mu_A o\psi$ and $\nu_B = \nu_A o\psi$ ;
$\mu_{B|_{N^*}} = \mu_A o\varphi$ and $\nu_{B|_{N^*}} = \nu_A o\varphi$ .
Thus, we notice that

$$\mu_B(k(m)) = (\mu_B ok)(m) = \mu_{B|_{N^*}}(m) = (\mu_A o\varphi)(m)$$
$$= \mu_A(\varphi(m)) = \mu_A((\psi ok)(m)) = \mu_A(\psi(k(m)))$$

and

$$\nu_B(k(m)) = (\nu_B ok)(m) = \nu_{B|_{N^*}}(m) = (\nu_A o\varphi)(m)$$
$$= \nu_A(\varphi)(m)) = \nu_A((\psi ok)(m)) = \nu_A(\psi(k(m))).$$

So, $\mu_B(k(m)) = \mu_A(\psi(k(m)))$ and $\nu_B(k(m)) = \nu_A(\psi(k(m)))$.
If $m \in N^*$, then $k(m) = m$. Thus $\mu_B(m) = \mu_A(\psi(m))$ and $\nu_B(m) = \nu_A(\psi(m))$.
If $m \in M^* \setminus N^*$, then

$$\mu_B(k(m)) = 0 \leq \mu_A(\psi(m))$$

and

$$\nu_B(k(m)) = 1 \geq \nu_A(\psi(k(m))).$$

Hence $\forall \psi \in Hom(M^*, M)$ and $m \in M^*$,

$$\mu_B(m) \leq \mu_A(\psi(m)) \text{ and } \nu_B(m) \geq \nu_A(\psi(m)).$$

Now, we are ready to define the injectivity of intuitionistic fuzzy G-module.

**Definition 3.11.** Let $M$ and $M^*$ be G-modules. Let $A = (\mu_A, \nu_A)$ be any intuitionistic fuzzy G-module on $M$ and $B = (\mu_B, \nu_B)$ be any intuitionistic fuzzy G-module on $M^*$. Then $A$ is $B$-injective if
(i) $M$ is $M^*$- injective and
(ii) $\mu_B(m) \leq \mu_A(\psi(m))$ and $\nu_B(m) \geq \nu_A(\psi(m))$, $\forall\ \psi \in Hom(M^*, M)$ and $m \in M^*$.

**Example 3.12.** Let G $= \{1, -1, i, -i\}$, $M = C$ and $M^* = Q(i)$. Then $M$ and $M^*$ are G-modules over $Q$. Define intuitionistic fuzzy sets $A$ and $B$ of $M$ and $M^*$ respectively as follows:

$$\mu_A(x+iy) = \begin{cases} 1, & \text{if } x = 0 \\ 1/2, & \text{if } x \in Q(i) - 0 \\ 1/4, & \text{if } x \in C - Q(i) \end{cases}, \quad \nu_A(x+iy) = \begin{cases} 0, & \text{if } x = 0 \\ 1/4, & \text{if } x \in Q(i) - 0 \\ 1/2, & \text{if } x \in C - Q(i). \end{cases}$$

$$\mu_B(x) = \begin{cases} 1/4, & \text{if } x = 0 \\ 1/5, & \text{if } x \neq 0 \end{cases}, \quad \nu_B(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1/4, & \text{if } x \neq 0. \end{cases}$$

Then, by Theorem 3.6, $A$ and $B$ are intuitionistic fuzzy G-modules on $M$ and $M^*$ respectively. Let $X$ be any G-submodule of $M^*$. Then either $X = 0$ or $X = M^*$. Let $\varphi : X \to M$ be any homomorphism.

Case (i): If $X = 0$, then $\varphi = 0$. Thus $\psi = 0 : M^* \to M$ extends $\varphi$.

Case (ii): If $X = M^*$, then $\psi = \varphi$ extends $\varphi$.

Thus $M$ is $M^*$-injective. Also it follows from the definition of $A$ and $B$ that
$\mu_B(m) \leq \mu_A(\psi(m))$ and $\nu_B(m) \geq \nu_A(\psi(m))$ $\forall \psi \in Hom(M^*, M)$ and $m \in M^*$.
So $A$ is $B$-injective.

