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Least directing congruence on fuzzy automata

V. Karthikeyan, M. Rajasekar

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Abstract. In this paper, we introduce directing congruence, trapping congruence, trap-directing congruence of a fuzzy automaton and a necessary and sufficient condition for congruence relation of a fuzzy automaton to be directing. We find the least directing congruence on a fuzzy automaton and a generalized directable fuzzy automaton. Finally, we provide an algorithm to find the least directing congruence on a fuzzy automaton and a generalized directable fuzzy automaton.

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Keywords: Necks, Congruence relation, Directing congruence, Generalized directable fuzzy automaton.

Corresponding Author: V. Karthikeyan (vkarthikau@gmail.com)

1. Introduction

The concept of fuzzy set was introduced by L. A. Zadeh in 1965 [19]. The mathematical formulation of a fuzzy automaton was first proposed by W.G. Wee in 1967 [18]. E. S. Santos proposed fuzzy automata as a model of pattern recognition [17]. The concept of fuzzy set is applied in different discipline including medical sciences, artificial intelligence, pattern recognition and automata theory. For instance, γ-synchronized fuzzy automaton were introduced by V. Karthikeyan and M. Rajasekar in [8]. Using the concept of synchronization the authors were introduced an application related to petrol passing through different pipelines in n cities with minimal flow capacity and minimum maintenance cost [11].

Regular expression is applied in different applications such as string matching, parsing text data files into sections for import into a database etc. Conversion of fuzzy regular expressions into fuzzy automata using the concept of follow automata were discussed in [14]. Similarly, Conversion of parallel fuzzy regular expression to its epsilon free fuzzy automaton were discussed in [4].

The notion of a generalized directable automata were introduced by T. Petkovic et.al [16]. Directability and stronger directability of fuzzy automata were introduced in [9, 10, 12] and Generalized directable fuzzy automata were introduced by V. Karthikeyan and M. Rajasekar [13].

Directing congruences on automata were considered in [6], and it was noted that every finite automaton has the least directing congruence, and an algorithm for finding this congruence was given in [5].

We introduce directing congruence, trapping congruence, trap-directing congruence of a fuzzy automaton.

The main purpose of this paper is to introduce the structural characterizations of fuzzy automata and generalized directable fuzzy automata. Also, we provide a a necessary and sufficient condition for congruence relation of a fuzzy automaton to be directing, provide an algorithm to find the least directing congruence on a fuzzy automaton and a generalized directable fuzzy automaton. Finally, we find the relation between the least directing congruence and the least trapping congruence.

2. Preliminaries

This section present basic concept and results to be used in the sequel.

Let X denote a universal set. Then a fuzzy set A in X is set of ordered pairs: $A = \{(x,\ \mu_A(x)|x \in X\}$, $\mu_A(x)$ is called the membership function or grade of membership of x in A which maps X to the membership space $[0, 1]$[20].

A finite fuzzy automaton is a system of 5 tuples, $M = (Q,\ \Sigma,\ f_M,\ q_0,\ F)$, where, Q is set of states, Σ is input symbols, f_M is transition function from $Q \times \Sigma \times Q \to [0, 1]$, q_0 is an initial state and $q_0 \in Q$, and $F \subseteq Q$ set of final states. The transition in a fuzzy automaton is as follows:

$f_M(q_i, a, q_j) = \mu$, $0 \le \mu \le 1$, means that when M is in state q_i and reads the input a will move to the state q_j with weight function μ.

f_M can be extended to $Q \times \Sigma^* \times Q \to [0, 1]$ by,

$$f_M(q_i,\ \epsilon,\ q_j) = \begin{cases} 1 & \text{if } q_i = q_j \\ 0 & \text{if } q_i \neq q_j \end{cases}$$

$$f_M(q_i, w, q_m) = Max\{Min\{f_M(q_i, a_1, q_1), f_M(q_1, a_2, q_2), .., f_M(q_{m-1}, a_m, q_m)\}\}$$

for $w = a_1\ a_2\ a_3\ ...\ a_m \in \Sigma^*$, where Max is taken over all the paths from q_i to q_m [7].

Throughout this paper, we consider a fuzzy automaton without initial state and final state and M denotes $M = (Q, \Sigma, f_M)$, f_M is transition function from $Q \times \Sigma \times Q \to [0, 1]$.

A fuzzy automaton M is called deterministic if for each $a \in \Sigma$ and $q_i \in Q$, there exists a unique state q_a such that $f_M(q_i,\ a,\ q_a) > 0$ otherwise it is called nondeterministic [3].

Let $M' = (Q',\ \Sigma,\ f_{M'})$, $Q' \subseteq Q$ and $f_{M'}$ is the restriction of f_M. The fuzzy automaton M' is called a subautomaton of M if

(i) $f_{M'} : Q' \times \Sigma \times Q' \to [0, 1]$ and

(ii) For any $q_i \in Q'$ and $f_{M'}(q_i, u, q_j) > 0$ for some $u \in \Sigma^*$, then $q_j \in Q'$.

M is said to be strongly connected if for every $q_i,\ q_j \in Q$, there exists $u \in \Sigma^*$ such that $f_M(q_i,\ u,\ q_j) > 0$. Equivalently, M is strongly connected if it has no proper subautomaton [15].

