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Fuzzy bi-ideals of near-subtraction semigroups

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Abstract. In this paper, we introduce the notion of fuzzy bi-ideals of near-subtraction semigroups. Some of its characterizations with examples were also given.

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1. Introduction

A system of the form $(\phi; \circ; \backslash)$ where ϕ is a set of functions closed under the composition '$\circ$'of function (hence $(\phi; \circ)$ is a function semigroup) and $\backslash$ is the set theoretic subtraction is called a subtraction algebra in the sence of [1]. Scheine [9] showed that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. Zelinka[11] studied a special type of subtraction algebra called atomic subtraction algebra. The study of ideals in subtraction algebra was initiated by Jun et al.[4], who also established some basic properties. Dheena et al.[2, 3] discussed and derived some properties of near-subtraction semigroups, a generalization of subtraction semigroup. The concept of fuzzy set was first initiated by Zadeh[10]. Narayanan et al.[7] defined the concept of generalized fuzzy ideals of near-rings. Mahalakshmi et al.[5] studied the notion of bi-ideals of near-subtraction semigroups. Manikandan[6] studied fuzzy fuzzy bi-ideals of near-ring and established some of their properties.

In this paper we introduce the notion of fuzzy bi-ideals of near-subtraction semigroups and prove some characterizations with examples.

2. Preliminaries

In this section, we reproduce some basic definitions which are essential for the development of the paper.

Definition 2.1 ([4]). A nonempty set X together with a binary operation "$-$" is said to be a subtraction algebra if it satisfies the following conditions:
(i) $x - (y - x) = x$,
(ii) $x - (x - y) = y - (y - x)$,
(iii) $(x - y) - z = (x - z) - y$ for every $x, y, z \in X$.
The last identity permits us to omit parenthesis in expressions of the form $(x-y)-z$.

Definition 2.2 ([2]). A non-empty set X together with the binary operations "$-$" and "$\cdot$" is said to be a right near-subtraction semigroup if it satisfies the following:
(i) $(X, -)$ is a subtraction algebra,
(ii) $(X, \cdot)$ is a semigroup,
(iii) $(x - y)z = xz - yz$ for all $x, y, z \in X$.

It is clear that $0x = 0$, for all $x \in X$. Similarly we can define a left near-subtraction semigroup. Here after a near-subtraction semigroup means only a right near-subtraction semigroup, unless mentioned otherwise.

Definition 2.3 ([2]). A near-subtraction semigroup X is said to be zero-symmetric if $x0 = 0$ for every $x \in X$.

Definition 2.4 ([2]). A non-empty subset S of a subtraction algebra X is said to be a subalgebra of X, if $x - y \in S$, for all $x, y \in S$.

Definition 2.5 ([2]). A non-empty subset S of a near-subtraction semigroup X is said to be a near-subtraction subsemigroup of X, if $x - y, xy \in S$, for all $x, y \in S$.

Definition 2.6 ([2]). Let $(X, -, \cdot)$ be a near-subtraction semigroup. A non-empty subset I of X is called:
(I1) a left ideal if I is a subalgebra of $(X, -)$ and $xi - x(y - i) \in I$ for all $x, y \in X$ and $i \in I$.
(I2) a right ideal if I is a subalgebra of $(X, -)$ and $IX \subseteq I$.
(I3) an ideal if I is both a left and right ideal.

Definition 2.7 ([5]). Let A and B be two subsets of X. Then the product and $\star$ product defined by $AB = \{ab | a \in A \text{ and } b \in B\}$ and $A \star B = \{ab - a(a' - b) | a, a' \in A \text{ and } b \in B\}$.

Definition 2.8 ([5]). An subalgebra B of X is said to be bi-ideal if $BXB \cap BX \star B \subseteq B$.

Definition 2.9 ([8]). A fuzzy subset μ is called a fuzzy ideal of X if it satisfies the following conditions:
(i) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$,
(ii) $\mu(xi - x(y - i)) \geq \mu(i)$,
(iii) $\mu(xy) \geq \mu(x)$, for all $x, y, i \in X$.