**Proposition 3.13.** *Let $M$ and $M^*$ be G-modules such that $M$ is finite dimensional and $M$ is $M^*$-injective. Let $\{\beta_1, \beta_2, ...., \beta_n\}$ be a basis for $M$. If $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are intuitionistic fuzzy G-module of $M$ and $M^*$ respectively such that $\mu_B(m) \leq min\{\mu_A(\beta_j) : j = 1, 2, 3, ..., n\}$ and $\nu_B(m) \geq max\{\nu_A(\beta j) : j = 1, 2, 3, ...., n\}$ for all $m \in M^*$ respectively. Then $A$ is $B$-injective.*

*Proof.* Let $A$ be an intuitionistic fuzzy of G-module $M$. Let $x, y \in M$ and $a, b \in K$, then

$$\mu_A(ax + by) \geq \mu_A(x) \wedge \mu_A(y) \text{ and } \nu_A(ax + by) \leq \nu_A(x) \vee \nu_A(y). \tag{3.13.1}$$

As $M$ is $M^*$-injective G-module and let $\psi \in Hom(M^*, M)$ be any G-homomorphism. For any $m \in M^*, \psi(m) \in M$. Thus $\psi(m) = \alpha_1\beta_1 + \alpha_2\beta_2 + .... + \alpha_n\beta_n$, where $\alpha_i \in K$ and $\beta_i \in M$. So, from our assumption and (3.13.1), we have
$\mu_A(\psi(m)) = \mu_A(\alpha_1\beta_1 + \alpha_2\beta_2 + ..... + \alpha_n\beta_n) \geq min\{\mu_A(\beta_j) : j = 1, 2, ...n\} \geq \mu_B(m)$
and
$\nu_A(\psi(m)) = \nu_A(\alpha_1\beta_1 + \alpha_2\beta_2 + ..... + \alpha_n\beta_n) \leq max\{\nu_A(\beta_j) : j = 1, 2, ...n\} \leq \nu_B(m)$.
Hence $\mu_A(\psi(m)) \geq \mu_B(m)$ and $\nu_A(\psi(m)) \leq \nu_B(m)$, $\forall\ \psi \in Hom(M^*, M)$ and $m \in M^*$. Therefore $A$ is $B$-injective. □

**Example 3.14.** Let $G = \{1, -1\}, M = Q(2^{1/2}, 2^{1/3})$ and $M^* = Q(i)$. Then $M$ and $M^*$ are G-modules over $Q$. Since G-module $M^*$ and $M$ are finite dimensional over $Q$ and the sets $\{\alpha_1 = 1, \alpha_2 = 2^{1/2}, \alpha_3 = 2^{1/3}, \alpha_4 = 2^{5/6}, \alpha_5 = 2^{2/3}, \alpha_6 = 2^{7/6}\}$ and $\{1, i\}$ are basis of $M$ and $M^*$ respectively. Thus, as in Case (ii) of Example 2.5, we have $M$ is $M^*$ - injective.

Now we define intuitionistic fuzzy sets $A$ on $M^*$ and $B$ on $M$ by

$$\mu_A(x + iy) = \begin{cases} 1/8, & \text{if } x = y = 0, \forall x, y \in Q \\ 1/9, & \text{if } x \neq 0, y = 0 \\ 1/10, & \text{if } y \neq 0 \end{cases}$$

$$\nu_A(x + iy) = \begin{cases} 1/10, & \text{if } x = y = 0, \forall x, y \in Q \\ 1/9, & \text{if } x \neq 0, y = 0 \\ 1/8, & \text{if } y \neq 0. \end{cases}$$

$$\mu_B(c_1\alpha_1 + c_1\alpha_2, ...., +c_6\alpha_6) = \begin{cases} 1, & \text{if } c_i = 0 \forall i \\ 1/2, & \text{if } c_1 \neq 0, c_2, c_3 = ......c_6 = 0 \\ 1/3, & \text{if } c_2 \neq 0, c_3, c_4 = ......c_6 = 0 \\ 1/4, & \text{if } c_3 \neq 0, c_4, c_5, c_6 = 0 \\ 1/5, & \text{if } c_4 \neq 0, c_5, c_6 = 0 \\ 1/6, & \text{if } c_5 \neq 0, c_6 = 0 \\ 1/7, & \text{if } c_6 \neq 0 \end{cases}$$

and

$$\nu_B(c_1\alpha_1 + c_1\alpha_2, ...., +c_6\alpha_6) = \begin{cases} 0, & \text{if } c_i = 0 \forall i \\ 1/7, & \text{if } c_1 \neq 0, c_2, c_3 = ......c_6 = 0 \\ 1/6, & \text{if } c_2 \neq 0, c_3, c_4 = ......c_6 = 0 \\ 1/5, & \text{if } c_3 \neq 0, c_4, c_5, c_6 = 0 \\ 1/4, & \text{if } c_4 \neq 0, c_5, c_6 = 0 \\ 1/3, & \text{if } c_5 \neq 0, c_6 = 0 \\ 1/2, & \text{if } c_6 \neq 0. \end{cases}$$