Let $q_i \in Q$. The subautomaton of M generated by q_i is denoted by $\langle q_i \rangle$ and is given by $\langle q_i \rangle = \{\ q_j\ /\ f_M(q_i,\ u,\ q_j) > 0,\ u \in \Sigma^*\}$. It is called a least subautomaton of M containing q_i and it is also called a monogenic subautomaton of M. For any non-empty $H \subseteq Q$, the subautomaton of M generated by H is denoted by $\langle H \rangle$ and is given by $\langle H \rangle = \{\ q_j\ /\ f_M(q_i,\ w,\ q_j) > 0,\ q_i \in H,\ w \in \Sigma^*\}$. It is called a least subautomaton of M containing H. The least subautomaton of a fuzzy automaton M is called the kernel of M [8].

A state $q_j \in Q$ is called a neck of M, for every $q_i \in Q$ if there exists $u \in \Sigma^*$ such that $f_M(q_i,\ u,\ q_j) > 0$. In that case q_j is also said to be a u-neck of M and the word u is called a directing word of M. If M has a directing word, then we say that M is a directable fuzzy automaton. The set of all necks of M is denoted by $N(M)$ and the set of all directing words of M is denoted by $DW(M)$. If $N(M) \neq \phi$, then $N(M)$ is a subautomaton of M [8].

A state $q_j \in Q$ is called local neck of M if it is neck of some directable subautomaton of M. The set of all local necks of M is denoted by $LN(M)$[8].

A state $q_i \in Q$ is called reversible if for every word $v \in \Sigma^*$, there exists a word $u \in \Sigma^*$ such that $f_M(q_i,\ vu,\ q_i) > 0$. The set of all reversible states of M are called the reversible part of M. It is denoted by $R(M)$. $R(M)$ is non empty, then $R(M)$ is a subautomaton of M. If each state of a fuzzy automaton M is reversible, then the fuzzy automaton M is called reversible fuzzy automaton [8].

A fuzzy automaton M is said to be a direct sum of its subautomata $M_\alpha,\ \alpha\ \in\ Y$, if $M = \cup_{\alpha\ \in\ Y}\ Q_\alpha$ and $Q_\alpha\ \cap\ Q_\beta = \phi$, for every $\alpha,\ \beta\ \in\ Y$ such that $\alpha \neq \beta$.

A subset I of a semigroup S is called an ideal if $SIS \subseteq I$ [8].

An equivalence relation R on Q in M is called a congruence relation if for all $q_i, q_j \in Q$ and $a \in \Sigma$, $q_i\ R\ q_j$ implies that, then there exists $q_l, q_k \in Q$ such that $f_M(q_i, a, q_l) > 0$, $f_M(q_j, a, q_k) > 0$ and $q_l\ R\ q_k$[1, 2].

Let M be a fuzzy automaton. The quotient fuzzy automaton determined by the congruence $\cong$ is a fuzzy automaton
$M/\cong = (Q/\cong, \Sigma, f_{M/\cong})$, where $Q/\cong = \{Q_i = [q_i]\}$ and
$f_{M/\cong}(Q_1, a, Q_2) = Min\,\{f_M(q_1, a, q_2) > 0\ /\ q_1\ \in\ Q_1,\ q_2\ \in\ Q_2\ \text{ and } a \in \Sigma\}$[10].

We say that two states $q_i, q_j\ \in\ Q$ are said to be mergeable or reducible if there exists a word $u \in \Sigma^*$ and $q_j \in Q$ such that $f_M(q_i, u, q_k) > 0 \Leftrightarrow f_M(q_j, u, q_k) > 0$ [9].

A state $q_j \in Q$ is called a trap of M if $f_M(q_j,\ u,\ q_j) > 0$, for every word $u \in \Sigma^*$ [9].

If M has exactly one trap, then M is called one-trap fuzzy automaton. The set of all traps of a fuzzy automaton M is denoted by $Tr(M)$ [9].

A fuzzy automaton M is called a trapped fuzzy automaton, for each $q_i \in Q$,if there exists a word $u \in \Sigma^*$ such that $f_M(q_i,\ u,\ q_j) > 0$, $\ q_j \in Tr(M)$ [9].

Example 2.1.

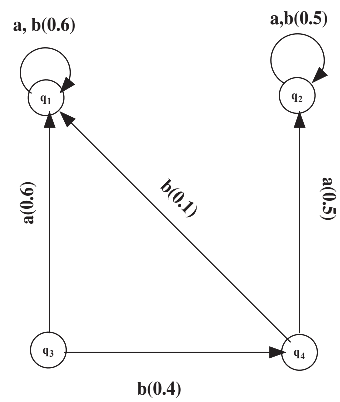


Fig - 2.1

In the above fuzzy automaton, the states q_1 and q_2 are traps. In this case, the above fuzzy automaton M is said to be a trapped fuzzy automaton. Since $f_M(q_i,\ u,\ q_1) > 0, f_M(q_i,\ u,\ q_2) > 0,\ q_1, q_2 \in Tr(M)$, for each $u \in \Sigma^*$ and $q_i \in Q$.