3. Fuzzy bi-ideals of Near-subtraction semigroup

In this section, we introduc the new concept of fuzzy bi-ideals of near-subtraction semigroup. Throughout this paper, f_I is the characteristic function of the subset I of X and the characteristic function of X is denoted by $\mathbf{X}$, that means, $\mathbf{X} : X \to [0,1]$ mapping every element of X to 1.

Definition 3.1. Let μ and λ be any two fuzzy subsets of X. Then $\mu \cap \lambda,\ \mu \cup \lambda,\ \mu - \lambda,\ \mu\lambda$ and $\mu \star \lambda$ are fuzzy subsets of X defined by:

$$(\mu \cap \lambda)(x) = \min\{\mu(x),\ \lambda(x)\}.$$

$$(\mu \cup \lambda)(x) = \max\{\mu(x),\ \lambda(x)\}.$$

$$(\mu - \lambda)(x) = \begin{cases} \sup\limits_{x=y-z} \min\{\mu(y),\ \lambda(z)\} & \text{if } x \text{ can be expressed as } x = y - z \\ 0 & \text{otherwise.} \end{cases}$$

$$(\mu\lambda)(x) = \begin{cases} \sup\limits_{x=yz} \min\{\mu(y),\ \lambda(z)\} & \text{if } x \text{ can be expressed as } x = yz \\ 0 & \text{otherwise.} \end{cases}$$

$$(\mu \star \lambda)(x) = \begin{cases} \sup\limits_{x=ac-a(b-c)} \min\{\mu(a),\ \lambda(c)\} & \text{if } x \text{ can be expressed as} \\ & x = ac - a(b-c) \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.2. A fuzzy subset μ of X is said to be a fuzzy subalgebra of X, if $x,\ y \in X$ implies $\mu(x - y) \geq \min\{\mu(x),\ \mu(y)\}$.

Definition 3.3. A fuzzy subalgebra μ of X is called a fuzzy bi-ideal of X, if $(\mu\mathbf{X}\mu) \cap (\mu\mathbf{X} \star \mu) \subseteq \mu$.

Example 3.4. Let $X = \{0, a, b, c\}$ in which " $-$ " and "**.**" are defined by:

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	b
c	0	0	0	c

Then $(X, -, \cdot)$ is a near-subtraction semigroup. Let $\mu : X \to [0,1]$ be a fuzzy subset of X defined as $\mu(0) = 0.9, \mu(a) = 0.7, \mu(b) = 0.6$ and $\mu(c) = 0.4$. Then μ is a fuzzy bi-ideal of X.

Lemma 3.5. *Let μ be a fuzzy subset of X. If μ is a fuzzy left ideal of X then μ is a fuzzy bi-ideal of X.*

Proof. Let $x' \in X$ be such that $x' = abc = xz - x(y-z)$, where a, b, c, x, y and z are in X. Then

$$
\begin{aligned}
&((\mu\mathbf{X}\mu)\cap(\mu\mathbf{X}\star\mu))(x')\\
&\quad= \min\{(\mu\mathbf{X}\mu)(x'), (\mu\mathbf{X}\star\mu)(x')\}\\
&\quad= \min\{\sup_{x'=abc} \min\{\mu(a), \mathbf{X}(b), \mu(c)\}, \sup_{x'=xz-x(y-z)} \min\{(\mu\mathbf{X})(x), \mu(z)\}\}\\
&\quad= \min\{\sup\min\{\mu(a), \mu(c)\}, \sup\min\{(\mu\mathbf{X})(x), \mu(z)\}\},\\
&\qquad\qquad [\text{since } \mu\mathbf{X} \subseteq \mathbf{X} \text{ and } \mu \text{ is a fuzzy left ideal, then}\\
&\qquad\qquad\qquad \mu(xz - x(y-z)) \geq \mu(z).]\\
&\quad\leq \min\{\mathbf{X}(a), \mathbf{X}(c), \mathbf{X}(x), \mu(xz - x(y-z))\}\\
&\quad= \min\{1, 1, 1, \mu(xz - x(y-z))\} = \mu(xz - x(y-z)) = \mu(x').
\end{aligned}
$$