Then $A$ and $B$ are intuitionistic fuzzy G-modules on $M^*$ and $M$ respectively.
Also from the definition of $A$ and $B$, we have
$\mu_B(m) \leq min\{\mu_A(\alpha_j) : j = 1, 2, ..., n\}$ and $\nu_B(m) \geq max\{\nu_A(\alpha_j) : j = 1, 2, ..., n\}$; $\forall m \in M^*$ and so,
$\mu_B(m) \leq \mu_A(\psi(m))$ and $\nu_B(m) \geq \nu_A(\psi(m)) \ \forall \ \psi \in Hom(M^*, M)$ and $m \in M^*$.
Hence $A$ is $B$-injective.

**Proposition 3.15.** *Let $M$ and $M^*$ be G-modules and $A, B$ be intuitionistic fuzzy G-module on $M$ and $M^*$ respectively such that $A$ is $B$-injective. If $N^*$ is a G-submodule of $M^*$ and $C$ is an intuitionistic fuzzy G-module on $N^*$, then $A$ is $C$-injective if $C \subseteq B|_{N^*}$.*

*Proof.* Since $M$ is $M^*$-injective and $N^*$ is a G-submodule of $M^*$, by Proposition 2.11, $M$ is $N^*$-injective. Let $\psi \in Hom(N^*, M)$. Since $M$ is $M^*$-injective, there exist an extension homomorphism $\varphi : M^* \to M$. Thus $\psi = \varphi|_{N^*}$. Since $A$ is $B$-injective, we have $\forall \ n \in N^*$,

(3.15.1) $$\mu_B(n) \leq \mu_A(\varphi(n)) = \mu_A(\psi(n)) \text{ and } \nu_B(n) \geq \mu_A(\varphi(n)) = \nu_A(\psi(n)).$$

Since $C \subseteq B|_{N^*}$,

(3.15.2) $$\mu_C(n) \leq \mu_B(n) \text{ and } \nu_C(n) \geq \nu_A(\psi(n)), \forall n \in N^*.$$

So, from (3.15.1) and (3.15.2), we have,

$$\mu_C(n) \leq \mu_A(\psi(n)) \text{ and } \nu_C(n) \geq \nu_A(\psi(n)) \ \forall \ \psi \in Hom(N^*, M) \text{ and } n \in N^*.$$

Hence $A$ is $C$-injective. □

**Proposition 3.16.** *Let $M$ and $M^*$ be G-modules and let $A$ and $B$ be intuitionistic fuzzy G-modules on $M$ and $M^*$ respectively such that $A$ is $B$-injective. Then for every $r \in [0, 1]$, the intuitionistic fuzzy set $B_r = (\mu_{B_r}, \nu_{B_r})$ where*

$$\mu_{B_r}(m) = \mu_B(m) \wedge r \text{ and } \nu_{B_r}(m) = \nu_B(m) \vee (1-r),\ \forall\ m \in M^*;$$

*is an intuitionistic fuzzy G-module on $M^*$ and $A$ is $B_r$-injective.*

*Proof.* It follows from Proposition 3.4 and Theorem 3.15 [Here $N^* = M^*$ and $B_r \subseteq B$]. □

**Proposition 3.17.** *Let $A$ and $B$ be intuitionistic fuzzy G-modules on the G-modules $M$ and $M^*$ respectively such that $A$ is $B$-injective. For any G-submodule $N^*$ of $M^*$, define the intuitionistic fuzzy set $B_{N^*}$ on $M^*/N^*$ by*

$$\mu_{B_{N^*}}(m + N^*) = \mu_B(m) \text{ and } \nu_{B_{N^*}}(m + N^*) = \nu_B(m),\ \forall\ m \in M^*.$$

*Then $B_{N^*}$ is an intuitionistic fuzzy G-module on $M^*/N^*$ and $A$ is $B_{N^*}$ -injective.*

*Proof.* It follows from Proposition 3.6 that $B_{N^*}$ is an intuitionistic fuzzy G-module on $M^*/N^*$. Since $N^*$is a G-submodule of $M^*$, from Proposition 2.13, $M$ is $M^*/N^*$-injective. Let $\varphi \in Hom(M^*/N^*, M)$. Since $M$ is $M^*$-injective, there exist an extension $\theta \in Hom(M^*, M)$. Since $A$ is $B$-injective and $\theta \in Hom(M^*, M)$, we have

$$\mu_B(m) \leq \mu_A(\theta(m)) \text{ and } \nu_B(m) \geq \nu_A(\theta(m)), \forall m \in M^*. \tag{3.17.1}$$