Let $M = (Q,\ \Sigma,\ f_M)$ be a fuzzy automaton. If M has a single neck, then M is called a trap-directable fuzzy automaton.

Example 2.2.

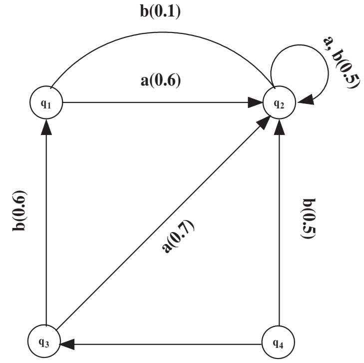


Fig - 2.2

In the above fuzzy automaton, there exists a word $bb \in \Sigma^*$ such that $f_M(q_i, bb, q_2) > 0$, for every $q_i \in Q$ and the state q_2 is a single neck. Hence, the above fuzzy automaton is a trap-directable fuzzy automaton.

Generalized directable fuzzy automaton 2.3 [11]**.**

Let $M = (Q,\ \Sigma,\ f_M)$ be a fuzzy automaton. M is called a generalized directable fuzzy automaton if for every $v \in \Sigma^*$ and $q_i \in Q$, there exists a word $u \in \Sigma^*$ and $q_j \in Q$ such that $f_M(q_i, uvu, q_j) > 0 \Leftrightarrow f_M(q_i, u, q_j) > 0$ and the word u is called generalized directing word of a fuzzy automaton M and the set of all generalized directing words of M are denoted by $GDW(M)$.

Example 2.4.

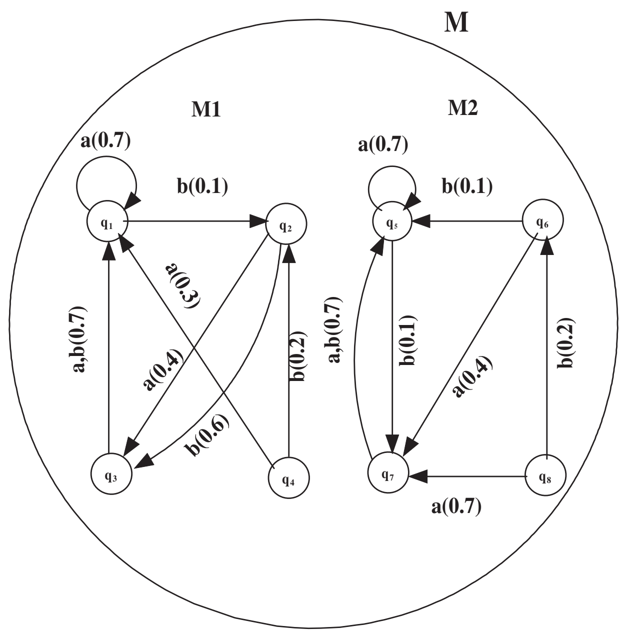


Fig - 2.3

In the above fuzzy automaton, for any $v \in \Sigma^*$, $\exists\, aa \in \Sigma^*$ such that $f_M(q_i, aavaa, q_j) > 0 \Leftrightarrow f_M(q_i, aa, q_j) > 0\ \forall q_i, q_j \in Q$. In that case, the word $aa \in \Sigma^*$ is a generalized directing word of M.

3. Least directing congruence on fuzzy automata

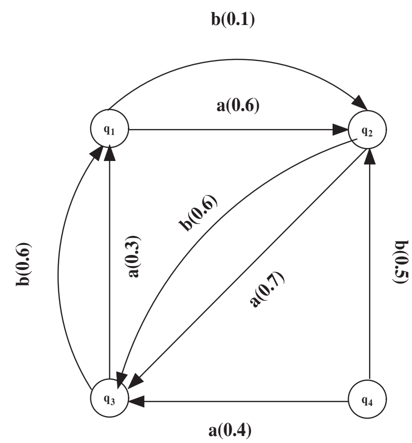
Nonmergeable Pair of a Fuzzy Automaton 3.1.

Let $M = (Q,\ \Sigma,\ f_M)$ be a fuzzy automaton. Two states q_i and q_j are said to be nonmergeable, if there is no $q_l \in Q$ such that

$$f_M(q_i, w, q_l) > 0 \Leftrightarrow f_M(q_j, w, q_l) > 0, \text{ for every } w \in \Sigma^*.$$

The set of nonmergeable pair is denoted by $M_{nmp} = \{\{q_i, q_j\} \ / \ q_i, q_j \ \in \ Q, \ \ q_i \neq q_j\}$.

Example 3.2.