If x' is not expressible as $x' = abc = xz - x(y-z)$ then $(\mu\mathbf{X}\mu \cap \mu\mathbf{X}\star\mu)(x') = 0 \leq \mu(x')$. Thus $\mu\mathbf{X}\mu \cap \mu\mathbf{X}\star\mu \subseteq \mu$. Hence μ is a fuzzy bi-ideal of X. □

Lemma 3.6. *Let μ be a fuzzy subset of X. If μ is a fuzzy right ideal of X then μ is a fuzzy bi-ideal of X.*

Proof. Let $x' \in X$ be such that $x' = ab = xz - x(y-z), a = a_1a_2$, where a, a_1, a_2, b, x, y and z are in X. Consider,

$$
\begin{aligned}
&((\mu\mathbf{X}\mu)\cap(\mu\mathbf{X}\star\mu))(x')\\
&\quad= \min\{(\mu\mathbf{X}\mu)(x'), (\mu\mathbf{X}\star\mu)(x')\}\\
&\quad= \min\{\sup_{x'=ab} \min\{(\mu\mathbf{X})(a), \mu(b)\}, (\mu\mathbf{X}\star\mu)(xz - x(y-z)\}\\
&\quad= \min\{\sup_{x'=ab} \min\{\sup_{a=a_1a_2} \min\{\mu(a_1), \mathbf{X}(a_2)\}, \mu(b)\}, (\mu\mathbf{X}\star\mu)(xz - x(y-z)\}\\
&\quad= \min\{\sup_{x'=ab} \min\{\sup_{a=a_1a_2} \{\mu(a_1)\}, \mu(b)\}, (\mu\mathbf{X}\star\mu)(xz - x(y-z)\}\\
&\quad= \min\{\mu(a_1), \mu(b), (\mu\mathbf{X}\star\mu)(xz - x(y-z)\},\\
&\qquad\qquad [\text{since } \mu \text{ is a fuzzy right ideal, we have}\\
&\qquad\qquad\qquad \mu(ab) = \mu(a_1a_2b) = \mu(a_1(a_2b)) \geq \mu(a_1).]\\
&\quad\leq \min\{\mu(ab), 1, 1\} = \mu(ab) = \mu(x').
\end{aligned}
$$

If x' is not expressible as $x' = ab = xz - x(y-z)$ then $(\mu\mathbf{X}\mu \cap \mu\mathbf{X}\star\mu)(x') = 0 \leq \mu(x')$. Thus $\mu\mathbf{X}\mu \cap \mu\mathbf{X}\star\mu \subseteq \mu$. Hence μ is a fuzzy bi-ideal of X. □

Theorem 3.7. *Let μ be a fuzzy subset of X. If μ is a fuzzy ideal of X then μ is a fuzzy bi-ideal of X.*

Proof. Straightforward from Lemma 3.5 and Lemma 3.6. □

Remark 3.8. The converse of the Theorem 3.7 is not true in general which is demonstrated by the following Example.

Example 3.9. Let $X = \{0, a, b, c\}$ in which "$-$" and "$\cdot$" are defined as in Example 3.4. Let $\mu : X \to [0, 1]$ be a fuzzy subset of X defined by $\mu(0) = 0.9, \mu(a) = 0.6 = \mu(c)$ and $\mu(b) = 0.3$. Then μ is a fuzzy bi-ideal of X but μ is not a fuzzy ideal of X, since $\mu(bc - b(0-c)) = \mu(b) = 0.3 \ngeq 0.6 = \mu(c)\}$.

Theorem 3.10. *Let μ be a fuzzy subset of X. Then μ is a fuzzy bi-ideal of X if and only if the level subset μ_t is a bi-ideal of X, for all $t \in [0,1]$.*