For any $m \in M^*, m + N^* \in M^*/N^*$, we have

$$\begin{aligned} \mu_A(\theta(m + N^*)) &= \mu_A(\theta(m) + 0) \\ &= \mu_A(1.\theta(m) + 1.0) \\ &\geq \mu_A(\theta(m)) \wedge \mu_A(0) \\ &\geq \mu_A(\theta(m)). \end{aligned}$$

Similarly,

$$\begin{aligned} \nu_A(\theta(m + N^*)) &= \nu_A(\theta(m) + 0) \\ &= \nu_A(1.\theta(m) + 1.0) \\ &\leq \nu_A(\theta(m)) \vee \nu_A(0) \\ &\leq \nu_A(\theta(m)). \end{aligned}$$

Thus,

$$\mu_A(\theta(m + N^*)) \geq \mu_A(\theta(m)) \text{ and } \nu_A(\theta(m + N^*)) \leq \nu_A(\theta(m)). \tag{3.17.2}$$

Also,

$$\begin{aligned} \mu_{B_{N^*}}(m + N^*) &= \mu_B(m) \\ &\leq \mu_A(\theta(m))[\text{By (3.17.1)}] \\ &\leq \mu_A(\theta(m + N^*))[\text{By (3.17.2)}] \\ &\leq \mu_A(\varphi(m + N^*)), \forall \varphi \in Hom(M^*/N^*, M) \end{aligned}$$

and

$$\begin{aligned} \nu_{B_{N^*}}(m + N^*) &= \nu_B(m) \\ &\geq \nu_A(\theta(m))[By1] \\ &\geq \nu_A(\theta(m + N^*))[\text{By (3.17.2)}] \\ &\geq \nu_A(\varphi(m + N^*)), \forall \varphi \in Hom(M^*/N^*, M). \end{aligned}$$

So, $\forall\ \varphi \in \text{Hom}(M^*/N^*, M)$,

$$\mu_{B_{N^*}}(m + N^*) \leq \mu_A(\varphi(m + N^*)) \text{ and } \nu_{B_{N^*}}(m + N^*) \geq \nu_A(\varphi(m + N^*)).$$

Hence $A$ is $B_{N^*} - injective$. □

**Definition 3.18.** Let M be a G-module and let A be an intuitionistic fuzzy G-module on M. Then A is quasi-injective if

(i) M is quasi-injective and

(ii) $\mu_A(m) \leq \mu_A(\psi(m))$ and $\nu_A(m) \geq \nu_A(\psi(m)), \forall \psi \in Hom(M, M)$ and $m \in M$.

**Example 3.19.** Every constant intuitionistic fuzzy set defined on a quasi-injective G-module M is always quasi-injective module i.e., if M is a quasi-injective G-module. Then an intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ on M defined by

$$\mu_A(x) = r \text{ and } \nu_A(x) = s, \forall\ x \in M, \text{ where } r, s \in [0, 1] \text{ such that } r + s \leq 1,$$

is quasi-injective.

**Example 3.20.** We have, the G-module M as defined in Example 2.13 is quasi-injective. On this M, If we define an intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = 0 \\ r, & \text{if } x \neq 0. \end{cases} \quad , \nu_A(x) = \begin{cases} 0, & \text{if } x = 0 \\ s, & \text{if } x \neq 0 \end{cases}$$

$\forall x \in M$, where $r, s \in [0, 1]$ such that $r + s \leq 1$, is quasi-injective.

## 4. Properties of intuitionistic fuzzy G-module injectivity

**Proposition 4.1** ([20]). *Let $M$ be a G-module and let $M = \oplus_{i=1}^n M_i$ , where $M_i^{'} s$ are G-submodules of $M$. If $A_i^{'} s$ $(1 \leq i \leq n)$ are intuitionistic fuzzy G- modules on $M_i^{'} s$, then an intuitionistic fuzzy set $A$ of $M$ defined by*
*$\mu_A(m) = \wedge\{\mu_{A_i}(m_i) : i = 1, 2, ......, n\}$ and $\nu_A(m) = \vee\{\nu_{A_i}(m_i) : i = 1, 2, ......, n\}$, where $m = \sum_{i=1}^n m_i \in M$, is an intuitionistic fuzzy G-module on M.*

**Definition 4.2** ([20]). An intuitionistic fuzzy G-module A on $M = \oplus_{i=1}^n M_i$, in Proposition 4.1 with $\mu_A(0) = \mu_{A_i}(0)$ and $\nu_A(0) = \nu_{A_i}(0)$, $\forall\ i$, is called the direct sum of $A_i$ and it is written as $A = \oplus_{i=1}^n A_i$.