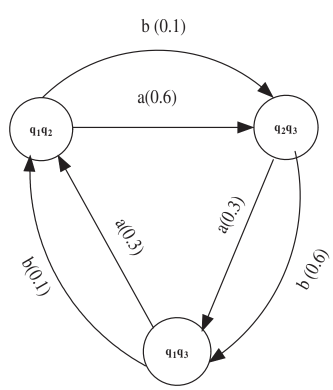


M

Nonmergeable pair of $M_{nmp} = \{\{q_1, q_2\}, \{q_2, q_3\}, \{q_1, q_3\}\}$.
We define a new fuzzy automaton M_1' by using nonmergeable pairs of M.
$M_1' = (Q_1', \ \Sigma, \ f_{M_{1'}})$, where,

$Q_1' = \{\{q_1, q_2\}, \ \{q_2, q_3\}, \ \{q_1, q_3\}\}, \ \ \Sigma = \{a, \ b\}$ and

$f_{M_{1'}}(\{q_i, q_j\}, \ a, \ \{q_k, q_l\}) \ = \ Min\{f_M(q_i, \ a, \ q_k), \ f_M(q_j, \ a, \ q_l)\} \ > \ 0$, for some $q_k, \ q_l \ \in \ Q$ and for every $a \ \in \ \Sigma$.



M1'

Remark 3.3. Let $M = (Q,\ \Sigma,\ f_M)$ be a fuzzy automaton. If $\{q_i, q_j\}$ is a nonmergeable pair, then $\{q_j, q_i\}$ is also a nonmergeable pair.

Directing, trapping and trap-directing congruence 3.4.

Let $M = (Q,\ \Sigma,\ f_M)$ be a fuzzy automaton. The set of all equivalence relations on a set Q is denoted by $Eq(Q)$. Let $\delta_M \in Eq(Q)$. If for any two states $q_i, q_j \in Q$ are called δ_M-Mergeable, then there exists $(q_k, q_l) \in \delta_M$ such that $f_M(q_i, w, q_k) > 0$ and $f_M(q_j, w, q_l) > 0$, for some $w \in \Sigma^*$.

Let ρ be the congruence relation on the states set Q in M. If ρ is called directing, then the quotient fuzzy automaton M/ρ is a directable fuzzy automaton.

If ρ is called trapping congruence, then the quotient fuzzy automaton M/ρ is a trapped fuzzy automaton.

If ρ is called trap-directing, then the quotient fuzzy automaton M/ρ is a trap-directable fuzzy automaton.

Compatible relation 3.5.

A relation R on Q is said to be compatible if $(q_i, q_j) \in R$, then there exists $(q_k, q_l) \in R$ such that $f_M(q_i, a, q_k) > 0$ and $f_M(q_j, a, q_l) > 0$, for some $a \in \Sigma$.

Least directing congruence on fuzzy automata

Theorem 3.1. *Let $M = (Q,\ \Sigma,\ f_M)$ be a fuzzy automaton and let δ_M be the transitive closure of the relation ρ_M defined on Q in M by*

$(q_i, q_j) \in \rho_M \Leftrightarrow q_i = q_j$

or

$\{\{q_i, q_j\} / (\forall v \in \Sigma^*)(\exists u \in \Sigma^*) \text{ such that } f_M(q_i, vu, q_i) > 0, f_M(q_j, vu, q_j) > 0\}.$

Then δ_M is the least directing congruence on the states set Q in M.

Proof. Clearly ρ_M is reflexive and symmetric. Let $(q_i, q_j) \in \rho_M$ and $a \in \Sigma$. Then for each $av_1 \in \Sigma^*$, there exists $u_1 \in \Sigma^*$ such that

$$f_M(q_i, av_1u_1, q_i) = Min_{q_k \in Q}\{f_M(q_i, a, q_k), f_M(q_k, v_1u_1, q_i)\} > 0$$

and

$$f_M(q_j, av_1u_1, q_j) = Min_{q_l \in Q}\{f_M(q_j, a, q_l), f_M(q_l, v_1u_1, q_j)\} > 0.$$

Since $f_M(q_i, a, q_k) > 0$ and $f_M(q_j, a, q_l) > 0$, we have $f_M(q_k, v_1u_1a, q_k) > 0$ and $f_M(q_l, v_1u_1a, q_l) > 0$. Thus, $(q_k, q_l) \in \rho_M$. So, ρ_M is a compatible relation on Q in M. Being the transitive closure of a reflexive, symmetric and compatible relation,δ_M has the same properties and is transitive. Hence it is a congruence relation on M.

To prove δ_M is a directing congruence, consider any $q_i, q_j \in Q$.

Suppose there exists a $q_m \in Q$ such that $f_M(q_i, w, q_m) > 0$ and $f_M(q_j, w, q_m) > 0$, for some $w \in \Sigma^*$. Then $(q_m, q_m) \in \delta_M$. In this case, q_i and q_j are δ_M-mergeable.

Suppose now there is no $q_n \in Q$ such that $f_M(q_i, w, q_n) > 0$ and $f_M(q_j, w, q_n) > 0$, for every $w \in \Sigma^*$. Clearly $\{q_i, q_j\}$ is a state of the nonmergeable pair of a fuzzy automaton M_{nmp} of M. By proof of the Theorem 3.1 [11], there exists $w \in \Sigma^*$

such that $f_M(q_i, w, q_r) > 0$ and $f_M(q_j, w, q_s) > 0$, $q_r \neq q_s$. Thus, $\{q_r, q_s\}$ is a reversible state of M_{nmp}. That is, $f_M(q_r, vu, q_r) > 0$ and $f_M(q_s, vu, q_s) > 0$. So, $(q_r, q_s) \in \rho_M \subseteq \delta_M$. Hence, all pairs of $q_i, q_j \in Q$ are δ_M-mergeable. Therefore, by Theorem 4.2 [12], δ_M is a directing congruence on M.