Proof. Assume that μ is a fuzzy bi-ideal of X. Let $t \in [0,1]$. Let $x, y \in \mu_t$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. Since μ is a fuzzy bi-ideal of X, we have

$$\mu(x-y) \geq \min\{\mu(x), \mu(y)\} \geq t.$$

It follows that $x - y \in \mu_t$. Let $x' \in X$ and $x' \in \mu_t \mathbf{X} \mu_t \cap \mu_t \mathbf{X} * \mu_t$. If there exist $a_1, b, x_1, z \in \mu_t$ and $a_2, a, x, x_2, y \in X$ such that

$$x' = ab = xz - x(y-z), a = a_1 a_2 \text{ and } x = x_1 x_2.$$

Then $\mu(a_1) \geq t, \mu(b) \geq t, \mu(z) \geq t$ and $\mu(x_1) \geq t$. Thus

$$\begin{aligned}
\mu(x') &\geq (\mu\mathbf{X}\mu \cap \mu\mathbf{X} \star \mu)(x') \\
&= \min\{(\mu\mathbf{X}\mu)(x'), (\mu\mathbf{X} \star \mu)(x')\} \\
&= \min\{\sup_{x'=ab} \min\{(\mu\mathbf{X})(a), \mu(b)\}, \sup_{x'=xz-x(y-z)} \min\{(\mu\mathbf{X})(x), \mu(z)\}\} \\
&= \min\{\sup_{x'=ab} \min\{\sup_{a=a_1a_2} \min\{\mu(a_1), \mathbf{X}(a_2)\}, \mu(b)\}, \\
&\qquad \sup_{x'=xz-x(y-z)} \min\{\sup_{x=x_1x_2} \min\{\mu(x_1), \mathbf{X}(x_2)\}, \mu(z)\}\} \\
&= \min\{\mu(a_1), \mu(b), \mu(x_1), \mu(z)\} \geq t.
\end{aligned}$$

This implies that $\mu(x') \geq t$. So $x' \in \mu_t$, that is, $\mu_t \mathbf{X} \mu_t \cap \mu_t \mathbf{X} \star \mu_t \subseteq \mu_t$. Hence μ_t is a bi-ideal of X.

Conversely, assume that μ_t is a bi-ideal of X for $t \in [0,1]$. Let $x' \in X$. Suppose that $(\mu\mathbf{X}\mu \cap \mu\mathbf{X} \star \mu)(x') > \mu(x')$. Choose $0 < t \leq 1$ such that

$$(\mu\mathbf{X}\mu \cap \mu\mathbf{X} \star \mu)(x') > t > \mu(x').$$

This implies that $(\mu\mathbf{X}\mu)(x') \geq t$ and $(\mu\mathbf{X} \star \mu)(x') \geq t$. Then

$$\begin{aligned}
(\mu\mathbf{X}\mu)(x') &= \sup_{x'=ab} \min\{\sup_{a=a_1a_2} \min\{\mu(a_1), \mathbf{X}(a_2)\}, \mu(b)\} \\
&= \min\{\mu(a_1), \mu(b)\} \geq t
\end{aligned}$$

and

$$\begin{aligned}
(\mu\mathbf{X} \star \mu)(x') &= \sup_{x'=xz-x(y-z)} \min\{(\mu\mathbf{X})(x), \mu(z)\} \\
&= \sup_{x'=xz-x(y-z)} \min\{\sup_{x=x_1x_2} \min\{\mu(x_1), \mathbf{X}(x_2)\}, \mu(z)\} \\
&= \min\{\mu(x_1), \mu(z)\} \geq t.
\end{aligned}$$

Thus $a_1, b, x_1, z \in \mu_t$. Since μ_t is a bi-ideal of X, we have

$$x' = a_1 a_2 b \in \mu_t \mathbf{X} \mu_t \text{ and } x' = x_1 x_2 z - x_1 x_2 (y - z) \in \mu_t \mathbf{X} \star \mu_t.$$

So $x' \in \mu_t \mathbf{X} \mu_t \cap \mu_t \mathbf{X} \star \mu_t$, implying, $x' \in \mu_t$, since μ_t is a bi-ideal of X. Hence $\mu(x') \geq t$ which is a contradiction. Therefore $\mu\mathbf{X}\mu \cap \mu\mathbf{X} \star \mu \subseteq \mu$ and hence μ is a fuzzy bi-ideal of X. □

Lemma 3.11. *Let A and B be two nonempty subsets of X. Then the following are true:*
(1) $f_A \cap f_B = f_{A\cap B}$.
(2) $f_A \cup f_B = f_{A\cup B}$.
(3) $f_A f_B = f_{AB}$.
(4) $f_A \star f_B = f_{A\star B}$.