**Theorem 4.3.** *Let $M$ be a G-module such that $M = \oplus_{i=1}^n M_i$ , where $M_i^{'} s$ are G-submodules of $M$. Let $B_i^{'} s$ be intuitionistic fuzzy G-modules on $M_i$ and let $B = \oplus_{i=1}^n B_i$. Let $A$ be any intuitionistic fuzzy G-modules on $M$. Then $A$ is $B$-injective if and only if $A$ is $B_i$-injective for all $i$ $(1 \leq i \leq n)$.*

*Proof.* (⇒): Assume that A is B-injective. Then

(i) $M$ is $M = \oplus_{i=1}^n M_i$-injective and

(ii) $\mu_B(m) \leq \mu_A(\psi(m))$ and $\nu_B(m) \geq \nu_A(\psi(m))$, $\forall\ \psi \in Hom(M, M)$ and $m \in M$.

To prove that $A$ is $B_i$-injective for all $i(1 \leq i \leq n)$, i.e., to prove that

(a) $M$ is $M_i$-injective and

(b) $\mu_{B_i}(m_i) \leq \mu_A(\psi(m_i))$ and $\nu_{B_i}(m_i) \geq \nu_A(\psi(m_i))$, $\forall\ \psi \in \text{Hom}(M_i, M)$ $\forall$ $m_i \in M_i$.

Proof of (a): Since $M_i$ is a G-submodule of $M$, from Proposition 2.11, it follows that $M$ is $M_i$-injective.

Proof of (b): Let $\psi \in Hom(M_i, M)$ and let $m_i \in M_i$. Then

$$m_i = 0 + 0 + ...... + m_i + 0 + ...... + 0.$$

Thus

$$\begin{aligned} \mu_B(m_i) &= \mu_B(0 + 0 + ...... + m_i + 0 + ...... + 0) \\ &= \mu_{B_1}(0) \wedge \mu_{B_2}(0) \wedge ...... \wedge \mu_{B_i}(m_i) \wedge ..... \wedge \mu_{B_n}(0) \\ &= \mu_{B_i}(m_i)[\because \mu_{B_i}(0) \geq \mu_{B_i}(m_i), \forall i] \end{aligned}$$

and

$$\begin{aligned} \nu_B(m_i) &= \nu_B(0 + 0 + ...... + m_i + 0 + ...... + 0) \\ &= \nu_{B_1}(0) \vee \nu_{B_2}(0) \vee ...... \vee \nu_{B_i}(m_i) \vee ..... \vee \nu_{B_n}(0) \\ &= \nu_{B_i}(m_i)[\because \nu_{B_i}(0) \leq \nu_{B_i}(m_i), \forall i]. \end{aligned}$$

Since $M$ is $M$-injective, $\exists$ an extension $\varphi : M \longrightarrow M$ of $\psi$. So, for each $m_i \in M_i$,

$$\mu_{B_i}(m_i) = \mu_B(m_i) \leq \mu_A(\varphi(m_i)) \leq \mu_A(\psi(m_i))$$

and

$$\nu_{B_i}(m_i) = \nu_B(m_i) \geq \nu_A(\varphi(m_i)) \geq \nu_A(\psi(m_i))[\text{by (ii)}].$$

Hence $\mu_{B_i}(m_i) \leq \mu_A(\psi(m_i))$ and $\nu_{B_i}(m_i) \geq \nu_A(\psi(m_i))$ $\forall$ $\psi \in \text{Hom}(M_i, M)$. Therefore, $A$ is $B_i$-injective for all $i(1 \leq i \leq n)$.

($\Leftarrow$): Assume that $A$ is $B_i$-injective for all $i(1 \leq i \leq n)$. To prove that $A$ is $B$-injective, i.e., to prove

(c) $M$ is $M$-injective and

(d) $\mu_B(m) \leq \mu_A(\psi(m))$ and $\nu_B(m) \geq \nu_A(\psi(m))$ $\forall$ $\psi \in$ Hom (M, M) and $m \in M$.

Proof of (c): Let $N$ be a be a G-submodule of $M$ and $\varphi : N \to M = \oplus_{i=1}^{n} M_i$ be a homomorphism. Then we have three cases:

(i) $N$ is a G-submodule of $M_i$ for some $i$.

(ii) $N = M_i$ , for some $i$.

(iii) $N = \oplus_{i=1}^{t} M_i$ , where $t \leq n$.