It remains to prove that δ_M is contained in an arbitrary directing congruence η on M.

Let $(q_i, q_j) \in \rho_M$. Then by the hypothesis and Theorem 4.2 [12], q_i and q_j are η-mergeable. That is, there exists a word $v \in \Sigma^*$ such that

$$f_M(q_i, v, q_t) > 0 \text{ and } f_M(q_j, v, q_y) > 0 \text{ and } (q_t, q_y) \in \eta.$$

On the other hand, $(q_i, q_j) \in \rho_M$ implies that for each $v \in \Sigma^*$, there exists $u \in \Sigma^*$ such that $\{\{q_i, q_j\} / f_M(q_i, vu, q_i) > 0, f_M(q_j, vu, q_j) > 0\}$, where

$$f_M(q_i, vu, q_i) = Min_{q_t \in Q} \{f_M(q_i, v, q_t), f_M(q_t, u, q_i)\} > 0.$$

Then

$$f_M(q_t, u, q_i) > 0$$

and

$$f_M(q_j, vu, q_j) = Max\left\{Min_{q_y \in Q} \{f_M(q_j, v, q_y), f_M(q_y, u, q_j)\}\right\} > 0.$$

Thus $f_M(q_y, u, q_j) > 0$. Since $(q_t, q_y) \in \eta$, by the property of congruence, we have $(q_i, q_j) \in \eta$. So, $\rho_M \subseteq \delta_M \subseteq \eta$. Hence, δ_M is the least directing congruence on Q in M. □

Algorithm for finding the least directing congruence on fuzzy automata 3.6.

Let $M = (Q, \Sigma, f_M)$ be a fuzzy automaton.

Step1: Compute Δ_Q in M, where Δ_Q is an identical relation on states set Q of M, i.e., $\Delta_Q = \{(q_i, q_i)/q_i \in Q\}$.

Step2: Find all nonmergeable pairs Q in M. That is, M_{nmp}.

Step3: Compute $\rho_M = \Delta_Q \cup M_{nmp}$.

Step4: Find the transitive closure of ρ_M which is called δ_M. δ_M is called the least directing congruence on the states set Q in M.

Example 3.7.

Step1: $\Delta_Q = \{(q_1, q_1), (q_2, q_2), (q_3, q_3), (q_4, q_4)\}$.

Step2: The nonmergeable pairs

$M_{nmp} = \{\ \{q_1, q_2\}, \{q_2, q_3\}, \{q_3, q_1\}, \{q_2, q_1\}, \{q_3, q_2\}, \{q_1, q_3\}\ \}$.

Step3: $\rho_M = \Delta_Q \cup M_{nmp}$

$$\rho_M = \{(q_1, q_1), (q_2, q_2), (q_3, q_3), (q_4, q_4), (q_1, q_2), (q_2, q_3), (q_3, q_1), (q_2, q_1), (q_3, q_2), (q_1, q_3)\}.$$

Step4:

$$\delta_M = \{(q_1, q_1), (q_2, q_2), (q_3, q_3), (q_4, q_4), (q_1, q_2), (q_2, q_3), (q_3, q_1), (q_2, q_1), (q_3, q_2), (q_1, q_3)\}.$$

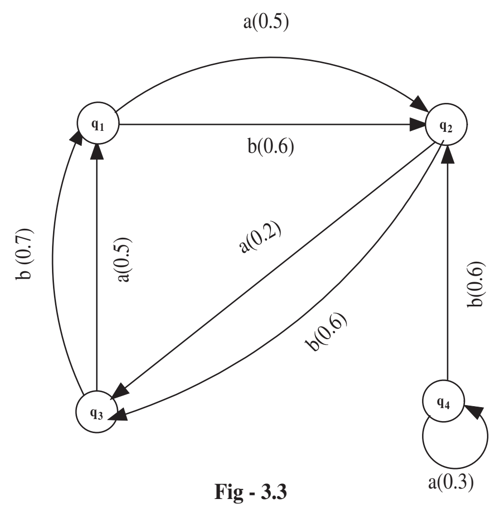


Fig - 3.3

This δ_M is the least directing congruence on the above fuzzy automaton M.

4. Least directing congruence on generalized directable fuzzy automata

Language associate by a state q_i 4.1.

Let $M = (Q,\ \Sigma,\ f_M)$ be a fuzzy automaton. Then to each state $q_i \in Q$, we can associate a language $G(q_i) \subseteq \Sigma^*$ and defined as follows:

$$G(q_i) = \{u \in \Sigma^* / (\forall v \in \Sigma^*), f_M(q_i, vu, q_i) > 0\}.$$

Remark 4.2. Let $M = (Q,\ \Sigma,\ f_M)$ be a fuzzy automaton. Let $q_i,\ q_j\ \in\ Q$. Then G_Q is defined as follows:

$$G_Q = \{(q_i, q_j)\ /\ G(q_i) \cap G(q_j) \neq \phi\}.$$

Lemma 4.1. *Let $M = (Q,\ \Sigma,\ f_M)$ be a fuzzy automaton and $q_i \in Q$. Then $G(q_i) \neq \phi$ if and only if $\langle q_i \rangle$ is a strongly directable fuzzy automaton. In that case the following conditions hold:*

(1) $G(q_i) = \{u \in \Sigma^* / q_i \ \ is\ a\ u - neck \ \ of \ \ \langle q_i \rangle\}$.
(2) $G(q_i)$ *is a left ideal of* Σ^*.
(3) $G(q_i)w \subseteq G(q_j)$ *such that* $f_M(q_i, w, q_j) > 0$, *for every* $w \in \Sigma^*$.