Proof. Straightforward. □

Lemma 3.12. *A nonempty subset B of X is a bi-ideal of X if and only if f_B is a fuzzy bi-ideal of X.*

Proof. Assume that B is a bi-ideal of X. Let f_B be a fuzzy subset of X defined by

$$f_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$$

Let $x, y \in X$. Suppose that $f_B(x-y) < \min\{f_B(x), f_B(y)\}$. Then

$$f_B(x-y) = 0 \text{ and } \min\{f_B(x), f_B(y)\} = 1.$$

This point out that $x, y \in B$ and $x - y \notin B$, which is a contradiction to our conjecture. Thus, f_B is a fuzzy subalgebra of X. For some $x \in X$, take $f_B(x) < \min\{(f_B\mathbf{X}f_B)(x), (f_B\mathbf{X} \star f_B)(x)\}$. Then

$$f_B(x) = 0 \text{ and } \min\{(f_B\mathbf{X}f_B)(x), (f_B\mathbf{X} \star f_B)(x)\} = 1,$$

that is,

$$(f_B\mathbf{X}f_B)(x) = 1 \text{ and } (f_B\mathbf{X} \star f_B)(x) = 1.$$

This means that $f_{BXB}(x) = 1$ and $f_{BX\star B}(x) = 1$. Thus by Lemma 3.11,

$$(f_{BXB} \cap f_{BX\star B})(x) = f_{BXB\cap BX\star B}(x) = 1.$$

So $x \in BXB \cap BX \star B$. This implies that $x \in B$, which is a contradiction. Hence,

$$f_B(x) \geq \min\{(f_B\mathbf{X}f_B)(x), (f_B\mathbf{X} \star f_B)(x)\}.$$

Therefore, f_B is a fuzzy bi-ideal of X.

Conversely, assume that f_B is a fuzzy bi-ideal of X. Let $x' \in BXB \cap BX \star B$. Then $x' \in BXB$ and $x' \in BX \star B$. Let $a_1, b, z, x_1 \in B$ and $a_2, x, y, x_2, z \in X$ be such that $x' = a_1a_2b = xz - x(y-z), x = x_1x_2$. Now,

$$\begin{aligned}
f_B(x') &\geq (f_B\mathbf{X}f_B \cap f_B\mathbf{X} \star f_B)(x') \\
&= \min\{(f_B\mathbf{X}f_B)(x'), (f_B\mathbf{X} \star f_B)(x')\} \\
&= \min\{\sup_{x'=ab} \min\{(f_B\mathbf{X})(a), f_B(b)\}, \sup_{x'=xz-x(y-z)} \min\{(f_B\mathbf{X})(x), f_B(z)\}\} \\
&= \min\{\sup_{x'=ab} \min\{\sup_{a=a_1a_2} \min\{f_B(a_1), \mathbf{X}(a_2)\}, f_B(b)\}, \\
&\qquad \sup_{x'=xz-x(y-z)} \min\{\sup_{x=x_1x_2} \min\{f_B(x_1), \mathbf{X}(x_2)\}, f_B(z)\}\} \\
&= \min\{f_B(a_1), f_B(b), f_B(x_1), f_B(z)\} = 1.
\end{aligned}$$

This implies that $f_B(x') = 1$. Thus $x' \in B$, that is, $BXB \cap BX \star B \subseteq B$. So B is a bi-ideal of X. □

Theorem 3.13. *Let μ be a fuzzy subalgebra of X. If $\mu\mathbf{X}\mu \subseteq \mu$ then μ is a fuzzy bi-ideal of X.*

Proof. Assume that μ is a fuzzy subalgebra of X and $\mu\mathbf{X}\mu \subseteq \mu$. Let $x \in X$. Then

$$(\mu\mathbf{X}\mu \cap \mu\mathbf{X} \star \mu)(x) = \min\{(\mu\mathbf{X}\mu)(x), (\mu\mathbf{X} \star \mu)(x)\} \leq (\mu\mathbf{X}\mu)(x) \leq \mu(x).$$