Case (i): Suppose $N$ is a G-submodule of $M_i$, for some $i$. Since $M$ is $M_i$-injective, $\exists$ an extension $\psi : M_i \to M$ of $\varphi$. Then $\eta : M \to M$ defined $\eta(m) = \psi(m_i)$, where $m = \sum_{i=1}^{n} m_i \in M$ is a homomorphism and $\eta|_{M_i} = \psi$. Then $\eta|_N = \psi|_N = \varphi$. Thus $\eta$ extends $\varphi$.

Case (ii): Suppose $N = M_i$, for some $i$. The function $\eta$ obtained as in Case (i) with $\psi = \varphi$ is an extension of $\varphi$.

Case (iii): Suppose $N = \oplus_{i=1}^{t} M_i$ , where $t \leq n$. Then the mapping $\eta : M \to M$ defined by $\eta(m) = \varphi(\sum_{i=1}^{t} m_i)$, where $m = \sum_{i=1}^{n} m_i \in M$ is a homomorphism and $\eta$ extends $\varphi$.

So, in all the cases, $\eta : M \to M$ extends $\varphi$. Hence $M$ is $M$-injective.

Proof of (d): Let $\psi \in$ Hom (M, M) and $m \in M$. Then $m = \sum_{i=1}^{n} m_i$ , where $m_i \in M_i$, for each $i$. Now,

$$\mu_B(m) = \mu_B(\textstyle\sum_{i=1}^{n} m_i) = \wedge\{\mu_{B_i}(m_i) : i = 1, 2, ..., n\} \leq \mu_{B_i}(m_i)$$

and

$$\nu_B(m) = \nu_B(\sum_{i=1}^{n} m_i) = \vee\{\nu_{B_i}(m_i) : i = 1, 2, ..., n\} \geq \nu_{B_i}(m_i), \text{ for all } i.$$

Thus

(4.3.1) $$\mu_B(m) \leq \mu_{B_i}(m_i) \text{ and } \nu_B(m) \geq \nu_{B_i}(m_i), \text{ for all } i.$$

Since $A$ is $B_i$- injective, for every $i$, we have

(4.3.2) $$\mu_{B_i}(m_i) \leq \mu_B(\psi_i(m_i)) \text{ and } \nu_{B_i}(m_i) \geq \nu_B(\psi_i(m_i)), \text{ where } \psi_i = \psi|_{M_i}.$$

Thus

(4.3.3) $$\mu_{B_i}(m_i) \leq \mu_B(\psi(m_i)) \text{ and } \nu_{B_i}(m_i) \geq \nu_B(\psi(m_i)), \text{ for all } i.$$

From (4.3.1) and (4.3.3), we have

$$\mu_B(m) \leq \mu_{B_i}(m_i) \leq \mu_B(\psi(m_i)) \text{ and } \mu_B(m) \geq \nu_{B_i}(m_i) \geq \nu_B(\psi(m_i)). \text{ for all } i.$$

So

$$\begin{aligned} \mu_B(m) &\leq \wedge\{\mu_A(\psi(m_i)) : i = 1, 2, ..., n\} \\ &\leq \mu_A(\psi(m_1) + \psi(m_2) + ...... + \psi(m_n))[\text{Since A is an IFGM}] \\ &\leq \mu_A(\psi(m_1 + m_2 + ....... + m_n)) \\ &\leq \mu_A(\psi(m)), \text{ since } m = \sum_{i=1}^{n} m_i. \end{aligned}$$

Similarly,

$$\begin{aligned} \nu_B(m) &\geq \vee\{\nu_A(\psi(m_i)) : i = 1, 2, ..., n\} \\ &\geq \nu_A(\psi(m_1) + \psi(m_2) + ...... + \psi(m_n))[\text{Since A is an IFGM}] \\ &\geq \nu_A(\psi(m_1 + m_2 + ....... + m_n)) \\ &\geq \nu_A(\psi(m)), \text{ since } m = \sum_{i=1}^{n} m_i. \end{aligned}$$

Hence $\mu_B(m) \leq \mu_A(\psi(m))$ and $\nu_B(m) \geq \nu_A(\psi(m))$, for all $\psi \in Hom(M, M)$.
Therefore $A$ is $B$-injective. □

**Theorem 4.4.** *Let $M_1$ and $M_2$ be two G-submodules of a G-module $M$ such that $M = M_1 \oplus M_2$. If $M$ is a quasi-injective, then $M_i$ is $M_j$-injective for $i, j \in \{1, 2\}$. Further if $B_i{}'s$ are intuitionistic fuzzy G-modules on $M_i(i = 1, 2)$ such that $B = B_1 \oplus B_2$ and if $B$ is quasi-injective, then $B_i$ is $B_j$-injective for $i, j \in \{1, 2\}$.*

*Proof.* Assume that $M = M_1 \oplus M_2$ is quasi-injective. Then by Proposition 2.11, $M$ is $M_j$-injective for $j = \{1, 2\}$. Also it follows from Proposition 2.9 that $M_i$ is $M_j$-injective for $i, j \in 1, 2$. This proves the first part of the theorem.