Proof. (1) Let $q_i \in Q$. If $G(q_i) \neq \phi$, then for every $v \in \Sigma^*$, there exists $u \in \Sigma^*$ such that $f_M(q_i, vu, q_i) > 0$. On the one hand,

$$f_M(q_i, vu, q_i) = Min_{q_k\ \in\ \langle q_i \rangle}\{f_M(q_i, v, q_k), f_M(q_k, u, q_i)\} > 0.$$

Then, this implies that $f_M(q_k, u, q_i) > 0$ for every $q_k \in \langle q_i \rangle$. Thus, $\langle q_i \rangle$ is a directable fuzzy automaton and u is a directing word. On the other hand, q_i is reversible, we can conclude that $\langle q_i \rangle$ is strongly connected. So, $\langle q_i \rangle$ is strongly directable fuzzy automaton.

Conversely, let $\langle q_i \rangle$ be strongly directable. Then q_i is u-neck of $\langle q_i \rangle$ for some $u \in \Sigma^*$. Thus $u \in G(q_i)$

(2) Let $w \in \Sigma^*$. Then for each vw with $v \in \Sigma^*$, there exist $u \in G(q_i)$ such that $f_M(q_i, vwu, q_i) > 0$. Thus $wu \in G(q_i)$. So $G(q_i)$ is a left ideal of Σ^*.

(3) Consider arbitrary $u \in G(q_i)$ and $w \in \Sigma^*$. Let $f_M(q_i, w, q_j) > 0$. Then for all $wv \in \Sigma^*$, there exists $u \in \Sigma^*$ such that $f_M(q_i, wvu, q_i) > 0$. Now,

$$f_M(q_i, wvuw, q_j) = Min_{q_j \in Q} \{f_M(q_i, w, q_j), f_M(q_j, vuw, q_j)\} > 0.$$

Thus $f_M(q_j, vuw, q_j) > 0$. So $uw \in G(q_j)$. Hence $G(q_i)w \subseteq G(q_j)$. □

Theorem 4.2. *Let $M = (Q, \Sigma, f_M)$ be an arbitrary generalized directable fuzzy automaton and let υ_M be the transitive closure of the relation ν_M defined on Q in M by $(q_i, q_j) \in \nu_M \Leftrightarrow q_i = q_j$ or $G(q_i) \cap G(q_j) \neq \phi$. Then υ_M is the least directing congruence on the states set Q in M.*

Proof. Clearly ν_M is reflexive and symmetric. Let $(q_i, q_j) \in \nu_M$ and $a \in \Sigma$. Then for each $av_1 \in \Sigma^*$, there exists $u_1 \in \Sigma^*$ such that

$$f_M(q_i, av_1u_1, q_i) = Min_{q_k \in Q} \{f_M(q_i, a, q_k), f_M(q_k, v_1u_1, q_i)\} > 0$$

and

$$f_M(q_j, av_1u_1, q_j) = Min_{q_l \in Q} \{f_M(q_j, a, q_l), f_M(q_l, v_1u_1, q_j)\} > 0.$$

Since $f_M(q_i, a, q_k) > 0$ and $f_M(q_j, a, q_l) > 0$, we have $f_M(q_k, v_1u_1a, q_k) > 0$ and $f_M(q_l, vu_1a, q_l) > 0$. Thus, $(q_k, q_l) \in \nu_M$. So ν_M is a compatible relation on M. Being the transitive closure of a reflexive, symmetric and compatible relation, υ_M has the same properties and is transitive. Hence it is a υ_M is a congruence relation on M.

To prove that υ_M is a directing congruence on M.

Consider an arbitrary $u \in GDW(M)$ and $q_i, q_j \in Q$. Since $u \in GDW(M)$, we have

$$f_M(q_i, uvu, q_k) > 0 \Leftrightarrow f_M(q_i, u, q_k) > 0, \text{ for some } q_k \in Q$$

and

$$f_M(q_j, uvu, q_l) > 0 \Leftrightarrow f_M(q_j, u, q_l) > 0, \text{ for some } q_l \in Q.$$

Now,

$$f_M(q_i, uvu, q_k) > 0 \Leftrightarrow Min_{q_k \in Q} \{f_M(q_i, u, q_k), f_M(q_k, vu, q_k)\} > 0.$$

Then $f_M(q_k, vu, q_k) > 0$. Thus $u \in G(q_k)$.
Also,

$$f_M(q_j, uvu, q_l) > 0 \Leftrightarrow Min_{q_l \in Q} \{f_M(q_j, u, q_l), f_M(q_l, vu, q_l)\} > 0.$$

Then $f_M(q_l, vu, q_l) > 0$. Thus $u \in G(q_l)$. So, $u \in G(q_k) \cap G(q_l)$. Hence, $(q_k, q_l) \in \nu_M \subseteq \upsilon_M$. Therefore, υ_M is a directing congruence on Q in M.