Thus $\mu\mathbf{X}\mu \cap \mu\mathbf{X} \star \mu \subseteq \mu$ and μ is a fuzzy bi-ideal of X. □

Theorem 3.14. *If X is a zero-symmetric near-subtraction semigroup and μ be a fuzzy bi-ideal of X then $\mu\mathbf{X}\mu \subseteq \mu$.*

Proof. Let μ be a fuzzy bi-ideal of X. Then $\mu\mathbf{X}\mu \cap \mu\mathbf{X} \star \mu \subseteq \mu$. Clearly $\mu(0) \geq \mu(x)$. Thus $(\mu\mathbf{X})(0) \geq (\mu\mathbf{X})(x)$ for all $x \in X$. Since X is a zero-symmetric near-subtraction semigroup, $\mu\mathbf{X}\mu \subseteq \mu\mathbf{X} \star \mu$. So $\mu\mathbf{X}\mu \cap \mu\mathbf{X} \star \mu = \mu\mathbf{X}\mu \subseteq \mu$, which is the required result. □

Theorem 3.15. *Let X be a zero-symmetric near-subtraction semigroup and μ be a fuzzy subalgebra of X. Then the following conditions are equivalent:*
(1) *μ is a fuzzy bi-ideal of X.*
(2) *$\mu\mathbf{X}\mu \subseteq \mu$.*

Proof. The proof is straightward from Theorem 3.13 and Theorem 3.14. □

Theorem 3.16. *Let μ be a fuzzy bi-ideal of a zero-symmetric near-subtraction semigroup X. Then $\mu(xyz) \geq \min\{\mu(x), \mu(z)\}$.*

Proof. Assume that μ is a fuzzy bi-ideal of zero-symmetric near-subtraction semigroup X. By Theorem 3.13, $\mu\mathbf{X}\mu \subseteq \mu$. Let $x, y, z \in X$. Then

$$\begin{aligned}
\mu(xyz) &\geq (\mu\mathbf{X}\mu)(xyz) \\
&= \sup_{xyz=ab} \min\{(\mu\mathbf{X})(a), \mu(b)\} \\
&\geq \min\{(\mu\mathbf{X})(xy), \mu(z)\} \\
&\geq \min\{\mu(x), \mathbf{X}(y), \mu(z)\} \\
&= \min\{\mu(x), 1, \mu(z)\} \\
&= \min\{\mu(x), \mu(z)\}.
\end{aligned}$$

Thus $\mu(xyz) \geq \min\{\mu(x), \mu(z)\}$. □

Theorem 3.17. *Let μ be a fuzzy bi-ideal of a zero-symmetric near-subtraction semigroup X. Then the following are equivalent:*
(1) *$\mu(xyz) \geq \min\{\mu(x), \mu(z)\}$.*
(2) *$\mu\mathbf{X}\mu \subseteq \mu$.*

Proof. Let μ be a fuzzy bi-ideal of zero-symmetric near-subtraction semigroup X. Let $x' \in X$.

(1)$\Longrightarrow$(2): If there exist $x, y, x_1, x_2 \in X$ such that $x' = xy$ and $x = x_1x_2$. Then by hypothesis, $\mu(x_1x_2y) \geq \min\{\mu(x_1), \mu(y)\}$. We have

$$\begin{aligned}(\mu\mathbf{X}\mu)(x') &= \sup_{x'=xy} \min\{(\mu\mathbf{X})(x), \mu(y)\}\\ &= \sup_{x'=xy} \min\{\sup_{x=x_1x_2} \min\{\mu(x_1), \mathbf{X}(x_2)\}, \mu(y)\}\\ &= \sup_{x'=xy} \min\{\sup_{x=x_1x_2} \min\{\mu(x_1), 1\}, \mu(y)\}\\ &= \sup_{x'=x_1x_2y} \min\{\mu(x_1), \mu(y)\}\\ &\leq \sup_{x'=x_1x_2y} \mu(x_1x_2y) = \mu(x').\end{aligned}$$

Thus, $\mu\mathbf{X}\mu \subseteq \mu$. So (2) holds.