Now assume that $B$ is quasi-injective. Then

(i) $M$ is $M$-injective and

(ii) $\mu_B(m) \leq \mu_B(\psi(m))$ and $\nu_B(m) \geq \nu_B(\psi(m)), \forall\, \psi \in Hom(M, M)$ and $m \in M$.

First to prove $B_1$ is $B_2$-injective, i.e., to prove

(a) $M_1$ is $M_2$-injective and

(b) $\mu_{B_2}(m_2) \leq \mu_{B_1}(m_2)$ and $\nu_{B_2}(m_2) \geq \nu_{B_1}(m_2)$ $\forall\, \psi \in Hom(M_2, M_1)$ and $m_2 \in M_2$.

Proof of (a): From (i), $M$ is $M$-injective. Then it follows from the first part of the theorem that $M_1$ is $M_2$-injective.

Proof of (b): Let $\psi \in$ Hom $(M_2, M_1)$. Consider the inclusion homomorphism $\varphi : M_1 \to M_1 \oplus M_2 = M$. Then $\psi^{'} = \varphi \circ \psi : M_2 \to M_1 \oplus M_2 = M$ is a homomorphism. Since $M$ is $M$- injective, $\exists$ an extension $\varphi^{'} = \varphi \circ \psi : M \to M$ of $\psi^{'}$. Thus

(4.4.1) $$\varphi^{'}|_{M_2} = \psi^{'}.$$

Since $\varphi^{'} \in Hom(M, M)$, from (ii),

(4.4.2) $$\mu_B(m) \leq \mu_B(\varphi^{'}(m)) \text{ and } \nu_B(m) \geq \nu_B(\varphi^{'}(m)), \text{ for all } m \in M.$$

Since $M = M_1 \oplus M_2$, if $m_2 \in M_2$, then $m_2 = 0 + m_2 \in M_1 \oplus M_2 = M$.
From (4.4.2), we get

(4.4.3) $$\mu_B(m_2) \leq \mu_B(\varphi^{'}(m_2)) \text{ and } \nu_B(m_2) \geq \nu_B(\varphi^{'}(m_2).$$

Also,

$$\mu_B(m_2) = \mu_B(0 + m_2) = \mu_{B_1}(0) \wedge \mu_{B_2}(m_2) = \mu_{B_2}(m_2)$$

and

$$\nu_B(m_2) = \nu_B(0 + m_2) = \nu_{B_1}(0) \vee \nu_{B_2}(m_2) = \nu_{B_2}(m_2).$$

Then,

(4.4.4) $$\mu_B(m_2) = \mu_{B_2}(m_2) \text{ and } \nu_B(m_2) = \nu_{B_2}(m_2).$$

From (4.4.1), $\varphi^{'}(m_2) = \psi^{'}(m_2) = \varphi(\psi(m_2)) = \psi(m_2)$. Thus,

$$\begin{aligned} \mu_B(\varphi^{'}(m_2)) &= \mu_B(\psi(m_2)) = \mu_B(\psi(m_2) + 0) \\ &= \mu_{B_1}(\psi(m_2)) \wedge \mu_{B_2}(0) \\ &= \mu_{B_1}(\psi(m_2)) \end{aligned}$$

$$\begin{aligned} \nu_B(\varphi^{'}(m_2)) &= \nu_B(\psi(m_2)) = \nu_B(\psi(m_2) + 0) \\ &= \nu_{B_1}(\psi(m_2)) \vee \nu_{B_2}(0) \\ &= \nu_{B_1}(\psi(m_2)) \end{aligned}$$

So,

(4.4.5) $$\mu_B(\varphi^{'}(m_2)) = \mu_{B_1}(\psi(m_2)) \text{ and } \nu_B(\varphi^{'}(m_2)) = \nu_{B_1}(\psi(m_2)).$$

From (4.4.3),(4.4.4)and (4.4.5), we get
$\mu_{B_2}(m_2) \leq \mu_{B_1}(\psi(m_2))$ and $\nu_{B_2}(m_2) \geq \nu_{B_1}(\psi(m_2))$ $\forall$ $\psi \in$ Hom $(M_2, M_1)$ and $m_2 \in M_2$.
Hence $B_1$ is $B_2$-injective.