It remains to prove that υ_M is contained in an arbitrary directing congruence θ on M. Let $(q_i, q_j) \in \nu_M$ and $q_i \neq q_j$. Then there exist $u \in G(q_i) \cap G(q_j)$. On the other hand, for an arbitrary $v \in \Sigma^*$ and $(q_i, q_j) \in Q$, we have $(q_k, q_l) \in \theta$ such that $f_M(q_i, v, q_k) > 0$ and $f_M(q_j, v, q_l) > 0$. Now,$u \in G(q_i) \cap G(q_j)$ implies that

$$f_M(q_i, vu, q_i) = Min_{q_k \in Q} \{f_M(q_i, v, q_k), f_M(q_k, u, q_i)\} > 0$$

and

$$f_M(q_j, vu, q_j) = Min_{q_l \in Q} \{f_M(q_j, v, q_l), f_M(q_l, u, q_j)\} > 0$$

implies that $f_M(q_k, u, q_i) > 0$ and $f_M(q_l, u, q_j) > 0$. By directing congruence of θ, $(q_i, q_j) \in \theta$. Thus, $\nu_M \subseteq \theta$. So, $\upsilon_M \subseteq \theta$. Hence, υ_M is the least directing congruence on M. □

Algorithm for finding the least directing congruence on generalized directable fuzzy automata 4.2.

Let $M = (Q, \Sigma, f_M)$ be a generalized directable fuzzy automaton.

Step1: Compute Δ_Q in M, where Δ_Q is an identical relation on states set Q of M.

Step2: Find $G_Q = \{(q_i, q_j)/G(q_i) \cap G(q_j) \neq \phi\}$.

Step3: Compute $\nu_M = \Delta_Q \cup G_Q$.

Step4: Find the transitive closure of ν_M which is called υ_M.

υ_M is called the least directing congruence on the states set Q in M.

Consider the Example 2.4.

Step1: $\Delta_Q = \{(q_1, q_1), (q_2, q_2), (q_3, q_3), (q_4, q_4), (q_5, q_5), (q_6, q_6), (q_7, q_7), (q_8, q_8)\}$.

Step2:

$$G_Q = \{(q_1, q_2), (q_1, q_3), (q_2, q_3), (q_2, q_1), (q_3, q_1), (q_3, q_2), (q_5, q_6), (q_5, q_7), (q_6, q_7), (q_6, q_5), (q_7, q_5), (q_7, q_6)\}.$$

Step3: $\nu_M = \Delta_Q \cup G_Q$.

$$\nu_M = \{(q_1, q_1), (q_2, q_2), (q_3, q_3), (q_4, q_4), (q_5, q_5), (q_6, q_6), (q_7, q_7), (q_8, q_8), (q_1, q_2), (q_1, q_3), (q_2, q_3), (q_2, q_1), (q_3, q_1), (q_3, q_2), (q_5, q_6), (q_5, q_7), (q_6, q_7), (q_6, q_5), (q_7, q_5), (q_7, q_6)\}.$$

Step4:

$$\upsilon_M = \{(q_1, q_1), (q_2, q_2), (q_3, q_3), (q_4, q_4), (q_5, q_5), (q_6, q_6), (q_7, q_7), (q_8, q_8), (q_1, q_2), (q_1, q_3), (q_2, q_3), (q_2, q_1), (q_3, q_1), (q_3, q_2), (q_5, q_6), (q_5, q_7), (q_6, q_7), (q_6, q_5), (q_7, q_5), (q_7, q_6)\}.$$

This υ_M is called the least directing congruence on the states set Q in M.

Time complexity 4.3.

The time complexity for finding the least directing congruence on fuzzy automata and generalized directable fuzzy automata with n states and m input symbols are

$O(mn^2 + n^3)$.

Relation between the least directing congruence and the least trapping congruence 4.4.

(1) Let $M = (Q,\ \Sigma,\ f_M)$ be a generalized directable fuzzy automaton. Then the relation τ_M defined on Q in M by

$$(q_i, q_j) \in \tau_M \Leftrightarrow q_i = q_j$$

or

$$(\forall u, v \in \Sigma^*)(\exists u_1, v_1 \in \Sigma^*) \text{ such that } f_M(q_i, uu_1, q_j) > 0 \text{ and } f_M(q_j, vv_1, q_i) > 0.$$

Then τ_M is the least trapping congruence on Q in M. In other words, $(q_i, q_j) \in \tau_M$ if and only if either $q_i = q_j$ or q_i and q_j belong to the same strongly connected subautomaton of M.

(2) Let $M = (Q,\ \Sigma,\ f_M)$ be a generalized directable fuzzy automaton. Then the relation γ_M defined on Q in M by

$$(q_i, q_j) \in \gamma_M \Leftrightarrow q_i = q_j$$

or

$$(\forall u, v \in \Sigma^*)(\exists u_1, v_1 \in \Sigma^*) \text{ such that } f_M(q_i, uu_1, q_i) > 0 \text{ and } f_M(q_j, vv_1, q_j) > 0.$$

Then γ_M is the least trap-directing congruence on Q in M. Equivalently, $(q_i, q_j) \in \gamma_M$ if and only if either $q_i = q_j$ or $q_i, q_j \in R(M)$.