(2)$\Longrightarrow$(1): Assume that $\mu\mathbf{X}\mu \subseteq \mu$. Let $x, y, z, x' \in X$ be such that $x' = xyz$. Then

$$\begin{aligned}\mu(xyz) = \mu(x') &\geq (\mu\mathbf{X}\mu)(x')\\ &= \sup_{x'=ab} \min\{(\mu\mathbf{X})(a), \mu(b)\}\\ &= \sup_{x'=ab} \min\{\sup_{a=a_1a_2} \min\{\mu(a_1), \mathbf{X}(a_2)\}, \mu(b)\}\\ &\geq \min\{\mu(x), \mathbf{X}(y), \mu(z)\}\\ &= \min\{\mu(x), 1, \mu(z)\} = \min\{\mu(x), \mu(z)\}.\end{aligned}$$

Thus (1) holds. □

Theorem 3.18. *Let μ be a fuzzy subalgebra of a zero-symmetric near-subtraction semigroup X. Then the following conditions are equivalent:*

(1) *μ is a fuzzy bi-ideal of X.*

(2) $\mu(xyz) \geq \min\{\mu(x), \mu(y)\}$.

(3) $\mu\mathbf{X}\mu \subseteq \mu$.

Proof. Straightforward. □

Theorem 3.19. *Let λ and μ be any two fuzzy bi-ideals of X. Then $\mu \cap \lambda$ is also a fuzzy bi-ideal of X.*

Proof. Let λ and μ be any two fuzzy bi-ideals of X. Let $x, y \in X$. Then

$$\begin{aligned}(\mu \cap \lambda)(x - y) &= \min\{\mu(x - y), \lambda(x - y)\}\\ &\geq \min\{\min\{\mu(x), \mu(y)\}, \min\{\lambda(x), \lambda(y)\}\}\\ &= \min\{\min\{\mu(x), \lambda(x)\}, \min\{\mu(y), \lambda(y)\}\}\\ &= \min\{(\mu \cap \lambda)(x), (\mu \cap \lambda)(y)\}.\end{aligned}$$

Let $x' \in X$. Choose $a, b, x, y, z \in X$ such that $x' = abc = xz - x(y - z)$. Since λ and μ are fuzzy bi-ideals of X, we have

$$\min\left\{\sup_{x'=abc} \min\{\mu(a), \mu(c)\}, \sup_{x'=xz-x(y-z)} \mu(z)\right\} \leq \mu(x) \tag{3.1}$$

and

$$\min\left\{\sup_{x'=abc}\min\{\lambda(a),\lambda(c)\},\ \sup_{x'=xz-x(y-z)}\lambda(z)\right\}\leq\lambda(x). \tag{3.2}$$

Now

$$\begin{aligned}
&\min\{((\mu\cap\lambda)\mathbf{X}(\mu\cap\lambda))(x'),((\mu\cap\lambda)\mathbf{X}\star(\mu\cap\lambda))(x')\}\\
&\quad=\min\{\sup_{x'=abc}\min\{(\mu\cap\lambda)(a),(\mu\cap\lambda)(c)\},\ \sup_{x'=xz-x(y-z)}(\mu\cap\lambda)(z)\}\\
&\quad=\min\{\sup_{x'=abc}\min\{\min\{\mu(a),\lambda(a)\},\min\{\mu(c),\lambda(c)\}\},\\
&\qquad\qquad\qquad\qquad\sup_{x'=xz-x(y-z)}\min\{\mu(z),\lambda(z)\}\}\\
&\quad=\min\{\min\{\sup_{x'=abc}\min\{\mu(a),\mu(c)\},\ \sup_{x'=xz-x(y-z)}\mu(z)\},\\
&\qquad\qquad\min\{\sup_{x'=abc}\min\{\lambda(a),\lambda(c)\},\ \sup_{x'=xz-x(y-z)}\lambda(z)\}\}\\
&\quad\leq\min\{\mu(x),\lambda(x)\},\ \text{[from (3.1) and (3.2)]}\\
&\quad=(\mu\cap\lambda)(x).
\end{aligned}$$

Thus $\mu\cap\lambda$ is a fuzzy bi-ideal of X. □

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