Similarly, we can show that $B_2$ is $B_1$-injective.

Now to prove $B_1$ is $B_1$-injective.

From (4.4.1), $M$ is $M$-injective. Then, from the first part of this theorem, we get $M_1$ is $M_1$-injective. Now, let $\psi \in$ Hom $(M_1, M_1)$ and let $\varphi : M_1 \to M$ be the inclusion homomorphism. Then $\varphi \circ \psi : M_1 \to M$ is a homomorphism. Since $M$ is $M$-injective, $\exists$ an extension $\varphi^{''} : M \to M$ of $\varphi \circ \psi$. Thus $\varphi^{''}|_{M_1} = \varphi \circ \psi$. Since $\varphi^{''} \in$ Hom $(M, M)$, from (ii), we get

$$\mu_B(m) \leq \mu_B(\varphi^{''}(m)) \text{ and } \nu_B(m) \geq \nu_B(\varphi^{''}(m)), \forall\ m \in M.$$

That is,

$$\text{(4.4.6)} \qquad \mu_B(m_1) \leq \mu_B(\varphi^{''}(m_1)) \text{ and } \nu_B(m_1) \geq \nu_B(\varphi^{''}(m_1)), \text{ for all } m_1 \in M_1.$$

If $m_1 \in M_1$, then we have

$$\mu_B(m_1) = \mu_B(m_1 + 0) = \mu_{B_1}(m_1) \wedge \mu_{B_2}(0) = \mu_{B_1}(m_1)$$

and

$$\nu_B(m_1) = \nu_B(m_1 + 0) = \nu_{B_1}(m_1) \vee \nu_{B_2}(0) = \nu_{B_1}(m_1).$$

That is,

$$\text{(4.4.7)} \qquad \mu_B(m_1) = \mu_{B_1}(m_1) \text{ and } \nu_B(m_1) = \nu_{B_1}(m_1).$$

Also, $\varphi^{''}(m_1) = (\varphi \circ \psi)(m_1) = \varphi(\psi(m_1)) = \psi(m_1) \in M_1$. So

$$\begin{aligned}\mu_B(\varphi^{''}(m_1)) &= \mu_B(\psi(m_1)) = \mu_B(\psi(m_1) + 0)\\ &= \mu_{B_1}(\psi(m_1)) \wedge \mu_{B_1}(0) = \mu_{B_1}(\psi(m_1))\end{aligned}$$

and

$$\begin{aligned}\nu_B(\varphi^{''}(m_1)) &= \nu_B(\psi(m_1)) = \nu_B(\psi(m_1) + 0)\\ &= \nu_{B_1}(\psi(m_1)) \vee \nu_{B_1}(0) = \nu_{B_1}(\psi(m_1)).\end{aligned}$$

That is,

$$\text{(4.4.8)} \qquad \mu_B(\varphi^{''}(m_1)) = \mu_{B_1}(\psi(m_1)) \text{ and } \nu_B(\varphi^{''}(m_1)) = \nu_{B_1}(\psi(m_1)).$$

From (4.4.6),(4.4.7) and (4.4.8), we get
$\mu_B(m_1) \leq \mu_{B_1}(\psi(m_1))$ and $\nu_B(m_1) \geq \nu_{B_1}(\psi(m_1))$ $\forall\ \psi \in$ Hom $(M_1, M_1)$ and $m_1 \in M_1$.
Hence $B_1$ is $B_1$-injective.

Similarly, we can show that $B_2$ is $B_2$-injective. This completes the proof. □

**Corollary 4.5.** *Let $M = \oplus_{i=1}^{n} M_i$ be a G-module, where $M_i\ '$s are G-submodules of $M$. If $M$ is quasi-injective, then $M_i$ is $M_j$-injective for $i, j \in \{1, 2, ....., n\}$. Also if $B_i\ '$s are intuitionistic fuzzy G-modules on $M_i\ '$s such that $B = \oplus_{i=1}^{n} B_i$ and if $B$ is quasi-injective, then $B_i$ is $B_j$-injective for every $i$ and $j$.*

## 5. Conclusions

In this paper, we have introduced the notion of injectivity and quasi injectivity of an intuitionistic fuzzy G-modules and have constructed some structure revealing examples. We have also analyzed the relative injectivity (quasi-injectivity) of an intuitionistic fuzzy G-module with regards to another intuitionistic fuzzy G-module.

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