Theorem 4.3. *Let $M = (Q, \Sigma, f_M)$ be a generalized directable fuzzy automaton. Then $\upsilon_M \circ \tau_M = \tau_M \circ \upsilon_M = \gamma_M$.*

Proof. Since $\upsilon_M \subseteq \gamma_M$ and $\tau_M \subseteq \gamma_M$, then $\upsilon_M \circ \tau_M \subseteq \gamma_M$ and $\tau_M \circ \upsilon_M \subseteq \gamma_M$.

It remains to prove the opposite inclusions.

Now consider an arbitrary pair $(q_i, q_j) \in \gamma_M$. If $q_i = q_j$, then clearly $(q_i, q_j) \in \upsilon_M \circ \tau_M$ and $(q_i, q_j) \in \tau_M \circ \upsilon_M$. Assume that $q_i \neq q_j$. Then $q_i, q_j \in R(M)$. Thus by proof of the Theorem 3.3 [11],$\langle q_i \rangle$ and $\langle q_j \rangle$ are strongly directable fuzzy automata, i.e. $G(q_i) \neq \phi$ and $G(q_j) \neq \phi$.

Take an arbitrary $u \in G(q_i)$ and $v \in G(q_j)$. Since $u \in G(q_i)$, $f_M(q_i, w_1u, q_i) > 0$, for some $w_1 \in \Sigma^*$. Then by (2) and (3) of Lemma 4.1, we have that

$$uv \in \Sigma^* G(q_j) \subseteq G(q_j) \text{ and } uv \in G(q_i)v \subseteq G(q_k), \tag{4.1}$$

where $f_M(q_i, v, q_k) > 0$. Thus by (4.1), $f_M(q_k, w_2uv, q_k) > 0$, for some $w_2 \in \Sigma^*$. On one hand,

$$f_M(q_i, w_1uv, q_k) > 0. \tag{4.2}$$

On the other hand, $f_M(q_i, vw_2u, q_i) = Min_{q_k \in Q} \{f_M(q_i, v, q_k), f_M(q_k, w_2u, q_i)\} > 0$. Then,

$$f_M(q_k, w_2u, q_i) > 0. \tag{4.3}$$

From (4.2) and (4.3),

$$(q_i, q_k) \in \tau_M. \tag{4.4}$$

From (4.1),$uv \in G(q_j) \cap G(q_k)$. Thus,

$$(q_k, q_j) \in \nu_M \subseteq \upsilon_M. \tag{4.5}$$

From (4.4) and (4.5), $(q_i, q_j) \in \tau_M \circ \upsilon_M$.

Now take an arbitrary $u \in G(q_i)$ and $v \in G(q_j)$. Since $v \in G(q_j)$, $f_M(q_j, w_3v, q_j) > 0$, for any $w_3 \in \Sigma^*$. Then by (2) and (3) of Lemma 4.1, we have that

$$vu \in \Sigma^* G(q_i) \subseteq G(q_i) \text{ and } vu \in G(q_j)u \subseteq G(q_l), \tag{4.6}$$

where $f_M(q_j, u, q_l) > 0$. From (4.6), $vu \in G(q_i) \cap G(q_l)$. Thus,

$$(q_i, q_l) \in \upsilon_M. \tag{4.7}$$

Also, from (4.6), since $vu \in G(q_l)$, $f_M(q_l, w_4vu, q_l) > 0$, for some $w_4 \in \Sigma^*$.

Now,

$$f_M(q_j, w_3vu, q_l) > 0. \tag{4.8}$$

Then

$$f_M(q_j, uw_4v, q_j) = Min_{q_l \in Q}\{f_M(q_j, u, q_l), f_M(q_l, w_4v, q_j)\} > 0.$$

Thus

$$f_M(q_l, w_4v, q_j) > 0. \tag{4.9}$$

So, from (4.8) and (4.9),

$$(q_l, q_j) \in \tau_M. \tag{4.10}$$

Hence, from (4.7) and (4.10), we have $(q_i, q_j) \in \upsilon_M \circ \tau_M$. □

5. Conclusion

The main aim this paper is to find the least directing congruence on a fuzzy automaton and a generalized directable fuzzy automaton. We introduce directing congruence, trapping congruence, trap-directing congruence of a fuzzy automaton and a necessary and sufficient condition for congruence relation of a fuzzy automaton to be directing. Finally, we provide an algorithm to find the least directing congruence on a fuzzy automaton and a generalized directable fuzzy automaton.

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V. Karthikeyan (vkarthikau@gmail.com)
Department of Mathematics, Annamalai University, Annamalai Nagar, Chidambaram, Tamilnadu, India - 608002

M. Rajasekar (mdivraj1962@gmail.com)
Mathematics Section, Faculty of Engineering and Technology, Annamalai University, Annamalai Nagar, Chidambaram, Tamilnadu, India - 608002